Nilpotent orbits in semisimple Lie algebras

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Lemma

Any $X \in \mathfrak{sl}_2(\mathbb{C})$ is similar to either $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{C}$.

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- The proof follows from the Jordan normal forms of X.

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Definition

Let V be a finite-dimensional complex vector space and let $X \in End(V)$.

- X is nilpotent if $X^r = 0$ for some $r > 0$.
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- For $\mathfrak{sl}_2(\mathbb{C})$, the lemma gives a complete description of conjugacy classes of nilpotent and semisimple elements.

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- $GL_2(\mathbb{C}), SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$-conjugacy classes of $\mathfrak{sl}_2(\mathbb{C})$ coincide because $AXA^{-1} = \left(\frac{A}{\sqrt{det(A)}}\right) X \left(\frac{A}{\sqrt{det(A)}}\right)^{-1}$.

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- More generally, the conjugacy classes of $\mathfrak{sl}_n(\mathbb{C})$ under the actions of $GL_n(\mathbb{C}), SL_n(\mathbb{C})$ and $PSL_n(\mathbb{C})$ coincide, which can be seen by replacing $\sqrt{det(A)}$ by n^{th} root of $det(A)$.

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Problem

What are the conjugacy classes of nilpotent (and semisimple) elements of $\mathfrak{g}$ under the adjoint action of G_{ad}?

- The set of semisimple orbits in $\mathfrak{sl}_2(\mathbb{C})$ is in bijective correspondence with $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\} / S_2$.

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Theorem

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and W the associated Weyl group. Then the set of semisimple orbits is in bijective correspondence with $\mathfrak{h}/W$.

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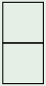
Example

In $\mathfrak{sl}_2(\mathbb{C})$, there are two nilpotent orbits given by

-  corresponding to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
-  corresponding to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

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Type	$\mathfrak{g}$	G_{ad}	W
A_n	$\mathfrak{sl}_{n+1}(\mathbb{C})$	$PSL_{n+1}(\mathbb{C})$	S_{n+1}
B_n	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$SO_{2n+1}(\mathbb{C})$	$(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$
C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	$PSp_{2n}(\mathbb{C})$	$(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$
D_n	$\mathfrak{so}_{2n}(\mathbb{C})$	$PSO_{2n}(\mathbb{C})$	$(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$

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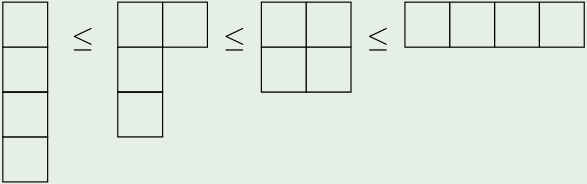
- Nilpotent orbits in $\mathfrak{so}_{2n}(\mathbb{C})$ are in one-to-one correspondence with the set of partitions of $2n$ in which even parts occur with even multiplicity except that "very even" partitions correspond to two nilpotent orbits.

Example

In $\mathfrak{sp}_4(\mathbb{C})$, the nilpotent orbits are ordered as follows:



$[1,1,1,1] \quad [2,1,1] \quad [2,2] \quad [4]$

$\mathit{dim} = 0 \quad 4 \quad 6 \quad 8$

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Example

The real nilpotent orbits of $\mathfrak{sl}_2(\mathbb{R})$ are parametrized by the following partitions of 2:

- $[2; I]$ and $[2; II]$ corresponding to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$
- $[1, 1]$ corresponding to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

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- $\boxed{+}\boxed{-}$, $\boxed{-}\boxed{+}$ corresponding to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$
- $\begin{array}{|c|}\hline + \\ \hline - \\ \hline \end{array}$ corresponding to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Example

Here are some signed Young diagrams that parametrize some real nilpotent orbits of $\mathfrak{sp}_4(\mathbb{R})$:

- $\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline \end{array}, \begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline \end{array}, \begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline \end{array}$
- $\begin{array}{|c|}\hline + \\ \hline - \\ \hline + \\ \hline - \\ \hline \end{array}$

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- By identifying $T^*(G)$ with $G \times \mathfrak{g}^*$, and projecting to $\mathfrak{g}^*$, we get a subset of $\mathfrak{g}^*$.
- $\mathfrak{g}$ can be identified with $\mathfrak{g}^*$.

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- The description of the wave front set as a nilpotent orbit is a combinatorial invariant.
- Application: For an induced module for G (from a parabolic subgroup P) with a fixed infinitesimal character, the wave front set imposes a constraint on the factors in the composition series.

Example

$G = GL_{2n}(\mathbb{R})$, P is the maximal parabolic subgroup whose Levi factor is $GL_n(\mathbb{R}) \times GL_n(\mathbb{R})$. The wave front set of the induced module $Ind_P^G(\chi)$ from one dimensional character χ corresponds to the closure of the nilpotent orbit $[2, 2, \cdots, 2]$.

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