Root System Basics

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Vertex Operator Algebras

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We say $\mathfrak{g}$ is a *Lie algebra* if it is a vector space with bilinear multiplication $[\cdot, \cdot]$ that satisfies:

- $[x, x] = 0$ for all $x$ (which implies $[x, y] = -[y, x]$)
- $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$.

For $x \in \mathfrak{g}$, define $\text{ad}_x : \mathfrak{g} \to \mathfrak{g}$ by $g \mapsto [x, g]$.

The second rule should be thought of as:

$$\text{ad}_{[x,y]} = \text{ad}_x \text{ad}_y - \text{ad}_y \text{ad}_x$$
The prototypical Lie algebra is $\text{End}(V)$ with $[A, B] = AB - BA$. Lie algebra homomorphisms are defined as you would expect. A representation is a Lie algebra homomorphism:

$$\phi : g \to \text{End}(V)$$

for some vector space $V$. $V$ is also said to be a $g$ module (corresponding to this representation):

$$x \cdot v = \phi(x)v$$

Putting these together, $V$ is a $g$ module if

$$x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, y] \cdot v$$
Any Lie algebra $\mathfrak{g}$ is a module over itself via:

$$x \cdot g = [x, g]$$

since

$$x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, [y, v]] - [y, [x, v]] = [[x, y], v]$$
Any Lie algebra has two trivial ideals - 0 and itself. The uninteresting one-dimensional Lie algebra that maps all brackets to 0 technically only has these two ideals; thus we say Lie algebra is *simple* if it has non-trivial ideals and is dimension $> 1$. From here on out, we always assume vector spaces and Lie algebras are over $\mathbb{C}$. 
Last week, we identified an important simple Lie algebra \( \mathfrak{sl}(2) \) - the subspace of Endomorphisms on \( \mathbb{C}^2 \) with trace 0. This is the smallest simple Lie algebra (over \( \mathbb{C} \); on other fields it is tied for the smallest). Under some choice of basis of \( \mathbb{C} \), we set

\[
\begin{align*}
    h &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
    e &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
    f &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\end{align*}
\]

and have relations

\[
[h, e] = 2e, [h, f] = -2f, [e, f] = h
\]

(recall \([x, x] = 0\) and \([x, y] = -[y, x]\) so this gives all basis relations). Note \(ad_h\) acts diagonally on \( \mathfrak{sl}(2) \). Thus we say \( h \) is a semisimple element.
We pointed out that all finite-dimensional modules of simple Lie algebras can be written as the direct sum of irreducible modules and that semisimple elements in $\mathfrak{g}$ act diagonally on such modules. We also found that there is exactly one irreducible $\mathfrak{sl}(2)$ module $V_{k-1}$ of each dimension $k > 0$, and it has the following properties:

- $V_{k-1}$ has a basis $\{v_{k-1}, v_{k-3}, \ldots, v_{-(k-1)}\}$
- $h \cdot v_i = iv_i$.
- $f \cdot v_i = v_{i-2}$ (or 0 if $i = -k - 1$)
- $e \cdot v_i = \frac{k+k^2/2+i-i^2/2}{2} v_{i+2}$
We also introduced an important bilinear form known as the Killing form \((x, y) = \text{tr}(ad_x ad_y)\) on \(g\). With just algebra manipulation, one can show:

- \((x, y) = (y, x)\) (symmetric)
- \((x, [y, z]) = ([x, y], z)\) (\(g\)-invariant)

but what’s a bit harder to show (and very important) is that if \(g\) is simple, the Killing form is non-degenerate.
Let \( g \) be a simple Lie algebra. One can show that \( g \) must contain some semisimple elements; take let \( \mathfrak{h} \) be a maximal subspace of commuting semisimple elements. This is called a Cartan Subalgebra.

Commuting diagonal operators have the same eigenspaces (with possibly different eigenvalues):

\[
BAv_{\lambda,B} = ABv_{\lambda,B} = A\lambda v_{\lambda,B}
\]

So \( A \) preserves eigenspaces of \( B \) and vice versa. Thus \( A \) has eigenspaces in the eigenspaces of \( B \) and vice versa, so their eigenspaces are the same.
With that being said, write

\[ \mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda \]

where each \( \mathfrak{g}_\lambda \) is an eigenspace for all \( \text{ad}_h \) and \( \text{ad}_h', h \in \mathfrak{h} \) has eigenvalue \( \lambda(h) \) on this space.
Note $\mathfrak{h} \subset \mathfrak{g}_0$ since $\mathfrak{h}$ is abelian. One can show we actually have $\mathfrak{h} = \mathfrak{g}_0$.

Let $\Phi \subset \mathfrak{h}^*$ be the set of $\lambda \neq 0$ for which $\mathfrak{g}_\lambda \neq 0$. These are called the roots. So in this notation:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$
Lemma

\( \Phi \) spans \( h^* \)

Proof.

Otherwise there is some \( h \in h \) for which \( \Phi(h) = 0 \). Then for all \( \alpha \in \Phi \), \( x_\alpha \in g_\alpha \) we have

\[
[h, x_\alpha] = \alpha(h)x_\alpha = 0
\]

and \( [h, h] \subset [h, h] = 0 \). So \( h \) spans a 1-dimensional ideal of \( g \), but \( g \) is simple (also can't happen in semisimple). \( \square \)
Lemma

\[[g_\alpha, g_\beta] \subseteq g_{\alpha + \beta}\]

Proof.
Take \(x_\alpha \in g_\alpha, x_\beta \in g_\beta\).

\[
[h, [x_\alpha, x_\beta]] = [[h, x_\alpha], x_\beta] + [x_\alpha, [h, x_\beta]]
\]

\[
= \alpha(h)[x_\alpha, x_\beta] + \beta(h)[x_\alpha, x_\beta] = (\alpha + \beta)(h)[x_\alpha, x_\beta]
\]
Lemma

\[(g_\alpha, g_\beta) = 0 \text{ if } \alpha \neq -\beta\]

Proof.

Take an \( h \) for which \( \alpha \) and \(-\beta\) disagree. Take \( x_\alpha \in g_\alpha, x_\beta \in g_\beta \).

\[([x_\alpha, h], x_\beta) = (x_\alpha, [h, x_\beta])\]

by \( g \) associativity of the killing form. Since \([x_\alpha, h] = -[h, x_\alpha] = -\alpha(h)x_\alpha \) and \([h, x_\beta] = \beta(h)x_\beta \), this is equivalent to

\[-\alpha(h)(x_\alpha, x_\beta) = \beta(h)(x_\alpha, x_\beta)\]

By our assumption on \( h \), this forces \((x_\alpha, x_\beta) = 0\). \(\square\)
By the non-degeneracy of $(\cdot, \cdot)$, this forces $g_\alpha$ and $g_{-\alpha}$ to pair non-degenerately and $(\cdot, \cdot)$ to be non-degenerate on $\mathfrak{h}$. In particular, $\alpha \in \Phi \iff -\alpha \in \Phi$.

This non-degeneracy on $\mathfrak{h}$ allows us to naturally pair $\mathfrak{h}$ with $\mathfrak{h}^*$ (isomorphically) via

$$h \rightarrow (h, \cdot)$$

For $\alpha \in \Phi$, let $t_\alpha$ be the corresponding element of $h$ in this association (so $(t_\alpha, h) = \alpha(h)$).

We can also lift $(\cdot, \cdot)$ to a form on $\mathfrak{h}^*$ via $(\alpha, \beta) = (t_\alpha, t_\beta)$
Lemma

For $e_\alpha \in g_\alpha, f_\alpha \in g_{-\alpha}$, we have

$$[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha) t_\alpha$$

Proof.

$$(h, [e_\alpha, f_\alpha]) = ([h, e_\alpha], f_\alpha) = \alpha(h)(e_\alpha, f_\alpha) = (h, (e_\alpha, f_\alpha) t_\alpha)$$
Let \( h_\alpha = \frac{2t_\alpha}{(t_\alpha, t_\alpha)} \) (one can show this denominator isn’t 0). Choose \( e_\alpha \) arbitrarily and choose \( f_\alpha \) such that \((e_\alpha, f_\alpha) = \frac{2}{(t_\alpha, t_\alpha)}\)

Then from this, we can see that \( e_\alpha, h_\alpha, f_\alpha \) forms an \( \mathfrak{sl}(2) \) (with the same relations as \( e, h, f \) from earlier):

\[
[e_\alpha, f_\alpha] = \frac{2t_\alpha}{(t_\alpha, t_\alpha)} = h_\alpha
\]

\[
[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = (t_\alpha, h_\alpha)e_\alpha = \frac{2(t_\alpha, t_\alpha)}{(t_\alpha, t_\alpha)}e_\alpha = 2e_\alpha
\]

(note we’ve shown here that \( \alpha(h_\alpha) = 2 \)).

\[
[h_\alpha, f_\alpha] = -\alpha(h_\alpha)f_\alpha = -2f_\alpha
\]
Thus the span of \( \{ h_{\alpha}, e_{\alpha}, f_{\alpha} \} \) form an \( \mathfrak{sl}(2) \) subalgebra (we’ll call it \( \mathfrak{sl}(2)_{\alpha} \))

Now for each \( \alpha \in \Phi \) and choice of associated \( \mathfrak{sl}(2)_{\alpha} \), we can view \( g \) as an \( \mathfrak{sl}(2)_{\alpha} \) module via adjoint. Since \( g \) is finite dimensional and \( \mathfrak{sl}(2)_{\alpha} \) simple, we know it decomposes uniquely into a direct sum of irreducible modules of the type we described earlier (and thus every submodule has a complement).
Lemma

We have shown if $\alpha \in \Phi$, then $-\alpha \in \Phi$. There are no other multiples of $\alpha$ in $\Phi$. Furthermore, the root space $g_\alpha$ is 1-dimensional.

Let's consider the $\mathfrak{sl}(2)_\alpha$ submodule

$$W_\alpha = \bigoplus_{c \in \mathbb{C}} g_{c\alpha}$$

This is an $\mathfrak{sl}(2)$ submodule (why?) and note $h_\alpha$ scales vectors in $g_{c\alpha}$ by $c\alpha(h_\alpha) = 2c$. Since we know $h_\alpha$ acts integrally on finite dimensional modules, we have $c \in \mathbb{Z}/2$

Now note the following is an $\mathfrak{sl}(2)_\alpha$ submodule of $W_\alpha$:

$$W_{\alpha,0} = \mathfrak{h} \oplus \mathbb{C}e_\alpha \oplus \mathbb{C}f_\alpha$$

The lemma is equivalent to showing $W_{\alpha,0} = W_\alpha$
Suppose not. Then there is some complement $W_{\alpha,1} \subset W_\alpha$. Note

$$W_{\alpha,1} \subset \bigoplus_{c \in \mathbb{Z}/2 \setminus \{0\}} g_{c\alpha}$$

Thus $h_\alpha$ has no 0 weight on $W_{\alpha,1}$. This forces all weights in $W_{\alpha,1}$ to be odd, since all irreducible submodules of $W_{\alpha,1}$ with even weights would contain a 0 weight. Thus

$$W_{\alpha,1} \subset \bigoplus_{c \in \mathbb{Z} + \frac{1}{2}} g_{c\alpha}$$

This immediately shows that $g_\alpha$ and $g_{-\alpha}$ have dimension 1, and there are no other integer multiples of $\alpha$ as roots. In particular, $\alpha$ and $2\alpha$ cannot both be roots. Hence $\alpha/2$ cannot be a root either.
So we have

\[ W_{\alpha,1} \subset \bigoplus_{c \in \mathbb{Z} + \frac{1}{2} \setminus \{ \frac{1}{2} \}} \mathfrak{g}_{c\alpha} \]

So \( W_{\alpha,1} \) does not have the weight 1. Thus it cannot contain any odd weights, as all its irreducible submodules with odd weights would contain the weight 1. So \( W_{\alpha,1} \) is a finite-dimensional \( \mathfrak{sl}(2)_\alpha \) submodule with no even or odd weights. So it must be 0.
Lemma

If \( \alpha, \beta \in \Phi \), then \( \beta(h_\alpha) \in \mathbb{Z} \) and \( \beta - \beta(h_\alpha) \alpha \in \Phi \).

If \( \alpha = \pm \beta \) this is clear. Assume otherwise.

Consider the \( \mathfrak{sl}(2)\alpha \) submodule of \( \mathfrak{g} \):

\[
W_\alpha^\beta = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}
\]

Since \( \beta \neq \pm \alpha \) and no other multiples of \( \alpha \) are roots, \( \mathfrak{g}_0 = \mathfrak{h} \) is not among these spaces. Thus they are all root spaces - 1 dimensional.

The weight of \( h_\alpha \) on \( \beta + i\alpha \) is \( \beta(h_\alpha) + i\alpha(h_\alpha) = \beta(h_\alpha) + 2i \).

So all weight spaces are 1-dimensional and all weights are the same parity. This means it is impossible for \( W_\alpha^\beta \) to be the sum of 2 or more irreducibles so \( W_\alpha^\beta \) is irreducible.
Let $q$ be largest such that $\beta + r\alpha \in \Phi$. Then the highest weight of $W_\alpha^\beta$ is $\beta(h_\alpha) + 2r$. So the lowest weight is

$-\beta(h_\alpha) - 2r = \beta(h_\alpha) - 2(\beta(h_\alpha) + r)$

and all integers of the same parity in between are weights. Thus

$$\beta + i\alpha \in \Phi \iff -\beta(h_\alpha) - r \leq i \leq r$$

In particular $\beta - \beta(h_\alpha)\alpha \in \Phi$

Note that we have also shown that the action of $\mathfrak{sl}(2)_\alpha$ on $\mathfrak{g}$ decomposes into irreducibles as follows:

$$\mathfrak{g} = \text{Ker}(\alpha) \oplus \mathfrak{sl}(2)_\alpha \oplus \bigoplus_{\beta \in \Phi / \mathbb{C} \alpha} \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}$$
Note $\beta - \beta(h_\alpha)\alpha = \beta - \frac{2(t_\beta, t_\alpha)}{(t_\alpha, t_\alpha)}\alpha = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$.

If these were vectors in Euclidean space and $(\cdot, \cdot)$ was dot product, this would mean the reflection of $\beta$ across the hyperplane orthogonal to $\alpha$ is in $\Phi$.

To get this realization, need to show the following:

**Lemma**

1. *All $\Phi$ lies in an $\mathbb{R}$ vector subspace of $\mathfrak{h}^*$ of the same dimension. Call this space $E$.*
2. $(\cdot, \cdot)$ is non-degenerate and positive-definite on $E$.
Since \( \Phi \) spans \( \mathfrak{h}^* \), choose a basis \( \{\alpha_1, \ldots \alpha_n\} \in \Phi \) for \( \mathfrak{h}^* \). We show that all roots \( \beta \in \Phi \) are in the \( \mathbb{R} \) span of the \( \{\alpha_i\} \).

We know

\[
\beta = \sum_i c_i \alpha_i
\]

for \( c_i \in \mathbb{C} \).

So for all \( \alpha_j \) we have

\[
\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_i c_i \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}
\]

Since the \( \alpha_j \) span, treating \( c_i \) as free variables, this set of equations has a unique solution (the actual \( c_i \)). And since all coefficients are integers, the solutions are rational; in particular real. So \( \Phi \subset E = \mathbb{R}\{\alpha_1, \ldots \alpha_n\} \).
Next, we show \( (\gamma, \gamma) > 0 \) for all \( \gamma \in E, \gamma \neq 0 \).

Note \( (\gamma, \gamma) = (t_\gamma, t_\gamma) = \text{tr}(\text{ad} t_\gamma)^2 \). Since all root spaces \( g_\alpha \) are one dimensional and \( \text{ad} t_\gamma \) kills \( \mathfrak{h} \), we have

\[
(t_\gamma, t_\gamma) = \sum_{\alpha \in \Phi} (\alpha(t_\gamma))^2 = \sum_{\alpha \in \Phi} (\alpha, \gamma)^2 = \sum_{\alpha \in \Phi} \left( \frac{2(\gamma, \alpha)}{(\alpha, \alpha)} \right)^2 (\alpha, \alpha)^2
\]

Since \( \gamma \in E \) and \( \frac{2(a_i, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \) for all \( \alpha_i \) in the basis and \( \alpha \in \Phi \), we know \( \frac{2(\gamma, \alpha)}{(\alpha, \alpha)} \in \mathbb{R} \).
All that remains to be shown is that \((\beta, \beta) \in \mathbb{R}\) for \(\beta \in \Phi\). We use a similar idea:

\[
(\beta, \beta) = \sum_{\alpha \in \Phi} (\alpha, \beta)^2 = \sum_{\alpha \in \Phi} \left(\frac{2(\alpha, \beta)}{(\beta, \beta)}\right)^2 (\beta, \beta)^2
\]

and divide through by \((\beta, \beta)^2\) to get

\[
\frac{1}{(\beta, \beta)} = \sum_{\alpha \in \Phi} \left(\frac{2(\alpha, \beta)}{(\beta, \beta)}\right)^2 \in \mathbb{Z} \subset \mathbb{R}
\]

So \((\beta, \beta) \in \mathbb{R}\) for all \(\beta \in \Phi\) (actually the inverse of a positive integer, from this argument).

Thus \(\sum_{\alpha \in \Phi} \left(\frac{2(\gamma, \alpha)}{(\alpha, \alpha)}\right)^2 (\alpha, \alpha)^2\) is the sum of squares of real numbers; hence non-negative. And since \(\Phi\) spans, not all terms are 0.
We summarize as follows: Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a maximal abelian semisimple subspace, and $(\cdot, \cdot)$ the killing form of $\mathfrak{g}$. Then there is a finite subset $\Phi \in \mathfrak{h}^*$ such that we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

with:
Recap
Cartan Decomposition
Root Systems

1. \( h \in \mathfrak{h} \) acts on \( \mathfrak{g}_\alpha \) (via bracket) with eigenvalue \( \alpha(h) \)
2. \( [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \)
3. All \( \mathfrak{g}_\alpha \) are 1-dimensional
4. For every \( \alpha \in \Phi \), we have \( -\alpha \in \Phi \) and no other multiples
5. \( (\cdot, \cdot) \) restricts non-degenerately to \( \mathfrak{h} \), so we can equip \( \mathfrak{h}^* \) with a form corresponding to \( (\cdot, \cdot) \) that we label the same way.
6. \( \Phi \) spans \( \mathfrak{h}^* \) and an \( \mathbb{R} \)-subspace \( E \) of \( \mathfrak{h}^* \) of the same dimension. On this subspace, \( (\cdot, \cdot) \) is positive definite and this subspace can therefore be realized as Euclidean space with \( (\cdot, \cdot) \) being dot product.
7. For all \( \alpha, \beta \in \Phi \), \( \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z} \)
8. On \( E \), define \( s_\alpha \) (\( \alpha \in \Phi \)) to be the map \( \gamma \mapsto \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\alpha \); in other words, reflection across the hyperplane orthogonal to \( \alpha \). \( \Phi \) is closed under \( s_\alpha \) for all \( \alpha \in \Phi \).
A subset $\Phi \subset E$ of Euclidean space with the properties

1. $\Phi$ spans $E$ and is finite.
2. $\Phi$ is closed under $s_{\alpha}$ for all $\alpha \in \Phi$.
3. For all $\alpha, \beta \in \Phi$, $\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$
4. For every $\alpha \in \Phi$, we have $-\alpha \in \Phi$ and no other multiples are in $\Phi$

is called a root system. These have beautiful structures that hint at the beauty of Lie theory as a whole. A root system is decomposable if we have $\Phi = \Phi_1 \cup \Phi_2$, $\Phi_1 \cap \Phi_2 = \emptyset$ and $(\Phi_1, \Phi_2) = 0$. Indecomposable otherwise. Simple Lie algebras will have indecomposable root systems; the idea being otherwise you could separate the root space decomposition based on this partition of $\Phi$ and each would be an ideal in $g$.

With a bit of work, one can also show the reverse direction - for any simple root system there is an associated simple Lie algebra.
Last time, we saw that any finite dimensional module of a simple Lie algebra is diagonalized by $\mathfrak{h}$:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda}$$

and note

$$h \cdot (x_\alpha \cdot v_\lambda) = (x_\alpha \cdot h \cdot v_\lambda) + [h, x_\alpha] \cdot v_\lambda = \lambda(h)x_\alpha \cdot v_\lambda + \alpha(h)x_\alpha \cdot v_\lambda$$

So $x_\alpha \cdot V_\lambda \subset V_{\lambda+\alpha}$.

Since $V_\lambda$ is also a module for the subalgebra $\mathfrak{sl}(2)_\alpha$ for each $\alpha \in \Phi$, we must have $\lambda(\mathfrak{h}_\alpha) \in \mathbb{Z}$ for all weights $\lambda$ in $V$ and $\alpha \in \Phi$. 
Here are the 2-dimensional indecomposable root systems:
Let $\theta$ be the angle between roots $\alpha$ and $\beta$. Then since

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} = 2 \frac{||\alpha||}{||\beta||} \cos(\theta),$$

we have

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 4 \cos^2(\theta)$$

In particular, $4 \cos^2(\theta) \in \mathbb{Z}$. So $\theta$ must be related (in the pre-calculus sense) to $0, \pi/6, \pi/3, \text{ or } \pi/4$.

Furthermore, if $\alpha$ and $\beta$ are non-proportional, this forces

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \{0, 1, 2, 3\} \text{ (and only 0 if they are orthogonal).}$$
**Lemma**

*For non-proportional root* \( \alpha, \beta, \) *if* \((\alpha, \beta) < 0\) *then* \(\alpha + \beta \in \Phi\). *If* \((\alpha, \beta) > 0\), *then* \(\alpha - \beta \in \Phi\)

**Proof.**

In both cases, we have \(\frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \{1, 2, 3\}\). Since both are integral, one must be \(\pm 1\). In the first case, one must be \(-1\) and in the second case, one must be \(1\). WLOG, let this be \(fr\alpha\beta\). Then \(s_\beta(\alpha) = \alpha - fr\alpha\beta\beta = \alpha \pm \beta\) and the conclusion follows. \(\square\)
Now cut $\Phi$ by some arbitrary hyperplane that does not intersect any root, and let $\gamma$ be a vector orthogonal to it. Let $\Phi^+$ be the roots acute with $\gamma$ (the positive roots) and $\Phi^-$ the roots obtuse with $\gamma$ (the negative roots).

Let $\Delta = \{\alpha_i\}$ be a minimal set of positive roots such that every positive root is a non-negative integral combination of the $\alpha_i$. We call these simple roots.

**Lemma**

$\Delta$ is linearly independent. Thus it is a basis for $E$ for which every root in $\Phi$ has either all coefficients non-negative or non-positive (based on whether it’s in $\Phi^+$ or $\Phi^-$)
We show that vectors in $\Delta$ are all mutually non-acute. Then if we had a dependence relation, we could write $\sum c_i \alpha_i = \sum c_j \alpha_j$, all coefficients non-negative and distinct simple roots on both sides. But by assumption

$$(\sum c_i \alpha_i, \sum c_i \alpha_i) = (\sum c_i \alpha_i, \sum c_j \alpha_j) \leq 0$$

so $\sum c_j \alpha_j = 0$ by positive definiteness (in otherwords, this expression says a non-negative sum of simple roots is 0). But such a sum must have a positive inner product with $\gamma$ as all $(\gamma, \alpha_i) > 0$, leading to a contradiction. So we just need to show that all $(\alpha_i, \alpha_j) \leq 0$. 
Suppose otherwise - \((\alpha_i, \alpha_j) > 0\). Then we know \(\alpha_i - \alpha_j\) and 
\(\alpha_j - \alpha_i\) are roots. Suppose \(\alpha_i - \alpha_j\) is positive without loss of 
generality. Then by assumption \(\alpha_i - \alpha_j\) is a non-negative integral 
combination of simple roots \(\alpha_i - \alpha_j = \sum c_k \alpha_k\). So any time we 
see \(\alpha_i\), we can replace it with \(\alpha_j + \sum c_k \alpha_k\), so the \(\alpha_i \in \Delta\) is not needed. This contradicts minimality.
This also shows that $\mathbb{Z}\Delta = \mathbb{Z}\Phi$. Since the former is linearly independent, this means the roots form a lattice. The matrix $A = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ is called a Cartan Matrix for this Root system (some define it to be the transpose of this). By construction, we will have

- 2s on the diagonal all off-diagonal entries in $\{0, -1, -2, -3\}$
- 0s symmetric
- If $a_{i,j} \in \{-2, -3\}$, $a_{j,i} = -1$
- A positive definite
To an $n \times n$ Cartan matrix, we associate a graph on $n$ nodes called the *Dynkin Diagram* as follows:

- If $a_{i,j} = 0$, no edges between nodes $i$ and $j$.
- If $a_{i,j} = a_{j,i} = -1$, draw 1 edge between nodes $i$ and $j$.
- If $a_{i,j} - n < -1$, draw $n$ edges between nodes $i$ and $j$, and an arrow from node $i$ to node $j$. 
The Dynkin Diagrams of all indecomposable root systems are as follows: