The Greene-Kleitman Correspondence

Jordan Forms & Flag Varieties.

1. GK-Correspondence

2. Robinson-Schensted algorithm

3. Jordan Form of Nilpotent matrices

4. Relative position of pairs of flags

Survey by Britz-Fomin: arxiv:9912126.
§ 0 Background on Posets & Dilworth’s theorem

Def A partially ordered set (poset) is a set with a binary relation ≤ that is anti-sym, reflexive, & transitive.

Def The Hasse diagram of a poset is a graph s.t. \( \frac{b}{a} \rightarrow a < b \)

Eg. Boolean Lattice:

```
\[ \begin{array}{c}
   123 \\
  / \ \ \\
/ \  \\
/ \ \\
/ \\
\emptyset \\
\end{array} \]
```
Some more definitions...

**Def** A chain is a totally ordered subset.

**Def** An antichain is a pair-wise incomparable subset.

---

**Eg**

![Saturated chain](image1)

![Chain](image2)

![Antichain](image3)
Dilworth's theorem

**Theorem (Dilworth 1950)**

The minimum number of chains to cover a poset $P = \text{the size of maximal anti chain in } P$.

**Example**
A Dual version

Thm (Dilworth 1950)

The minimum number of chains to cover a poset $P$

= the size of maximal anti chain in $P$. 
A Dual version

Thm (Mirsky)

The minimum number of antichains to cover a poset $P$ = the size of maximal chain in $P$.

E.g.
§1 The Greene-Kleitman Correspondence.

**Motivation**
Maximal size of a chain = min anti chain cover,
What about maximal size of union of more chains?

**Def**
A $k$-(anti)chain is a family of $k$ (anti)chains.

**Def**
For a poset $P$, define:

\[ c_k := \text{maximal size of a } k\text{-chain of } P \]

\[ a_k := \text{maximal size of a } k\text{-anti-chain of } P \]
**Def (GK-correspondence)**

For a poset $P$, define

\[ \lambda_1 = c_1 - c_0 \quad \text{and} \quad \mu_1 = a_1 - a_0 \]
\[ \lambda_2 = c_2 - c_1 \quad \text{and} \quad \mu_2 = a_2 - a_1 \]
\[ \lambda_3 = c_3 - c_2 \quad \text{and} \quad \mu_3 = a_3 - a_2 \]

\[ \vdots \]

Then $\lambda(P) := (\lambda_1, \lambda_2, \ldots)$ are weakly decreasing, and are both integer partitions of $\#P$.

**E.g.**

\[ \lambda_1 = 4 - 0 = 4 \]
\[ \lambda_2 = 6 - 4 = 2 \]
\[ \lambda_3 = 7 - 6 = 1 \]
\[ \lambda_4 = 7 - 7 = 0 \]

\[ \vdots \]

\[ a_1 = 3 - 0 = 3 \]
\[ a_2 = 5 - 3 = 2 \]
\[ a_3 = 6 - 5 = 1 \]
\[ a_4 = 7 - 6 = 1 \]
\[ a_5 = 7 - 7 = 0 \]

\[ \vdots \]
Theorem 1 (Greene, Greene-Kleitman)

\( \lambda(p) \) and \( \mu(p) \) are conjugate (dual) integer partitions of \#P.

Theorem 2 (Fomin)

If \( \alpha \) is a minimal or maximal element of \( \mathcal{P} \) then \( \lambda(\mathcal{P} - \{\alpha\}) \subseteq \lambda(\mathcal{P}) \)

E.g. \( \lambda(\downarrow \uparrow) = \begin{array}{ccc} & & \\ & \downarrow & \\ \uparrow & & \end{array} \subseteq \begin{array}{ccc} & & \\ & & \\ & \downarrow & \end{array} \)
Some remarks.

Thm 1 (duality) generalize Dilworth's thin & Minsky's thin simultaneously.

Thm 2 (monotonicity) means that we can calculate $\lambda(p)$ recursively by adding maximal elements.


§2 Tableaux Theory

**Def** A Semi Standard Young tableau is a filling of a Young diagram by $[n]$ st. columns are increasing rows are weakly increasing.

A Standard Young Tableau (SYT) is is a SSYT without repeated entries.

The underlying Young diagram is the shape of a tableau.

**Eg.**

\[
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 4 & 3 \\
3 & & \\
\end{array}
\] is a SSYT

\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & \\
6 & & \\
\end{array}
\] is a SYT
SYTs are flags of Young diagrams, so...

\[
\begin{array}{cccccc}
1 & \subset & 2 & \subset & 3 & \subset \\
\quad & \quad & \quad & \quad & \quad & \\
\quad & \quad & \quad & \quad & \quad & \\
\end{array}
\]

\[
\begin{array}{cccccc}
4 & \subset & 5 & \subset & 6 & \subset \\
\quad & \quad & \quad & \quad & \quad & \\
\quad & \quad & \quad & \quad & \quad & \\
\end{array}
\]

\[
\begin{array}{cccccc}
7 & = \\
\quad & \\
\quad & \\
\end{array}
\]

**Def.** A linear extension of \( P \) is a numbering of \( P \) by \( \{1, \ldots, \#P\} \) which preserves the partial order.

**Eq.**

\[
\begin{array}{c}
\star \quad \star \quad \star \\
\quad \checkmark \quad \times \\
\times \quad \checkmark \quad \times
\end{array}
\]

**Cor.** Thm 2 \( \Rightarrow \) Every linear extension of \( P \) gives a SYT whose shape is \( \lambda(P) \)
Robinson–Schensted (RS) Algorithm.

**Def/Thm** Every permutation $w \in S_n$ is in bijection with a pair of SYT, $\{P(w), Q(w)\}$ of the same shape, defined via the RS insertion algorithm.
Fun Facts about RS ...

• length of 1st column of $P(w)$ is the length of longest increasing subsequence.

• $\sum_{\lambda+n} \# \{ T : \text{shape}(T) = \lambda \}^2 = n!$

• $\{ w : P(w) = T \}$ are Kazhdan-Lusztig left cells

• $\{ w : Q(w) = T \}$ are Kazhdan-Lusztig right cells

• $\{ w : \text{shape} P(w) = \lambda \}$ are two-sided cells
From GK to RS

**Def** There is a poset attached to each permutation \( W \), called the **inversion poset**.

\[ 6451732 \]
From GK to RS

**Def** There is a poset attached to each permutation \( w \), called the inversion poset, denoted \( P_w \)

\[ 4126573 \]

**Thm** (Greene, Fomin) by example
There are two natural linear extensions on $P_w$ by row & column index.
Another way to see this...

In each box of the $[n] \times [n]$ board, we put a subposet of $P_w$

$$P_w^{(i,j)} = [i] \times [j] \cap P_w$$

And replace it with $\lambda_{ij} = \lambda(P_w^{(i,j)})$

E.g.

Get an $n \times n$ array called the growth diagram (Fomin).
§ 3  Jordan Blocks.

Def For every partial order $\mathcal{P}$ on $[n]$, its incidence algebra $I(\mathcal{P})$ contains $n \times n$ nilpotent matrices $M$ s.t.

$M_{ij} = \text{generic non-zero if } i < j$

$M_{ij} = 0 \text{ if } i, j \text{ incomparable.}$

E.g. 

[Diagram with nodes and edges, and a corresponding matrix with stars indicating non-zero entries.]
Thm (Gansner 1981, Saks 1980)

The Jordan blocks of any $M \in I(p)$ is determined by $\lambda(p)$.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]
§ Complete Flag Varieties

A (complete) flag is \( \phi = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = \mathbb{C}^n \).

The flag variety \( \mathcal{F}l_n(K) \) is the alg. variety containing all such flags.

\[ \mathcal{F}l_n(K) \cong \text{GL}_n(K) / B \]

Fix a basis \( \{e_1, e_2, \ldots, e_n\} \), the standard flag \( E^{id} \) is

\[ E^{id}_\cdot = \phi \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \ldots \subseteq \mathbb{C}^n \]

For \( w \in S_n \), the permutation flag \( E^w \) is

\[ E^w_\cdot = \phi \subseteq \langle e_{w(1)} \rangle \subseteq \langle e_{w(1)}, e_{w(2)} \rangle \subseteq \ldots \subseteq \mathbb{C}^n \]
Relative Position

For a pair of flags \( E : \bigcirc CE_0 CE_1 \ldots CE_n = C^n \)
\( F : \bigcirc CF_0 CF_1 \ldots CF_n = C^n \),

their relative position \( d(E,F) \) is the matrix \( D \) where

\[
D_{ij} = \dim(E_i \cap F_j)
\]

this is used to define Schubert cells:

\[
X_o^w = \{ F \in Fl_n(C) : d(E_i^w, F) = d(E_i^w, E^w) \} = B \cdot w B / B
\]
Fact: $d(E, F)$ is the north-west rank matrix of some permutation, thus can identify $d(E, F)$ with that permutation.

E.g.: \[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 2 \\
1 & 2 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\rightarrow
312
\]

Alternatively, the relative position of $E \& F$ is the permutation $\mathcal{w}$ such that if $E \triangleq \{0 \subset \{v_1\} \subset \{v_1, v_2\} \subset \ldots \subset \{v_n\}\}$

then $F \triangleq \{0 \subset \{v_{w(1)}\} \subset \{v_{w(1)}, v_{w(2)}\} \subset \ldots \subset \{v_n\}\}$
Get combinatorial data from flags!

**Def** We say a nilpotent $\mathbf{x}$ contracts a flag $\mathbf{F}$ if

$$\mathbf{x} \mathbf{F}_i \subseteq \mathbf{F}_{i+1} \quad \forall i$$

* For a nilpotent $\mathbf{x}$, can get a Young diagram $\lambda(x)$ from its Jordan blocks.

* Moreover, by restricting $\mathbf{x}$ on components of $\mathbf{F}_0$, can get a sequence of Young diagrams:

$$\emptyset \subseteq \lambda(x|\mathbf{F}_1) \subseteq \lambda(x|\mathbf{F}_2) \subseteq \ldots \subseteq \lambda(x)$$

i.e. a Young tableau!

**Note:** The $\subseteq$'s are strict $\not\subseteq$ when $\mathbf{x}$ is generic, in which case the tableau is a SYT.
Relative Position Via RS (steinberg)

Take two flags $E$ & $F$, let $x$ be a generic nilpotent which contracts both $E$ and $F$.

Get two SYTs:

$T_E: \varnothing \subset \lambda(x|_{E_1}) \subset \lambda(x|_{E_2}) \subset \cdots \subset \lambda(x)$

$T_F: \varnothing \subset \lambda(x|_{F_1}) \subset \lambda(x|_{F_2}) \subset \cdots \subset \lambda(x)$

Then the RS-correspondence tells us ...

$\begin{pmatrix} T_E \quad T_F \end{pmatrix} \overset{RS}{\leftrightarrow} \omega = d(E_0, F_0)$

the relative position of $E$ & $F$. 
... Why?

Let $\lambda_{ij} = \lambda(x | E_i \cap E_j)$, then the $n \times n$ array $\{\lambda_{ij}\}$ is the same as the "growth diagram".

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Pf idea  Given $\lambda(x | E_i \cap F_j)$, find $\lambda(x | E_i \cap F_j)$ by case-by-case analysis of the Jordan chains. The rules agree with the recursive calculation for GK-shape (Thm 2).
E: \( \emptyset \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \mathbb{C}^* \)
F: \( \emptyset \subset \langle e_2 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \mathbb{C}^* \)
Thank You for Listening!