BGG category $\mathcal{O}$ and BGG resolution

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Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$, with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$, universal enveloping algebra $U(\mathfrak{g})$, root system $\Phi \subset \mathfrak{h}^*$, simple roots (base) $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, and positive roots $\Phi^+$. Let $\mathfrak{b} := \mathfrak{n}_+ + \mathfrak{h}$ be the (standard) Borel subalgebra of $\mathfrak{g}$.

For any root $\alpha \in \Phi$, let $s_\alpha$ be the reflection through $\alpha$, i.e.

$$s_\alpha(\lambda) := \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha.$$ 

The Weyl group is defined to be the group generated by reflections through simple roots, i.e.

$$W = \langle s_\alpha; | i = 1, \ldots, l \rangle.$$ 

$s_{\alpha_i}$ is called a simple reflection.
Background setting and notations

- Partial order on \( \mathfrak{h}^* \): for \( \lambda, \mu \in \mathfrak{h}^* \), \( \lambda \leq \mu \) iff

\[
\lambda - \mu = \sum_{i=1}^{\ell} a_i \alpha_i, \quad a_i \in \mathbb{Z}_{\geq 0}.
\]

- Dot action. Let \( \rho := \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha_i \). For any \( \lambda \in \mathfrak{h}^* \), \( w \in W \), define the dot action:

\[
w \cdot \lambda = w(\lambda + \rho) - \rho.
\]
Background setting and notations

- **Length.** For \( w \in W \), write \( \ell(w) = n \) if \( w = s_1 \ldots s_n \) with \( s_i \) simple reflection and \( n \) as small as possible; such an expression is called reduced.

- **Bruhat order on \( W \).** If \( w_1 = s_\alpha w_2 \), with \( \alpha \in \Phi_+ \) and \( \ell(w_1) < \ell(w_2) \), we write \( w_1 \overset{s_\alpha}{\rightarrow} w_2 \). Define \( w < w' \) if \( w = w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_n = w' \).
Consider the category $\mathcal{C}$ of finite dimensional $\mathfrak{g}$-modules. We know that

$$\mathcal{C} = \bigoplus_{\lambda \in \Lambda_+} \mathcal{C}_\lambda$$

where $\mathcal{C}_\lambda$ is a full subcategory of $\mathcal{C}$ with

$$\text{Obj}(\mathcal{C}_\lambda) = \{ L(\lambda)^{\oplus n} \mid n \geq 0 \}$$

By Schur lemma, we know $\text{Hom}(L(\lambda), L(\mu)) = 0$ if $\lambda \neq \mu$, and

$$\text{End}(L(\lambda)) = \mathbb{C} \text{id}_{L(\lambda)}$$
Definition 1 (BGG category).

The BGG category $\mathcal{O}$ is defined to be the full subcategory of $U(\mathfrak{g})$-Mod whose objects are the modules $M$ satisfying the following three conditions:

- (O1) $M$ is finitely generated.
- (O2) $M$ is $\mathfrak{h}$-diagonalizable, i.e. there exists a basis of $M$ consisting of common eigenvectors of $\mathfrak{h}$. That is,

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_{\lambda},$$

where $M_{\lambda} := \{v \in M \mid h.v = \lambda(h)v, \forall h \in \mathfrak{h}\}$.
- (O3) $M$ is locally $\mathfrak{n}_+$-finite, i.e. $U(\mathfrak{n}_+).v$ is finite dimensional, $\forall v \in M$. 
Highest weight module

**Definition 2 (Maximal vector).**

Let $M \in U(g)$-Mod, $\lambda \in \mathfrak{h}^*$. A nonzero vector $v \in M_\lambda$ is called a maximal vector of weight $\lambda$ if $n_+v = 0$.

**Definition 3 (Highest weight module).**

$M \in \mathcal{O}$ is called a highest weight module (h.w.m.) with highest weight $\lambda \in \mathfrak{h}^*$ if it is generated by a maximal vector. That is, there exists a nonzero $v^+ \in M_\lambda$, such that $M = U(g).v^+$.

Any highest weight module is in category $\mathcal{O}$.
Fix $\lambda \in \mathfrak{h}^*$, let $C_\lambda = \mathbb{C}v_\lambda$ be a one dimensional $\mathfrak{h}$-module such that 

$$\begin{align*}
n_+ \cdot v_\lambda &= 0, \\
\mathfrak{h} \cdot v_\lambda &= \lambda(h)v_\lambda, \quad \forall h \in \mathfrak{h}.
\end{align*}$$

Define $M(\lambda) = \text{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})} C_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda$ to be the induced module from $U(\mathfrak{b})$-Mod to $U(\mathfrak{g})$-Mod.
Verma module $M(\lambda)$ is an h.w.m. of highest weight $\lambda$ with maximal vector $1 \otimes v_\lambda$. For any h.w.m. $M = U(g).v^+$ of highest weight $\lambda$, $M$ is a quotient of $M_\lambda$: $1 \otimes v_\lambda \mapsto v^+$. Notice:

$$\text{Hom}_{U(g)}(M(\lambda), M) = \text{Hom}_{U(g)}(\text{Ind}_b^g C_\lambda, M)$$
$$\cong \text{Hom}_{U(b)}(C_\lambda, \text{Res}_b^g M)$$
$$= \{v \in M_\lambda \mid v \text{ is a maximal vector or 0}\}.$$
**Theorem 4.**

Let $M = U(\mathfrak{g}).\mathfrak{v}^+$ be an h.w.m., then $M$ has a unique maximal submodule.

**Proof.**

For any submodule $N$ of $M$, $N = M$ iff $\mathfrak{v}^+ \in N$ iff $N_\lambda \neq 0$. So, one can prove $\sum_{N \subsetneq M} N \subsetneq M$ is the unique maximal submodule of $M$. Notice:

$$\left( \sum_{N \subsetneq M} N \right)_\lambda = \sum_{N \subsetneq M} N_\lambda = \sum_{N \subsetneq M} 0 = 0.$$
As a corollary, we know that there is a unique simple h.w.m. of weight $\lambda$. Namely, $L(\lambda) := M(\lambda)/N(\lambda)$, where $N(\lambda)$ is the maximal submodule of $M(\lambda)$. Also, every simple module $S$ is an h.w.m. So we know $S \cong L(\lambda)$ for some $\lambda \in \mathfrak{h}^\ast$. 
Define $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ to be the center of $U(\mathfrak{g})$. We know that 
$[Z(\mathfrak{g}), U(\mathfrak{h})] = 0$ implies $Z(\mathfrak{g}).M_\lambda \subset M_\lambda$, $\forall \lambda \in \mathfrak{h}^*$.

For an h.w.m. $M = U(\mathfrak{g}).v$ of weight $\lambda$, we know $M_\lambda = \mathbb{C}v$. So, 
$\forall z \in Z(\mathfrak{g}), \exists \chi_\lambda(z) \in \mathbb{C}$, such that $z.v = \chi_\lambda(z)v$. This $\chi_\lambda$ is an 
algebra homomorphism from $Z(\mathfrak{g})$ to $\mathbb{C}$. Moreover, $z$ acts on $M$ as 
a scalar multiplication $\chi_\lambda(z) \cdot id_M$:

$$z.(u.v) = u.(z.v) = u.(\chi_\lambda(z)v) = \chi_\lambda(z)u.v, \quad \forall u \in U(\mathfrak{g}).$$

Since every h.w.m. of highest weight $\lambda$ is a quotient of $M(\lambda)$, $\chi_\lambda$ is independent of the choice of $M$.

In fact, every algebra homomorphism $\chi$ is equal to $\chi_\lambda$ for some $\lambda$, 
i.e. $\mathfrak{h}^* \rightarrow \text{Hom}_{\text{Alg}}(Z(\mathfrak{g}), \mathbb{C})$. 
Let $\chi \in \text{Hom}_{\text{Alg}}(Z(g), \mathbb{C})$. Define a full subcategory of $\mathcal{O}$:

$$\mathcal{O}_\chi := \{ M \in \mathcal{O} : M = M^\chi \}$$

where

$$M^\chi := \{ x \in M : \forall z \in Z(g), \exists n \in \mathbb{N}, \text{s.t. } (z - \chi(z))^n x = 0 \}.$$ 

**Theorem 5.**

$\mathcal{O} = \bigoplus \chi \mathcal{O}_\chi$. In other words, $\forall M \in \mathcal{O}$, $M = \bigoplus \chi M^\chi$. Also, $\text{Hom}_\mathcal{O}(M, N) = \bigoplus \chi \text{Hom}_{\mathcal{O}_\chi}(M^\chi, N^\chi)$. 
Lemma 6.

Let \( \{x_i, y_i|1 \leq i \leq n\} \cup \{h_i|1 \leq i \leq \ell\} \) be a standard basis of \( \mathfrak{g} \).
For all \( k \geq 0 \), and \( 1 \leq i, j \leq \ell \), we have:

(a) \( [x_j, y_i^{k+1}] = 0 \) whenever \( j \neq i \).
(b) \( [h_j, y_i^{k+1}] = -(k + 1)\alpha_i (h_j) y_i^{k+1} \).
(c) \( [x_i, y_i^{k+1}] = -(k + 1)y_i^k (k \cdot 1 - h_i) \).

Theorem 7.

Given \( \lambda \in \mathfrak{h}^* \) and \( \alpha \in \Delta \), suppose \( n := \langle \lambda, \alpha^\vee \rangle \) lies in \( \mathbb{Z}_{\geq 0} \). If \( \nu^+ \) is a maximal vector of weight \( \lambda \) in \( M(\lambda) \), then \( y_\alpha^{n+1} \cdot \nu^+ \) is a maximal vector of weight \( \mu := \lambda - (n + 1)\alpha < \lambda \). Thus there exists a nonzero homomorphism \( M(\mu) \to M(\lambda) \) whose image lies in the maximal submodule \( N(\lambda) \).
A nonzero homomorphism $M(\mu) \rightarrow M(\lambda)$ implies $\chi_\mu = \chi_\lambda$. So, we get a conclusion: if $\lambda \in \Lambda_+$, then $\chi_{s_\alpha \cdot \lambda} = \chi_\lambda$.

By considering the reduced form of $w \in W$, we can show that if $\lambda \in \Lambda$, $\mu \in W \cdot \lambda$, then $\chi_\lambda = \chi_\mu$.

**Theorem 8 (Harish-Chandra).**

*Suppose $\lambda, \mu \in \mathfrak{h}^*$, then $\chi_\lambda = \chi_\mu$ if and only if $\lambda \in W \cdot \mu$.***
As an important corollary, $\mathcal{O}$ is of finite length (or to say, both Noetherian and Artinian), i.e., every $M \in \mathcal{O}$ has a finite length. Namely, $M$ has a filtration:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

with each $M_i / M_{i-1}$ simple.

**Proof.**

1. *$M$ has a finite filtration with every factor is an h.w.m.*  
   Since $M$ is finitely generated, we can assume $M$ is generated by one weight vector: $M = U(\mathfrak{g}).v$. Induction on $\dim V$, where $V := U(\mathfrak{n}_+).v$ is a finite dimensional vector space by the definition of $\mathcal{O}$.

2. *Every h.w.m. has a finite length.*  
   By the Harish-Chandra theorem. (cf. Humphreys, page 28)
For any $M \in \mathcal{O}$, we can define $[M : L(\lambda)]$ to be the multiplicity of $L(\lambda)$.

A natural question: when is $[M(\lambda) : L(\mu)] \neq 0$?

1. **Necessary condition 1:** $\mu \leq \lambda$.
   
   $[M(\lambda) : L(\mu)] \neq 0$ implies $M(\lambda)_{\mu} \neq 0$, so $\mu \leq \lambda$.

2. **Necessary condition 2:** $\chi_{\mu} = \chi_{\lambda}$.

   $[M(\lambda) : L(\mu)] \neq 0$ implies $L(\mu)$ is isomorphic to a subquotient of $M(\lambda)$.

By the Harish-Chandra theorem, condition 2 $\iff \mu = \mathcal{W} \cdot \lambda$. 
Let \( \lambda, \mu \in \mathfrak{h}^* \) and write \( \mu \uparrow \lambda \) if \( \mu = \lambda \) or there is a root \( \alpha > 0 \) such that \( \mu = s_\alpha \cdot \lambda < \lambda \); in other words, \( \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0} \).

More generally, if \( \mu = \lambda \) or there exist \( \alpha_1, \ldots, \alpha_r \in \Phi^+ \) such that

\[
\mu = (s_{\alpha_1} \ldots s_{\alpha_r}) \cdot \lambda \uparrow (s_{\alpha_2} \ldots s_{\alpha_r}) \cdot \lambda \uparrow \ldots \uparrow s_{\alpha_r} \cdot \lambda \uparrow \lambda
\]

we say that \( \mu \) is strongly linked to \( \lambda \) and write \( \mu \uparrow \lambda \).

This is a partial order on \( \mathfrak{h}^* \).
Theorem 9.

Let $\lambda, \mu \in \mathfrak{h}^*$. 

(a) (Verma) If $\mu \uparrow \lambda$, then $M(\mu) \hookrightarrow M(\lambda)$; in particular, we know $[M(\lambda) : L(\mu)] \neq 0$.

(b) (BGG) If $[M(\lambda) : L(\mu)] \neq 0$, then $\mu \uparrow \lambda$. 

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Theorem 10.

If \( \lambda \in \Lambda^+ \), the unique maximal submodule \( N(\lambda) \) of \( M(\lambda) \) is the sum of the submodules \( M(s_{\alpha_i} \cdot \lambda) \) for \( 1 \leq i \leq \ell \).

We can express this result by an exact sequence:

\[
\bigoplus_{w \in \mathcal{W}, \ell(w)=1} M(w \cdot \lambda) \rightarrow M(\lambda) \rightarrow L(\lambda) \rightarrow 0
\]
Theorem 11.

If $\lambda \in \Lambda^+$, the unique maximal submodule $N(\lambda)$ of $M(\lambda)$ is the sum of the submodules $M(s_{\alpha_i} \cdot \lambda)$ for $1 \leq i \leq \ell$.

We can express this result by an exact sequence:

$$\cdots \rightarrow \bigoplus_{w \in W, \ell(w) = 1} M(w \cdot \lambda) \rightarrow M(\lambda) \rightarrow L(\lambda) \rightarrow 0$$
The Weyl group $W$ has a unique maximal element $w_0$, and $\ell(w_0) = |\Phi_+|$. We define the BGG resolution of $L(\lambda)$ is an exact sequence of the following form.

$$0 \rightarrow M(w_0 \cdot \lambda) \xrightarrow{\delta_2} \bigoplus_{\ell(w) = |\Phi_+| - 1} M(w \cdot \lambda) \rightarrow \cdots$$

$$\xrightarrow{\delta_3} \bigoplus_{\ell(w) = 1} M(w \cdot \lambda) \xrightarrow{\delta_1} M(\lambda) \xrightarrow{\varepsilon} L(\lambda) \rightarrow 0.$$ 

**Theorem 12.**

*For $\lambda \in \Lambda_+$, there exists a BGG resolution for $L(\lambda)$.***
Application: Weyl character formula

We can prove the Weyl character formula from the existence of BGG resolution. Write $C_k := \bigoplus_{w \in W} M(w \cdot \lambda)$, then

$$\text{ch } C_k = \sum_{w \in W, \ell(w) = k} \text{ch } M(w \cdot \lambda).$$

We know the alternating sum of dimension of each weight space of $C_k$ is 0. The alternating sum of character should also be 0, i.e.

$$\text{ch } L(\lambda) + \sum_{i=0}^{\left| \Phi_+ \right|} (-1)^{i+1} \text{ch } C_k = 0$$
\[
\text{ch } L(\lambda) = \sum_{i=0}^{\lvert \Phi_+ \rvert} (-1)^i \text{ch } C_k
\]

\[
= \sum_{i=0}^{\lvert \Phi_+ \rvert} \sum_{w \in W, \ell(w) = i} (-1)^i \text{ch } M(w \cdot \lambda)
\]

\[
= \sum_{w \in W} (-1)^{\ell(w)} \text{ch } M(w \cdot \lambda).
\]

This is the Weyl character formula for finite dimensional irreducible module \(L(\lambda)\).
compute the dimension of Lie algebra cohomology group:

\[ H^k(n_-, L(\lambda)) := \text{Ext}^k_{n_-}(\mathbb{C}, L(\lambda)) \] (It is actually an \( \mathfrak{h} \)-module.)

**Theorem 13 (Bott).**

If \( \lambda \in \Lambda^+ \), then \( \text{dim} \ H^k(n^-, L(\lambda)) = |W^{(k)}| \), where \( W^{(k)} \) denotes the set of elements in \( W \) having length \( k \).
Proof.

By using property of dual functor \((-)^*\), we know
\[
\text{Ext}^k_{n-}(\mathbb{C}, L(\lambda)) \cong \text{Ext}^k_{n-}(L(\lambda)^*, \mathbb{C}), \quad L(\lambda)^* \cong L(-w_0 \cdot \lambda).
\]
We know \(\lambda^* := -w_0 \cdot \lambda \in \Lambda_+\) so, \(L(\lambda^*)\) has a BGG resolution, which is a free resolution as \(U(n_-)\)-module. By applying functor \(\text{Hom}_{n_-}(\cdot, \mathbb{C})\) to the BGG resolution, then taking the homology, we get \(\text{Ext}^*_{n_-}(L(\lambda^*, \mathbb{C}))\).

we can naturally identify \(\text{Hom}_{n_-}(M, \mathbb{C})\) with \((M/n_-M)^*\). For \(M\) is a Verma module \(M(\mu)\), \((M/n_-M)^* \cong \mathbb{C}_\mu^* = \mathbb{C}_{-\mu}\) is a one dimensional space. Thus, we know \(\dim \text{Hom}_{n_-}(C_k, \mathbb{C}) = |\mathcal{W}(k)|\).

Also, because all \(-w \cdot \lambda^*\) are distinct, all maps in this chain are zero.
Other applications of BGG resolution:

1. projective dimension & global dimension
2. compute $\text{Ext}^n_{\mathcal{O}} (M(w' \cdot \lambda), M(w \cdot \lambda)), \text{Ext}^n_{\mathcal{O}} (M(\mu), M(\lambda)^\vee)$. 
Thank you!