A Brief Introduction to D-modules

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Seeing the title, one asks the following natural questions: what are D-modules? and why study D-modules?

- D-modules are modules over a ring of differential operators.
- D-module theory has many interesting applications, like Kazhdan–Lusztig conjecture, Riemann–Hilbert correspondence, and geometric representation theory.

- 1920s, Born, Dirac, Heisenberg, Weyl, Littlewood, etc. Interested in system with one degree of freedom,

$$pq - qp = 1,$$

where p denotes momentum, q denotes position; or p, q are infinite matrices, leading to the first Wely algebra A_1.
- 1960-70s, the modern age in the theory of Weyl algebra arrived when its connections with Lie algebras were realized. The quotient of $U(\mathfrak{n})$ by a primitive ideal(i.e., annihilator of a simple left module) is always isomorphic to a Weyl algebra.
  Dixmier introduced the notation A_n that corresponds to physicist's system with n degree of freedom, and the name Wely algebra(suggested by Segal).
- 1971, Kashiwara systematically applied the idea of considering a differential equation as a module over a ring of differential operators.

- 1970s, I.N. Bernstein developed the theory of modules over the Weyl algebra from a different starting point.
  In 1954, I.M. Gelfand asked whether a complex variable function that is analytic in the upper half plane $\Re(z) > 0$ can be extended to a meromorphic function defined in the whole complex plane.
  Atiyah and, independently, Bernstein and I.S. Gelfand gave affirmative answers in 1968. Both proofs used resolution of singularities, a very deep and difficult result.
  Four years later, Bernstein discovered a new proof that was elementary. The key to the proof was a clever use of the Weyl algebra.

The theory of D-modules has two branches: analytic and algebraic, depending on the base variety. Highly sophisticated machinery is required in the study of general D-modules, and the most important results cannot be introduced without derived categories and sheaves.
Perhaps the most spectacular result of the theory is Riemann–Hilbert correspondence obtained independently by Kashiwara and Mebkhout in 1984. Very roughly speaking, the correspondence establishes an (anti-)equivalence between certain differential equations and their solution spaces. Unfortunately, the correspondence requires deep results of category theory.

Let K be a field of characteristic 0, $x_1, \ldots, x_n$ commuting indeterminates, and $K[X] = K[x_1, \ldots, x_n]$ the polynomial ring. $K[X]$ is an infinite dimensional vector space over K. Consider $\operatorname{End} K[X]$, the algebra of linear operators on $K[X]$.
Consider the following operators:

$$\hat{x}_i : f \mapsto x_i f, \quad \partial_i : f \mapsto \frac{\partial f}{\partial x_i}, \tag{1}$$

where $f \in K[X]$, and $i = 1, \ldots, n$.

Definition

The n-th Weyl algebra over K, denoted by $A_n(K)$, is the K-subalgebra of $\operatorname{End} K[X]$ generated by $\hat{x}_i$ and ∂_i, $i = 1, \ldots, n$.

The Weyl algebra is an associate algebra under composition, hence the commutator bracket $[D, D'] := DD' - D'D$ gives a Lie algebra structure.

Proposition

The generators $\hat{x}_i$ and ∂_i satisfy the following relations:

$$[\partial_i, \hat{x}_j] = \delta_{ij}, \tag{2}$$

$$[\hat{x}_i, \hat{x}_j] = 0 = [\partial_i, \partial_j]. \tag{3}$$

Proof.

$$[\partial_i, \hat{x}_j](f) = \partial_i(\hat{x}_j(f)) - \hat{x}_j(\partial_i(f)) = \frac{\partial}{\partial x_i}(x_j f) - x_j \frac{\partial f}{\partial x_i} = \frac{\partial x_j}{\partial x_i} f. \qquad \square$$

By abusing notation, we will write x_i for $\hat{x}_i$; in other words, we have the embedding $K[X] \hookrightarrow \operatorname{End} K[X]$, by sending f to the operator of multiplication by f.

Multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, write x^α for $\prod_{i=1}^n x_i^{\alpha_i}$, and ∂^β for $\prod_{i=1}^n \partial_i^{\beta_i}$, then $x^\alpha \partial^\beta = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$.

Proposition

The set $B = \{x^\alpha \partial^\beta \mid (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n\}$ is a basis of A_n.

Sketch of Proof.

- B is a generating set: A_n is spanned as a vector space by (non-commuting) monomials in x_i and ∂_i. Use (2) and (3) to switch order.
- B is linear independent. For any $D = \sum c_{\alpha\beta} x^\alpha \partial^\beta$ with non-zero coefficients, find certain x^σ with $D(x^\sigma) \neq 0$, hence $D \neq 0$. □

A lemma used in the proof: for $\sigma, \beta \in \mathbb{N}^n$ with $|\sigma| \leqslant |\beta|$, we have $\partial^\beta(x^\sigma) = \delta_{\sigma\beta} \cdot \beta!$, where $|\alpha| = \sum \alpha_i$ and $\alpha! = \prod_i \alpha_i!$.

Denote by F_{2n} the free algebra in $2n$ generators $z_1, \ldots, z_{2n}$. Define a map $\phi : F_{2n} \to A_n$ by sending z_i to x_i and z_{n+i} to ∂_i, for $1 \leqslant i \leqslant n$. Let J be the two-sided ideal generated by

$$[z_{n+i}, z_j] - \delta_{ij}, \quad [z_i, z_j], \quad [z_{n+i}, z_{n+j}],$$

where $1 \leqslant i, j \leqslant n$.
Then by (2) and (3), $J \subset \ker \phi$.

Theorem

$J = \ker \phi$, *hence* $\hat{\phi} : F_{2n}/J \xrightarrow{\sim} A_n$ *(as K-algebras).*

Sketch of Proof.

Arguing similarly as in the previous proposition, one can show that $C = \{z_1^{\lambda_1} \ldots z_{2n}^{\lambda_{2n}} + J \mid \lambda = (\lambda_1, \ldots, \lambda_{2n}) \in \mathbb{N}^{2n}\}$ is a generating set for F_{2n}/J. Since $\hat{\phi}(C) = B$ is a basis for A_n, C is a basis for F_{2n}/J. Hence, $\hat{\phi}$ is an isomorphism of vector spaces and, a fortiori, an isomorphism of rings. □

We can apply the theorem to construct automorphisms of Weyl algebras. One of the most important is the following Fourier transform $\mathcal{F}$.

Proposition

There exists an automorphism $\mathcal{F} : A_n \to A_n$ sending x_i to ∂_i and ∂_i to $-x_i$.

Proof.

Define $\psi : F_{2n} \to A_n$ by sending z_i to ∂_i and z_{n+i} to $-x_i$.
Check that $J \subset \ker \psi$. In fact, $\psi([z_{n+i}, z_j]) = [-x_i, \partial_j] = \delta_{ij}$, $\psi([z_i, z_j]) = [x_i, x_j] = 0$, and $\psi([z_{n+i}, z_{n+j}]) = [\partial_i, \partial_j] = 0$ for $1 \leqslant i, j \leqslant n$.
Hence $\mathcal{F} := \hat{\psi} \circ \hat{\phi}^{-1} : A_n \to F_{2n}/J \to A_n$ is an algebra homomorphism. Note that $\mathcal{F}^2 = -\operatorname{id}_{A_n}$, thus $\mathcal{F}$ is invertible, hence an automorphism. $\square$

Let $\mathfrak{gl}_n$ denote the associate/Lie algebra of $n \times n$ matrices over K. Consider the following embedding $\phi_n^{n+k} : \mathfrak{gl}_n \to \mathfrak{gl}_{n+k}$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. Then $\mathfrak{gl}_\infty := \lim_n \mathfrak{gl}_n$ is also a associate/Lie algebra.
Consider the following two matrices in $\mathfrak{gl}_\infty$:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Then the K-subalgebra generated by P and Q is isomorphic to the first Weyl algebra A_1.

For $D = \sum c_{\alpha\beta} x^\alpha \partial^\beta$ in canonical form, the degree of D is the largest $|\alpha| + |\beta|$ with $c_{\alpha\beta} \neq 0$, where $|\alpha| = \sum \alpha_i$. By convention, $\deg(0) = -\infty$. The degree satisfies the following properties.

Proposition

For $D, D' \in A_n$,

$$\deg(D + D') \leqslant \max\{\deg(D), \deg(D')\}. \tag{4}$$

$$\deg(DD') = \deg(D) + \deg(D'). \tag{5}$$

$$\deg([D, D']) \leqslant \deg(D) + \deg(D') - 2. \tag{6}$$

Theorem

The Weyl algebra A_n is a simple (non-commutative) domain.

Corollary

Every non-zero endomorphism of A_n is injective.

Conjecture ([Dixmier68])

Every non-zero endomorphism of A_n is an automorphism.

This conjecture implies the famous Jacobian conjecture.

Conjecture (Jacobian conjecture)

Let $F : K^n \to K^n$ be a polynomial map. If $\det J(F) = 1$ on K^n, then F has a polynomial inverse on the whole of K^n.

A map $F = (F_1, \ldots, F_n) : K^n \to K^n$ is polynomial if when all its components F_i are polynomials. F has a polynomial inverse if it has an inverse map, and the inverse map is also polynomial.

$J(F) = \left(\frac{\partial F_i}{\partial x_j}\right)_{ij}$ is the Jacobian matrix of F.

The Jacobian conjecture for $n = 1$ is trivial, and for $n \geqslant 2$ remains open, whereas Dixmier's conjecture is still open for every $n \geqslant 1$.

In general, let R be a commutative K-algebra(e.g., $K[X]$), and consider $\operatorname{End} R$ with the embedding $R \hookrightarrow \operatorname{End} R$, $a \mapsto (\lambda_a : r \mapsto ar)$. The **order** of an operator $P \in \operatorname{End} R$ is defined inductively as follows:

Definition

$P \in \operatorname{End} R$ has order 0 if $[P, a] = 0$ for all $a \in R$; suppose we have defined order $< k$, then P is of order k, if P is not of order $< k$, and $[P, a]$ is of order $< k$ for all $a \in R$.

Let $D^n(R)$ denote the space of all operators of order at most $n \geqslant 0$, and $D(R) = \bigcup_n D^n(R)$ is the **ring/algebra of differential operators**.

Proposition

- *If* $P \in D^n(R)$, $Q \in D^m(R)$, *then* $P + Q \in D^{\max\{m,n\}}(R)$, $PQ \in D^{m+n}(R)$.
- $D^0(R) = C_{\operatorname{End} R}(R) = \operatorname{End}_R R = R$, $D^1(R) = R \oplus \operatorname{Der} R$.

For $K[X] = K[x_1, \dots, x_n]$, define

$$C_r = \left\{ \sum f_\alpha \partial^\alpha \;\middle|\; |\alpha| \leqslant r, f_\alpha \in K[X] \right\}.$$

Then $C_0 \subset C_1 \subset C_2 \subset \cdots \subset A_n$, and $A_n = \bigcup_r C_r$. In fact, this defines a filtration of A_n, called the **order filtration**.

Theorem

$C_r = D^r(K[X])$. In other words, the ring of differential operators of $K[X]$ is A_n.

Modules over the Weyl Algebra

All modules in this talk will be left modules. $K[X]$ is naturally a left A_n-module, where x_i acts as multiplication and ∂_i acts on x_j giving δ_{ij}.

Proposition

$K[X]$ is an irreducible torsion A_n-module and $K[X] \simeq A_n/J$, where $J := \sum_i A_n \cdot \partial_i$ is the module generated by ∂_i.

Proof.

1 is clearly a generator of $K[X]$. Consider any submodule generated by some $0 \neq f \in K[X]$, with leading term ax^α(in the length-lexicographical order). Then $\partial^\alpha(f) = a\alpha! \neq 0$, hence $A_n \cdot f$ is $K[X]$.

$K[X]$ is clearly torsion.

It can be shown that $\mathrm{Ann}_{A_n}(1) = J$, hence $K[X] \simeq A_n/J$. □

A closely related module is A_n/J' where $J' = \sum_i A_n \cdot x_i$. As a vector space, it is isomorphic to $K[\partial] = K[\partial_1, \ldots, \partial_n]$. Under this isomorphism, ∂_i acts by multiplication and x_i acts on ∂_j giving $-\delta_{ij}$.

Apart from the obvious similarities, the modules $K[X]$ and $K[\partial]$ are related in a deeper way.

Suppose σ is an automorphism of A_n, and M is an A_n-module, then the **twisted module** of M by σ, denoted by M_σ, is defined as follows. $M_\sigma = M$ as an abelian group(or vector space), with the new action $a * m := \sigma(a) \cdot m$, where $a \in A_n$, $m \in M$.

Lemma

If J is a left ideal of A_n, then $(A_n/J)_\sigma = A_n/\sigma^{-1}(J)$.

Proposition

$(K[X])_{\mathcal{F}} = K[\partial]$, *where $\mathcal{F}$ is the Fourier transform.*

Let P be an operator in A_n, then P can be written as $\sum g_\alpha \partial^\alpha$, for $g_\alpha \in K[X]$. This gives rise to the differential equation

$$P(f) = \sum g_\alpha \partial_\alpha(f) = 0.$$

More generally, for operators $P_1, \ldots, P_m$, we have a system of differential equations

$$P_1(f) = \cdots = P_m(f) = 0. \tag{7}$$

The A_n-module **associated** to the system of differential equations (7) is A_n/J, where $J := \sum_i A_n \cdot P_i$ is the left ideal generated by P_i. A **polynomial solution** of (7) is a polynomial $f \in K[X]$ such that $P_i(f) = 0$ for all i.

Theorem

The vector space of polynomial solutions of (7) is isomorphic to $\mathrm{Hom}_{A_n}(A_n/J, K[X])$.

Proof.

Let $f \in K[X]$ be a polynomial solution. Consider the following A_n-homomorphism, $\sigma_f : A_n \to K[X]$, $\sigma_f(D) = D(f)$ for $D \in A_n$. Then $J \subset \ker \sigma_f$ and σ_f induces an A_n-homomorphism $\overline{\sigma}_f : A_n/J \to K[X]$, since $Q \in J = \sum_i A_n \cdot P_i$, then $\sigma_f(Q) = Q(\sigma_f(1)) = Q(f) = 0$.
Check that the following maps are the desired isomorphisms.

$$\{\text{Polynomial solutions}\} \overset{\sim}{\leftrightarrow} \mathrm{Hom}_{A_n}(M, K[X])$$
$$f \mapsto \overline{\sigma}_f$$
$$\tau(1 + J) \leftarrow\!\shortmid \tau.$$

Although we restricted ourselves to polynomial solutions of the system (7), this is not necessary.
Suppose $K = \mathbb{R}$, then the space of smooth function defined on an open set $U \subset \mathbb{R}^n$, denoted by $C^\infty(U)$, is a $A_n(\mathbb{R})$-module.
Similarly, when $K = \mathbb{C}$, the space of holomorphic functions defined on U is an $A_n(\mathbb{C})$-module.
These examples inspire us to make the following definition.

Definition

Let S be an A_n-module, and M be a finitely generated A_n-module, then $\mathrm{Hom}_{A_n}(M, S)$ is the **solution space** of M in S.

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This definition allows us to introduce generalized solutions of differential equations in a natural way. All one has to do is to choose an appropriate A_n-module S. This includes solutions in terms of distributions, hyperfunctions and microfunctions.
For example, consider the operator $x \in A_1(\mathbb{C})$. The “differential equation” $xf = 0$ does not have a non-zero solution even in the space of continuous functions. But it has a solutions in terms of microfunctions, the famous Dirac δ-function.

References

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Thanks!