Introduction to Real Analysis I

(Mathematics 311)

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These slides are available at http://www.math.rutgers.edu/~sussmann/311Fall2020.html

Lecture 1

September 2, 2020

All the information about the course will be posted on

H. J. Sussmann's Math 311 Web page:

http://www.math.rutgers.edu/ ~sussmann/311Fall2020.html

The Web page is not ready yet, but will be soon (not later than Friday September 4).

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Mathematical Analysis is the branch of mathematics that deals with limits and related concepts such as continuous functions, differentiation, integration, and differential equations.

It is possible to do Analysis in different settings, such as the real numbers, the complex numbers, the spaces $\mathbb{R}^n$, etc.

Real Analysis is the branch of Mathematical Analysis that studies limits, continuity of functions, differentiation and integration, in the setting of $\mathbb{R}$, the field of real numbers.

One could do Analysis on other number systems, for example $\mathbb{Q}$ (the system of rational numbers), or $\mathbf{Q}_p$, the system of "p-adic numbers" (if p is a prime number).

But Analysis on, for example, the rational numbers, is not very interesting, because most limits that ought to exist (such as, for example, $\sqrt{2}$) don't exist.

The real numbers are invented precisely for the purpose of being able to guarantee the existence of lots of limits, so that a rich theory can be developed.

This course is an introduction to Real Analysis.

HERE IS EVERYTHING

YOU NEED TO KNOW ABOUT

THE REAL NUMBER SYSTEM:

$\mathbb{R}$, the real number system, is a complete ordered field.

There are lots of different complete ordered fields. But they are all "the same", so we can work on a complete ordered field and we can exchange messages about it, even if we don't really know which complete ordered field we are dealing with.

It's like our textbook. I have a copy of the book here, and you have a copy over there. They are **not** the same object. (For example: my copy is here, yours is there. My copy has a coffee stain on page 17, yours does not. So they are clearly different.) But they are "the same", because they have matching parts that we can talk about. Mine has a page 17, an on page 17 there is a Lemma 0.12 that says that the composite of two one-to-one functions is one-to-one. Yours also has a page 17, an on page 17 there is also a Lemma 0.12 that says that the composite of two one-to-one functions is one-to-one.

So we can talk about "the book", as if your book was the same as my book. I can tell you, based on looking at *my* copy of the book, "look at page 17 and read Lemma 0.12", and then you can look at *your* copy of the book and you will find a page 17 and a Lemma 0.12. And when you read Lemma 0.12 it will say the same thing as the Lemma 0.12 in my copy of the book.

In other words, your copy of the book and my copy of the book are **isomorphic**, in the sense that every object that is part of my copy (every page, every lemma, every theorem, every word, every picture) can be matched with a similar part (page, lemma, theorem, word, picture) in your copy.

Thanks to the fact that your copy of the book and my copy of the book are isomorphic, you and I can communicate about them. I can look at Figures 3.2 and 3.3 on page 99 of *my* book, and tell you to look at them, and then you will be able to look at them, because you will find Figures 3.2 and 3.3 on page 99 of *your* book.

The same is true of complete ordered fields. They are all **the same**, meaning that any two of the are **isomorphic**. (I will give you the precise definition of "isomorphic" later in the course.)

So, for example, every complete ordered field has a 1, and a 2, and a 3, and a π, and a $\sqrt{2}$ in it. And every complete ordered field $\mathbb{F}$ has within it its own set $\mathbb{Q}_{\mathbb{F}}$ of "rational numbers" and its its own set $\mathbb{F}-\mathbb{Q}_{\mathbb{F}}$ of "irrational numbers"

So, if you have in your mind one particular complete ordered field $\mathbb{F}$, and I have in my mind a different complete ordered field $\mathbb{G}$, we can talk and exchange views about them as if they were the same. I can say to you, for example, that "$\sqrt{2}$ is irrational", meaning that in **my** complete ordered field $\mathbb{G}$, $\sqrt{2}$ is not a member of $\mathbb{Q}_{\mathbb{G}}$, and you will understand perfectly well what I mean, by taking it to mean that in **your** complete ordered field $\mathbb{F}$, $\sqrt{2}$ is not a member of $\mathbb{Q}_{\mathbb{F}}$.

Things are totally different if we talk about ordered fields.

There are, of course, lots and lots of ordered fields, but they are **not** all isomorphic to each other.

For example, the rational number system $\mathbb{Q}$ is an ordered field, and the real number system $\mathbb{R}$ is an ordered field, but they are truly different. (The mathematical way to say that is: $\mathbb{Q}$ and $\mathbb{R}$ are not isomorphic.)

In particular, for example, in $\mathbb{R}$ there is a $\sqrt{2}$. (We will prove later in the course that $(\exists r \in \mathbb{R})(r^2 = 2 \wedge r > 0)$.)

But in $\mathbb{Q}$ there is no such thing as $\sqrt{2}$. (You must have proved in Math 300 that "$\sqrt{2}$ is irrational", that is, that $\sim (\exists r \in \mathbb{R})(r^2 = 2 \wedge r > 0)$.)

This shows that $\mathbb{Q}$ and $\mathbb{R}$ are not just "different"; they are truly not the same; **they are not isomorphic**.

Definition. A field is a set $\mathbb{F}$, equipped with two binary operations $+ : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$ and $\times : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$ (called "addition" and "multiplication", respectively) and two special members of $\mathbb{F}$ called "0" ("zero") and "1" ("one"), such that (writing "$x + y$" rather than "$+((x, y))$" and "$x \times y$", or "$x \cdot y$", or "xy" rather than "$\times((x, y))$") the following **field axioms** hold:

F1: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})u + v = v + u.$

F2: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})u + (v + w) = (u + v) + w.$

F3: $0 \in \mathbb{F}.$

F4: $(\forall u \in \mathbb{F})u + 0 = u.$

F5: $(\forall u \in \mathbb{F})(\exists v \in \mathbb{F})u + v = 0.$

F6: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})uv = vu.$

F7: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})u(vw) = (uv)w.$

F8: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})(u + v)w = uw + vw.$

F9: $1 \in \mathbb{F}.$

F10: $\sim 0 = 1.$

F11: $(\forall u \in \mathbb{F})u.1 = u.$

F12: $(\forall u \in \mathbb{F})(\sim u = 0 \Rightarrow (\exists v \in \mathbb{F})uv = 1).$

Definition. An ordered field is a field $\mathbb{F}$, equipped —in addition to 0, 1, and the binary operations $+$ and $\times$— with a binary relation $<$ on $\mathbb{F}$ (i.e., a subset $<$ of $\mathbb{F} \times \mathbb{F}$), such that (writing "$x < y$" rather than "$(x, y) \in <$") the following **ordered field axioms** hold:

F13: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})\Big((u < v \wedge v < w) \Rightarrow u < w\Big)$.

F14: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})\Big((u < v \vee u = v \vee v < u) \wedge \sim (u < v \wedge u = v)\Big)$.

F15: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})(u < v \Rightarrow u + w < v + w)$.

F16: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})(u < v \Rightarrow uw < vw)$.

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F15: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})(u < v \Rightarrow u + w < v + w)$.

F16: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})(u < v \Rightarrow uw < vw)$.

This is wrong! F16 as written is false! If $u = 1$, $v = 2$, $w = -1$, then "$u < v$" is true (because $1 < 2$) but "$uw < vw$" is false, because "$-1 < -2$" is false!

Definition. An ordered field is a field $\mathbb{F}$, equipped —in addition to 0, 1, and the binary operations $+$ and $\times$— with a binary relation $<$ on $\mathbb{F}$ (i.e., a subset $<$ of $\mathbb{F} \times \mathbb{F}$), such that (writing "$x < y$" rather than "$(x, y) \in <$") the following **ordered field axioms** hold:

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F16: $(\forall u \in \mathbb{F})(\forall v \in \mathbb{F})(\forall w \in \mathbb{F})\Big((u < v \wedge 0 < w) \Rightarrow uw < vw\Big)$.

Now it is correct!

There are lots and lots of different fields. (And I mean, *really* different.)

Each field has a 1, and then a 2 (because $2 = 1 + 1$), and then a 3 (because $3 = 2 + 1$), and so on.

So each field $\mathbb{F}$ has its own "system of natural numbers", let's call it $\mathbb{N}_{\mathbb{F}}$.

The members of $\mathbb{N}_{\mathbb{F}}$ are 1, $1 + 1$, $1 + 1 + 1$, ..., and so on.

Here is a precise definition:

1. In a field $\mathbb{F}$, a subset S of $\mathbb{F}$ is said to be inductive if:

   (a) $1 \in S$,

   (b) whenever $n \in S$, it follows that $n+1 \in S$. (That is, precisely: $(\forall n)(n \in S \Rightarrow n + 1 \in S)$.)

2. A natural number in $\mathbb{F}$ is a member n of $\mathbb{F}$ such that n belongs to every inductive subset of $\mathbb{F}$.

3. The set of natural numbers of the field $\mathbb{F}$ is called $\mathbb{N}_{\mathbb{F}}$.

Example. The integers modulo 7 are the numbers 0, 1, 2, 3, 4, 5, and 6. So the set of all integers modulo 7 is the set $\mathbb{Z}_7$ given by

$$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}\,.$$

Addition and multiplication are defined as for ordinary integers, except that every multiple of 7 is thrown away. So, for example, the following are true in $\mathbb{Z}_7$:

$$\begin{aligned} 3+5 &= 1\,, \\ 3\times 5 &= 1\,, \\ 6\times 5+4 &= 6 \\ &= -1\,. \end{aligned}$$

Clearly, $\mathbb{N}_{\mathbb{Z}_7}$ **is all of** $\mathbb{Z}_7$. In particular, the set of natural numbers of $\mathbb{Z}_7$ is a finite set, with exactly seven members.

And $0 \in \mathbb{N}_{\mathbb{Z}_7}$. (That is: in $\mathbb{Z}_7$, 0 is a natural number)

As the previous example shows, the set $\mathbb{N}_{\mathbb{F}}$ of the natural numbers of the field $\mathbb{F}$ does not need at all to look like the set $\mathbb{N}$ of natural numbers that you are used to, that is, the set of natural numbers of the field $\mathbb{R}$ of real numbers.

For example, $\mathbb{N}_{\mathbb{Z}_7}$ is a finite set, with 7 members, whereas $\mathbb{N}_{\mathbb{R}}$ (the set of natural numbers in the field $\mathbb{R}$ of "true" real numbers) is, as you know very well, an infinite set.

But it turns out that for **ordered fields**, all the natural number systems look the same, that is, exactly like the natural numbers you know.

The reason for this is very simple.

First of all, in an ordered field $\mathbb{F}$, $0 < 1$.

(Proof: use Axioms F10, F13, F14, F15, and F16. By F14, either $0 < 1$ or $0 = 1$ or $1 < 0$. But "$0 = 1$" is excluded by F10. So either $0 < 1$ or $1 < 0$. If $0 < 1$ then of course $0 < 1$. If $1 < 0$, then by F15 we can add -1 to both sides of "$1 < 0$, and get $0 < -1$. Then by F16 $0 < (-1) \times (-1)$, so $0 < 1$. So $0 < 1$ in all possible cases, and then $0 < 1$.)

(Exercise: Write a rigorous proof from the axioms of the proposition "$(-1) \times (-1) = 1$". First you will need to *define* -1, because -1 is not mentioned in the axioms. You have to prove that there is a *unique* $r \in \mathbb{F}$ such that $1 + r = 0$ and then -1 is, by definition, this r.)

Then $0 < 1 + 1$, by F15, so $0 < 2$. Similarly, $0 < 2 + 1$ by F15, so $0 < 3$. And so on, so we get $0 < n$ for every natural number $n \in \mathbb{N}_{\mathbb{F}}$. (That is $(\forall n \in \mathbb{N}_{\mathbb{F}})0 < n$.)

(Exercise: Write a rigorous proof of this, **without using the very fishy "and so on"**. Use the definition of $\mathbb{N}_{\mathbb{F}}$. Prove that the set $\{x \in \mathbb{F} : 0 < x\}$ is inductive, and conclude that "$0 < x$" is true for every $x \in \mathbb{N}_{\mathbb{F}}$.)

And then in particular we can prove that, for every natural number $n \in \mathbb{N}_{\mathbb{F}}$, $n \neq 0$. (That is $(\forall n \in \mathbb{N}_{\mathbb{F}})n \neq 0$.)

(Exercise: Write a rigorous proof of this.)

So, in an ordered field $\mathbb{F}$, when you count ("1,2,3,4,5,...") **you never get back to zero**, unlike what happens in, for example, $\mathbb{Z}_7$, where when you count you get 1,2,3,4,5,6,0.

That's why the natural number systems of all **ordered fields** "look alike", or "look the same", in the sense that they are all like the natural number system $\mathbb{N}$ (the natural numbers corresponding to $\mathbb{R}$, the real number system) that you are most familiar with.

AN IMPORTANT DECISION FOR US

Our textbook calls the set of all natural numbers "J".
I think that's awful. Nobody I know uses "J" for the set of natural numbers.
But of course we can call anything any name we want.
Exactly as it us up to people to decide whether or not to call Pluto a "planet", or whether we will call 1 a "prime number", it is up to us to decide whether in this course we want to follow the book and use "J" as the name for the set of natural numbers.
My inclination is to call it "$\mathbb{N}$".
But some students may want me to follow the book.
So I am putting this to a vote.
Send me your vote. Tell me if you want me to use "J" or "$\mathbb{N}$" as the name of the set of all natural numbers.
Send me your vote by e-mail, to hectorjsussmann@gmail.com.
Use the subject "Vote on the natural numbers". **Deadline: Monday, September 7, at 5:00PM.**

Ordered fields can look very different from one another. (For example, as explained before, the rational number system $\mathbb{Q}$ and the real number system $\mathbb{R}$ are very different: in $\mathbb{Q}$ there is no $\sqrt{2}$, but in $\mathbb{R}$ there is.)

One important way in which ordered fields can differ from one another is that some ordered fields can be **Archimedean** while other ordered fields are **non-Archimedean**.

Definition 1. In an ordered field $\mathbb{F}$,

1. an <u>infinitesimal</u> is a member r of $\mathbb{F}$ such that $0 < r$ but $r < \frac{1}{n}$ for every $n \in \mathbb{N}_{\mathbb{F}}$;
2. an <u>infinitely large</u> member of $\mathbb{F}$ is a member r of $\mathbb{F}$ such that $n < r$ for every $n \in \mathbb{N}_{\mathbb{F}}$.

Clearly, if r is an infinitesimal then $\frac{1}{r}$ is infinitely large, and if r is infinitely large then $\frac{1}{r}$ is infinitesimal.

(This is very easy to prove. Make sure you know how to prove it.)

Definition 2. An ordered field $\mathbb{F}$ is Archimedean if it contains no infinitesimals.

Another way to see this is as follows: **An ordered field $\mathbb{F}$ is Archimedean if and only if it has no infinitely large members.**

Theorem. In an Archimedean ordered field $\mathbb{F}$, the following is true: for every member r of $\mathbb{F}$ there exists a unique integer $n \in \mathbb{Z}_{\mathbb{F}}$ such that $n \leq r < n + 1$.

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Mmmm... Something is wrong here.

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Mmmm... Something is wrong here.

Maybe the problem is that I have never defined what the "integers" of $\mathbb{F}$ are.

Definition. In a field $\mathbb{F}$, an integer is a member n of $\mathbb{F}$ such that either $n \in \mathbb{N}_{\mathbb{F}}$ or $-n \in \mathbb{N}_{\mathbb{F}}$ or $n = 0$.

We use "$\mathbb{Z}_{\mathbb{F}}$" to denote the set of all integers of $\mathbb{F}$.

Theorem A. In an Archimedean ordered field $\mathbb{F}$, the following is true: for every member r of $\mathbb{F}$ there exists a unique integer $n \in \mathbb{Z}_{\mathbb{F}}$ such that $n \leq r < n + 1$.

Now it's O.K.

Exercise: A commutative ring with unity is a set $\mathcal{R}$ equipped with binary operations $+ : \mathcal{R} \times \mathcal{R} \mapsto \mathcal{R}$ and $\times : \mathcal{R} \times \mathcal{R} \mapsto \mathcal{R}$, as well as members $0, 1$ of $\mathcal{R}$, that satisfy all the axioms F1 to F11, that is, all the field axioms except possible for Axiom F12.

Prove, using the definition I have given of $\mathbb{Z}_{\mathbb{F}}$, that if $\mathbb{F}$ is any field then $\mathbb{Z}_{\mathbb{F}}$ is a commutative ring with unity. NOTE: Most of the axioms are very easy to prove. The hardest things to prove are probably that the operations $+$ and $\times$ actually do map $\mathbb{Z}_{\mathbb{F}} \times \mathbb{Z}_{\mathbb{F}}$ to $\mathbb{Z}_{\mathbb{F}}$, that is, that the sum and the product of two integers is an integer.

Proof of Theorem A. Let r be an arbitrary member of $\mathbb{F}$.

Suppose first that $0 < r$.

Since $\mathbb{F}$ is Archimedean, r is not infinitely large.

So we can find a natural number $k \in \mathbb{N}_{\mathbb{F}}$ such that $r < k$.

Let $S = \{n \in \mathbb{N}_{\mathbb{F}} : r < n\}$. Then S is nonempty, because $k \in S$.

So by the Well-ordering Principle, S has a smallest member. Let's us call this smallest member m. Then $m \in \mathbb{N}_{\mathbb{F}} S$, because S is a set of natural numbers, i.e., $S \subseteq \mathbb{N}_{\mathbb{F}}$.

Since $m \in \mathbb{N}_{\mathbb{F}}$, it follows that $1 \leq m$.

(Why? Make sure you know how to prove that $(\forall n \in \mathbb{N}_{\mathbb{F}})1 \leq n$. The proof is by induction, that is, using the definition of $\mathbb{N}_{\mathbb{F}}$ that was given earlier: Let $A = \{n \in \mathbb{N}_{\mathbb{F}} : 1 \leq n\}$. Prove that A is inductive. Conclude that $\mathbb{N}_{\mathbb{F}} \subseteq A$, so $1 \leq n$ for every natural number n. Naturally, you must first **define** "$\leq$", but that's easy: if $a, b \in \mathbb{F}$, we say that $\underline{a \leq b}$ if $a = b \vee a < b$.)

Since $1 \leq m$, that is, $1 = m \vee 1 < m$, we will consider the two cases, $m = 1$ and $1 < m$.

First, suppose $m = 1$. Then of course $m \in \mathbb{Z}_{\mathbb{F}}$ and $m - 1 \in \mathbb{Z}_{\mathbb{F}}$ as well, because $m = 1$ and $m - 1 = 0$.

Next, suppose that $1 < m$. Then $m \in \mathbb{N}_{\mathbb{F}}$, so $m \in \mathbb{Z}_{\mathbb{F}}$ and $m \neq 1$. Therefore $m - 1 \in \mathbb{N}_{\mathbb{F}}$.

(Why? Make sure you know how to prove that $(\forall n \in \mathbb{N}_{\mathbb{F}})(n \neq 1 \Rightarrow n - 1 \in \mathbb{N}_{\mathbb{F}})$.)

Then $m - 1 \in \mathbb{Z}_{\mathbb{F}}$ as well.

So, in both cases ($1 = m$ and $1 < m$), we have $m \in \mathbb{Z}_{\mathbb{F}}$ and $m-1 \in \mathbb{Z}_{\mathbb{F}}$. Let $n = m - 1$. Then n is an integer $n \leq r$ and $r < n + 1$.

(It is clear that $r < n+1$, because $n+1 = m$ and $r < m$, since $m \in S$. Why is $n \leq r$? Because it cannot happen that $r < n$. Why? Because if $r < n$ then n would belong to S. But $n = m - 1$, and m was the **smallest** member of S, so $m-1$ cannot be in S. So "$r < n$" cannot be true, and then the trichotomy law —i.e., Axiom F14— implies that $n \leq r$.)

So if $0 < r$ we have found an integer $n \in \mathbb{Z}_{\mathbb{F}}$ such that $n \leq r < n+1$.

Next, we have to look at the case when $r = 0$. But in that case we can take n to be 0, and it is clear that $n \leq r < n+1$.

Finally, what if $r < 0$? Exercise: you should prove that in this case it is also true that $(\exists n \in \mathbb{Z}_{\mathbb{F}}) n \leq r < n+1$. It's easy, but you have to be careful. You can "almost" do it this way: Let $s = -r$. Using the first part of our proof, find $m \in \mathbb{Z}_{\mathbb{F}}$ such that $m \leq s < m+1$. Then $-(m+1) < r \leq -m$. Let $n = -(m+1)$, so $n = -m-1$. Then $n+1 = -m$. So $n < r \leq n+1$. This is almost what we want. Now you need a small extra step, and you are done.

All this was to prove the **existence** of n. We have proved that

$$(\forall r \in \mathbb{F})(\exists n \in \mathbb{Z}_{\mathbb{F}})n \leq r < n + 1.$$

How about uniqueness?

Exercise: Prove that for every $r \in \mathbb{F}$ the integer n that has just been proved to exist is unique. NOTE: This is easy, but precisely for this reason you should be able to write down a detailed, rigorous proof.

The previous theorem is exactly the book's Theorem 0.21, on page 23, except for two details:

1. The book does **not** state or prove the uniqueness of n.

   Uniqueness is **very** important. This integer, the unique integer n such that $n \leq r < n+1$, has a **name**, as you know: it's called the integer part of r. And you probably know the notation for it: it's "$[r]$". (Examples: $[1.3] = 1$, $[\sqrt{2}] = 1$, $[e] = 2$, $[\pi] = 3$, $[-4.7] = -5$.)

   And in mathematics we can only give a name to *the thing such that blahblahblah* if there exists a **unique** *thing such that blah-blablah*.

   For example, we cannot give a name to "the square root of 2" (and, say, call it "square root of 2"), because there are two of them. But we can give a name to the **positive** square root of 2, and call it "$\sqrt{2}$", because there is only one positive square root of 2.

So it is important to know that the integer n such that $n \leq r < n + 1$ is unique, because it's thanks to this that we can talk about **the** integer part of a real number r.

2. The book states and proves the theorem for $\mathbb{R}$, using the fact that $\mathbb{R}$ is a **complete ordered field**. We have stated and prove proved the theorem for **a general Archimedean field**.

   Our theorem is more general, because there are lots of Archimedean fields other than $\mathbb{R}$, such as for example $\mathbb{Q}$, the field of rational numbers, and our theorem applies to all of them.

Is our theorem **really** more general? The book's theorem applies to $\mathbb{R}$, which is the case that we all care about.

Does our theorem apply to $\mathbb{R}$?

Yes, because:

1. $\mathbb{R}$ is a **complete** ordered field,

2. Every complete ordered field is Archimedean.

But that's the stuff of Lecture No. 2, on Tuesday September 8.

A reminder: on Tuesday September 8 we have **Monday classes**, so in particular we **do** have a meeting of our Math 311 class.

See you on Tuesday.

Have a nice weekend.

Lectures 2 and 3

September 8 and 9, 2020

WHAT HAVE WE BEEN DOING?

We have been trying to **start doing Analysis**, by introducing the most basic objects of elementary mathematics:

- the natural numbers,
- the integers,
- the rational numbers,
- the real numbers.

There are (at least) three ways to do this.

THE FIRST WAY

We start with the **natural numbers** and the **integers**.

We then construct all the other things.

The natural numbers are characterized by some system of axioms such as the **Peano axioms**, formulated by the Italian mathematician **Giuseppe Peano** (1858-1932).

I will not follow this approach here. But let me point out that the most important one of the Peano axioms is the **Principle of Mathematical Induction (PMI)**.

In this approach, one starts with $\mathbb{N}$, then constructs $\mathbb{Z}$ (the integers), then constructs $\mathbb{Q}$ (the rational numbers), and then constructs $\mathbb{R}$ (the real numbers).

The German mathematician **Leopold Kronecker** (1823 1891) famously was quoted as having said **God made the integers, all else is the work of man"**.

The Peano axioms:

1. $1 \in \mathbb{N}$,
2. $(\forall n \in \mathbb{N})\sigma(n) \in \mathbb{N}$,
3. $(\forall m, n \in \mathbb{N})(\sigma(m) = \sigma(n) \Rightarrow m = n)$,
4. $(\forall n \in \mathbb{N})\sigma(n) \neq 1$,
5. *(the Principle of Mathematical Induction)* If S is a set such that $1 \in S$ and for every natural number n, $n \in S \Rightarrow \sigma(n) \in S$, then $\mathbb{N} \subseteq S$.

Here "σ" must be thought of as "the successor function". That is, $\sigma(n) = n + 1$ for $n \in \mathbb{N}$.

THE SECOND WAY

We start with the **real numbers**.

This is the approach that the book tries to follow, but in my view it only does so half-way.

I am trying to follow the same approach as the book, but I am trying to do it better and fill in the many gaps in the book's presentation.

In this approach, the real number system $\mathbb{R}$ is characterized as a **complete ordered field**.

WHAT DOES "CHARACTERIZE" MEAN?

To **characterize** a thing T means "to say enough about T to single it out uniquely". Or, if you prefer, to answer the question "what is it about T that makesit different from all other things?"

Example: If I say

> Rutgers is a university in New Jersey,

then this is a true statement about Rutgers, but is not a **characterization** of Rutgers, because there are lots of other universities in New Jersey.

But if I say

> Rutgers is the state university of New Jersey,

then **that** is a characterization of Rutgers, because it singles out Rutgers from all other universities: there is only one university that is **the** state university of New York Jersey.

Similarly, if I say

$\mathbb{R}$ is a field,

or

$\mathbb{R}$ is an ordered field,

or

$\mathbb{R}$ is an Archimedean ordered field,

then these are all true statements about $\mathbb{R}$. But they are not characterizations of $\mathbb{R}$, because there are lots of fields other than $\mathbb{R}$, lots of ordered fields other than $\mathbb{R}$, and even lots of Archimedean ordered fields other than $\mathbb{R}$.

But if I say

$\boxed{\boxed{\mathbb{R} \text{ is a complete ordered field}}}$,

then **that** is a characterization of the real number system, because **there is only one complete ordered field**.

(Actually, as I explained before, there are lots of different complete ordered fields, but they are are all "the same", exactly in the same way as there are lots of different copies of our textbook, but they are all the same, in the sense that they are all identical copies of each other. Or, as we would say in mathematics, they are all isomorphic.)

In this approach, we **start** with $\mathbb{R}$, and then we **define** $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

The way we do this is by defining, for every field $\mathbb{F}$, the systems $\mathbb{N}_{\mathbb{F}}$, $\mathbb{Z}_{\mathbb{F}}$, $\mathbb{Q}_{\mathbb{F}}$.

And then, in particular, when $\mathbb{F}$ is $\mathbb{R}$ (that is, when $\mathbb{F}$ is a complete ordered field) then the corresponding $\mathbb{N}_{\mathbb{F}}$, $\mathbb{Z}_{\mathbb{F}}$, $\mathbb{Q}_{\mathbb{F}}$ are the natural number system $\mathbb{N}$, the integer system $\mathbb{Z}$, and the rational number system $\mathbb{Q}$.

Here is the precise definition of the natural number system that we gave before:

1. In a field $\mathbb{F}$, a subset S of $\mathbb{F}$ is said to be inductive if:

   (a) $1 \in S$,

   (b) whenever $n \in S$, it follows that $n+1 \in S$. (That is, precisely: $(\forall n)(n \in S \Rightarrow n + 1 \in S)$.)

2. A natural number in $\mathbb{F}$ is a member n of $\mathbb{F}$ such that n belongs to every inductive subset of $\mathbb{F}$.

3. The set of natural numbers of the field $\mathbb{F}$ is called $\mathbb{N}_{\mathbb{F}}$.

From this definition we can easily **prove** the "Principle of Mathematical Induction":

Theorem. *(The Mathematical Induction Theorem)*
Let S be a subset of a field $\mathbb{F}$, such that

1. $1 \in S$,
2. whenever $n \in S$, it follows that $n + 1 \in S$. (That is, precisely: $(\forall n)(n \in S \Rightarrow n + 1 \in S)$.)

Then $\mathbb{N}_{\mathbb{F}} \subseteq S$. That is, every natural number of $\mathbb{F}$ belongs to S.

Proof. We want to prove that $(\forall n \in \mathbb{N}_{\mathbb{F}})n \in S$. So let $n \in \mathbb{N}_{\mathbb{F}}$ be arbitrary.

Since $n \in \mathbb{N}_{\mathbb{F}}$, n belongs to every inductive subset of $\mathbb{F}$.

And our assumptions on S tell us that S is an inductive subset of $\mathbb{F}$.

So $n \in S$. **QED**

THE THIRD WAY

We start with **sets**.

In this approach, the world of mathematical objects consists of sets, only sets, and nothing but sets.
The natural numbers (starting with 0, because in this approach it is more convenient to start with 0 rather than 1) are then constructed as follows:

- the number "zero" is the empty set (that is, $0 = \emptyset$),
- the number "one" is the one-member set whose only member is 0 (that is, $1 = \{0\}$, i.e., $1 = \{\emptyset\}$),
- the number "two" is the two-member set whose members are 0 and 1 (that is, $2 = \{0, 1\}$, i.e., $2 = \{\emptyset, \{\emptyset\}\}$),
- the number "three" is the three-member set whose members are 0, 1, and 2 (that is, $3 = \{0, 1, 2\}$, i.e., $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$),
- and so on.

The phrase "and so on" above is problematic. But this can be made precise and a rigorous general definition of "natural number" can be given. without using vague phrases such as "and so on".

If this worries you, here is the definition: a *natural number* is a finite transitive set which is well-ordered by "$\in$". (And the words here have a precise meaning: (1) a *transitive set* is a set S such that every member of S is a subset of S; (2) a *finite set* is a set S such that every nonempty set $\mathcal{A}$ of subsets of S has a maximal member, (3) a set S is *well-ordered by "$\in$"* if every nonempty subset of S has an $\in$-first member, (4) a *maximal member* of a set $\mathcal{A}$ of subsets of S is a member M of $\mathcal{A}$ such that there exists no member N of $\mathcal{A}$ such that $M \subseteq N$ and $M \neq N$, and (5) an *$\in$-first member* of a set T is a member t of T that belongs to every member s of T such that $s \neq t$.)

And if you find this definition too abstruse and hard to swallow, try doing this exercise: (a) prove that the sets 0, 1, 2, 3 defined above are natural numbers in this sense, and then (b) prove that if S is any set with three or fewer members that is a natural number in this sense, then $S = 0$ or $S = 1$, or $S = 2$, or $S = 3$.

Here is the program for this week's lectures:

1. Proof of the well-ordering principle (WOP).

   (This is needed because in Lecture 1 I used the WOP but had not proved it.)

2. The rational number system $\mathbb{Q}_{\mathbb{F}}$ of a field $\mathbb{F}$.

3. Density of $\mathbb{Q}_{\mathbb{F}}$ in an Archimedean field $\mathbb{F}$.

4. Completeness.

5. Complete ordered fields are Archimedean.

6. Existence of $\sqrt{2}$.

1. PROOF OF THE WELL-ORDERING PRINCIPLE (WOP)

We want to prove the following:

Theorem WOP. If $\mathbb{F}$ is an ordered field, and S is a nonempty subset of $\mathbb{N}_{\mathbb{F}}$, then S has a smallest member.

And, to be completely precise, let me explain what we means by a "smallest member" of a set:

Let $\mathbb{F}$ be an ordered field and let S be a subset of $\mathbb{F}$. A **smallest member of** S is a member s_* of S such that $s_* \leq s$ for every $s \in S$.

Theorem. *If $\mathbb{F}$ is an ordered field, and a subset S of has a smallest member, then that smallest member is unique.*

Proof. This is a recommended exercise for you. It's very easy. (HINT: Prove that if s_1, s_2 are smallest members of S, then $s_1 = s_2$.)

Before we prove the WOP, we review the definition of $\mathbb{N}_{\mathbb{F}}$:

1. In a field $\mathbb{F}$, a subset S of $\mathbb{F}$ is said to be inductive if:

   (a) $1 \in S$,

   (b) whenever $n \in S$, it follows that $n + 1 \in S$. (That is, precisely: $(\forall n)(n \in S \Rightarrow n + 1 \in S)$.)

2. A natural number in $\mathbb{F}$ is a member n of $\mathbb{F}$ such that n belongs to every inductive subset of $\mathbb{F}$.

3. The set of natural numbers of the field $\mathbb{F}$ is called $\mathbb{N}_{\mathbb{F}}$.

Proof of Theorem WOP:

We have to prove several lemmas first.

Lemma WOP1. *The set $\mathbb{N}_{\mathbb{F}}$ is an inductive subset of of $\mathbb{F}$.*

Proof of Lemma WOP1. First, we have to prove that $1 \in \mathbb{N}_{\mathbb{F}}$, and for that purpose we have to show that 1 belongs to every inductive subset of $\mathbb{F}$.

So let S be an inductive subset of $\mathbb{F}$.

Then $1 \in S$, by the definition of "inductive set".

Since this is true for an arbitrary inductive subset S of $\mathbb{F}$, it follows that 1 belongs to every inductive subset of $\mathbb{F}$.

Hence $1 \in \mathbb{N}_{\mathbb{F}}$.

Now we have to prove that if $n \in \mathbb{N}_{\mathbb{F}}$ then $n + 1 \in \mathbb{N}_{\mathbb{F}}$.

So let n be such that $n \in \mathbb{N}_{\mathbb{F}}$. We have to prove that $n + 1 \in \mathbb{N}_{\mathbb{F}}$, and for that purpose we have to show that $n + 1$ belongs to every inductive subset of $\mathbb{F}$.

So let S be an inductive subset of $\mathbb{F}$. Then $n \in S$, because $n \in \mathbb{N}_{\mathbb{F}}$, so n belongs to every inductive subset of $\mathbb{F}$.

Then, by the definition of "inductive set", it follows that $n + 1 \in S$.

So we have proved that if $n \in \mathbb{N}_{\mathbb{F}}$ then $n + 1 \in \mathbb{N}_{\mathbb{F}}$, for an arbitrary $n \in \mathbb{F}$.

This completes the proof that $\mathbb{N}_{\mathbb{F}}$ is an inductive subset of of $\mathbb{F}$.

QED *(Lemma WOP1)*

Lemma WOP2. *If $\mathbb{F}$ is an ordered field, a, b, c, d belong to $\mathbb{F}$, $a \leq b$ and $c < d$, then $a + c < b + d$. This is one of the several ways in which you can "add inequalities".*

Proof of Lemma WOP2. Since $c < d$, we have $c + a < d + a$ by Axiom F15. And also $c + a = a + c$ and $d + a = a + d$, so $a + c < a + d$.

Since $a \leq b$, we have $a = b$ or $a < b$.

Suppose first that $a = b$. Then, from "$a + c < a + d$", we get $a + c < b + d$.

Now suppose that $a < b$. Then $a + d < b + d$ by Axiom F15. So $a + c < a + d$ and $a + d < b + d$, and then $a + c < b + d$ by Axiom F13.

So in both cases we have proved that $a + c < b + d$.

QED *(Lemma WOP2)*

Lemma WOP3. $0 < 1$.

Proof of Lemma WOP3. We will use Axioms F10, F13, F14, F15, and F16.

By F14, either $0 < 1$ or $0 = 1$ or $1 < 0$.

But "$0 = 1$" is excluded by F10.

So either $0 < 1$ or $1 < 0$.

If $0 < 1$ then of course $0 < 1$.

If $1 < 0$, then by F15 we can add -1 to both sides of "$1 < 0$, and get $0 < -1$. Then by F16 $0 < (-1) \times (-1)$, so $0 < 1$. So $0 < 1$ in all possible cases, and then $0 < 1$. **QED***(Lemma WOP3)*

Lemma WOP4. $1 \leq m$ *for every* $m \in \mathbb{N}_{\mathbb{F}}$. *(This was proved before, in slide no. 30, but we are repeating the proof here.)*

Proof of Lemma WOP4.

Call a natural number $m \in \mathbb{N}_{\mathbb{F}}$ "nice" if $1 \leq m$.

Let Γ be the set of all $m \in \mathbb{N}_{\mathbb{F}}$ such that m is nice.

We prove that Γ is inductive.

First we have to show that $1 \in \Gamma$. But this is trivial, because $1 \in \mathbb{N}_{\mathbb{F}}$ by Lemma WOP1, and it is obvious that $1 \leq 1$.

Next, we have to show that $(\forall n)(n \in \Gamma \Rightarrow n + 1 \in \Gamma)$ (i.e., that whenever $n \in \Gamma$ it follows that $n + 1 \in \Gamma$).

So let n be an arbitrary member of Γ. Then n is nice. Let us show that $n + 1$ is nice.

First, $n + 1 \in \mathbb{N}_{\mathbb{F}}$ by Lemma WOP1, since $n \in \mathbb{N}_{\mathbb{F}}$ because n is nice, and $\mathbb{N}_{\mathbb{F}}$ is inductive.

Furthermore, since n is nice, we have $1 \leq n$. And in addition $0 < 1$. So by Lemma WOP2, $1 \leq n + 1$. Hence $n + 1$ is nice. So $n + 1 \in \Gamma$.

So Γ is an inductive subset of $\mathbb{F}$.

If $m \in \mathbb{N}_{\mathbb{F}}$, then m belongs to every inductive subset of $\mathbb{F}$. So $m \in \Gamma$. So m is nice. Hence $1 \leq m$. **QED***(Lemma WOP4)*

In our next lemma we will make use of " $-u$ ", for $u \in \mathbb{F}$, so we have to explain what that means.

What is the meaning of "$-n$"? Axiom F5 says that if $u \in \mathbb{F}$ then there exists a $v \in \mathbb{F}$ such that $u + v = 0$. And then we can easily prove that **for every $u \in \mathbb{F}$, the $v \in \mathbb{F}$ such that $u+v = 0$ is unique**.

(*Proof:* Let $u \in \mathbb{F}$. Let v_1, v_2 be such that $u+v_1 = 0$ and $u+v_2 = 0$. Then

$$v_1 = v_1+0 = v_1+(u+v_2) = (v_1+u)+v_2 = (u+v_1)+v_2 = 0+v_2 = v_2+0 = v_2$$

So $v_1 = v_2$. **QED**)

Definition. For $u \in \mathbb{F}$, the unique $v \in \mathbb{F}$ such that $u+v = 0$ is called **minus** u, and the symbol "$-u$" is used to denote it.

Definition. For $u, v \in \mathbb{F}$, "$u-v$" stands for "$u+(-v)$". The function $\mathbb{F} \times \mathbb{F} \ni (u, v) \mapsto u - v \in \mathbb{F}$ is called the subtraction operation of $\mathbb{F}$.

Lemma WOP5. *If $\mathbb{F}$ is a field, and m, n are integers of $\mathbb{F}$, then $m + n$ and $-n$ are integers of $\mathbb{F}$.*

Proof of Lemma WOP5. Let $\nu = -n$. Since $n \in \mathbb{Z}_{\mathbb{F}}$, either $n \in \mathbb{N}_{\mathbb{F}}$ or $n = 0$, or $-n \in \mathbb{N}_{\mathbb{F}}$. Then $-\nu \in \mathbb{N}_{\mathbb{F}}$ or $\nu = 0$, or $\nu \in \mathbb{N}_{\mathbb{F}}$. So $\nu \in \mathbb{Z}_{\mathbb{F}}$.

Proving that $m + n \in \mathbb{Z}_{\mathbb{F}}$ whenever $m, n \in \mathbb{Z}_{\mathbb{F}}$ was a homework problem. **QED** *(Lemma WOP5)*

Lemma WOP6. *If k, n are integers, then either $k \leq n$ or $n+1 \leq k$. (This says that there are no integers k, n such that $n < k < n+1$, because every integer k is either $\leq n$ or $\geq n+1$.)*

Proof of Lemma WOP6. Suppose k, n are integers such that $n < k < n+1$.

Let $\kappa = k - n$. Then $0 < \kappa < 1$. By Lemma WOP5, κ is an integer. Then $\kappa \in \mathbb{N}_{\mathbb{F}}$ or $\kappa = 0$ or $-\kappa \in \mathbb{N}_{\mathbb{F}}$, by the definition of $\mathbb{Z}_{\mathbb{F}}$. Since $0 < \kappa$, it follows that $\kappa \neq 0$ by Axiom F14. (If $\kappa = 0$, then we would have $0 < 0$, because $0 < \kappa$.) So $\kappa \in \mathbb{N}_{\mathbb{F}}$ or $-\kappa \in \mathbb{N}_{\mathbb{F}}$.

If $-\kappa \in \mathbb{N}_{\mathbb{F}}$ then we would have $1 \leq -\kappa$ by Lemma WOP3. So $0 < \kappa$ and $1 \leq -\kappa$, and then by Lemma WOP3 $1 < 0$. But $0 < 1$ by Lemma WOP2. So $1 < 0$ and $0 < 1$ and then, by Axiom F13, $0 < 0$, contradicting Axiom F14. So "$-\kappa \in \mathbb{N}_{\mathbb{F}}$" is excluded.

Hence $\kappa \in \mathbb{N}_{\mathbb{F}}$. So by Lemma WOP5, $1 \leq \kappa$. But $\kappa < 1$, so we have $1 \leq \kappa \wedge \kappa < 1$, and this easily leads to a contradiction. **QED(Lemma WOP6)**

Proof of the WOP.

Let us call a member of $\mathbb{N}_{\mathbb{F}}$ "good" if the following is true:

(G) *Every subset S of $\mathbb{N}_{\mathbb{F}}$ such that $n \in S$ has a smallest member.*

Let G be the set of all good members of $\mathbb{N}_{\mathbb{F}}$.

We prove that G is inductive.

To prove this, we have to show first that $1 \in G$, i.e. that 1 is good.

So let S be such that $1 \in S$ and $S \subseteq \mathbb{N}_{\mathbb{F}}$. We have to show that S has a smallest member.

We know that $1 \in S$, and Lemma WOP5 tellsus that $1 \leq n$ for every $n \in \mathbb{N}_{\mathbb{F}}$. Therefore, since $S \subseteq \mathbb{N}_{\mathbb{F}}$, $1 \leq n$ for every $n \in S$. So 1 is the smallest member of S. So S has a smallest member. So 1 is good.

Next we have to show that if $n \in G$ then $n + 1 \in G$.

Assume that $n \in G$, i.e., that n is good. We want to prove that $n + 1 \in G$. Let $S \subseteq \mathbb{N}_{\mathbb{F}}$ be such that $n + 1 \in S$. We want to prove that S has a smallest member.

Clearly, either $n \in S$ or $n \notin S$.

Consider first the case when $n \in S$. Since n is good, every subset of $\mathbb{N}_{\mathbb{F}}$ containg n has a smallest member. So S has a smallest member.

Now let us look at the case when $n \notin S$. Let $T = S \cup \{n\}$. (We are **adding** n to S, forming a new set T that consists of all the members of S and also n.) Then $n \in T$. So T is a subset of $\mathbb{N}_{\mathbb{F}}$ containing n. Since n is good. T has a smallest member t. Then $t \in T$ and $t \leq u$ for every $u \in T$. Since $S \subseteq T$, it follows that $t \leq u$ for every $u \in S$.

This almost tells us that t is the smallest member of S. But there is one problem: *t might not be in S.*

Now, the only way for t not to be in S is if $t = n$, because $t \in T$, and n the only member of T that is not in S, since $T = S \cup \{n\}$.

So there are two possibilities: $t \in S$ or $t = n$. If $t \in S$, then, since $t \leq u$ for every $u \in S$, we conclude that t is the smallest member of S, so S has a smallest member.

What if $t \notin S$? Then t must be n, so we can conclude that $n \leq u$ for every member of S. Let u be an arbitrary member of S. Then $n \leq u$, so $n = u$ or $n < u$. But $n \neq u$, because $u \in S$ and $n \notin S$. So $n < u$.

And then Lemma WOP6 comes to our rescue, and tells us that $n+1 \leq u$, because "$u < n+1$" would imply that $n < u < n+1$. So we have proved that $n+1 \leq u$ for every $u \in S$. And now we know in addition that $n+1 \in S$. (When we introduced S a long time ago, we said "Let $S \subseteq \mathbb{N}_{\mathbb{F}}$ be such that $n+1 \in S$.")

So $n+1$ is a member of S that is $\leq u$ for every $u \in S$. Hence $n+1$ is the smallest member of S.

So S has a smallest member.

Now, S was an arbitrary subset of $\mathbb{N}_{\mathbb{F}}$ such that $n+1 \in S$.

So we have proved that every subset S of $\mathbb{N}_{\mathbb{F}}$ such that $n+1 \in S$ has a smallest member.

Hence we have proved that $n+1$ is good, i.e., that $n+1 \in G$.

And we have proved that $n+1 \in G$ assuming that $n \in G$. And this was done for an aribtrary n.

Since we have also proved that $1 \in G$, it follows that G is an inductive subset of $\mathbb{F}$.

We now come to the last step of our proof. We want to prove that every nomempty subset S of $\mathbb{N}_{\mathbb{F}}$ has a smallest member.

So let $S \subseteq \mathbb{N}_{\mathbb{F}}$ be such that $S \neq \emptyset$.

Since $S \neq \emptyset$, we can pick a member s of S.

Then $s \in \mathbb{N}_{\mathbb{F}}$, because $S \subseteq \mathbb{N}_{\mathbb{F}}$. So S is a subset of $\mathbb{N}_{\mathbb{F}}$ containg s.

Since $s \in \mathbb{N}_{\mathbb{F}}$, s belongs to every inductive subset of $\mathbb{F}$.

Since G is an inductive subset of $\mathbb{F}$, $s \in G$.

So s is good. And this means that every subset of $\mathbb{F}$ contaning s has a smallest member.

But S is a subset of $\mathbb{N}_{\mathbb{F}}$ containg s.

So S has a smallest member.

And S was an **arbitrary** nonempty subset of $\mathbb{N}_{\mathbb{F}}$.

So we have proved that **every** nonempty subset of $\mathbb{N}_{\mathbb{F}}$ has a smallest member.

That is, we have proved the WOP. **QED***(Theorem WOP)*

1. Proof of the well-ordering principle (WOP).

   (This is needed because in Lecture 1 I used the WOP but had not proved it.)

2. The rational number system $\mathbb{Q}_{\mathbb{F}}$ of a field $\mathbb{F}$.

3. Density of $\mathbb{Q}_{\mathbb{F}}$ in an Archimedean field $\mathbb{F}$.

4. Completeness.

5. Complete ordered fields are Archimedean.

6. Existence of $\sqrt{2}$.

2. THE RATIONAL NUMBER SYSTEM $\mathbb{Q}_{\mathbb{F}}$ OF A FIELD $\mathbb{F}$.

Definition. Let $\mathbb{F}$ be a field.

1. A member r of $\mathbb{F}$ is a rational member of $\mathbb{F}$ if $(\exists n \in \mathbb{Z}_{\mathbb{F}})(n \neq 0 \wedge rn \in \mathbb{Z}_{\mathbb{F}})$, that is, if rn is an integer for some nonzero integer n.
2. We use "$\mathbb{Q}_{\mathbb{F}}$" to denote the set of all rational members of $\mathbb{F}$.

It is easy to prove from this definition that

1. if $\mathbb{F}$ is a field then $\mathbb{Q}_{\mathbb{F}}$ is a field,
2. if $\mathbb{F}$ is an ordered field then $\mathbb{Q}_{\mathbb{F}}$ is an ordered field,
3. if $\mathbb{F}$ is an ordered field then $\mathbb{Q}_{\mathbb{F}}$ is an Archimedean ordered field (even if $\mathbb{F}$ is not Archimedean).

1. Proof of the well-ordering principle (WOP).

   (This is needed because in Lecture 1 I used the WOP but had not proved it.)

2. The rational number system $\mathbb{Q}_{\mathbb{F}}$ of a field $\mathbb{F}$.

3. Density of $\mathbb{Q}_{\mathbb{F}}$ in an Archimedean field $\mathbb{F}$.

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THE IRRATIONALITY OF $\sqrt{2}$: AN EXAMPLE OF A PROOF USING WELL-ORDERING

Theorem. Let $\mathbb{F}$ be an ordered field. Then there does not exist an $r \in \mathbb{Q}_{\mathbb{F}}$ such that $r^2 = 2$.

I am sure you have seen the proof of this before, probably in Math 300. Here I am giving you the proof **primarily for the purpose of showing you an important application of the WOP**.

Proof. Suppose there exists a member ρ of $\mathbb{Q}_{\mathbb{F}}$ such that $\rho^2 = 2$. Pick one, and call it r. Then $r \in \mathbb{Q}_{\mathbb{F}}$ and $r^2 = 2$.

Let $S = \{n \in \mathbb{N}_{\mathbb{F}} : rn \in \mathbb{Z}_{\mathbb{F}}\}$.

Then S is, by definition, a set of natural numbers, i.e., a subset of $\mathbb{N}_{\mathbb{F}}$.

Also, $S \neq \emptyset$. (Why? Because $r \in \mathbb{Q}_{\mathbb{F}}$, and this means that we can pick a nonzero integer $n \in \mathbb{Z}_{\mathbb{F}}$ such that $rn \in \mathbb{Z}_{\mathbb{F}}$. Pick one such n. Then $r(-n) \in \mathbb{Z}_{\mathbb{F}}$ as well. And either n or $-n$ is in $\mathbb{N}_{\mathbb{F}}$. Let ν be the one of $n, -n$ that is a natural number. Then $r\nu \in \mathbb{Z}_{\mathbb{F}}$ and $\nu \in \mathbb{N}_{\mathbb{F}}$. Therefore $\nu \in S$, so $S \neq \emptyset$.)

By the Well-ordering principle, S has a smallest member s. Let $t = rs$.

Then $t \in \mathbb{Z}_{\mathbb{F}}$, $s \in \mathbb{N}_{\mathbb{F}}$, and $r = \frac{t}{s}$.

Since $rs = t$, we have $r^2 s^2 = t^2$. But $r^2 = 2$, so $2s^2 = t^2$.

This shows that t^2 is even, and then t must be even (because if t was odd then t^2 would be odd, because the product of two odd integers is odd). So we can write $t = 2\tau$, with τ an integer.

Then $2s^2 = t^2 = (2\tau)^2 = 4\tau^2$, so $s^2 = 2\tau^2$. Hence s^2 is even, and then s must be even (because if s was odd then s^2 would be odd, because the product of two odd integers is odd). So we can write $s = 2\sigma$, with σ an integer. And in fact σ is a natural number, because $\sigma > 0$ since $s > 0$.

Then $2r\sigma = rs = t = 2\tau$, so $r\sigma = \tau$. Since $\sigma \in \mathbb{N}_{\mathbb{F}}$ and $\tau \in \mathbb{Z}_{\mathbb{F}}$, it follows that $\sigma \in S$. But $\sigma < s$, because $2\sigma = s$. So s is not the smallest member of S, and we have reached a contradiction, since s is the smallest member of S. **QED**

NOTE: Using a similar argument, it can be proved that *if* $m \in \mathbb{N}_{\mathbb{F}}$ *and* m *is not a perfect square* (that is, there does not exist $\mu \in \mathbb{N}_{\mathbb{F}}$ such that $\mu^2 = m$) then there does not exist $\mu \in \mathbb{Q}_{\mathbb{F}}$ such that $\mu^2 = m$.

3. DENSITY OF $\mathbb{Q}_{\mathbb{F}}$ IN AN ARCHIMEDEAN FIELD $\mathbb{F}$.

Definition. Let $\mathbb{F}$ be an Archimedean ordered field and let S be a subset of $\mathbb{F}$.

We say that S is dense in $\mathbb{F}$ if

(D) *Every nonempty open interval of $\mathbb{F}$ contains a member of S.*

(That is, *whenever $a \in \mathbb{F}$, $b \in \mathbb{F}$ are such that $a < b$, it follows that there exists $s \in S$ such that $a < s < b$.*)

Theorem D. *If $\mathbb{F}$ is an Archimedean ordered field, then $\mathbb{Q}_{\mathbb{F}}$ is dense in $\mathbb{F}$.*

Proof. Let a, b be members of $\mathbb{F}$ such that $a < b$.

Then $0 < b - a$. Since $\mathbb{F}$ is Archimedean, $b - a$ is not infinitesimal. This means that there exists a natural number n such that $\frac{1}{n} < b - a$. Then $1 < nb - na$.

By Theorem A (see slide No. 28), we can pick an integer $k \in \mathbb{Z}_{\mathbb{F}}$ such that $k \leq an < k + 1$. Then

$$k + 1 \leq an + 1 < an + (nb - na) = nb\,.$$

So $an < k + 1 < bn$. Therefore $a < \frac{k+1}{n} < b$.

Hence, if we let $r = \frac{k+1}{n}$, we find that $r \in \mathbb{Q}_{\mathbb{F}}$ and $a < r < b$.

QED *(Theorem D)*

1. Proof of the well-ordering principle (WOP).

   (This is needed because in Lecture 1 I used the WOP but had not proved it.)

2. The rational number system $\mathbb{Q}_{\mathbb{F}}$ of a field $\mathbb{F}$.

3. Density of $\mathbb{Q}_{\mathbb{F}}$ in an Archimedean field $\mathbb{F}$.

4. **Completeness.**

5. Complete ordered fields are Archimedean.

6. Existence of $\sqrt{2}$.

4. COMPLETENESS.

Definition. Let $\mathbb{F}$ be an ordered field and let S be a subset of $\mathbb{F}$.

1. An upper bound for S in $\mathbb{F}$ is a member u of $\mathbb{F}$ such that $s \leq u$ for every $s \in S$.
2. S is bounded above if it has an upper bound, i.e., if there exists a member u of $\mathbb{F}$ which is an upper bound for S.
3. A least upper bound for S in $\mathbb{F}$ is a member u of $\mathbb{F}$ such is an upper bound for S and is such that $u \leq v$ for every uppper bound v for S.

Easy to prove: If a set S has a lest upper bound, then it has a unique least upper bound.

Example. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}\,.$$

Then 2 is an upper bound for S.

Example. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}.$$

Then 2 is an upper bound for S.

(Proof: Let $x \in S$. Then either $x \leq 1$ or $1 < x$. If $x \leq 1$ then of course $x \leq 2$. If $1 < x$ then we can multiply both sides of "$1 < x$" by x, using F16, and get $x < x^2$. Since $x^2 < 2$, it follows that $x < 2$. Since x is an arbitrary member of S, 2 is an upper bound for S.)

Example. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}\,.$$

Then 1.5 is an upper bound for S.

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$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}.$$

Then 1.5 is an upper bound for S.

(Proof: Let $x \in S$. Then $x^2 < 2$. If x was ≥ 1.5, we would have $x^2 \geq 1.5 \times 1.5 = 2.25$, so $x^2 > 2$. Hence $x < 1.5$. Since x is an arbitrary member of S, 1.5 is an upper bound for S.)

Example. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}\,.$$

Then 1.42 is an upper bound for S.

Example. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}.$$

Then 1.42 is an upper bound for S.

(Proof: Let $x \in S$. Then $x^2 < 2$. If x was ≥ 1.42, we would have $x^2 \geq 1.42 \times 1.42 = 2.0164$, so $x^2 > 2$. So $x < 1.42$. Since x is an arbitrary member of S, 1.42 is an upper bound for S.)

Example. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}\,.$$

Then 1.415 is an upper bound for S.

Example. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}\,.$$

Then 1.415 is an upper bound for S.

(Proof: Let $x \in S$. Then $x^2 < 2$. If x was ≥ 1.415, we would have $x^2 \geq 1.415 \times 1.415$, **and** $1.415 \times 1.415 > 2$, so $x^2 > 2$. So $x < 1.415$. Since x is an arbitrary member of S, 1.415 is an upper bound for S.)

Definition. An ordered field $\mathbb{F}$ is complete if

(C) *Every nonempty subset S of $\mathbb{F}$ that is bounded above has a least upper bound.*

Statement (C) is the **completeness axiom**, or **completeness property**.

Recall that our goal all along has been to understand and characterize $\mathbb{R}$, the real number system. Our characterization of $\mathbb{R}$ is as follows:

$\mathbb{R}$, the real number system, is a complete ordered field.

And now, finally, we know what that means.

From now on, I will stop talking about "a complete ordered field $\mathbb{F}$", and will just talk about $\mathbb{R}$.

This is really the same thing, **because the only thing we know about $\mathbb{R}$ is that it is a complete ordered field**, so everything we say about $\mathbb{R}$ is going to apply to any complete ordered field.

And $\mathbb{N}_{\mathbb{R}}$, $\mathbb{Z}_{\mathbb{R}}$, $\mathbb{Q}_{\mathbb{R}}$ will just be called $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$.

Now that we know what $\mathbb{R}$ is, our first two theorems about $\mathbb{R}$ are going to be

1. That $\mathbb{R}$ **is an Archimedean ordered field**,

2. That **in $\mathbb{R}$ every positive member has exactly two square roots, one of which is positive while the other one is negative**.

NOTE: In this course, "positive" means "> 0", and "nonnegative" means "≥ 0". So 0 is nonnegative but not positive.

1. Proof of the well-ordering principle (WOP).

   (This is needed because in Lecture 1 I used the WOP but had not proved it.)

2. The rational number system $\mathbb{Q}_{\mathbb{F}}$ of a field $\mathbb{F}$.

3. Density of $\mathbb{Q}_{\mathbb{F}}$ in an Archimedean field $\mathbb{F}$.

4. Completeness.

5. Complete ordered fields are Archimedean.

6. Existence of $\sqrt{2}$.

5. COMPLETE ORDERED FIELDS ARE ARCHIMEDEAN

Theorem. A complete ordered field is Archimedean. That is, $\mathbb{R}$ is an Archimedean field.

Proof. Let $\mathbb{F}$ be a complete ordered field. In order to prove that $\mathbb{F}$ is Archimedean, it suffices to prove that $\mathbb{F}$ does not have any infinitely large members.

Suppose $\mathbb{F}$ did have an infinitely large member. Pick one and call it L.

Then $n \leq L$ for every natural number n of $\mathbb{F}$.

This means that L is an upper bound for the set $\mathbb{N}_{\mathbb{F}}$ of all natural numbers of $\mathbb{F}$.

So $\mathbb{N}_{\mathbb{F}}$ is bounded above.

Since $\mathbb{F}$ is complete, $\mathbb{N}_{\mathbb{F}}$ has a least upper bound L_*.

Then in particular L_* is an upper bound for $\mathbb{N}_{\mathbb{F}}$, so $n \leq L_*$ for every $n \in \mathbb{N}_{\mathbb{F}}$.

Let $M = \frac{L_*}{2}$. Then, if $n \in \mathbb{N}_{\mathbb{F}}$, we have $L_* \geq 2n$, because $2n \in \mathbb{N}_{\mathbb{F}}$. So $\frac{L_*}{2} \geq n$, i.e. $M \geq n$. Since this is true for every natural number n of $\mathbb{F}$, we see that M is an upper bound for $\mathbb{N}_{\mathbb{F}}$. But $M < L_*$, so L_* is not the least upper bound of $\mathbb{N}_{\mathbb{F}}$, and we have arrived at a contradiction. **QED**

We have already proved the following two theorems:

Theorem A. In an Archimedean ordered field $\mathbb{F}$, the following is true: for every member r of $\mathbb{F}$ there exists a unique integer $n \in \mathbb{Z}_{\mathbb{F}}$ such that $n \leq r < n + 1$.

Theorem D. *If $\mathbb{F}$ is an Archimedean ordered field, then $\mathbb{Q}_{\mathbb{F}}$ is dense in $\mathbb{F}$.*

Now that we know that a complete ordered field is Archimedean and that $\mathbb{R}$ is a complete ordered field, we can write the following particular cases of Theorems A and D:

Theorem A'. For every real number r there exists a unique integer $n \in \mathbb{Z}$ such that $n \leq r < n + 1$.

Theorem D'. $\mathbb{Q}$ *is dense in* $\mathbb{R}$.

SQUARES AND SQUARE ROOTS

Definition. In a field $\mathbb{F}$, if r is a member of $\mathbb{F}$, then

1. The square of r is the product $r \times r$.
2. If $r \in \mathbb{F}$, we write "r^2" to denote the square of r.
3. A square root of r is a member s of $\mathbb{R}$ such that $s^2 = r$.
4. If $\mathbb{F}$ is an ordered field, the square root of r (which may or may not exist) is the unique member s of $\mathbb{R}$ such that $s^2 = r$ and $0 < s$.
5. If $\mathbb{F}$ is an ordered field, we use "$\sqrt{r}$" to denote the square root of r.

A VERY IMPORTANT REMARK

In the previous definition, we talk about **the** square of r and **a** square root of r.

You should make sure that you understand **exactly why** we use "the" for "the square" and "a" for "a square root".

EXISTENCE OF $\sqrt{2}$

Theorem. There exists a unique positive real number r such that $r^2 = 2$.

The proof is exactly as you would imagine: we define a set S of real numbers, prove that it is nonempty and bounded above, and then use the completeness property to get a least upper bound L of S.

Finally, we show that $L^2 = 2$, and that will conclude our proof.

The obvious choice for us to make for S is the set $\{x \in \mathbb{R} : x < \sqrt{2}\}$. But we cannot define a set like that because we don't know that there is a $\sqrt{2}$. So we have to say "$S = \{x \in \mathbb{R} : x < \sqrt{2}\}$" without mentioning $\sqrt{2}$.

But that's very easy: instead of saying "$x < \sqrt{2}$", we say "$x^2 < 2$", which basically amounts to the same thing, but has the advantage of not mentioning $\sqrt{2}$.

Actually, "$x^2 < 2$" is equivalent to "$x < \sqrt{2}$" for positive x, but not for all x. (For example, if $x = -4$ then $x < \sqrt{2}$, but it's not true that $x^2 < 2$.) So we will take S to be the set $\boxed{\{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}}$.

Proof of the existence of a positive square root of 2. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < 2\}.$$

Then S is nonempty, because $1 \in S$, since $0 < 1$ and $1^1 = 1 < 2$.

Also, S is bounded above. We proved this before. But let us repeat the proof: if $x \in S$ then either $x \leq 1$, in which case $x < 2$, or $2 \leq x$, in which case $x < x^2 < 2$, so $x < 2$ as well. So 2 is an upper bound for S.

So by the completeness property S has a least upper bound. Call this least upper bound L.

Now we will prove that $L^2 = 2$.

By Axiom F14, either $L^2 < 2$ or $L^2 = 2$ or $2 < L^2$.

We will exclude the possibilities that $L^2 < 2$ and that $2 < L^2$, and this will imply that $L^2 = 2$.

(This is how we prove equalities most of the time in Analysis: to prove that $a = b$ we prove that the inequalities "$a < b$" and "$b < a$" are not true. **In Analysis most of the mathematicians' time and effort is spent working with inequalities.**)

Proof that "$L^2 < 2$" is not true.

Suppose that $L^2 < 2$.

We will pick a real number δ such that $0 < \delta$ and $(L + \delta)^2 < 2$.

This is the hard part, where most of the work is needed. How do we pick such a δ?

Picking a positive δ such that $(L + \delta)^2 < 2$.

We want $(L + \delta)^2 < 2$. But $(L + \delta)^2 = L^2 + 2L\delta + \delta^2$.

So we want $L^2 + 2L\delta + \delta^2 < 2$.

At this point it will be convenient to use one of the most powerful tricks in Analysis. **You have to learn to use this trick, because we will be needing it millions of times in this course.**

The trick is: **restrict your choice**. If you have to choose something, and you don't know what to do, it is often a good idea to restrict your choice in advance, because that will make it easier to choose. (For example, if you want to buy a car, it helps to pick in advance a dealer that sells the kind of car you are interested in, and go there and just look at the cars for sale there. Then you wil have fewer cars to look at, but that will make it easier for you to choose)

Since we have to choose a positive real number δ such that $L^2 + 2\delta + \delta^2 < 2$, it is clear that you want δ to be "small", whatever that means. So we can restrict our choice by deciding in advance that we are only going to look for δ in the interval $\{x \in \mathbb{R} : 0 < x \leq 1\}$. That is, in addition to being positive, we are going to pick δ such that $\delta \leq 1$.

So we agree to choose δ such that $0 < \delta \leq 1$.

Then $L^2 + 2L\delta + \delta^2 \leq L^2 + 2L\delta + \delta = L^2 + (2L+1)\delta$.

So if we make the number $L^2 + (2L+1)\delta$ smaller than 2, then *a fortiori* $L^2 + 2L\delta + \delta^2$ will be smaller than 2.

And, to get $L^2 + (2L+1)\delta$ to be < 2, it suffices to make sure that $(2L+1)\delta < 2 - L^2$. If we pick, for example, δ to be $\frac{2-L^2}{2L+2}$, or smaller, then we will have $(2L+1)\delta < (2L+2)\delta \leq 2-L^2$, so $L^2+(2L+1)\delta < 2$, and then, if δ is also ≤ 1, we will have $L^2 + 2L\delta + \delta^2 < 2$, that is, $(L+\delta)^2 < 2$..

Notice that what makes this possible is the fact that $L^2 < 2$. It is thanks to this fact that $2 - L^2$ is positive, so $\frac{2-L^2}{2L+2}$ comes out positive.

So we want δ to be such that $0 < \delta$, $\delta \leq 1$, and $\delta \leq \frac{2-L^2}{2L+2}$. And to achieve these things, we can choose

$$\boxed{\delta = \min\left(1, \frac{2-L^2}{2L+2}\right).}$$

("min(a,b)" stands for "the smaller of a and b". The precise definition is $\min(a,b) = a$ if $a \leq b$, $\min(a,b) = b$ if $b \leq a$.)

If you want to be fancy, and do not like using the "min" function, you can achieve the same thing by picking

$$\boxed{\delta = \frac{2-L^2}{2-L^2+2L+2}.}$$

This choice of δ certainly guarantees that $\delta < 1$ (because the denominator $2-L^2+2L+2$ is larger than the numerator $2-L^2$).

And it also guarantees that $\delta < \frac{2-L^2}{2L+2}$ (because, clearly, the denominator $2 - L^2 + 2L + 2$ of the fraction $\frac{2-L^2}{2-L^2+2L+2}$ is larger than the denominator $2L + 2$ of the fraction $\frac{2-L^2}{2L+2}$, so the first fraction is smaller than the second one).

The point is that, whichever choice you make, δ satisfies:

$$\begin{aligned} \delta &> 0\,, \\ \delta &< 1\,, \\ \delta &< \frac{2-L^2}{2L+2}\,. \end{aligned}$$

Then

$$\begin{aligned} (L+\delta)^2 &= L^2 + 2L\delta + \delta^2 \\ &\leq L^2 + 2L\delta + \delta \\ &= L^2 + (2L+1)\delta \\ &< L^2 + (2L+2)\delta \\ &\leq L^2 + (2L+2)\frac{2-L^2}{2L+2} \end{aligned}$$

$$\begin{aligned} &= L^2 + (2 - L^2) \\ &= 2\,. \end{aligned}$$

So $(L+\delta)^2 < 2$.

It is clear that $L + \delta > L$ and $L > 0$. (Why $L > 0$? Because L is an upper bound for S and $1 \in S$, so $L \geq 1$, and then $L > 0$.)

Since $(L+\delta)^2 < 2$, the number $L + \delta$ belongs to S.

But L is the least upper bound for S, so in particular L is an upper bound for S. Since $L + \delta \in S$, it follows that $L \geq L + \delta$. But then $\delta \leq 0$, contradicting the fact that $\delta > 0$.

So we have derived a contradiction from the assumption that $L^2 < 2$.

Hence "$L^2 < 2$" is not true.

Proof that "$2 < L^2$" is not true.

Suppose that $2 < L^2$.

We will pick a real number δ such that $0 < \delta$ and $2 < (L - \delta)^2$.

Picking δ such that $0 < \delta$ and $2 < (L - \delta)^2$.

We want $2 < (L - \delta)^2$. But $(L - \delta)^2 = L^2 - 2L\delta + \delta^2$.

So we want $2 < L^2 - 2L\delta + \delta^2$.

Suppose we find a positive δ such that $2 < L^2 - 2L\delta$.

Then *a fortiori* $2 < L^2 - 2L\delta + \delta^2$, because $L^2 - 2L\delta < L^2 - 2L\delta + \delta^2$.

To guarantee that $2 < L^2 - 2L\delta$, it suffices to have $2 + 2L\delta < L^2$, i.e., $2L\delta < L^2 - 2$, i.e., $\delta < \frac{L^2-2}{2L}$.

And we can achieve this by taking $\boxed{\delta = \frac{L^2-2}{3L}}$.

Then

$$\begin{aligned}
(L-\delta)^2 &= L^2 - 2L\delta + \delta^2 \\
&> L^2 - 2L\delta \\
&= L^2 - 2L\frac{L^2-2}{3L} \\
&= L^2 - \frac{2}{3}(L^2-2) \\
&= L^2 - \frac{2}{3}(L^2-2) \\
&= \frac{1}{3}\times L^2 + \frac{2}{3}\times 2 \\
&> \frac{1}{3}\times 2 + \frac{2}{3}\times 2 \\
&= 2.
\end{aligned}$$

So $(L-\delta)^2 > 2$.

Notice that what makes this possible is the fact that $2 < L^2$. It is thanks to this fact that $\frac{1}{3}\times L^2 + \frac{2}{3}\times 2$ is larger than $\frac{1}{3}\times 2 + \frac{2}{3}\times 2$, which is a key step in our calculation.

Furthermore, δ is clearly positive, because $2 < L^2$.

Finally, $\delta < L$. (We need this because we need $L - \delta$ to be positive.)

(Reason: $\delta = \frac{L^2-2}{3L}$, so $\frac{\delta}{L} = \frac{L^2-2}{3L^2}$. And the fraction $\frac{L^2-2}{3L^2}$ is clearly < 1, because $L^2 - 2 < L^2 < 3L^2$. So $\frac{\delta}{L} < 1$, and then $\delta < L$.)

So $L - \delta > 0$, because $\delta < L$.

We claim that $L - \delta$ is an upper bound for S.

To see this, let $x \in S$. Then $x > 0$ and $x^2 < 2$. But $L - \delta > 0$ and $(L - \delta)^2 > 2$. Hence x and $L - \delta$ are positive real numbers such that $x^2 < (L - \delta)^2$. And this implies that $x < L - \delta$. (Proof: $L - \delta - x = \frac{(L-\delta-x)(L-\delta+x)}{L-\delta+x} = \frac{(L-\delta)^2-x^2}{L-\delta+x}$, and both $(L - \delta)^2 - x^2$ and $L - \delta + x$ are positive.)

So $x < L - \delta$ for every $x \in S$. Hence $L - \delta$ is an upper bound for S.

So we have arrived at a contradiction, because L is supposed to be the **least** upper bound of S, $L - \delta$ is an upper bound, and $L - \delta < L$, since $\delta > 0$.

Hence **the possibility that $2 < L^2$ is excluded as well**.

Therefore

$$L^2 = 2$$

This proves the **existence** of a positive real number L such that $L^2 = 2$.

To prove the **uniqueness**, suppose L_1, L_2 are positive real numbers such that $L_1^2 = 2$ and $L_2^2 = 2$. Then

$$L_1^2 - L_2^2 = (L_1 + L_2)(L_1 - L_2)\,,$$

so

$$L_1 - L_2 = \frac{L_1^2 - L_2^2}{L_1 + L_2}\,.$$

But $L_1^2 - L_2^2 = 0$, because $L_1^2 = 2$ and $L_2^2 = 2$.

So $L_1 - L_2 = 0$, and then $L_1 = L_2$.

Lecture 4

September 14, 2020

EXISTENCE OF $\sqrt{a}$ FOR AN ARBITRARY POSITIVE REAL NUMBER a

Theorem. Let a be a positive real number. Then there exists a unique positive real number r such that $r^2 = a$.

Proof of the existence of a positive square root of a. Let

$$S = \{x \in \mathbb{R} : 0 < x \wedge x^2 < a\}.$$

Then S is nonempty. Why is $S \neq \emptyset$? When $a = 2$, we just argued that $1 \in S$, because $1^2 = 1 < 2$. But now we cannot use the same argument, because a is an **arbitrary** positive real number, so in particular a could be smaller than 1, and then 1 would not be in S. So we need to reason differently. Here is how we can do it: either $a > 1$ or $a \leq 1$. If $a > 1$, then $1^2 = 1 < a$, so $1 \in S$, and then $S \neq \emptyset$. If $a \leq 1$, then $a^2 \leq a$, and $\frac{a^2}{4} < a^2 \leq a$, so if we let $\alpha = \frac{a}{2}$, we see that $\alpha > 0$ and $\alpha^2 < a$, so $\alpha \in S$ and then, once again, $S \neq \emptyset$. So $S \neq \emptyset$ in both cases.

Also, S is bounded above. Proof: if $x \in S$ then either $x \leq 1$ or $1 \leq x$. If $x \leq 1$ then of course $x \leq \max(1, a)$. If $1 \leq x$, then $x \leq x^2$, and $x^2 < a$, so $x < a$, and then again $x \leq \max(1, a)$. So $x \leq \max(1, a)$. So $\max(1, a)$ is an upper bound for S. So S has an upper bound. So S is bounded above. **Notice that this is an example of a proof by cases.**

So by the completeness property S has a least upper bound. Call this least upper bound L.

Now we will prove that $L^2 = a$.

By Axiom F14, either $L^2 < a$ or $L^2 = a$ or $a < L^2$.

We will exclude the possibilities that $L^2 < a$ and that $a < L^2$, and this will imply that $L^2 = a$.

Proof that "$L^2 < a$" is not true.

Suppose that $L^2 < a$.

We will pick a real number δ such that $0 < \delta$ and $(L + \delta)^2 < a$.

This is the hard part, where most of the work is needed. How do we pick such a δ?

Picking a positive δ such that $(L + \delta)^2 < a$.

We want $(L + \delta)^2 < a$. But $(L + \delta)^2 = L^2 + 2L\delta + \delta^2$.

So we want $L^2 + 2L\delta + \delta^2 < a$.

We agree to choose δ such that $0 < \delta \leq 1$. (Here we are using the trick of **restricting our choice**.)

Then $L^2 + 2L\delta + \delta^2 \leq L^2 + 2L\delta + \delta = L^2 + (2L+1)\delta$.

So if we make the number $L^2 + (2L+1)\delta$ smaller than a, then *a fortiori* $L^2 + 2L\delta + \delta^2$ will be smaller than a.

And, to get $L^2 + (2L+1)\delta$ to be $< a$, it suffices to make sure that $(2L+1)\delta < a - L^2$. If we pick, for example, δ to be $\frac{a-L^2}{2L+2}$, or smaller, then we will have $(2L+1)\delta < (2L+2)\delta \leq a - L^2$, so $L^2 + (2L+1)\delta < a$, and then, if δ is also ≤ 1, we will have $L^2 + 2L\delta + \delta^2 < a$, that is, $(L+\delta)^2 < a$.

Notice that what makes this possible is the fact that $L^2 < a$. It is thanks to this fact that $a - L^2$ is positive, so $\frac{a-L^2}{2L+2}$ comes out positive.

So we want δ to be such that $\boxed{0 < \delta}$, $\boxed{\delta \leq 1}$, and $\boxed{\delta \leq \frac{a-L^2}{2L+2}}$. And to achieve these things, we can choose

$$\boxed{\delta = \min\left(1, \frac{a - L^2}{2L + 2}\right).}$$

If you want to be fancy, and do not like using the "min" function, you can achieve the same thing by picking

$$\boxed{\delta = \frac{a - L^2}{a - L^2 + 2L + 2}.}$$

This choice of δ certainly guarantees that $\delta < 1$ (because the numerator $a - L^2$ is positive —since $L^2 < a$— and the denominator $a - L^2 + 2L + 2$ is larger than the numerator, so the quotient is smaller than 1).

And it also guarantees that $\delta < \frac{a-L^2}{2L+2}$ (because, clearly, the denominator $a - L^2 + 2L + 2$ of the fraction $\frac{a-L^2}{a-L^2+2L+2}$ is larger than the denominator $2L + 2$ of the fraction $\frac{a-L^2}{2L+2}$, so the first fraction is smaller than the second one).

The point is that, whichever choice you make, δ satisfies:

$$\begin{aligned} \delta &> 0\,, \\ \delta &< 1\,, \\ \delta &< \frac{a-L^2}{2L+2}\,, \end{aligned}$$

so it satisfies the three boxed conditions we had stated before.

Then

$$\begin{aligned}
(L+\delta)^2 &= L^2 + 2L\delta + \delta^2 \\
&\leq L^2 + 2L\delta + \delta \\
&= L^2 + (2L+1)\delta \\
&< L^2 + (2L+2)\delta \\
&\leq L^2 + (2L+2)\frac{a-L^2}{2L+2} \\
&= L^2 + (a-L^2) \\
&= a\,.
\end{aligned}$$

So $(L+\delta)^2 < a$.

It is clear that $L + \delta > L$ and $L > 0$.

Why $L > 0$? Here is why: we have already proved that $S \neq \emptyset$. Pick a member x of S. Then the definition of S tells us that $x > 0$. And L is the least upper bound of S, so S is an upper bound for S. Hence $L \geq x$. So $\geq x > 0$. Hence $L > 0$.

Since $(L+\delta)^2 < a$, and $L+\delta > 0$, the number $L+\delta$ belongs to S.

But L is the least upper bound for S, so in particular L is an upper bound for S. Since $L+\delta \in S$, it follows that $L \geq L+\delta$. But then $\delta \leq 0$, contradicting the fact that $\delta > 0$.

So we have derived a contradiction from the assumption that $L^2 < a$.

Hence $\boxed{\text{“}L^2 < a\text{” is not true}}$.

Proof that "$a < L^2$" is not true.

Suppose that $a < L^2$.

We will pick a real number δ such that $0 < \delta$ and $a < (L - \delta)^2$.

Picking δ such that $0 < \delta$ and $a < (L - \delta)^2$.

We want $a < (L - \delta)^2$. But $(L - \delta)^2 = L^2 - 2L\delta + \delta^2$.

So we want $a < L^2 - 2L\delta + \delta^2$.

Suppose we find a positive δ such that $a < L^2 - 2L\delta$.

Then *a fortiori* $a < L^2 - 2L\delta + \delta^2$, because $L^2 - 2L\delta < L^2 - 2L\delta + \delta^2$.

To guarantee that $a < L^2 - 2L\delta$, it suffices to have $a + 2L\delta < L^2$, i.e., $2L\delta < L^2 - a$, i.e., $\delta < \frac{L^2 - a}{2L}$.

And we can achieve this by taking $\boxed{\delta = \frac{L^2 - a}{3L}}$.

Then

$$\begin{aligned}
(L-\delta)^2 &= L^2 - 2L\delta + \delta^2 \\
&> L^2 - 2L\delta \\
&= L^2 - 2L\frac{L^2-a}{3L} \\
&= L^2 - \frac{2}{3}(L^2-a) \\
&= L^2 - \frac{2}{3}(L^2-a) \\
&= \frac{1}{3}\times L^2 + \frac{2}{3}\times a \\
&> \frac{1}{3}\times a + \frac{2}{3}\times a \\
&= a\,.
\end{aligned}$$

So $(L-\delta)^2 > a$.

Notice that what makes this possible is the fact that $a < L^2$. It is thanks to this fact that $\frac{1}{3}\times L^2 + \frac{2}{3}\times a$ is larger than $\frac{1}{3}\times a + \frac{2}{3}\times a$, which is a key step in our calculation.

Furthermore, δ is clearly positive, **because** $a < L^2$.

Finally, $\delta < L$. (We need this because we need $L - \delta$ to be positive.)

(Reason: $\delta = \frac{L^2-a}{3L}$, so $\frac{\delta}{L} = \frac{L^2-a}{3L^2}$. And the fraction $\frac{L^2-a}{3L^2}$ is clearly < 1, because $L^2 - a < L^2 < 3L^2$. So $\frac{\delta}{L} < 1$, and then $\delta < L$.)

So $L - \delta > 0$, because $\delta < L$.

We claim that $L - \delta$ is an upper bound for S.

To see this, let $x \in S$. Then $x > 0$ and $x^2 < 2$. But $L - \delta > 0$ and $(L - \delta)^2 > a$. Hence x and $L - \delta$ are positive real numbers such that $x^2 < (L - \delta)^2$ (because $x^2 < a$, since $xinS$, and $(L - \delta)^2 > a$.) And this implies that $x < L - \delta$. (Proof: $L - \delta - x = \frac{(L-\delta-x)(L-\delta+x)}{L-\delta+x} = \frac{(L-\delta)^2-x^2}{L-\delta+x}$, and both $(L - \delta)^2 - x^2$ and $L - \delta + x$ are positive.)

So $x < L - \delta$ for every $x \in S$. Hence $L - \delta$ is an upper bound for S.

So we have arrived at a contradiction, because L is supposed to be the **least** upper bound of S, $L - \delta$ is an upper bound, and $L - \delta < L$, since $\delta > 0$.

Hence **the possibility that $a < L^2$ is excluded as well**.

Therefore

$$L^2 = a$$

This proves the **existence** of a positive real number L such that $L^2 = a$.

To prove the **uniqueness**, suppose L_1, L_2 are positive real numbers such that $L_1^2 = a$ and $L_2^2 = a$. Then

$$L_1^2 - L_2^2 = (L_1 + L_2)(L_1 - L_2)\,, \tag{1}$$

so

$$L_1 - L_2 = \frac{L_1^2 - L_2^2}{L_1 + L_2}\,. \tag{2}$$

But $L_1^2 - L_2^2 = 0$, because $L_1^2 = a$ and $L_2^2 = a$.

So $L_1 - L_2 = 0$, and then $L_1 = L_2$.

In this argument, the assumption that L_1 and L_2 are positive plays a crucial role. It is thanks to this that we can conclude that $L_1 + L_2 > 0$, so that $L_1 + L_2 \neq 0$, so we can divide both sides of (1) by $L_1 + L_2$, and get (2).

If we had **not** assumed that $L_1 > 0$ and $L_2 > 0$, what would have happened?

We would have proved (1), and concluded that

$$(L_1 + L_2)(L_1 - L_2) = 0\,. \tag{3}$$

And then, using a very important property of the real numbers:

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\Big(xy = 0 \Rightarrow (x = 0 \text{ or } y = 0)\Big) \tag{4}$$

we would have concluded that $L_1 + L_2 = 0$ or $L_1 - L_2 = 0$. If $L_1 > 0$ and $L_2 > 0$, then $L_1 + L_2$ cannot be zero, so we end up with $L_1 = L_2$ as before.

But if we do not assume that, then we have another possibility. It could happen that $L_1 = -L_2$. (And this **does** happen: the two square roots of a are $\sqrt{a}$ and $-\sqrt{a}$.)

Make sure you know how to prove (4). In homework no. 1, several people who submitted the assignment got this proof wrong.

THE SQUARE ROOT

If a is a nonnegative real number, then the unique nonnegative real number r such that $r^2 = a$ is called **the** square root of a, and we use the expression

$$\sqrt{a}$$

to denote it.

Definition. **A** square root of a real number a is a real number r such that $r^2 = a$.

Theorem. If a is a real number, then

- if $a > 0$ then a has two square roots, namely, $\sqrt{a}$ and $-\sqrt{a}$,
- if $a = 0$ then a has one square root, namely, 0,
- if $a < 0$ then a has no square roots.

EXISTENCE OF $\sqrt[n]{a}$ FOR AN ARBITRARY POSITIVE REAL NUMBER a AND AN ARBITRARY NATURAL NUMBER n

Theorem. Let a be a positive real number and let n be a natural number. Then there exists a unique positive real number r such that $r^n = a$.

The proof of this is very similar to the one we gave for $\sqrt{a}$.

This proof is a homework problem.

The only significant difference between the case of $\sqrt{a}$ and the general case is this:

- in the proof for $\sqrt{a}$, at the point when we had a positive number L such that $L^2 < a$, and we want a positive δ such that $(L+\delta)^2 < a$, we used the formula

  $$(L+\delta)^2 = L^2 + 2L\delta + \delta^2\,,$$

  and then derived the inequality

  $$(L+\delta)^2 \le L^2 + (2L+1)\delta \qquad \text{if} \quad \delta \le 1\,,$$

  which we used to conclude that, in order to make $(L+\delta)^2 < a$, it suffices to make $\delta < \frac{a-L^2}{2L+1}$, and for this it suffices to make $\delta \le \frac{a-L^2}{2L+2}$.

- in the general proof, we need an inequality for $(L+\delta)^n$, and this can be achieved either by using the **binomial formula**

$$(L+\delta)^n = \sum_{k=0}^{n} \binom{n}{k} L^k \delta^{n-k} \,,$$

or by deriving the inequality

$$(L+\delta)^n \leq L^n + n(L+\delta)^{n-1}\delta$$

(which follows from the Mean Value Theorem but can be proved directly by induction without using Calculus), and deriving from it the corollary that

$$(L+\delta)^n \leq L^n + n(L+1)^{n-1}\delta \,, \qquad \text{if} \quad \delta \leq 1 \,.$$

n-th ROOTS

Definition. If n is a natural number, an n-th root of a real number a is a real number r such that $r^n = a$.

Theorem. If a is a real number and n is a natural number, then

- if n is odd, then a has a unique n-root, and we use "$\sqrt[n]{a}$" to denote it,
- if n is even and $a > 0$, then
  - a has a unique positive n-th root, and we use $\sqrt[n]{a}$ to denote it,
  - a has exactly two n-th roots, namely, $\sqrt[n]{a}$ and $-\sqrt[n]{a}$,
- if n is even and $a < 0$, then a has no n-th roots,
- if n is even and $a = 0$, then a has a unique n-th root, namely, 0.

THESE THINGS ARE VERY EASY TO PROVE, BUT YOU SHOULD KNOW HOW TO PROVE THEM

ABSOLUTE VALUE

Definition. The absolute value of a real number x is the real number $|x|$ given by

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x \leq 0 \end{cases} .$$

An equivalent version of this definition is

Definition. The absolute value of a real number x is the real number $|x|$ given by

$$|x| = \max(x, -x) .$$

Another equivalent version of this definition is

Definition. The absolute value of a real number x is the real number $|x|$ given by

$$|x| = \sqrt{x^2} .$$

BASIC PROPERTIES OF THE ABSOLUTE VALUE

Theorem. Let a, b be real numbers. Then:

i. $-|a| \leq a \leq |a|$

ii. $|ab| = |a||b|$

iii. $|a + b| \leq |a| + |b|$ (the triangle inequality)

iv. $||a| - |b|| \leq |a - b|$

All these are proved in the book.

Since it is so important, let me repeat the **proof of the triangle inequality**:

- we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$,
- so $-|a| - |b| \leq a + b \leq |a| + |b|$,
- and then $-|a| - |b| \leq -(a + b) \leq |a| + |b|$,
- so in particular $a + b \leq |a| + |b|$ and $-(a + b) \leq |a| + |b|$,
- so both numbers $a + b$ and $-(a + b)$ are $\leq |a| + |b|$,
- and one of the numbers $a + b$, $-(a + b)$, is $|a + b|$,
- so $|a + b| \leq |a| + |b|$. **QED**

The triangle inequality $|a+b| \leq |a| + |b|$ is very important. We will be using it all the time in our proofs about limits.

The inequality $||a| - |b|| \leq |a - b|$ is also useful (though not as much as the triangle inequality).

So let's prove it:

- $a = a - b + b$ (using an important trick that will be used repeatedly: add and subtract the same thing);
- so $|a| = |a-b+b| \leq |a-b|+|b|$ (by the triangle inequality);
- since $|a| \leq |a - b| + |b|$, we have $|a| - |b| \leq |a - b|$;
- similarly $|b| - |a| \leq |b - a|$,
- but $|b - a| = |a - b|$,
- and $|b| - |a| = -(|a| - |b|)$,
- so $|a| - |b| \leq |a - b|$ and $-(|a| - |b|) \leq |a - b|$,
- and one of the numbers $|a| - |b|$, $-(|a| - |b|)$ is $||a| - |b||$,
- hence $||a| - |b|| \leq |a - b|$. **QED**

Lecture 5

September 16, 2020

SEQUENCES

Definition. A sequence is a function whose domain is the set $\mathbb{N}$ of all natural numbers.
If a is a sequence, then

- we write a_n rather than $a(n)$, for $n \in \mathbb{N}$,
- we write "$a = (a_n)_{n=1}^{\infty}$", or "$a = (a_n)_{n\in\mathbb{N}}$",
- if $n \in \mathbb{N}$, then a_n is **the n-th entry** of the sequence a.

Notice that **the a_n are not "members" of the sequence**. They are **entries** of the sequence. (The sequence a also has members, of course, because a sequence is a function, and a function is a set. But the members of a are not the a_n. The members of a are the ordered pairs (n, a_n), because the sequence a is the set $\{(n, a_n) : n \in \mathbb{N}\}$.)

Example. The sequence

$$\left(\frac{1}{n}\right)_{n=1}^{\infty}$$

is the function a with domain $\mathbb{N}$ such that $a(n) = \frac{1}{n}$ for $n \in \mathbb{N}$.

In other words, a is the set

$$\left\{ u : (\exists n \in \mathbb{N}) u = \left(n, \frac{1}{n}\right) \right\}.$$

The **entries** of a are the numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, and one writes sometimes

$$a = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right),$$

but I think this is **very bad** notation and should not be used.

For example, 1 is the first entry of a, $\frac{1}{2}$ is the second entry of a, and so on.

The **members** of a are the ordered pairs $(1, 1)$, $(2, \frac{1}{2})$, $(3, \frac{1}{3})$, etc.

A sequence of real numbers is a sequence all whose entries are real numbers.

Equivalently, a sequence of real numbers is a sequence a such that $a_n \in \mathbb{R}$ for every $n \in \mathbb{N}$.

THE MODERN DEFINITION OF LIMIT OF A SEQUENCE OF REAL NUMBERS

The modern definition of a limit (for every positive ε there exists an index $N \in \mathbb{N}$ such that ...) was given by

- the Bohemian mathematician and philosopher **Bernhard Bolzano** (1781-1848), in *Der binomische Lehrsatz*, an 1816 book little noticed at the time,

- the German mathematician **Karl Weierstrass** (1815-1897) in the 1870s.

In 1734, **Bishop George Berkeley** published a book called

The Analyst

subtitled

> *A DISCOURSE Addressed to an Infidel MATHEMATICIAN. WHEREIN It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith*

In section 16 of his book, Berkeley criticizes

> *...the fallacious way of proceeding to a certain Point on the Supposition of an Increment, and then at once shifting your Supposition to that of no Increment . . . Since if this second Supposition had been made before the common Division by o, all had vanished at once, and you must have got nothing by your Supposition. Whereas by this Artifice of first dividing, and then changing your Supposition, you retain 1 and nxn-1. But, notwithstanding all this address to cover it, the fallacy is still the same.*

Berkeley's purpose was not to attack mathematics. It was to defend religion.

Religion was under attack by people who argued that religion was superstitious nonsense, based on faulty incoherent reasoning.

Berkely wanted to defend religion using a common strategy:

Maybe we do all that, but everybody else does it too.

In particular, Berkeley said that mathematicians also use faulty logic, they have numbers that are not zero and then suddently they become zero, and so on.

Berkeley's specific target were arguments like this:

Let $y = x^2$. Let us compute the derivative of y with respect to x.

We add to x an infinitely small number dx, and let dy be the amount by which y changes when we do that, so

$$y + dy = (x + dx)^2 = x^2 + 2x\,dx + dx^2$$

so

$$dy = 2x\,dx + dx^2\,.$$

We divide both sides by dx (which is possible **if $dx \neq 0$**), and get

$$\frac{dy}{dx} = 2x + dx\,,$$

and then set $dx = 0$ **(blatantly contradicting the assumption that $dx \neq 0$)** and get

$$\frac{dy}{dx} = 2x\,,$$

which is the formula we all know for the derivative of $y = x^2$.

Mathematicians, from Newton and Leibniz to Euler, Lagrange, and other great minds of the 17th and 18th centuries, did that kind of thing all the time.

They knew it was wrong, and they worried about the problem, but they did not know how to do it better.

Leonhard Euler (1707-1783), one of the greatest, said very openly something like

> *I know this is wrong; but it works, and gives the right results, so I keep doing it; I am sure that some day somebody will figure out how to do these things rigorously.*

Eventually, **somebody** (Weierstrass, although in fact Bolzano had done it about 55 years earlier) **did** do it rigorously, in the 19th century.

The solution to the problem was to give a **rigorous definition of limit**, and observe that **a "quantity" —that is, a sequence, or a function— can be nonzero but have a limit that is zero.**

And here it is, first for sequences.

We will define "limit of a sequence of real numbers" in four steps:

1. First, we will define "convergence of a sequence $a = (a_n)_{n=1}^{\infty}$ of real numbers to a real number L", and we will write "$a \to L$" to indicate that a converges to L.
   This is a **two-argument** predicate. It is a statement about two things: the sequence a and the number L.
2. Second, we will **prove** that the L such that a converges to L, if it exists, is **unique**.
   And we will do that the way we always prove uniqueness in mathematics, by proving that **if the sequence a converges to L_1 and also converges to L_2, then it follows that $L_1 = L_2$.**
3. We will call a sequence a convergent if there exists $L \in \mathbb{R}$ such that a converges to L.
   Convergence is a **one-argument** predicate. It is a statement about one thing: the sequence a.
4. Finally, we will define **the** limit of a convergent sequence a to be the real number L such that $a \to L$. This will be justified, thanks to the fact that L is unique.
   And we will use the expression $\boxed{\lim_{n\to\infty} a_n}$ to denote the limit of a sequence a.

CONVERGENCE OF A SEQUENCE TO A REAL NUMBER

If

$$a = (a_n)_{n=1}^{\infty}$$

is a sequence of real numbers, and $L \in \mathbb{R}$, then we say that

a converges to L

if

for every positive real number ε there exists a natural number N such that $|a_n - L| < \varepsilon$ whenever $n \in \mathbb{N}$, $n \geq N$,

that is, if

$$(\forall \varepsilon \in \mathbb{R})\Big(\varepsilon > 0 \Rightarrow (\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq N \Rightarrow |a_n - L| < \varepsilon)\Big)$$

Let us try to understand the condition for a to converge to L.

The condition is

For every positive real number ε there exists a natural number N such that $|a_n - L| < \varepsilon$ whenever $n \in \mathbb{N}$, $n \geq N$.

Think of it as follows:

- we would like to be able to prove that

  $|a_n - L|$ is small, very small, really truly very very small.

- But we have a problem: the words "small" and "very small" are meaningless. Nobody has ever figured out what they mean. We know what "smaller than 1" means, what "smaller than 0.1" means, what "smaller than 0.000000001" means. But "small"? There is no such a thing.

- And we cannot prove something if we don't know what it means.

- So we give up on trying to prove that "$|a_n - L|$ is small", and settle for something that does have a clear meaning: we try to prove that $|a_n - L|$ smaller than a positive number ε.

- But ..., which number ε shall we choose? Since there is no good reason to single out any particular number and saying "small" means "smaller than 0.1" or "small" means "smaller than 0.00001" or "small" means "smaller than 0.0000000000001", we try to show that we can do it for **all** positive numbers ε.

- That is, we try to prove that we can do it for an **arbitrary** positive ε.

- So we pick ε to be a **completely arbitrary** positive real number.

WHAT DOES "ARBITRARY" MEAN?

An **arbitrary** real number is a real number that could be any number.

The best way to think about this is as follows: **an arbitrary number is a number that someone else has picked, and you don't know which number it is. You have to reason about it, and say true things about it, but you don't know which number it is; so there are lots of things that you cannot say.**

For example, if ε is an arbitrary positive real number, then

- you cannot truthfully say that $\varepsilon < 1$ (because ε could be 5, for example),

- you cannot truthfully say that $\varepsilon^2 \leq \varepsilon$ (because ε could be 2),

but you can say truthfully that

- $(1+\varepsilon)^2 = 1 + 2\varepsilon + \varepsilon^2$,

- if $\varepsilon < 1$ then $\varepsilon^2 < \varepsilon$.

- So maybe we want to prove that "$|a_n - L| < \varepsilon$ for a completely arbitrary positive ε".

- But we soon realize that this is ridiculous. The only way $|a_n - L|$ can be "$< \varepsilon$ for an arbitrary positive ε" is if $|a_n - L| = 0$, and we don't want to insist that $|a_n - L|$ actually be zero.

- On the other hand, we don't want $|a_n - L|$ to be smaller than ε for every n. What we really want is for $|a_n - L|$ to be smaller than ε **eventually**.

- We don't want $|a_n - L|$ to **be** small. We want $|a_n - L|$ to **become** small.

WHAT DOES "EVENTUALLY" MEAN?

Eventually means "after a sufficiently long time".

For example, if you think of the entries of a sequence $(a_n)_{n=1}^{\infty}$ as being presented or displayed in order (first a_1, then a_2, then a_3, and so on) then the statement

$$a_n \text{ is eventually smaller than } 0.1$$

means

If you wait long enough, up to some index N, then the a_n, from $n = N$ on, will be smaller than 0.1.

Example. Let a be the sequence $(\frac{1}{3n} - \frac{1}{n^2})_{n=1}^{\infty}$. Which of the following statements is eventually true?

1. $a_n > 0$,
2. $a_n < 0$,
3. $a_n < 0.1$.

Answer to 1. The statement "$a_n > 0$" is eventually true. (**Proof**: If $n \geq 4$, then $\frac{1}{n^2} = \frac{1}{n} \times \frac{1}{n} < \frac{1}{3} \times \frac{1}{n} = \frac{1}{3n}$, so $\frac{1}{3n} - \frac{1}{n^2} > 0$. So if we let $N = 4$ then it is true that $a_n > 0$ whenever $n \geq N$.)

Notice that a_n is **not** positive for all n. (For example, $a_1 = -\frac{2}{3}$, $a_2 = -\frac{1}{12}$.) What is happening here is that initially a_n is negative, but eventually a_n becomes positive.

Answer to 2. The statement "$a_n < 0$" is not eventually true. (**Proof**: We already know that "$a_n > 0$" **is** eventually true. So no matter what $N \in \mathbb{N}$ you pick there will exist an $n \in \mathbb{N}$ such that $n \geq N$ and $a_n > 0$.)

Answer to 3. The statement "$a_n < 0.1$" is eventually true. (**Proof**: Let $N = 4$. Then if $n \geq N$ we have $\frac{1}{3n} \leq \frac{1}{3N} = \frac{1}{12} < 0.1$. So $\frac{1}{3n} - \frac{1}{n^2} < 0.1$ as well.)

So we can rephrase our definition of convergence of a sequence a of real numbers to a limit L as follows:

A sequence $(a_n)_{n=1}^{\infty}$ of real numbers converges to the real number L if for any arbitrary positive real number ε the inequality

$$|a_n - L| < \varepsilon$$

is eventually true.

Example 1. The sequence $(\frac{1}{n})_{n=1}^{\infty}$.

Theorem 1. The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to 0.

Proof. Let $a_n = \frac{1}{n}$. Then $|a_n - 0| = |a_n| = a_n$.

Let ε be an arbitrary positive real number.

We want to find $N \in \mathbb{N}$ such that, if $n \in \mathbb{N}$ and $n \geq N$, then $a_n < \varepsilon$.

Now, if $n \in \mathbb{N}$, $N \in \mathbb{N}$, $n \geq N$, we have

$$\frac{1}{n} \leq \frac{1}{N}.$$

So if we choose N so that $\frac{1}{N} < \varepsilon$, then we have $a_n = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. So

$$a_n < \varepsilon \qquad \text{whenever } n \in \mathbb{N}, n \geq N. \tag{5}$$

To get $\frac{1}{N} < \varepsilon$, it suffices to have $N > \frac{1}{\varepsilon}$. And for that purpose we can take $N = 1 + \left[\frac{1}{\varepsilon}\right]$.

With this choice of N, (5) clearly holds. **QED**

Lecture 6

September 21, 2020

In today's lecture:

1. Examples of limits, with proofs based directly on the $\varepsilon - N$ definition of limit.

2. Uniqueness of the limit.

3. Convergence of a sequence.

4. The sandwiching theorem.

5. Limits of sums, differences, products and quotients of two sequences.

But first,

here is a question

that I would like

to discuss with you.

A DISCUSSION QUESTION FOR THE STUDENTS

The book, on page 35, has a lemma:

> **LEMMA** A sequence $(a_n)_{n=1}^{\infty}$ converges to A iff each neighborhood of A contains all but a finite number of terms of the sequence.

My question is: I don't undertand this lemma and I do not know whether it is true.

Specifically, I would like to know:

- What is the precise definition of "term" of a sequence?

- What is the precise definition of "contains"?

I know the precise definition of "sequence". (It's in the book: *A sequence is a function whose domain is* $\mathbb{N}$. And *a function is a set* S *of ordered pairs such that whenever* $(x, y) \in S$ *and* $(x, z) \in S$ *it follows that* $y = z$.)

And I know what a member of a sequence is. (Since a sequence $a = (a_n)_{n=1}^{\infty}$ is a set of ordered pairs, *the members of the sequence* $a = (a_n)_{n=1}^{\infty}$ *are the ordered pairs* (n, a_n), *for* $n = 1$, 2, *etc.*)

And I also know what an entry of a sequence is. (I explained that in class. *The entries of the sequence* $(a_n)_{n=1}^{\infty}$ *are the* a_n.)

But **I don't know what a "term" of a sequence is.**

- Is it an **entry**?
- Is it a **member**?
- Is it something else, and if so what is it?

If by "term" we mean "entry", then the Lemma as stated in the book is false.

REASON: Look at the sequence $a = (a_n)_{n=1}^{\infty}$ given by $a_n = 1$ for every $n \in \mathbb{N}$. (So a is the sequence $(1, 1, 1, 1, 1, 1, \ldots,)$

This sequence clearly does not converge[a] to 5.

But every neighborhood of 5 contains all but a finite number of terms of the sequence, if by "term" we mean "entry". (REASON: **The sequence only has one entry**. The only entry of the sequence is 1.)

So it is true that *each neighborhood of 5 contains all but a finite number of terms of the sequence.*

So, if the lemma was true, the sequence a would converge to 5.

But the sequence a obviously does not converge to 5.

So **the lemma is false.**

[a] I am picking 5 just to have a specific example. Of course, the sequence does not converge to 23 either, and I could have picked 23. Or 7. Or -11.

So **maybe "term" does not mean "entry"?**
Could it mean "member"? And maybe "A contains x" means "$x \in A$"?
If so, then I claim that **according to the lemma the sequence a does not converge to 1.**
Let us see. Is is true that *each neighborhood of 1 contains all but a finite number of terms of the sequence*?
Well, let us look at the interval $(0, 2)$, which is certainly a neighborhood of 1. This interval does not contain **any** term of the sequence (if by "term" we mean "member"), because the terms (i.e., the members of a) are the ordered pairs $(n, 1)$, for $n = 1, 2, 3, \ldots$, whereas the members of the interval $(0, 2)$ are real numbers, not ordered pairs of real numbers. So no member of a is belongs to the interval $(0, 2)$. So it's not true that "every neighborhood of 1 contains all but finitely many terms of the sequence". So **the sequence does not converge to 1, if the lemma is true** (if "term" means "member" and "contains" means what I said it might).

Could "contain" mean something else? If so, what does it mean?

The word "contain" in mathematics is ambiguous, so in my view it should never be used.

I think everybody would agree that "A contains x" means the same as "x is contained in A".

So far so good.

But what on Earth does "x is contained in A" mean?

Sometimes, people say "x is contained in A" to mean "x belongs to A".

And sometimes people say "x is contained in A" to mean "x is a subset of A".

I have seen it used in both ways. And often it is not clear what is meant.

On the other hand, "belongs to" and "is a subset of" are perfectly clear.

So I think the word "contains" should never be used.

But, in any case, when the author uses it in the lemma, I think it is clear that he must mean what I just said it does. And, if so, the Lemma is false.

So the questions I want you to discuss are:

- What does "term" mean?
- What does "contain" mean?
- Is the lemma on page 35 just plain wrong, or is there a way to make it right by giving precise definitions of "term" and "contains"?

And, even better, **I would like you to fix the lemma.**

After all, it is rather clear what the author wants to say. So I am asking you to

Write a correct statement of the lemma of page 35 of the book, saying what the author is trying to say, but saying it correctly, using words and phrases that have a clear, precise, unambiguous meaning, such as "belongs to", "is a member of", "is a subset of", etc.

In today's lecture:

1. **Examples of limits, with proofs based directly on the $\varepsilon - N$ definition of limit.**
2. Uniqueness of the limit.
3. Convergence of a sequence.
4. The sandwiching theorem.
5. Limits of sums, differences, products and quotients of two sequences.

Example 2. The sequence $(\frac{1}{\sqrt{n}})_{n=1}^{\infty}$.

Theorem 2. **The sequence $\left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty}$ converges to 0.**

Proof. Let $a_n = \frac{1}{\sqrt{n}}$. Then $|a_n - 0| = |a_n| = a_n$.
Let ε be an arbitrary positive real number.
We want to find $N \in \mathbb{N}$ such that, if $n \in \mathbb{N}$ and $n \geq N$, then $a_n < \varepsilon$.
Now, if $n \in \mathbb{N}$, $N \in \mathbb{N}$, $n \geq N$, we have

$$\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}}.$$

So if we choose N so that $\frac{1}{\sqrt{N}} < \varepsilon$, then we have $a_n = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon$. So

$$a_n < \varepsilon \qquad \text{whenever } n \in \mathbb{N}, n \geq N. \tag{6}$$

To get $\frac{1}{\sqrt{N}} < \varepsilon$, it suffices to have $\frac{1}{N} < \varepsilon^2$. And for that to happen, it suffices to have $N > \frac{1}{\varepsilon^2}$. And for that purpose we can take $N = 1 + \left[\frac{1}{\varepsilon^2}\right]$.
With this choice of N, (6) clearly holds. **QED**

Example 3. The sequence $(\frac{4n+3}{5n+8})_{n=1}^{\infty}$.

Theorem 3. **The sequence $\left(\frac{4n+3}{5n+8}\right)_{n=1}^{\infty}$ converges to $\frac{4}{5}$.**

Proof. Let $a_n = \frac{4n+3}{5n+8}$. Let $L = \frac{4}{5}$. Let $b_n = a_n - L$. Then $|a_n - L| = |b_n|$.
We compute b_n. We have

$$\begin{aligned}
b_n &= \frac{4n+3}{5n+8} - \frac{4}{5} \\
&= \frac{5 \times (4n+3)}{5 \times (5n+8)} - \frac{4 \times (5n+8)}{5 \times (5n+8)} \\
&= \frac{20n+15}{25n+40)} - \frac{20n+32}{25n+40} \\
&= \frac{20n+15-(20n+32)}{25n+40} \\
&= \frac{15-32}{25n+40} \\
&= -\frac{17}{25n+40}.
\end{aligned}$$

It follows that $|b_n| = \frac{17}{25n+40}$.

Let ε be an arbitrary positive real number.

We want to find $N \in \mathbb{N}$ such that, $(n \in \mathbb{N} \wedge n \geq N) \Rightarrow |b_n| < \varepsilon$.

Now, if $n \in \mathbb{N}$, $N \in \mathbb{N}$, $n \geq N$, we have

$$|b_n| = \frac{17}{25n+40} \leq \frac{17}{25N+40}\,.$$

So if we choose N so that $\frac{17}{25N+40} < \varepsilon$, then we have

$$|b_n| \leq \frac{17}{25N+40} < \varepsilon\,.$$

To get $\frac{17}{25N+40}$ to be $< \varepsilon$, it suffices to have $\frac{25N+40}{17}$ to be $> \frac{1}{\varepsilon}$, and for that to happen it suffices to have

$$25N + 40 > \frac{17}{\varepsilon}\,. \tag{7}$$

If we pick N such that $25N > \frac{17}{\varepsilon}$, then *a fortiori* (7) will hold.

And for that to happen, it suffices to have $N > \frac{17}{25\varepsilon}$.

And for that purpose we can take $N = 1 + \left[\frac{17}{25\varepsilon}\right]$.

With this choice of N, (7) clearly holds, and then $|a_n - L| < \varepsilon$ whenever $n \in \mathbb{N}$, $n \geq N$. **QED**

Example 4. The sequence $(\sqrt{n+1}-\sqrt{n})_{n=1}^{\infty}$.

Theorem 4. **The sequence $\left(\sqrt{n+1}-\sqrt{n}\right)_{n=1}^{\infty}$ converges to 0.**

Proof. Let $a_n = \sqrt{n+1}-\sqrt{n}$. Then $a_n > 0$ for all $n \in \mathbb{N}$.
Let $L = 0$. Then $|a_n - L| = a_n$.
We get an alternative formula for a_n. We have

$$\begin{aligned} a_n &= \sqrt{n+1}-\sqrt{n} \\ &= \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{(\sqrt{n+1})^2-(\sqrt{n})^2)}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}}. \end{aligned}$$

Then, for all $n \in \mathbb{N}$,

$$a_n \leq \frac{1}{2\sqrt{n}}.$$

Now that we know that

$$a_n \leq \frac{1}{2\sqrt{n}}\,,$$

we go on as in the previous examples.

Let ε be an arbitrary positive real number.

We want to find $N \in \mathbb{N}$ such that, $(n \in \mathbb{N} \wedge n \geq N) \Rightarrow a_n < \varepsilon$.

Now, if $n \in \mathbb{N}$, $N \in \mathbb{N}$, $n \geq N$, we have

$$a_n \leq \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}}\,.$$

So if we choose N so that $\frac{1}{2\sqrt{N}} < \varepsilon$, then we have

$$a_n < \varepsilon\,.$$

To get $\frac{1}{2\sqrt{N}}$ to be $< \varepsilon$, it suffices to have $\frac{1}{4N}$ to be $< \varepsilon^2$. And for that to happen it suffices to have

$$4N > \frac{1}{\varepsilon^2}\,. \tag{8}$$

If we pick N such that $N > \frac{1}{\varepsilon^2}$, then *a fortiori* (8) will hold.

And for that purpose we can take $N = 1 + \left[\frac{1}{\varepsilon^2}\right]$.

With this choice of N, (8) clearly holds, and then $|a_n - L| < \varepsilon$ whenever $n \in \mathbb{N}$, $n \geq N$. . **QED**

Example 5. The sequence $(\sqrt[n]{n})_{n=1}^{\infty}$.

Theorem 5. The sequence $\left(\sqrt[n]{n}\right)_{n=1}^{\infty}$ converges to 1.

Proof. Let $a_n = \sqrt[n]{n}$. Then $a_n > 1$ for all $n \in \mathbb{N}$.
Let $L = 1$. Let $b_n = a_n - L$, so $b_n = \sqrt[n]{n} - 1$.
Then $|a_n - L| = b_n$.
We are going to use a **very important inequality:**

If $n \in \mathbb{N}$, $x \in \mathbb{R}$, and $x > 0$, then

$$(1+x)^n \geq 1 + nx + \frac{1}{2}n(n-1)x^2\,. \tag{9}$$

This inequality is an easy consequence of the binomial formula.
And it can be proved directly by induction. (Homework problem.)

Let's apply (9) plugging in b_n for x. We get

$$(1+b_n)^n \geq 1 + nb_n + \frac{1}{2}n(n-1)b_n^2\,. \tag{10}$$

Now recall that $b_n = \sqrt[n]{n} - 1$. So $1 + b_n = \sqrt[n]{n}$, and then $(1+b_n)^n = n$. So (10) says:

$$n \geq 1 + nb_n + \frac{1}{2}n(n-1)b_n^2\,. \tag{11}$$

Clearly, then,

$$\begin{aligned} n &\geq \frac{1}{2}n(n-1)b_n^2\,, \\ 1 &\geq \frac{1}{2}(n-1)b_n^2\,, \\ 1 &\geq \frac{n-1}{2}b_n^2\,, \\ \frac{n-1}{2}b_n^2 &\leq 1\,, \\ b_n^2 &\leq \frac{2}{n-1}\,, \end{aligned}$$

and then, finally,

$$b_n \leq \sqrt{\frac{2}{n-1}}\,. \tag{12}$$

Recall that $b_n = \sqrt[n]{n} - 1$. So what we are trying to prove is that b_n converges to 0. Clearly, Inequality (12) does it, and we could stop here. But, just to make sure everything is clear, let us finish the problem properly.

We go on as in the previous examples.

Let ε be an arbitrary positive real number.

We want to find $N \in \mathbb{N}$ such that, $(n \in \mathbb{N} \wedge n \geq N) \Rightarrow b_n < \varepsilon$.

Now, if $n \in \mathbb{N}$, $N \in \mathbb{N}$, $n \geq N$, we have

$$b_n \leq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{N-1}}\,.$$

So if we choose N so that $\sqrt{\frac{2}{N-1}} < \varepsilon$, then we have

$$b_n < \varepsilon\,.$$

To get $\sqrt{\frac{2}{N-1}}$ to be $< \varepsilon$, it suffices to have $\frac{2}{N-1} < \varepsilon^2$. And for that to happen it suffices to have

$$N - 1 > \frac{2}{\varepsilon^2}\,. \tag{13}$$

If we pick N such that $N > 1 + \frac{2}{\varepsilon^2}$, then *a fortiori* (13) will hold.

And for that purpose we can take $N = 2 + \left[\frac{2}{\varepsilon^2}\right]$.

With this choice of N, (13) clearly holds, and then $|a_n - L| < \varepsilon$ whenever $n \in \mathbb{N}$, $n \geq N$. . **QED**

Example 6. The sequence $(2^{-n})_{n=1}^{\infty}$.

Theorem 6. **The sequence $\left(2^{-n}\right)_{n=1}^{\infty}$ converges to 0.**

(Recall that 2^{-n} is just the same as $\frac{1}{2^n}$.)

Proof. Let $a_n = 2^{-n}$, and let $L = 0$. Then $|a_n - L| = a_n = 2^{-n}$. We are going to use another **very important inequality:**

If $n \in \mathbb{N}$, then

$$n < 2^n \,. \tag{14}$$

This inequality can easily be proved by induction. (Homework problem.)

Let's apply (14). Since $n < 2^n$, we have $a_n = 2^{-n} < \frac{1}{n}$.

Let ε be an arbitrary positive real number.

We want to find $N \in \mathbb{N}$ such that, $(n \in \mathbb{N} \wedge n \geq N) \Rightarrow b_n < \varepsilon$.

Now, if $n \in \mathbb{N}$, $N \in \mathbb{N}$, $n \geq N$, we have

$$a_n = 2^{-n} < \frac{1}{n} \leq \frac{1}{N}\,.$$

So if we choose N so that $\frac{1}{N} < \varepsilon$, then we have

$$a_n < \varepsilon \qquad \text{whenever } n, N \in \mathbb{N},\ n \geq N\,.$$

To get $\frac{1}{N}$ to be $< \varepsilon$, it suffices to have $N > \frac{1}{\varepsilon}$.

If we pick $N = 1 + \left[\frac{1}{\varepsilon}\right]$, $|a_n - L| < \varepsilon$ whenever $n \in \mathbb{N}$, $n \geq N$. **QED**

From now on, I will write **real sequence** for **sequence of real numbers**.

In today's lecture:

1. Examples of limits, with proofs based directly on the $\varepsilon - N$ definition of limit.
2. **Uniqueness of the limit.**
3. Convergence of a sequence.
4. The sandwiching theorem.
5. Limits of sums, differences, products and quotients of two sequences.

PASSING TO THE LIMIT IN AN INEQUALITY

Theorem. Suppose that

1. $a = (a_n)_{n=1}^{\infty}$ **and** $b = (b_n)_{n=1}^{\infty}$ **are real sequences,**
2. L **and** M **are real numbers,**
3. $a_n \leq b_n$ **for every** $n \in \mathbb{N}$

4. a **converges to** L **and** b **converges to** M**.**

Then $L \leq M$**.**

PASSING TO THE LIMIT IN AN INEQUALITY

Theorem. Suppose that

1. $a = (a_n)_{n=1}^{\infty}$ **and** $b = (b_n)_{n=1}^{\infty}$ **are real sequences,**
2. L **and** M **are real numbers,**
3. $a_n \leq b_n$ **eventually (that is,** $(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq N \Rightarrow a_n \leq b_n)$**),**
4. a **converges to** L **and** b **converges to** M**.**

Then $L \leq M$**.**

PASSING TO THE LIMIT IN AN INEQUALITY

Theorem. Suppose that

1. $a = (a_n)_{n=1}^{\infty}$ **and** $b = (b_n)_{n=1}^{\infty}$ **are real sequences,**
2. L **and** M **are real numbers,**
3. $a_n \leq b_n + \varepsilon$ **eventually for every positive** ε **(that is,**

$$(\forall \varepsilon)(\varepsilon > 0 \Rightarrow (\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq N \Rightarrow a_n \leq b_n))),$$

4. a **converges to** L **and** b **converges to** M**.**

Then $L \leq M$**.**

Proof. Assume that $L > M$. Let $\varepsilon = \frac{L-M}{3}$.

Then, eventually, we have:

$$\begin{aligned} |a_n - L| &< \varepsilon\,, \\ |b_n - M| &< \varepsilon\,, \\ a_n &\leq b_n + \varepsilon\,, \end{aligned}$$

So we have, eventually:

$$\begin{aligned} L - a_n &< \varepsilon\,, \\ b_n - M &< \varepsilon\,, \\ a_n &\leq b_n + \varepsilon\,, \end{aligned}$$

and then, eventually:

$$\begin{aligned} L &< a_n + \varepsilon \\ &< b_n + \varepsilon + \varepsilon \\ &< M + \varepsilon + \varepsilon + \varepsilon \\ &= M + 3\varepsilon \\ &= M + 3\frac{L-M}{3} \\ &= M + (L - M) \\ &= L\,. \end{aligned}$$

So $L < L$. And we have reached a contradiction. **QED**

UNIQUENESS OF THE LIMIT

Theorem. Suppose that

1. $a = (a_n)_{n=1}^{\infty}$ **is a real sequence,**
2. L_1 **and** L_2 **are real numbers,**
3. a **converges to** L_1 **and also converges to** L_2**.**

Then $L_1 = L_2$**.**

Proof. Let $b_n = a_n$ for $n \in \mathbb{N}$.

Then $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$ are real sequences that satisfy $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Furthermore, a converges to L_1 and B converges to L_2.

So by the passage to the limit theorem, $L_1 \leq L_2$.

Also, by exactly the same argument, $L_2 \leq L_1$.

So $L_1 = L_2$. **QED**

In today's lecture:

1. Examples of limits, with proofs based directly on the $\varepsilon - N$ definition of limit.

2. Uniqueness of the limit.

3. **Convergence of a sequence.**

4. The sandwiching theorem.

5. Limits of sums, differences, products and quotients of two sequences.

CONVERGENT SEQUENCES AND THEIR LIMITS

Definition. A real sequence $a = (a_n)_{n=1}^{\infty}$ converges (or is convergent) if there exists a real number L such that a converges to L.

If a real sequence $a = (a_n)_{n=1}^{\infty}$ converges, then the real number L such that a converges to L is unique, by the uniqueness theorem.

Definition. If $a = (a_n)_{n=1}^{\infty}$ is a convergent real sequence, then the unique real number L such that a converges to L is called the limit of a, and we use the expression

$$\lim_{n\to\infty} a_n$$

to denote it.

Examples:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} \frac{4n+3}{5n+8} = \frac{4}{5}$$

$$\lim_{n\to\infty} \frac{n^k}{2^n} = 0 \qquad \text{if } k \in \mathbb{N}$$

$$\lim_{n\to\infty} n^k r^n = 0 \qquad \text{if } -1 < r < 1 \wedge k \in \mathbb{N}.$$

In today's lecture:

1. Examples of limits, with proofs based directly on the $\varepsilon - N$ definition of limit.

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3. Convergence of a sequence.

4. **The sandwiching theorem.**

5. Limits of sums, differences, products and quotients of two sequences.

THE SANDWICHING THEOREM

Theorem. Let $a = (a_n)_{n=1}^{\infty}$, $b = (b_n)_{n=1}^{\infty}$ be convergent sequences of real numbers that have the same limit L, and are such that

$$a_n \leq b_n \qquad \text{for every } n \in \mathbb{N}.$$

Let $c = (c_n)_{n=1}^{\infty}$ be a real sequence such that

$$a_n \leq c_n \leq b_n \qquad \text{for every } n \in \mathbb{N}.$$

Then c converges to L.

Proof. Let ε be an arbitrary positive real number.

Since $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L$, we can pick natural numbers N_1, N_2 such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N_1\,,$$

and

$$|b_n - L| < \varepsilon \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N_2\,.$$

Let $N = \max(N_1, N_2)$. Then, if $n \geq N$, we have

$$-\varepsilon < a_n - L < \varepsilon \quad \text{and} \quad -\varepsilon < b_n - L < \varepsilon\,,$$

and $a_n \leq c_n \leq b_n$, so

$$-\varepsilon < a_n - L \leq c_n - L \leq b_n - L < \varepsilon\,,$$

and then

$$-\varepsilon < c_n - L < \varepsilon\,.$$

Therefore

$$|c_n - L| < \varepsilon \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N\,.$$

Hence $|c_n - L| < \varepsilon$ eventually. Since this is true for arbitrary positive ε, we have shown that c converges to L. **QED**

In today's lecture:

1. Examples of limits, with proofs based directly on the $\varepsilon - N$ definition of limit.

2. Uniqueness of the limit.

3. Convergence of a sequence.

4. The sandwiching theorem.

5. **Limits of sums, differences, products and quotients of two sequences.**

LIMITS OF SUMS, DIFFERENCES, AND PRODUCTS

Theorem. Let $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$ be convergent sequences. Then

1. The sequences $(a_n + b_n)_{n=1}^{\infty}$, $(a_n - b_n)_{n=1}^{\infty}$, $(a_n b_n)_{n=1}^{\infty}$, are convergent.
2. The following hold:

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n\,,$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n\,,$$

$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n).(\lim_{n\to\infty} b_n)\,,$$

LIMITS OF QUOTIENTS

Theorem. Let $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$ be convergent sequences. Assume that

A1. $b_n \neq 0$ for every n,

A2. $\lim_{n\to\infty} b_n \neq 0$.

Then

1. The sequence $(\frac{a_n}{b_n})_{n=1}^{\infty}$ is convergent,

and

2.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

I will not do the proofs for the sum and the difference, because they are very easy. **But you should make sure you know how to do them.**

Proof that $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n).(\lim_{n\to\infty} b_n)$**.**

Let

$$\begin{aligned} L &= \lim_{n\to\infty} a_n\,, \\ M &= \lim_{n\to\infty} b_n\,. \end{aligned}$$

(We don't introduce a name for $\lim_{n\to\infty}(a_n b_n)$ because we don't know that this limit exists.)

Let ε be an arbitrary positive real number. We will prove that the inequality $|a_n b_n - LM| < \varepsilon$ is eventually true.

For that purpose, we will use the fact that $|a_n - L|$ and $|b_n - M|$ are eventually smaller than any arbitrary positive real number δ, and we will choose δ so that $|a_n b_n - LM|$ will have to be smaller than ε eventually.

Let δ be a positive real number, to be chosen later.

Since $\lim_{n\to\infty} a_n = L$, we can pick a natural number N_1 such that

$$|a_n - L| < \delta \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N_1\,. \tag{15}$$

Since $\lim_{n\to\infty} b_n = M$, we can pick a natural number N_2 such that

$$|b_n - M| < \delta \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N_2\,. \tag{16}$$

Let $N = \max(N_1, N_2)$. Then

$$|a_n - L| < \delta \text{ and } |b_n - M| < \delta \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N\,. \tag{17}$$

Let $n \in \mathbb{N}$ be such that $n \geq N$. Then

$$a_n b_n - LM = a_n b_n - Lb_n + Lb_n - LM$$

Here we are using the trick of **adding and subtracting the same thing**.

Therefore

We had: $a_n b_n - LM = a_n b_n - Lb_n + Lb_n - LM$.

$$\begin{aligned}
|a_n b_n - LM| &= |a_n b_n - Lb_n + Lb_n - LM| \\
&\leq |a_n b_n - Lb_n| + |Lb_n - LM| \\
&= |(a_n - L)b_n| + |L(b_n - M)| \\
&= |a_n - L|.|b_n| + |L|.|b_n - M| \\
&\leq \delta.|b_n| + |L|.\delta \\
&\leq (|b_n| + |L|).\delta\,.
\end{aligned}$$

Here we are using the **triangle inequality** and **the identity** $|xy| = |x|.|y|$.

But

$$\begin{aligned}
|b_n| &= |b_n - M + M| \\
&\leq |b_n - M| + |M| \\
&\leq \delta + |M|\,,
\end{aligned}$$

Adding and subtracting the same thing, and **triangle inequality**.

so

$$|a_n b_n - LM| \leq (\delta + |M| + |L|)\delta\,.$$

We had: $|a_n b_n - LM| \leq (\delta + |M| + |L|)\delta$.

We want: $|a_n b_n - LM| < \varepsilon$.

Next, we use the trick of **restricting our choice**:

We agree to choose δ such that $0 < \delta \leq 1$.

Then

$$|a_n b_n - LM| \leq (1 + |M| + |L|)\delta\,.$$

So, finally, if we choose δ to be equal to $\frac{\varepsilon}{2+|M|+|L|}$ or smaller, we get

$$\begin{aligned} |a_n b_n - LM| &\leq (1 + |M| + |L|)\delta \\ &\leq (1 + |M| + |L|) \times \frac{\varepsilon}{2 + |M| + |L|} \\ &= \frac{1 + |M| + |L|}{2 + |M| + |L|}\varepsilon \\ &< \varepsilon\,. \end{aligned}$$

We agreed to choose δ such that $\delta \leq \frac{\varepsilon}{2+|M|+|L|}$.

We want: $|a_n b_n - LM| < \varepsilon$.

But we had also agreed to choose δ so that $0 < \delta \leq 1$. In order to achieve both objectives (that is, $\delta \leq \frac{\varepsilon}{2+|M|+|L|}$ and $\delta \leq 1$), we choose

$$\delta = \min\left(1, \frac{\varepsilon}{2+|M|+|L|}\right).$$

With this choice of δ, we have proved that

$$|a_n b_n - LM| < \varepsilon \qquad \text{whenever } n \in \mathbb{N},\, n \geq N\,.$$

So we have shown that $|a_n b_n - LM| < \varepsilon$ eventually.

Since this is has been shown for an arbitrary positive ε, we can conclude that **the sequence** $(a_n b_n)_{n=1}^{\infty}$ **converges to** LM. **QED**

Proof that $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Let

$$\begin{aligned} L &= \lim_{n\to\infty} a_n \,, \\ M &= \lim_{n\to\infty} b_n \,. \end{aligned}$$

(We don't introduce a name for $\lim_{n\to\infty} \frac{a_n}{b_n}$ because we don't know that this limit exists.)

Let $c_n = \frac{a_n}{b_n}$. Then c_n is well defined because we are asuming that $b_n \neq 0$.

Let $C = \frac{L}{M}$. We want to prove that for every positive ε $|c_n - C| < \varepsilon$ eventually.

Let ε be an arbitrary positive real number.

For that purpose, we will use the fact that $|a_n - L|$ and $|b_n - M|$ are eventually smaller than any arbitrary positive real number δ, and we will choose δ so that $|c_n - C|$ will have to be smaller than ε eventually.

Let δ be a positive real number, to be chosen later.

Since $\lim_{n\to\infty} a_n = L$, we can pick a natural number N_1 such that

$$|a_n - L| < \delta \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N_1\,. \tag{18}$$

Since $\lim_{n\to\infty} b_n = M$, we can pick a natural number N_2 such that

$$|b_n - M| < \delta \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N_2\,. \tag{19}$$

Let $N = \max(N_1, N_2)$. Then

$$|a_n - L| < \delta \text{ and } |b_n - M| < \delta \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N\,. \tag{20}$$

Let $n \in \mathbb{N}$ be such that $n \geq N$.

Then

$$\begin{aligned} c_n - C &= \frac{a_n}{b_n} - \frac{L}{M} \\ &= \frac{a_n M - b_n L}{b_n M} \\ &= \frac{a_n M - a_n b_n + a_n b_n - b_n L}{b_n M} \\ &= \frac{a_n(M - b_n) + (a_n - L)b_n}{b_n M}\,. \end{aligned}$$

Here we are using the trick of **adding and subtracting the same thing**.

We had: $c_n - C = \frac{a_n(M-b_n)+(a_n-L)b_n}{b_n M}$.

Therefore

$$\begin{aligned}
|c_n - C| &= \left|\frac{a_n(M-b_n)+(a_n-L)b_n}{b_n M}\right| \\
&= \frac{|a_n(M-b_n)+(a_n-L)b_n|}{|b_n M|} \\
&\leq \frac{|a_n(M-b_n)|+|(a_n-L)b_n|}{|b_n M|} \\
&\leq \frac{|a_n|.|M-b_n|+|a_n-L|.|b_n|}{|b_n M|} \\
&\leq \frac{|a_n|.\delta+|b_n|.\delta}{|b_n M|} \\
&= \frac{(|a_n|+|b_n|).\delta}{|b_n M|}\,.
\end{aligned}$$

Here we are using the **triangle inequality**, **the identity** $|xy| = |x|.|y|$ and **the identity** $|\frac{x}{y}| = \frac{|x|}{|y|}$, **valid if** $y \neq 0$.

so

$$|c_n - C| \leq \frac{(|a_n|+|b_n|).\delta}{|b_n M|}\,.$$

We had: $|c_n - C| \leq \frac{(|a_n|+|b_n|).\delta}{|b_n M|}$.

But

$$\begin{aligned} |a_n| &= |a_n - L + L| \\ &\leq |a_n - L| + |L| \\ &\leq \delta + |L| \,, \end{aligned}$$

Adding and subtracting the same thing, and **triangle inequality**.

and

$$\begin{aligned} |b_n| &= |b_n - M + M| \\ &\leq |b_n - M| + |M| \\ &\leq \delta + |M| \,, \end{aligned}$$

so

$$|a_n| + |b_n| \leq |L| + |M| + 2\delta \,.$$

We had: $|c_n - C| \leq \frac{(|a_n|+|b_n|).\delta}{|b_n M|}$ and $|a_n| + |b_n| \leq |L| + |M| + 2\delta$.

Then

$$\begin{aligned} |c_n - C| &\leq \frac{(|L| + |M| + 2\delta).\delta}{|b_n M|} \\ &= \frac{(|L| + |M| + 2\delta).\delta}{|b_n|.|M|}\,. \end{aligned}$$

Next, we use the trick of **restricting our choice**:

We agree to choose δ such that $0 < \delta \leq 1$.

Then

$$|c_n - C| \leq \frac{2 + |M| + |L|}{|b_n|.|M|} \times \delta\,.$$

We had: $|c_n - C| \leq \frac{2+|M|+|L|}{|b_n|.|M|} \times \delta$.

Now we have to get rid of the "$|b_n|$" in the denominator. We know that $|b_n \leq |M|+1$, but this does not do us any good now. In order to get rid of the $|b_n|$ in the denominator we need a bound of the form $|b_n| \geq X$, because then we will get $\frac{2+|M|+|L|}{|b_n|.|M|} \leq \frac{2+|M|+|L|}{X.|M|}$.
So we ask **what mechanism causes $|b_n|$ to be larger than something?** And the obvious answer is that this happens because $|M| > 0$ and $\lim_{n\to\infty} b_n = M$, which means that eventually $|b_n|$ should be larger than, for example, $\frac{|M|}{2}$.
Now recall that we have guaranteed that $|b_n - M| \leq \delta$ when $n \geq N$.
We then **restrict our choice of δ a second time**, by requiring that $\delta \leq \frac{|M|}{2}$.
Now, if $n \in \mathbb{N}$ and $n \geq N$, we have

$$|M| = |M - b_n + b_n| \leq |M - b_n| + |b_n| \leq \delta + |b_n| \leq \frac{|M|}{2} + |b_n| \,.$$

We had: $|c_n - C| \leq \frac{2+|M|+|L|}{|b_n|.|M|} \times \delta$, and $|M| \leq \frac{|M|}{2} + |b_n|$.

So $|M| \leq \frac{|M|}{2} + |b_n|$, and then $\frac{|M|}{2} \leq |b_n|$.
So we have the bound $|b_n| \geq \frac{|M|}{2}$, and from that we can now conclude that, if $n \geq N$, then

$$|c_n - C| \leq \frac{2+|M|+|L|}{|b_n|.|M|} \times \delta \leq \frac{2+|M|+|L|}{\frac{|M|}{2}.|M|} \times \delta \leq \frac{4+2|M|+2|L|}{|M|^2} \times \delta .$$

It is now clear what to do: we choose

$$\delta = \min\Big(1\,,\, \frac{|M|}{2}\,,\, \frac{|M|^2.\varepsilon}{5+2|M|+2|L|}\Big) .$$

Then

$$|c_n - C| < \varepsilon \qquad \text{whenever } n \in \mathbb{N} \wedge n \geq N .$$

So we have shown that $|c_n - C| < \varepsilon$ eventually.
Since this is has been shown for an arbitrary positive ε, we can conclude that the sequence $(c_n)_{n=1}^{\infty}$ converges to C. That is, **the sequence $(\frac{a_n}{b_n})_{n=1}^{\infty}$ converges to $\frac{L}{M}$. QED**