Lecture 12

October 12, 2020

In today's lecture:

- Sequential limits of functions.
- How sequential limits of functions of a real variable are related to limits of functions in the ε-δ sense.
- Limits as x goes to infinity.
- Infinite limits.

In today's lecture:

- **Sequential limits of functions.**
- How sequential limits of functions of a real variable are related to limits of functions in the ε-δ sense.
- Limits as x goes to infinity.
- Infinite limits.

SEQUENTIAL LIMITS OF FUNCTIONS

DEFINITION. Assume that

1. S is a set of real numbers (that is, $S \subseteq \mathbb{R}$),
2. $f : S \mapsto \mathbb{R}$ is a function,
3. p is a point (that is, a real number) that may or may not belong to S,
4. p can be approached by points of S (precisely: p is an accumulation point of S),
5. L is a real number.

We say that

- $\underline{f(x) \text{ converges to } L \text{ sequentially as } x \text{ approaches } p \text{ along } S}$

if

Whenever $x = (x_n)_{n=1}^{\infty}$ is a sequence of real numbers such that $x_n \in S$ and $x_n \neq p$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = p$, it follows that

$$\lim_{n\to\infty} f(x_n) = L\,.$$

FUNCTION LIMITS

VS. SEQUENTIAL LIMITS

THEOREM. Assume that

1. S is a set of real numbers (that is, a subset of $\mathbb{R}$),
2. $f : S \mapsto \mathbb{R}$ is a function,
3. p is a point (that is, a real number) that may or may not belong to S,
4. p can be approached by points of S (precisely: p is an accumulation point of S),
5. L is a real number.

Then

$f(x)$ converges to L sequentially as x approaches p along S if and only if

$$\lim_{x \to p\,,\, x \in S} f(x) = L\,.$$

In order to prove the theorem, it will be conventient to review first what the ε-δ definition of “limit” says:

LIMITS OF FUNCTIONS OF A REAL VARIABLE

DEFINITION. Assume that

1. S is a set of real numbers (that is, a subset of $\mathbb{R}$),
2. $f : S \mapsto \mathbb{R}$ is a function,
3. p is a point (that is, a real number) that may or may not belong to S,
4. p can be approached by points of S (precisely: p is an accumulation point of S),
5. L is a real number.

We say that

- $\underline{f(x) \text{ converges to } L \text{ as } x \text{ approaches } p \text{ along } S}$,

and we write
$$\lim_{x \to p\,,\ x \in S} f(x) = L\,,$$

$$\boxed{\begin{array}{l}\text{For every positive real number } \varepsilon \text{ there exists a} \\ \text{positive real number } \delta \text{ such that} \\ |f(x) - L| < \varepsilon \ \text{ whenever } \ x \in S\,,\ x \neq p \ \text{ and } \ |x-p| < \delta.\end{array}}$$

Proof. First we prove that

$$\text{If } \lim_{x \to p\,,\, x \in S} f(x) = L\,, \text{ then } f(x) \to L \text{ sequentially as } x \to p\,,\, x \in S\,. \tag{1}$$

To prove (1), assume that

$$\lim_{x \to p\,,\, x \in S} f(x) = L\,. \tag{2}$$

We want to prove that

$$f(x) \to L \text{ sequentially as } x \to p\,,\, x \in S\,. \tag{3}$$

To prove (3), let $x = (x_n)_{n=1}^{\infty}$ be a sequence of points of S such that $x_n \neq p$ for all n and x converges to p.

We want to prove that

$$\text{The sequence } (f(x_n))_{n=1}^{\infty} \text{ converges to } L\,. \tag{4}$$

To prove (4), we have to show that for every positive real number ε there exists a natural number N such that

$$|f(x_n) - L| < \varepsilon \text{ whenever } n \in \mathbb{N}\,,\ n \geq N\,. \tag{5}$$

We know from the fact that (2) holds, that there exists a positive real number δ such that

$$|f(x) - L| < \varepsilon \text{ whenever } x \in S\,,\ x \neq p\,,\ |x - p| < \delta\,. \tag{6}$$

Also, since

$$\lim_{n\to\infty} x_n = p\,, \tag{7}$$

we can find a natural number N such that

$$|x_n - p| < \delta \text{ whenever } n \in \mathbb{N} \text{ and } n \geq N\,. \tag{8}$$

Then, if $n \in \mathbb{N}$ and $n \geq N$, we have $|x_n - p| < \delta$ (because of (8)), and also $x_n \in S$ and $x_n \neq p$. So we can apply (6) and conclude that

$$|f(x_n) - L| < \varepsilon\,. \tag{9}$$

So we have proved that (9) holds whenever $n \geq N$. This means that (5) holds.

So we have proved that for an arbitrary positive ε there exists $N \in \mathbb{N}$ such that (5) holds.

This means that the sequence $(f(x_n))_{n=1}^{\infty}$ converges to L.

The sequence x was an arbitrary sequence of points of S such that $x_n \neq p$ and x converges to p.

So we have proved that for every such sequence the sequence $(f(x_n))_{n=1}^{\infty}$ converges to L.

Hence $f(x)$ converges to L sequentially as $x \to p$ via values in S.

We now have to prove the other implication. That is, we have to prove that

$$\lim_{x \to p\,,\, x \in S} f(x) = L\,, \tag{10}$$

assuming that

$$f(x) \to L \text{ sequentially as } x \to p\,,\, x \in S\,. \tag{11}$$

For this purpose, let us assume that (10) is not true.

So we are going to prove that (11) implies (10) by contradiction.

Since (10) says that "for every positive ε there exists a positive δ such that ...", the assumption that (10) is not true tells us that "for some positive ε there exists no positive δ such that ...".

So we may pick a positive real number ε such that

$$\text{there is no positive } \delta \in \mathbb{R} \text{ that works for } \varepsilon\,, \tag{12}$$

where "δ works for ε" means

$$|f(x) - L| < \varepsilon \text{ whenever } x \in S\,,\ x \neq p\,,\ |x - p| < \delta\,. \tag{13}$$

Since there is no δ that works for our ε, it follows that in particular $\frac{1}{n}$ does not work for ε, no matter who n is.

So for every $n \in \mathbb{N}$ (13) is not true if $\delta = \frac{1}{n}$.

This means that the sentence "$|f(x) - L| < \varepsilon$" is not true for every x such that $x \in S$, $x \neq p$, and $|x - p| < \frac{1}{n}$.

So we may pick x_n such that

$$x_n \in S \quad x_n \neq p\,,\ |x_n - p| < \frac{1}{n}\,,\ \text{and}\ \ |f(x_n) - L| \geq \varepsilon\,. \tag{14}$$

We can pick such an x_n for each n. So the x_n are the entries of a sequence $x = (x_n)_{n=1}^{\infty}$. And this sequence satisfies:

$$x_n \in S \text{ for every } n \in \mathbb{N}\,, \tag{15}$$

$$x_n \neq p \text{ for every } n \in \mathbb{N}\,, \tag{16}$$

$$|x_n - p| < \frac{1}{n} \text{ for every } n \in \mathbb{N}\,, \tag{17}$$

$$|f(x_n) - L| \geq \varepsilon \text{ for every } n \in \mathbb{N}\,. \tag{18}$$

It follows from (15), (16), and (17) that

(I) *x is a sequence of points of S that are all different from p,*

(II) *x converges to p.*

(III) *the sequence $(f(x_n))_{n=1}^{\infty}$ does not converge to L.*

On the other hand, since $f(x)$ converges to L sequentially as $x \to p$ via values in S, it follows that

(IV) *the sequence $(f(x_n))_{n=1}^{\infty}$ converges to L.*

Clearly, (III) and (IV) add up to a contradiction.

So the assumption that it is not true that $\lim_{x \to p\,,\, x \in S} f(x) = L$ has led us to a contradiction.

Hence

$$\lim_{x \to p\,,\, x \in S} f(x) = L\,.$$

So we have proved the two implications that we had to prove. Therefore our proof is now complete. **QED**

We have already seen two definitions of “limit” of a function:

1. The definition of **limit of a sequence as $n \to \infty$** (the “ε-N definition”: for every positive ε there exists N such that ...).
2. The definition of **limit of a function of a real variable as x goes to a point p** (the “ε-N definition”: for every positive ε there exists N such that ...) (the “ε-δ definition”: for every positive ε there exists a positve δ such that ...).

If two definitions are too many, and it upsets you that you have to remember so much stuff, I’ll make it worse for you: **I will give you several more definitions of limits of various sorts**.

First, we need to talk about limits $\lim_{x\to+\infty} f(x)$.

In today's lecture:

- Sequential limits of functions.
- How sequential limits of functions of a real variable are related to limits of functions in the ε-δ sense.
- **Limits as x goes to infinity.**
- Infinite limits.

LIMITS OF FUNCTIONS OF A REAL VARIABLE AS x GOES TO $+\infty$

DEFINITION. Assume that

1. S is a set of real numbers (that is, a subset of $\mathbb{R}$),
2. $f : S \mapsto \mathbb{R}$ is a function,
3. $+\infty$ can be approached by points of S (precisely: $+\infty$ is an accumulation point of S),
4. L is a real number.

We say that

- <u>$f(x)$ converges to L as x approaches $+\infty$ along S</u>,

and we write
$$\lim_{x \to +\infty\,,\; x \in S} f(x) = L\,,$$

if

$$\boxed{\begin{array}{l}\text{For every positive real number } \varepsilon \text{ there exists a} \\ \text{real number } M \text{ such that} \\ \qquad |f(x) - L| < \varepsilon \;\text{ whenever }\; x \in S\,, \text{ and } x > M\,.\end{array}}$$

What does "$+\infty$ is an accumulation point of S" mean?

First, we need to define "neighborhoods of $+\infty$":

DEFINITION. A basic neighborhood of $+\infty$ is a set V such that, for some $M \in \mathbb{R}$, $V = \{x : x \in \mathbb{R} \text{ and } x > M\}$.

DEFINITION. A neighborhood of $+\infty$ is a set V that contains a basic neighborhood of $+\infty$.

DEFINITION. If S is a subset of $\mathbb{R}$, we say that

$+\infty$ is an accumulation point of S

if every neighborhood of $+\infty$ contains a point of S.

Compare this with the definitions of "neighborhood" and "accumulation point" for points $p \in \mathbb{R}$:

The definition of "basic neighborhood of a point $p \in \mathbb{R}$ is different: a basic neighborhood of p is a set V such that, for some positive $\varepsilon \in \mathbb{R}$,

$$V = \{x : x \in \mathbb{R} \ p - \varepsilon < x + p + \varepsilon\}.$$

But then, **once we know what "basic negiborhood of p" means, the definitions of "neighborhood" and "accumulation point" are exactly the same.**

A minor point:

In the definition of what it means for a point $p \in \mathbb{R}$ to be an “accumulation point” of a subset S of $\mathbb{R}$, we say that every neighborhood of p has to contain a point q of S **other than** p.

But in the definition of what it means for $+\infty$ to be an “accumulation point” of a subset S of $\mathbb{R}$, we say that every neighborhood of $+\infty$ has to contain a point q of S, and we do not say **a point** q **of** S **other than** $+\infty$.

But this is not really a difference: a point q of S is automatically “other than $+\infty$’, because $+\infty$ (whatever it may be) is not a real number, so $+\infty$ is not in S (since $S \subseteq \mathbb{R}$), so if $q \in S$ then $q \neq +\infty$.

Example. Let us prove that

$$\lim_{x \to +\infty} \frac{x}{\sqrt{1+x^2}} = 1\,.$$

Proof. Let ε be an arbitrary positive real number.

We want to find $M \in \mathbb{R}$ such that

$$\left| \frac{x}{\sqrt{1+x^2}} - 1 \right| < 1 \text{ whenever } x \in \mathbb{R}\,,\ \ x > M\,. \tag{19}$$

Let us restrict our choices by agreeing in advance that we are to take M to be positive.

We will choose M to be a positive number that will be specified later.

Then, if $x > M$, we have

$$\begin{aligned} \frac{x}{\sqrt{1+x^2}} - 1 &= \frac{x}{\sqrt{1+x^2}} - \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \\ &= \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}}. \end{aligned}$$

But the identity $(a-b)(a+b) = a^2 - b^2$ implies that

$$(x - \sqrt{1+x^2})(x + \sqrt{1+x^2}) = x^2 - (1+x^2) = -1\,,$$

so

$$x - \sqrt{1+x^2} = -\frac{1}{x + \sqrt{1+x^2}}\,,$$

and then

$$\begin{aligned} \frac{x}{\sqrt{1+x^2}} - 1 &= \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}} \\ &= -\frac{1}{\sqrt{1+x^2} \times (x + \sqrt{1+x^2})} \, . \end{aligned}$$

Hence, since $x > 0$ (because $x > M$ and $M > 0$),

$$\begin{aligned} \left| \, \frac{x}{\sqrt{1+x^2}} - 1 \, \right| &\leq \frac{1}{\sqrt{1+x^2} \times (x + \sqrt{1+x^2})} \\ &\leq \frac{1}{\sqrt{1+x^2} \times \sqrt{1+x^2}} \\ &= \frac{1}{1+x^2} \\ &< \frac{1}{1+M^2} \\ &< \frac{1}{M^2} \, . \end{aligned}$$

So, if we take M such that $\frac{1}{M^2} < \varepsilon$, we get the desired inequality

$$\left| \frac{x}{\sqrt{1+x^2}} - 1 \right| < \varepsilon \, .$$

And, to get $\frac{1}{M^2}$ to be $< \varepsilon$, it suffices to to have $M^2 > \frac{1}{\varepsilon}$, for which it suffices to take $M > \frac{1}{\sqrt{\varepsilon}}$. So choosing

$$M = 1 + \frac{1}{\sqrt{\varepsilon}}$$

suffices.

With this choice of M, (19) holds, and this completes our proof. **QED**

Obviously, we can also define limits of the form

$$\lim_{x\to-\infty} f(x)\,.$$

LIMITS OF FUNCTIONS OF A REAL VARIABLE AS x GOES TO $-\infty$

DEFINITION. Assume that

1. S is a set of real numbers (that is, a subset of $\mathbb{R}$),
2. $f : S \mapsto \mathbb{R}$ is a function,
3. $-\infty$ can be approached by points of S (precisely: $-\infty$ is an accumulation point of S),
4. L is a real number.

We say that

- $\underline{f(x) \text{ converges to } L \text{ as } x \text{ approaches } -\infty \text{ along } S}$,

and we write
$$\lim_{x \to -\infty\,,\; x \in S} f(x) = L\,,$$

if

$$\boxed{\begin{array}{l}\text{For every positive real number } \varepsilon \text{ there exists a} \\ \text{real number } M \text{ such that} \\ \qquad |f(x) - L| < \varepsilon \ \text{ whenever } \ x \in S\,, \text{ and } x < M\,.\end{array}}$$

And, instead of having the limit be a real number, we can define what it means for the limit to be $+\infty$ or $-\infty$:

In today's lecture:

- Sequential limits of functions.
- How sequential limits of functions of a real variable are related to limits of functions in the ε-δ sense.
- Limits as x goes to infinity.
- **Infinite limits.**

INFINITE LIMITS OF FUNCTIONS OF A REAL VARIABLE

DEFINITION. Assume that

1. S is a set of real numbers (that is, a subset of $\mathbb{R}$),
2. $f : S \mapsto \mathbb{R}$ is a function,
3. p is a point (that is, a real number) that may or may not belong to S,
4. p can be approached by points of S (precisely: p is an accumulation point of S).

We say that

- $f(x)$ goes to $+\infty$ as x approaches p along S,

and we write $$\lim_{x \to p\,,\ x \in S} f(x) = +\infty\,,$$

if

For every real number M there exists a positive real number δ such that

$$f(x) > M \text{ whenever } x \in S\,,\ x \neq p \text{ and } |x - p| < \delta.$$

Example. Let

$$S = \{ x : x \in \mathbb{R} \wedge x \neq 0 \} .$$

Let us prove that

$$\lim_{x \to 0 \, , x \in S} \frac{1}{x^2} = +\infty \, . \tag{20}$$

Let M be an arbitrary real number. We want to find a positive real number δ such that

$$\frac{1}{x^2} > M \text{ whenever } x \in \mathbb{R} \, , \; x \neq 0 \text{ and } |x| < \delta. \tag{21}$$

For any given positive δ, of $x \in \mathbb{R}$, $x \neq 0$, and $|x| < \delta$, we have

$$\frac{1}{x^2} > \frac{1}{\delta^2} \, .$$

So, if we want to guarantee that $\frac{1}{x^2}$ will be $> M$, it suffices to take δ such that

$$\frac{1}{\delta^2} > M \, . \tag{22}$$

How can we achieve (22)?

If $M \leq 0$ then any positive number δ will do.

Now suppose $M > 0$. Then, in order to achieve (22), it suffices to make sure that

$$\delta^2 < \frac{1}{M}\,,$$

i.e., that

$$\delta < \frac{1}{\sqrt{M}}\,.$$

So we may take

$$\delta = \frac{1}{1+\sqrt{M}}\,,$$

which is the same as

$$\delta = \frac{1}{1+\sqrt{|M|}}\,. \tag{23}$$

The δ given by (23) works if $M > 0$. And it also works if $M \geq 0$, because in that casy any positive δ works, as we explained before.

So, finally, we can choose δ according to formula (23), and that works for every $M \in \mathbb{R}$.

So for every $M \in \mathbb{R}$ there exists a positve real number δ such that (21) holds. This prove that (20) holds. **QED**

So far, we have defined the meaning of

$$\begin{aligned} \lim_{x\to p} f(x) &= L && \text{if } p \in \mathbb{R}\,,\ L \in \mathbb{R}\,, \\ \lim_{x\to +\infty} f(x) &= L && \text{if } L \in \mathbb{R}\,, \\ \lim_{x\to -\infty} f(x) &= L && \text{if } L \in \mathbb{R}\,, \\ \lim_{x\to p} f(x) &= +\infty && \text{if } p \in \mathbb{R}\,. \end{aligned}$$

Obviously, there are other possibilities.

Actually, there are **nine** possibilities:

$$\begin{aligned}
\lim_{x\to p} f(x) &= L && \text{if } p\in\mathbb{R},\ L\in\mathbb{R},\\
\lim_{x\to+\infty} f(x) &= L && \text{if } L\in\mathbb{R},\\
\lim_{x\to-\infty} f(x) &= L && \text{if } L\in\mathbb{R},\\
\lim_{x\to p} f(x) &= +\infty && \text{if } p\in\mathbb{R},\\
\lim_{x\to+\infty} f(x) &= +\infty\,,\\
\lim_{x\to-\infty} f(x) &= +\infty\,,\\
\lim_{x\to p} f(x) &= -\infty && \text{if } p\in\mathbb{R},\\
\lim_{x\to+\infty} f(x) &= -\infty\,,\\
\lim_{x\to-\infty} f(x) &= -\infty\,.
\end{aligned}$$

I will not do the remaing five possibilities, because they are too boring, and you can figure them out by yourself.

The good thing to know is that they are all just special cases of one, single, unified, definition of “limit”.

Before we get there, I need to tell you what the **extended real line** is.

Fix, one and for all, two objects, called “$+\infty$” and “$-\infty$”.

It does not matter what these two objects are. For example, $+\infty$ could be the matrix

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix},$$

or the function f given by $f(x) = \sqrt{x^2+7}$ for $x \in \mathbb{R}$, or my cousin Dorothy, or a zebra in the Bronx Zoo. It does not matter who “$+\infty$” is, **as long as it is not a real number**.

And “$-\infty$ should be another object. Any object, **as long as it is not a real number and it is not $+\infty$.**

DEFINITION. The <u>extended real line</u> is the set $\bar{\mathbb{R}}$ whose members are the real numbers, as well as $+\infty$ and $-\infty$.

That is:

$$\bar{\mathbb{R}} = \mathbb{R} \bigcup \{+\infty, -\infty\}\,.$$

In other words, the extended real line is the set $\mathbb{R}$ with two extra points, one to the right of all the real numbers, and one to the left of all the real numbers.

Next, I have to tell you what **neighborhood** of a point p of $\mathbb{R}$ is.
Actually, I have already told you, but I am going tot ell you again.

DEFINITION.

1. If $p \in \mathbb{R}$, a basic neighborhood of $+\infty$ is a set V such that, for some positive real number ε,

$$V = \{\, x : x \in \mathbb{R} \text{ and } p - \varepsilon < x < p + \varepsilon \,\}\,.$$

2. If $p = +\infty$, a basic neighborhood of p is a set V such that, for some $M \in \mathbb{R}$,

$$V = \{\, x : x \in \mathbb{R} \text{ and } x > M \,\}\,.$$

3. If $p = -\infty$, a basic neighborhood of p is a set V such that, for some $M \in \mathbb{R}$,

$$V = \{\, x : x \in \mathbb{R} \text{ and } x < M \,\}\,.$$

4. If $p \in \bar{\mathbb{R}}$, a neighborhood of p is a subset V of $\mathbb{R}$ that contains a basic neigborhood of p.

And I have to tell you what an **accumulation point** of a subset of $\mathbb{R}$ is.

Actually, I have already told you this too, but I am going tot ell you again.

DEFINITION.

If $S \subseteq \mathbb{R}$, an accumulation point of S is a point p of $\bar{\mathbb{R}}$ such that

$$\boxed{\text{Every neighborhood of } p \text{ contains a point of } S \text{ other than } p.}$$

THE GENERAL DEFINITION OF LIMIT

DEFINITION. Assume that

1. S is a set of real numbers (that is, $S \subseteq \mathbb{R}$),
2. $f : S \mapsto \mathbb{R}$ is a function,
3. p is a point of $\bar{\mathbb{R}}$ (that is, an extended real number) that may or may not belong to S,
4. p can be approached by points of S (precisely: p is an accumulation point of S),
5. L is an extended real number, i.e., a member of $\bar{\mathbb{R}}$.

We say that

- $\underline{f(x)\ \text{converges to}\ L\ \text{as}\ x\ \text{approaches}\ p\ \text{along}\ S}$

if

Whenever V is a neigborhood of L, there exists a neighborhood W of p such that

$$f(x) \in V \quad \text{for every} \quad x \in (W \cap S) - \{p\}\,.$$

Lecture 13

October 14, 2020

In today's lecture:

1. Real functions of a real variable (RFRVs).
2. The comparison theorem for functions.
3. Uniqueness of the limit of a function.
4. The sandwiching theorem for functions.
5. Limits of sums, differences, products and quotients of two functions.

In today's lecture:

1. **Real functions of a real variable (RFRVs).**

2. The comparison theorem for functions.

3. Uniqueness of the limit of a function.

4. The sandwiching theorem for functions.

5. Limits of sums, differences, products and quotients of two functions.

From now on, I will be using the following terminology and notation:

- A **real function** is a function with real values, that is, a function $f : S \mapsto \mathbb{R}$, where S is a set (*any* set).
- A **function of a real variable** is a function whose domain is a subset of $\mathbb{R}$, that is, a function $f : S \mapsto T$, where S is a subset of $\mathbb{R}$ and T is a set (*any* set).
- A **real function of a real variable** (RFRV) is real function which is a function of a real variable, i.e., a function $f : S \mapsto \mathbb{R}$, where S is a subset of $\mathbb{R}$.

- If
  - f is a function,
  - $E(x)$ is any expression for the value $f(x)$,
  - S is the domain of f,

  then the expression

  $$\text{the function } S \ni x \mapsto E(x)$$

  (read as "the function that sends (or 'takes', or 'maps') x in S to $E(x)$")

  is another name for f.

So, for example:

- "the function $\mathbb{R} \ni x \mapsto x^2$" is a name for the function f with domain $\mathbb{R}$ such that $f(x) = x^2$ for each $x \in \mathbb{R}$;

- "the function $\{x : x \in \mathbb{R} \wedge x \geq 0\} \ni x \mapsto x^2$" is a name for the function g with domain $\{x : x \in \mathbb{R} \wedge x \geq 0\}$ such that $g(x) = x^2$ for each $x \in \mathbb{R}$ such that $x \geq 0$;

- "the function $\{x : x \in \mathbb{R} \wedge x \geq 0\} \ni x \mapsto \sqrt{x}$" is a name of the square root function.

- "the function $\mathbb{N} \ni n \mapsto x_n$" is a name for the sequence $(x_n)_{n=1}^{\infty}$,

- if f is a function, and S is the domain of f, then "the function $S \ni x \mapsto f(x)$" is another name for f,

- but if f is a function, and S is **not** the domain of f, then
  - if S is a subset of the domain D of f, then the expression “the function $S \ni x \mapsto f(x)$” is a name for the **restriction** of f to S (see the remark below on the meaning of “restriction”),
  - and if S is not a subset of the domain D of f, then the expression “the function $S \ni x \mapsto f(x)$” is not acceptable, because the expression “$f(x)$” does not make sense if $x \notin D$, and there are members of S that are not in D.

REMARK. If $f : A \mapsto B$ is a function, and C is a subset of A, then the restriction of f to C is the function $C \ni x \mapsto f(x)$.

We use “f_C” to denote the restriction of f to C, so “f_C” is just another name for the function $C \ni x \mapsto f(x)$.

In today's lecture:

1. Real functions of a real variable (RFRVs).
2. **The comparison theorem for functions.**
3. Uniqueness of the limit of a function.
4. The sandwiching theorem for functions.
5. Limits of sums, differences, products and quotients of two functions.

PASSING TO THE LIMIT IN AN INEQUALITY

Theorem. Suppose that

1. f and g are RFRVs with domain S,
2. p is an accumulation point of S,
3. L and M are real numbers,
4. $f(x) \leq g(x)$ for every $x \in S$
5. $\lim_{x \to p\,,\, x \in S} f(x) = L$ and $\lim_{x \to p\,,\, x \in S} g(x) = M$.

Then $L \leq M$.

Proof. Assume that $L > M$. Let $\varepsilon = \frac{L-M}{2}$.

Then there exist positive real numbers δ_1, δ_2 such that

$$\begin{aligned} |f(x) - L| &< \varepsilon \text{ whenever } x \in S\,,\ x \neq p\,,\ |x-p| < \delta_1\,, \\ |g(x) - M| &< \varepsilon \text{ whenever } x \in S\,,\ x \neq p\,,\ |x-p| < \delta_2\,. \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$.

Then, if $x \in S$, $x \neq p$, $|x-p| < \delta$, we have $|f(x) - L| < \varepsilon$ and $|g(x) - M| < \varepsilon$.

Furthermore, **since p is an accumulation point of** S, we can pick a point q such that $q \in S$, $q \neq p$ and $|q-p| < \delta$.

Notice that the key point here is that p is an accumulation point of S, and that makes it possible for us to pick q. If p was not an accumulation point of S, then the sentence

$$(\forall x \in S)(|x-p| < \delta \Rightarrow (|f(x) - L| < \varepsilon \wedge |g(x) - M| < \varepsilon))$$

might be true vacuously, in which case there would not exist a q such that $q \in S$, $q \neq p$ and $|q-p| < \delta$, and we would not be able to argue as we do in the next page.

Then

$$\begin{aligned} L - f(q) &< \varepsilon\,, \\ g(q) - M &< \varepsilon\,, \\ f(q) &\le g(q)\,, \end{aligned}$$

so

$$\begin{aligned} L &< f(q) + \varepsilon \\ &\le g(q) + \varepsilon \\ &< M + \varepsilon + \varepsilon \\ &= M + 2\varepsilon \\ &= M + 2\frac{L - M}{2} \\ &= M + (L - M) \\ &= L\,. \end{aligned}$$

So $L < L$. And we have reached a contradiction. **QED**

In today's lecture:

1. Real functions of a real variable (RFRVs).
2. The comparison theorem for functions.
3. **Uniqueness of the limit of a function.**
4. The sandwiching theorem for functions.
5. Limits of sums, differences, products and quotients of two functions.

UNIQUENESS OF FUNCTION LIMITS

Theorem. Suppose that

1. f is a RFRV,
2. L_1 and L_2 are real numbers,
3. f has the limit L_1 and also the limit L_2 as $x \to p$, $x \in S$.

Then $L_1 = L_2$.

Proof. Let $g = f$.

Then f and g are RFRVs with domain S that satisfy $f(x) \leq g(x)$ for all $x \in S$.

Furthermore, f converges to L_1 and g converges to L_2.

So by the passage to the limit theorem, $L_1 \leq L_2$.

Also, by exactly the same argument, $L_2 \leq L_1$.

So $L_1 = L_2$. **QED**

A CONFESSION. I am guilty of an unforgivable sin. And I beg for forgiveness.

I have given a name to “the” limit of a function f as $x \to p$, $x \in S$, **before** I knew that the limit was unique.

One should never do that.

I did it temporarily, because I knew that I was going to prove uniqueness very soon. Instead of writing

$$\lim_{x \to p\ x \in S} f(x) = L$$

(which implies that if $\lim_{x \to p\ x \in S} f(x) = L$ and $\lim_{x \to p\ x \in S} f(x) = M$ then $L = M$), I should have written something like

$$f(x) \to L \text{ as } x \to p\,,\ x \in S$$

which does not have that implication.

But in any case now we have proved uniqueness, so maybe I can be forgiven after all.

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THE SANDWICHING THEOREM

Theorem. Let f, g be RFRVs with the same domain S, let p be an accumulation point of S, and assume that f and g have the same limit L as $x \to p$, $x \in S$.

Let $h : S \mapsto \mathbb{R}$ be a RFRV such that

$$f(x) \leq h(x) \leq g(x) \qquad \text{for every } x \in S\,.$$

Then $\lim_{x \to p\,,\, x \in S} h(x) = L$.

Proof.
This follows easily using either the ε-δ definition of limit, or using sequential convergence. **YOU DO IT.**

In today's lecture:

1. Real functions of a real variable (RFRVs).
2. The comparison theorem for functions.
3. Uniqueness of the limit of a function.
4. The sandwiching theorem for functions.
5. **Limits of sums, differences, products and quotients of two functions.**

LIMITS OF SUMS, DIFFERENCES, PRODUCTS, AND QUOTIENTS

Theorem. Let f, g be RFRVs with the same domain S, let L, M be real numbers, and let p be an accumulation point of S such that

$$\lim_{x\to p\,,\,x\in S} f(x) = L \quad \text{and} \quad \lim_{x\to p\,,\,x\in S} g(x) = M\,.$$

Then

$$\begin{aligned}
\lim_{x\to p\,,\,x\in S}(f(x)+g(x)) &= L+M\,,\\
\lim_{x\to p\,,\,x\in S}(f(x)-g(x)) &= L-M\,,\\
\lim_{x\to p\,,\,x\in S}(f(x).g(x)) &= L.M\,.
\end{aligned}$$

Furthermore, if $g(x)\neq 0$ for all $x\in S$, and $M\neq 0$, then

$$\lim_{x\to p\,,\,x\in S}\frac{f(x)}{g(x)} = \frac{L}{M}\,.$$

I will not do the proofs for the sum and the difference, because they are very easy. **(But you should make sure you know how to do them.)**

The proofs for the product and the quotient take more work, but a lot of work can be saved using sequential convergence.

The book

- does a direct (i.e., ε-δ) proof of the result for the product,
- gives a proof using sequential convergence of the result for the quotient,
- and leaves the direct proof of the result for the quotient to "the adventurous reader".

I will give you here the proof for the product using sequential convergence, and concur with the author of the book in asking you to be adventurous and do the direct proof of the quotient result

Proof that $\lim_{x \to p\,,\, x \in S}(f(x).g(x)) = L.M$ **using sequential convergence.**

We will use the fact that $\lim_{x \to p\,,\, x \in S}(f(x).g(x)) = L.M$ if and only if $f(x).g(x)$ goes to LM as $x \to p$, $x \in S$, sequentially.

So we are going to prove that $f(x).g(x)$ goes to LM sequentially as $x \to p$, $x \in S$.

To prove this, let $x = (x_n)_{n=1}^{\infty}$ be a sequence of points of $S - \{p\}$ (that is, points of S that are different from p) such that the sequence x converges to p, and let us prove that the sequence $(f(x_n)g(x_n))_{n=1}^{\infty}$ converges to LM.

Since $\lim_{x \to p\,,\, x \in S} f(x) = L$, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to L.

And, since $\lim_{x \to p\,,\, x \in S} g(x) = M$, the sequence $(g(x_n))_{n=1}^{\infty}$ converges to M.

Then, by the theorem of the limit of a product of sequences, the sequence $(f(x_n)g(x_n))_{n=1}^{\infty}$ converges to LM.

And that completes the proof.

Is there a **Cauchy condition** for limits of functions?

Yes, there is.

DEFINITION. Suppose that

- f **is a RFRV with domain** S**,**
- p **is an accumulation point of** S**.**

We say that $\underline{f \text{ satisfies the Cauchy condition at } p \text{ along } S}$ **if**

(C) for every positive real number ε **there exists a positive real number** δ **such that**

$|f(x)-f(y)| < \varepsilon$ whenever $x \in S$, $y \in S$, $x \neq p$, $y \neq p$, $|x-p| < \delta$, $|y-p| < \delta$.

And, not suprisingly:

- There is a **sequential Cauchy condition**: **for every sequence $x = (x_n)_{n=1}^{\infty}$ of points of $S - \{p\}$ that converges to p, the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy.**

- There are two theorems:
  - $\lim_{x \to p\,,\, x \in S} f(x)$ **exists if and only if f satisfies the Cauchy condition,**
  - **f satisfies the Cauchy condition if and only if f satisfies the sequential Cauchy condition.**

All these things are easy to prove, using the same arguments we have used several times.