Lecture 24

November 30, 2020

The purpose of today's lecture is

- To show how the methods of Differential Calculus make it possible to **construct new functions and study their properties**.

- To do this using the tools developed so far, especially
  - **the Mean Value Theorem**,
  - **the Inverse Function Theorem**,
  - **Monotone Convergence**.

Specifically, we will

- **construct the natural logarithm function**

$$\mathbb{R}_+ \ni x \mapsto \ln x \, ,$$

(where $\mathbb{R}_+ \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x > 0\}$)

- **study its properties**,

- **construct the exponential function**

$$\mathbb{R} \ni x \mapsto e^x \, .$$

and study its properties.

Since the Mean Value Theorem was proved using Rolle's Theorem, which in turn depends on the Extreme Value Theorem, which depends on the Bolzano-Wierstrass Theorem, **we will in fact be using practically all the ideas we have developed in the course**.

Let I be a real interval, and let $f : I \mapsto \mathbb{R}$.

An antiderivative of f is a function $F : I \mapsto \mathbb{R}$ such that

(AD) *F is differentiable at every point of I, and*

$$F'(x) = f(x) \qquad \text{for every } x \in I\,.$$

Our goal is

- To constuct an antiderivative F of f, where f is the function

$$\mathbb{R}_{+} \ni x \mapsto \frac{1}{x}\,.$$

And we also want to answer the following questions:

1. Is F unique?

2. Why do we want to do this?

3. Does F have any interesting properties?

THE UNIQUENESS QUESTION

It is obvious that F **is not unique**.

REASON: Let F be an antiderivative of the function $\mathbb{R}_+ \ni x \mapsto \frac{1}{x}$.

Then $F + 1$ is also an antiderivative of f. So the antiderivative is not unique.

On the other hand:

THEOREM. If F and G are two antiderivatives of a function f on an interval I, then $F - G$ is a constant.

In particular, if there is a point $p \in I$ such that $F(p) = G(p)$, then $F = G$.

Proof. Let $H = F - G$. Then H is differentiable everywhere in I and $H'(x) = 0$ for every $x \in I$. So H is a constant by the Fundamental Theorem of Calculus. **QED**

So we reformulate our goal by adding en extra condition on F:

Our new goal is

- **To consrtuct an antiderivative F of f such that $F(1) = 0$, where f is the function**

$$\mathbb{R}_+ \ni x \mapsto \frac{1}{x}.$$

Now it is clear that F, if it exists, is unique.

Form now on, until further notice, "f" stands for the function

$$\mathbb{R}_+ \ni x \mapsto \frac{1}{x}.$$

IS F INTERESTING?

A solution to our problem, if it exists, would have lots of interesting properties.

THEOREM. If F is an antiderivative of f and $F(1) = 0$, then

- **F is strictly increasing,**
- **$F(x) < 0$ if $0 < x < 1$ and $F(x) > 0$ if $x > 1$,**
- **$F(xy) = F(x) + F(y)$ whenever $x > 0$ and $y > 0$,**
- **$F\left(\frac{1}{x}\right) = -F(x)$ whenever $x > 0$.**
- $\lim_{x\to\infty} F(x) = +\infty$,
- $\lim_{x\to 0} F(x) = -\infty$,
- $\lim_{x\to\infty} \frac{F(x)}{x^r} = 0$ **for every $r \in \mathbb{Q}$ such that $r > 0$.**
  (So $F(x)$ goes to infinity more slowly than any power of x.)

Proof. That F **is strictly increasing** follows from the fact that $F'(x) = \frac{1}{x}$, and $\frac{1}{x} > 0$ for $x > 0$.

That $F(x) < 0$ **if** $0 < x < 1$**, and** $F(x) > 0$ **if** $x > 1$ follows from the fact that F is strictly increasing and $F(1) = 0$.

Let us prove that

$$F(xy) = F(x) + F(y) \qquad \text{whenever } x > 0,\, y > 0\,. \tag{1}$$

Fix a positive real number y. Let

$$g(x) = F(xy) - F(x) - F(y)\,.$$

Then $g : \mathbb{R}_+ \mapsto \mathbb{R}$, g is differentiable on $\mathbb{R}_+$, and

$$g'(x) = yF'(xy) - F'(x) = y\frac{1}{xy} - \frac{1}{x} = \frac{1}{x} - \frac{1}{x} = 0\,.$$

So g is a constant by the Fundamental Theorem of Calculus. But $g(1) = F(y) - F(1) - F(y) = 0$, since $F(1) = 0$. So $g = 0$. So (1) holds.

The formula

$$F\left(\frac{1}{x}\right) = -F(x) \qquad \text{whenever } x > 0 \tag{2}$$

follows easily, because

$$0 = F(1) = F\left(\frac{1}{x} \times x\right) = F\left(\frac{1}{x}\right) + F(x)\,,$$

so

$$F(x) = -F\left(\frac{1}{x}\right)\,.$$

We now prove that

$$\lim_{x\to\infty} F(x) = +\infty\,. \tag{3}$$

Let $a = F(2)$. Then $a > 0$, because $F(1) = 0$ and F is strictly increasing.

The product law $F(xy) = F(x)+F(y)$ then implies $F(4) = F(2\times 2) = F(2) + F(2) = 2F(2) = 2a$, $F(8) = F(4) + F(2) = 3F(2) = 3a$, and so on, so $F(2^n) = na$ for every $n \in \mathbb{N}$. Given a positive real number M, pick $N \in \mathbb{N}$ such that $Na > M$. (For example, let $N = 1 + \left[\, \frac{M}{a} \,\right]$.) Then $F(2^N) = Na > M$. So $F(x) > M$ whenever $x > 2^N$. Hence we have shown that for every $M \in \mathbb{R}$ there exists $L \in \mathbb{R}_+$ such that $F(x) > M$ whenever $x > L$. This proves (3).

Finally, we prove that, if $r \in \mathbb{Q}$ and $r > 0$, then

$$\lim_{x\to\infty} \frac{F(x)}{x^r} = 0\,. \tag{4}$$

Since both $F(x)$ and x^r go to $+\infty$ as $x \to +\infty$, we can apply L'Hôpital's rule. The derivative of $F(x)$ is $\frac{1}{x}$, and the derivative of x^r is rx^{r-1}, so

$$\lim_{x\to\infty} \frac{F(x)}{x^r} = \lim_{x\to\infty} \frac{\frac{1}{x}}{rx^{r-1}} = \lim_{x\to\infty} \frac{1}{r}\frac{1}{x} \times x^{1-r} = \lim_{x\to\infty} \frac{1}{r} x^{-r} = 0\,,$$

since $r > 0$.

The product rule

$$F(xy) = F(x) + F(y) \qquad \text{whenever } x > 0,\ y > 0 \tag{5}$$

made logarithms very important and very useful in the days before calculators and computers.

That's because adding numbers is much easier and can be done much faster than multiplying them.

To multiply two large numbers, say to find $A \times B$, where

$$A = 792,890,335,612 \qquad \text{and} \qquad B = 601,254,018,324$$

you would use a *table of logarithms*: you would find

$$a = \ln 792,890,335,612\,, \qquad b = \ln 601,254,018,324\,,$$

then you would add them, and then you would look for the "antilogarithm" of $a + b$, and that would be the desired product $A \times B$, because

$$\ln(AB) = \ln A + \ln B = a + b\,.$$

Slide rules were based on the same priciple.

And in the old days every engineer had one.

CONSTRUCTING F

BACKGROUND

We started this course with some axioms about the algebraic operations on the real numbers.

Using these axioms, we were able to construct some special functions:

1. First of all, there are the **constant functions**: for each $c \in \mathbb{R}$, we let k_c be the function

   $$\mathbb{R} \ni x \mapsto c\,.$$

   So $k_c(x) = c$ for each $x \in \mathbb{R}$.

2. Next, there is the **identity function of $\mathbb{R}$**, i.e., the function

$$\mathbb{R} \ni x \mapsto x \,.$$

3. Using the multiplication operation, we can then form the **monomials**: for each $n \in \mathbb{N}$, we let p_n be the function

$$\mathbb{R} \ni x \mapsto x^n \,,$$

so $p_n(x) = x^n$ for $x \in \mathbb{R}$. The functions p_n are also called **power functions**, or just **powers**.

4. Using addition and multiplication, we can then form the **polynomial functions**: for each $n \in \mathbb{N}$, and each $n+1$-tuple $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ of real numbers, we let $q_{n,\mathbf{a}}$ be the function

$$\mathbb{R} \ni x \mapsto a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \,,$$

so $q_{n,\mathbf{a}}(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ for $x \in \mathbb{R}$.

5. Then, using division, we can form the **rational functions**, i.e., functions of the form

$$\mathbb{R} - S \ni x \mapsto \frac{q_{m,\mathbf{a}}(x)}{q_{n,\mathbf{b}}(x)}\,.$$

(Here S is the set $\{x \in \mathbb{R} : q_{n,\mathbf{b}}(x) = 0\}$. The quotient in the above expression is not defined for $x \in S$, so we have to exclude S from the domain of our function.)

6. Special examples of rational functions are the **negative integer powers**, i.e., the functions p_n defined, for for negative integers n, by letting p_n to be the function

$$\mathbb{R} - \{0\} \ni x \mapsto \frac{1}{x^{|n|}}\,.$$

And, if $m \in \mathbb{N}$, and $n = -m$, we write x^n (i.e., x^{-m}, to denote $p_n(x)$, so $x^{-m} = \frac{1}{x^m}$.

7. Up to this point, all we have done is use the four arithmetic operations (addition, subtraction, multiplication, and division). And to do this it is not necessary to worry about exactly what "numbers" are. All we need to know is that the four arithmetic operations can be carried out. But now we need to construct the **n-th root functions**,

$$\mathbb{R}_+ \ni x \mapsto \sqrt[n]{x}$$

for $n \in \mathbb{N}$. (We only look at positive real numbers x because roots of negative numbers are problematic. For example, negative numbers do not have a square root. We call this function $p_{\frac{1}{n}}$, so $p_{\frac{1}{n}}(x) = \sqrt[n]{x}$ for $n \in \mathbb{N}$, $x \in \mathbb{R}_+$.

We constructed these functions in two ways:

(a) First, we did it directly, using the completeness axiom to prove that if $x \in \mathbb{R}_+$ then there exists a unique $y \in \mathbb{R}_+$ such that $y^n = x$, and we called this y $\sqrt[n]{x}$.

(b) Second, we did it using the inverse function theorem: the function p_n is strictly increasing on $\mathbb{R}_+$, so it has an inverse $g_n : \mathbb{R}_+ \mapsto \mathbb{R}_+$. This function satisfies $g_n(p_n(x)) = x$ for all $x \in \mathbb{R}_+$, so if we write $u = p_n(x)$ we have $g_n(u) = x$, and then $g_n(u)^n = x^n = p_n(x) = u$, so $g_n(u)^n = u$ for every $u \in \mathbb{R}_+$, and then $g_n(x)^n = x$ for every $x \in \mathbb{R}_+$, so $y^n = x$, of we write $y = g_n(x)$.

8. We then defined the power functions $p_r : \mathbb{R}_+ \mapsto \mathbb{R}_+$, for rational r, as follows: if $r = \frac{m}{n}$, $n \in \mathbb{N}$, $m \in \mathbb{Z}$, then

$$p_r(x) = \left(\sqrt[n]{x} \right)^m \qquad \text{for } x \in \mathbb{R}_+ .$$

9. And then, using the theorems about derivatives, we proved that all the power functions are differentiable, and that the derivative p_r' of p_r, for r rational, is given by

$$p_r(x) = r p_{r-1}(x) .$$

(This is the familiar formula

$$\frac{d}{dx}\left(x^r \right) = r x^{r-1} \,.$$

The formula is proved first for $r = n$, $n \in \mathbb{N}$, by induction. Then it is proved for $r = n$, $n \in \mathbb{Z}$, $n < 0$, using the formula for the derivative of a quotient. Then it is proved for $r = \frac{1}{n}$, $n \in \mathbb{N}$, using the Inverse Function Theorem for derivatives. And then it is proved for general r.)

10. In particular, if $r \in \mathbb{Q}$ and $r \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{r+1}}{r+1} \right) = x^r \,.$$

So **the function**

$$\mathbb{R}_+ \ni x \mapsto \frac{x^{r+1}}{r+1} \quad \left(= \frac{p_{r+1}(x)}{r+1} \right) \tag{6}$$

is an antiderivative of the rational power function

$$\mathbb{R}_+ \ni x \mapsto x^r \quad \Big(= p_r(x) \Big) \, . \tag{7}$$

11. It follows that

> **Every power function p_r, for every rational number r, has an antiderivative, except for $r = -1$.**

So the function $\mathbb{R}_+ \ni x \mapsto \frac{1}{x}$ is left alone among all the power functions, as the only power function that does not appear to have an antiderivative.

How do we find an antiderivative for the function p_1, i.e., for the function $x \mapsto \frac{1}{x}$?

That is the question.

We will do it by taking a **limit.**

We will look at the functions $p_{-1-\frac{1}{n}}$, which do have antiderivatives, and then take the limit of these antiderivatives as $n \to \infty$.

But first we have to choose our antiderivatives carefully.

We could choose our antiderivatives by letting, for $r = -1 - \frac{1}{n}$,

$$A_n(x) = \frac{x^{r+1}}{r+1},$$

i.e.,

$$A_n(x) = \frac{x^{-\frac{1}{n}}}{-\frac{1}{n}},$$

that is,

$$A_n(x) = -nx^{-\frac{1}{n}}.$$

But if we do that then $A_n(1) = -n$, so there is no hope that the limit $\lim_{n\to\infty} A_n(x)$ will exist for all $x \in \mathbb{R}_+$, because it does not exist even for $x = 1$.

So a better idea is to add a constant to A_n. That will not change its derivative, but may give us a better chance that the limit will exist.

So we let

$$B_n(x) = A_n(x) + n.$$

Then B_n is also an antiderivative of $p_{-1-\frac{1}{n}}$, but we now also have

$$B_n(1) = 0.$$

And this is very good, because we want our antiderivative of $x \mapsto \frac{1}{x}$ to have the value 0 at $x = 1$.

So we define:

$$B_n(x) = -nx^{-\frac{1}{n}} + n\,,$$

that is,

$$B_n(x) = n\left(1 - \frac{1}{x^{\frac{1}{n}}}\right),$$

and try to prove that

(i) the limit $B(x) = \lim_{n\to\infty} B_n(x)$ exists for every $x \in \mathbb{R}_+$,

(ii) the function B is an antiderivative of the function $\mathbb{R}_+ \ni x \mapsto \frac{1}{x}$,

(iii) $B(1) = 0$.

(Proving (iii) is going to be easy, because $B_n(1) = 0$ for every n. So the real problem is proving (i) and (ii).)

Actually, we are going to prove (i) and (ii) for $x \geq 1$. (The proof for $0 < x \leq 1$ is similar, and I am leaving it as an exercise for you. But we will not need the result for $x < 1$ anyhow, as I will show you in a little while.)

Let x be such that $x \in \mathbb{R}$ and $x \geq 1$.

Then the Mean value Theorem tells us that, for each $n \in \mathbb{N}$,

$$\frac{B_n(x) - B_n(1)}{x-1} = B_n'(c)$$

for some $c \in [1, x]$. But $B_n(1) = 0$, so

$$B_n(x) = B_n'(c)(x-1)\,. \tag{8}$$

Now B_n' is given by $B_n'(u) = u^{-1-\frac{1}{n}}$. So $B_n'(u) \leq 1$ whenever $u \in \mathbb{R}$, $u \geq 1$.

In particular, $B_n'(c) \leq 1$. So $B_n(x) \leq x - 1$.

So we have proved that, for every x such that $1 \leq x$, the inequalities

$$0 \leq B_n(x) \leq x - 1$$

hold for every n. So **the sequence $\left(B_n(x) \right)_{n=1}^{\infty}$ is bounded.**

And, of course, this is very good. A good way to prove that a sequence is convergent is to prove that it is bounded and monotone. We have proved that it is bounded. So we are halfway through.

To prove that the sequence $\left(B_n(x) \right)_{n=1}^{\infty}$ converges, it suffices to prove that it is increasing.

That is, we want to prove that

$$B_n(x) \leq B_{n+1}(x) \qquad \text{if } x \in \mathbb{R},\, x \geq 1,\, n \in \mathbb{N}\,. \tag{9}$$

And we now we use another omprtant tool: the Cauchy Mean Value Theorem (CMVT).

According to the CMVT, if $x \geq 1$ and $n \in \mathbb{N}$,

$$\frac{B_n(x) - B_n(1)}{B_{n+1}(x) - B_{n+1}(1)} = \frac{B'_n(c)}{B'_{n+1}(c)} \qquad \text{for some } c \in [1, x]\,.$$

But $B_n(1) = 0$ and $B_{n+1}(1) = 0$, so

$$\frac{B_n(x)}{B_{n+1}(x)} = \frac{B'_n(c)}{B'_{n+1}(c)}\,.$$

But

$$\begin{aligned} B'_n(c) &= c^{-1-\frac{1}{n}} = \frac{1}{c^{1+\frac{1}{n}}}\,, \\ B'_{n+1}(c) &= c^{-1-\frac{1}{n+1}} = \frac{1}{c^{1+\frac{1}{n+1}}}\,, \end{aligned}$$

so

$$\frac{B'_n(c)}{B'_{n+1}(c)} = \frac{c^{-1-\frac{1}{n}}}{c^{-1-\frac{1}{n+1}}} = c^{\frac{1}{n+1}-\frac{1}{n}} = c^{-\frac{1}{(n+1)n}} \leq 1\,.$$

So

$$\frac{B_n'(c)}{B_{n+1}'(c)} \le 1\,,$$

and then

$$\frac{B_n(x)}{B_{n+1}(x)} \le 1\,,$$

so, finally,

$$B_n(x) \le B_{n+1}(x),.$$

Hence we have proved that **the sequence $\left(\ B_n(x)\ \right)_{n=1}^{\infty}$ is increasing.**

Since the sequence $\left(\ B_n(x)\ \right)_{n=1}^{\infty}$ is both bounded and increasing, it follows that it is convergent.

We can now define

$$B(x) = \lim_{n\to\infty} B_n(x)\,,$$

for every $x \in \mathbb{R}$ such that $x \geq 1$.

Now we have to prove that B is an antiderivative of p_{-1}.

We are going to prove that, if $1 \leq x$, then

$$\lim_{y\to x, y\geq 1} \frac{B(y)-B(x)}{y-x} = \frac{1}{x}\,. \tag{10}$$

(The condition "$y \geq 1$" is included because we have only defined $B(y)$ for $y \geq 1$, so if $x = 1$ we cannot take the two-sided limit.)

Using the Mean Value Theorem, we write, for each x, y, n such that $x, y \in \mathbb{R}$, $x \geq 1$, $y \geq 1$, $y \neq x$, $n \in \mathbb{N}$,

$$\frac{B_n(y)-B_n(x)}{y-x} = B_n'(c)$$

for some c that lies between x and y.

Also, $B_n'(c) = c^{-1-\frac{1}{n}}$, so

$$\frac{B_n(y) - B_n(x)}{y - x} = c^{-1-\frac{1}{n}}.$$

Subtracting $x^{-1-\frac{1}{n}}$ from both sides, we get

$$\begin{aligned} \frac{B_n(y) - B_n(x)}{y - x} - x^{-1-\frac{1}{n}} &= c^{-1-\frac{1}{n}} - x^{-1-\frac{1}{n}} \\ &= p_{-1-\frac{1}{n}}(c) - p_{-1-\frac{1}{n}}(x). \end{aligned}$$

Using the Mean Value Theorem again, we get

$$\frac{p_{-1-\frac{1}{n}}(c) - p_{-1-\frac{1}{n}}(x)}{c - x} = \left(-1 - \frac{1}{n} \right) p_{-2-\frac{1}{n}}(d)$$

for some d that lies between c and x.

Then

$$\left| \, p_{-1-\frac{1}{n}}(c) - p_{-1-\frac{1}{n}}(x) \, \right| \leq |c - x| \left(1 + \frac{1}{n} \right) p_{-2-\frac{1}{n}}(d)$$

so

$$\left|\ p_{-1-\frac{1}{n}}(c) - p_{-1-\frac{1}{n}}(x)\ \right| \leq 2|c-x| \leq 2|y-x|\,.$$

Therefore

$$\left|\ \frac{B_n(y)-B_n(x)}{y-x} - x^{-1-\frac{1}{n}}\ \right| \leq 2|y-x|\,. \tag{11}$$

Since (11) holds for every n, we can take the limit as $n \to \infty$, and get

$$\left|\ \frac{B_(y)-B_(x)}{y-x} - \frac{1}{x}\ \right| \leq 2|y-x|\,. \tag{12}$$

Since $\lim_{y\to x}|y-x| = 0$, it follows that

$$\lim_{y\to x}\left|\ \frac{B_(y)-B_(x)}{y-x} - \frac{1}{x}\ \right| = 0\,. \tag{13}$$

And this implies that

$$\lim_{y\to x}\left(\ \frac{B_(y)-B_(x)}{y-x}\ \right) = \frac{1}{x}\,. \tag{14}$$

So B is differentiable at x and $B'(x) = \frac{1}{x}$.

So B indeed is an antiderivative of the function $\mathbb{R}_+ \ni x \mapsto \frac{1}{x}$.

This almost completes our job.

But there is one minor thing missing. We have to construct the antiderivative $B(x)$ for $0 < x < 1$.

But this is easy:

For $0 < x < 1$, we define:

$$B(x) = -B\left(\frac{1}{x}\right). \tag{15}$$

(The right-hand side of (15) is well defined because $B(u)$ has already been defined for $u \geq 1$, and if $x0 < x < 1$ then $\frac{1}{x}$ is ≥ 1.)

Then B is differentiable on $(0, 1]$, because it is the composite of two differentiable functions. And the derivative of B can be computed using the chain rule. The result is:

$$\begin{aligned} B'(x) &= -B'\left(\frac{1}{x}\right) \times \left(-\frac{1}{x^2}\right) \\ &= -\frac{1}{\frac{1}{x}} \times \left(-\frac{1}{x^2}\right) \\ &= x \times \frac{1}{x^2} \\ &= \frac{1}{x}. \end{aligned}$$

So B is also an antiderivative of $\mathbb{R}_+ \ni x \mapsto \frac{1}{x}$ on the interval $(0, 1]$.

And, of course, $B(1) = 0$.

Lecture 25

December 7, 2020

Finally, we have constructed a function $B : (0, +\infty) \mapsto \mathbb{R}$ such that

$$\begin{aligned} B & \quad \text{is differentiable everywhere on } (0, +\infty)\,, & (16) \\ B'(x) &= \frac{1}{x} \text{ for all } x \in (0, +\infty)\,, & (17) \\ B(1) = 0\,, & & (18) \end{aligned}$$

and we have proved that the function $B : (0, +\infty) \mapsto \mathbb{R}$ that satisfies (16), (17), (18) is unique.

This function is called "the natural logarithm", and we write "$\ln x$" rather than $B(x)$.

And we have proved that

- ln is strictly increasing,
- $\ln x < 0$ if $0 < x < 1$ and $\ln x > 0$ if $x > 1$,
- $\ln(xy) = \ln x + \ln y$ whenever $x > 0$ and $y > 0$,
- $\ln\left(\frac{1}{x}\right) = -\ln x$ whenever $x > 0$.
- $\lim_{x\to\infty} \ln x = +\infty$,
- $\lim_{x\to 0} \ln x = -\infty$,
- $\lim_{x\to\infty} \frac{\ln x}{x^r} = 0$ for every $r \in \mathbb{Q}$ such that $r > 0$.
  (So $\ln x$ goes to infinity more slowly than any power of x.)

The function B is strictly increasing, so it is one-to-one.
Since $\lim_{x\to\infty} B(x) = +\infty$ and $\lim_{x\to 0} B(x) = -\infty$, if follows that B maps $\mathbb{R}_+$ onto $\mathbb{R}$. So B has an inverse function $E : \mathbb{R} \mapsto \mathbb{R}_+$.
Since $B'(x) \neq 0$ for $x \in \mathbb{R}_+$, the function E is differentiable, and its derivative is given by the inverse function theorem: we have, for $v \in \mathbb{R}_+$, if we write $u = B(v)$ (so $v = E(u)$),

$$E'(u) = \frac{1}{B'(v)} = \frac{1}{\frac{1}{v}} = v = E(u)\,.$$

So the function E satisfies:

$E : \mathbb{R} \mapsto \mathbb{R}_+\,,\quad E$ is differentiable, $E(0) = 1\,,$ and $E' = E\,.$

THE EXPONENTIAL FUNCTION

The function E is called **the exponential function.**

Since $E(\ln(x)) = x$ for every x, we have, if $x, y \in \mathbb{R}$, and we let $a = E(x)$, $b = E(y)$:

$$\ln(E(x)E(y)) = \ln(ab) = \ln a + \ln b = x + y\,,$$

so

$$E(x)E(y) = E(x+y) \text{ for all } x, y \in \mathbb{R}\,. \qquad (19)$$

We define the number e to be $E(1)$. Then

$$E(2) = E(1)E(1) = e^2, \quad E(3) = e^3,\ E(4) = e^4, \text{ and so on.}$$

Thus $E(n) = e^n$ for each $n \in \mathbb{N}$.

Since $E(n)E(-n) = E(n + (-n)) = E(0) = 1$, we have $E(-n) = \frac{1}{E(n)} = \frac{1}{e^n} = e^{-n}$ for $n \in \mathbb{N}$. So $E(n) = e^n$ for all integers n.

Then, if $n \in \mathbb{N}$, we have

$$E\left(\frac{1}{n}\right)^n = E\left(\times\frac{1}{n}\right)^n = E(1) = e\,,$$

so

In the homework, we proved that for each positive real number a there exists a unique continuous function $f_a : \mathbb{R} \mapsto \mathbb{R}$ such that $f_a(r) = a^r$ for r rational. Now we see that the function E is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ whose value for rational r is e^r. So E is the function f_e.
And we agreed to write $f_a(x) = a^x$. It follows that

$$E(x) = e^x \qquad \text{for all } x \in \mathbb{R}\,.$$

So the function E we have just constructed isn't actually new. It is none other than the function $\mathbb{R} \ni x \mapsto e^x$.
But we have done something new: we have discovered a number e having the property that

(EP) The function $\mathbb{R} \ni x \mapsto e^x$ is differentiable everywhere and is equal to its own derivative.

And then it follows that

(EP*) If $a > 0$, then the function $\mathbb{R} \ni x \mapsto a^x$ is differentiable everywhere and its derivative is the function $\mathbb{R} \ni x \mapsto (\ln a)a^x$.

(Proof: $a^x = \left(e^{\ln a} \right)^x = e^{x \ln a}$.)

So we have found the solutions to the differential equation

$$f' = f\,.$$

(The solutions are" all the functions of the form $\mathbb{R} \ni x \mapsto ke^x$, for k a constant.

And, using these functions, we can solve other differential equations. For example, for a constant a, let $F_a(x) = e^{ax^2}$.

Then $F_a' = 2axF_a(x)$. So F_a is a solution of the differential equation

$$f' = 2axf\,.$$

So: **every time we find an antiderivative F of a function $x \mapsto f(x)$ that we did not know how to antidfferentiate before, we can use this new function F, and combine it in various ways with other functions, to form new functions that solve new differential equations**.

THE ARC TANGENT FUNCTION

We now want to do for the function $\frac{1}{1+x^2}$ the same thing we did before for the function $\frac{1}{x}$

That is:

We want to find an antiderivative F of the function $\mathbb{R} \ni x \mapsto \frac{1}{1+x^2}$.

By the same arguments as before, any two such functions differ by a constant. So, if we fix the value of F at some x, then F will be unique. So we will require that $F(0) = 0$.

$F(x)$ is the angle θ whose tangent is x. $\boxed{d\theta = \frac{1}{1+x^2}dx}$.

So $\boxed{\frac{d\theta}{dx} = \frac{1}{1+x^2}}$.

So we want to find a function $F : \mathbb{R} \mapsto \mathbb{R}$ such that

1. F is differentiable everywhere,

2. $F'(x) = \frac{1}{1+x^2}$ for every $x \in \mathbb{R}$,

3. $F(0) = 0$.

Before we do that, we prove some properties of F.

THEOREM. If $F : \mathbb{R} \mapsto \mathbb{R}$ and F satisfies conditions 1,2,3 above, then

1. F **is strictly increasing.**

2. $F(x) > 0$ **whenever** $x > 0$ **and** $F(x) < 0$ **whenever** $x < 0$**.**

3. $F(-x) = -F(x)$ **for every** $x \in \mathbb{R}$**.**

4. $F(x) + F\left(\frac{1}{x}\right) = 2F(1)$ whenever $x > 0$ **.**

5. $F(x) + F\left(\frac{1}{x}\right) = -2F(-1)$ whenever $x < 0$ **.**

6. $\lim_{x\to+\infty} F(x) = 2F(1)$ **and** $\lim_{x\to-\infty} F(x) = -2F(1)$**.**

Proof. F is strictly increasing because $F'x) > 0$ for every x, since $F'x) = \frac{1}{1+x^2}$.

Since $F(0) = 0$ and F is strictly increasing, it follows that $F(x) > 0$ whenever $x > 0$ and $F(x) < 0$ whenever $x < 0$.

Let $G(x) = -F(-x)$. Then $G(0) = 0$ and $G'(x) = \frac{1}{1+x^2}$. So $G = F$. So $-F(-x) = F(x)$, and then $F(-x) = -F(x)$.

Let $H(x)F(x) + F\left(\frac{1}{x}\right)$. Then H is differentiable on $(0, +\infty)$, and

$$\begin{aligned} H'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right) \\ &= \frac{1}{1+x^2} - \frac{1}{x^2+1} \\ &= 0\,, \end{aligned}$$

so the function H is a constant c on the interval $(0, +\infty)$. The value of c can be computed by taking $x = 1$, and we get $c = H(1) = 2F(1)$.

So $F(x) + F\left(\frac{1}{x}\right) = 2F(1)$ whenever $x > 0$.

A similar argument shows that $F(x) + F\left(\frac{1}{x}\right) = 2F(-1) = -2F(1)$ whenever $x < 0$.

Now, we have

$$\begin{aligned}
\lim_{x\to+\infty} F(x) &= \lim_{x\to+\infty}\left(2F(1) - F\left(\frac{1}{x}\right)\right)\\
&= 2F(1) - \lim_{x\to+\infty}\left(F\left(\frac{1}{x}\right)\right)\\
&= 2F(1) - \lim_{y\to 0} F(y)\\
&= 2F(1) - F(0)\\
&= 2F(1)\,.
\end{aligned}$$

A similar argument shows that

$$\lim_{x\to-\infty} F(x) = -2F(1)\,.$$

CONSTRUCTION OF F

Let $f(x) = \frac{1}{1+x^2}$. We approximate the function f by functions that we know have antiderivatives.

We let

$$f_n(x) = \frac{1 + x^{4n+2}}{1 + x^2} \text{ for } x \in \mathbb{R},\, n \in \mathbb{N}\,.$$

REASON: Remember the **very important formula**

$$1 + r + r^2 + r^3 + \cdots + r^m = \frac{1 - r^{m+1}}{1 - r}\,. \tag{20}$$

(Proof: Let $S = 1 + r + r^2 + \cdots + r^m$. Then $rS = r + r^2 + r^3 \cdots + r^{m+1}$. So $S - rS = 1 - r^{m+1}$. So $S = \frac{1-r^{m+1}}{1-r}$.)

In (20), plug in $r = -x^2$. We get:

$$1 - x^2 + x^4 - x^6 + \cdots + (-x^2)^m = \frac{1 - (-x^2)^{m+1}}{1 + x^2}\,, \tag{21}$$

$$1 - x^2 + x^4 - x^6 + \cdots + (-1)^m x^{2m} = \frac{1 - (-1)^{m+1} x^{2m+2}}{1 + x^2}\,. \tag{22}$$

Now plug in $m = 2n$, so $(-1)^m = 1$, and $(-1)^{m+1} = -1$. Then

$$1 - x^2 + x^4 - x^6 + \cdots + x^{4n} = \frac{1 + x^{4n+2}}{1 + x^2}\,. \tag{23}$$

So $f_n(x)$ is the polynomial $1 - x^2 + x^4 - x^6 + \cdots + x^{4n}$.

Does the sequence $(f_n(x))_{n=1}^{\infty}$ converge to $f(x)$ for every x?

NO!!! But it does converge to $f(x)$ if $|x| < 1$, because if $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$.

So we are going to construct our antiderivative F just on the interval $[0,1]$. (After we do that, using the formula $F(x) = 2F(1) - F\left(\frac{1}{x}\right)$ we will also get F on the whole half-line $[0,+\infty)$. And then, using $F(-x) = -F(x)$, we will define $F(x)$ for every $x \in \mathbb{R}$.)

We let $F_n(x)$ be the antiderivative of $f_n(x)$ such that $F_n(0) = 0$. Then

$$F_n(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{4n+1}}{4n+1}\,.$$

The following are easy to prove:

(i) If $0 \leq x < 1$ **then** $0 < f_n(x) \leq 1$ **for every** $n \in \mathbb{N}$**.**

(ii) If $0 \leq x \leq 1$ **then** $0 \leq F_n(x) \leq x$ **for every** $n \in \mathbb{N}$**.**

(iii) If $0 \leq x < 1$ **then** $f_{n+1}(x) \leq f_n(x)$ **for every** $n \in \mathbb{N}$**.**

(iv) If $0 \leq x < 1$ **then** $F_{n+1}(x) \leq F_n(x)$ **for every** $n \in \mathbb{N}$**.**

Proof of (i): $f_n(x) = \frac{1+x^{4n+2}}{1+x^2}$, so clearly $f_n(x) \geq 0$. And $x^{4n+2} \leq x^2$, so $1 + x^{4n+2} \leq 1 + x^2$, so $f_n(x) \leq 1$.

Proof of (ii): By the Mean Value Theorem, $F_n(x) = F_n(x) - F(0) = f_n(c)x$ for some $c \in (0, x)$. Since $0 \leq f_n(c) \leq 1$, we have $0 \leq F_n(x) \leq x$.

Proof of (iii): $f_{n+1}(x) = \frac{1+x^{4n+6}}{1+x^2} = \frac{1+x^4x^{4n+2}}{1+x^2} \leq \frac{1+x^{4n+2}}{1+x^2} = f_n(x)$, since $x^4 \leq 1$.

Proof of (iv): By the Cauchy Mean Value Theorem,

$$\frac{F_{n+1}(x)}{F_n(x)} = \frac{F_{n+1}(x) - F_{n+1}(0)}{F_n(x) - F_n(0)} = \frac{f_{n+1}(c)}{f_n(c)} \quad \text{for some } c \in (0, x)\,.$$

Since $f_{n+1}(c) \leq f_n(c)$, the inequality $F_{n+1}(x) \leq F_n(x)$ follows.

So for each $x \in [0,1]$, the sequence $(F_n(x))_{n=1}^{\infty}$ is decreasing and bounded.

So by the monotone convergence theorem, the limit

$$F(x) = \lim_{n\to\infty} F_n(x)$$

exists.

This defines a function $F : [0,1] \mapsto \mathbb{R}$.

Since $0 \le F_n(x) \le x$ for every $x \in [0,1]$, the function F satisfies $0 \le F(x) \le x$ for every $x \in [0,1]$.

Now we have to prove that F is differentiable on $[0,1]$ and its derivative is f.

Given $x, y \in [0,1]$ such that $x \neq y$, we have

$$\frac{F_n(y) - F_n(x)}{y - x} = f_n(c) \quad \text{for some } c \text{ strictly between } x \text{ and } y\,.$$

So

$$\frac{F_n(y) - F_n(x)}{y - x} - f_n(x) = f_n(c) - f_n(x)\,.$$

But

$$\frac{f_n(c) - f_n(x)}{c - x} = f_n'(d) \quad \text{for some } d \text{ strictly between } x \text{ and } c\,.$$

Now a technical problem arises: **we cannot find a bound for $f_n'(d)$**.

Let us compute f_n':

Since $f_n(x) = \frac{1+x^{4n+2}}{1+x^2}$, we have, for $0 \le x \le 1$,

$$f_n'(x) = \frac{(4n+2)x^{4n+1}(1+x^2) - (1+x^{4n+2})2x}{(1+x^2)^2}$$

so

$$|f_n'(x)| \le (4n+2)x^{4n+1} + (1+x^{4n+2})2x \le (4n+2)x^{4n+1} + 4\,.$$

Since $x \leq 1$, we can say that $|f_n'(x)| \leq 4n + 6$.

But this is bad. We do not get a bound independent of n.

To take care of this, let us do something clever.

Instead fo working in the whole interval $[0, 1]$, let us pick a real number a such that $0 < a < 1$, and work in the interval $[0, a]$.

Then, if $z \in [0, a]$, we have

$$|f_n'(z)| \leq (4n + 2)z^{4n+1} + 4 \leq (4n + 2)a^{4n+1} + 4\,.$$

We saw earlier in the course that

$$\lim_{m \to \infty} ma^m = 0\,,$$

since $0 \leq a < 1$. So $\lim_{n \to \infty}(4n + 1)a^{4n+1} = 0$.

Hence

$$\lim_{n\to\infty}(4n+2)a^{4n+1} = \lim_{n\to\infty}(4n+1)a^{4n+1} = \lim_{n\to\infty} a^{4n+1} = 0+0=0\,.$$

So the sequence $((4n+2)a^{4n+1})_{n=1}^{\infty}$ is bounded. That means that there exists a positive number b_a (I call it b_a because it depends on a) such that

$$(4n+2)a^{4n+1} \le b_a \qquad \text{for every } n \in \mathbb{N}\,.$$

Using this number, we can go back to our calculation:

Assume that x and y are in $[0,a]$, and $x \neq y$. Then we have

$$\frac{F_n(y)-F_n(x)}{y-x} - f_n(x) = f_n(c) - f_n(x) \quad \text{and} \quad \frac{f_n(c)-f_n(x)}{c-x} = f_n'(d)$$

for some c that lies between x and y and some d that lies between x and c.

But now $d \in [0, a]$, so $|f_n'(d)| \le (4n+2)a^{4n+1} + 4 \le b_a + 4$. Let $B_a = b_a + 4$. Then $|f_n'(d)| \le B_a$. So

$$|f_n(c) - f_n(x) \le B_a|c - x| \le B_a|y - x\,.$$

So

$$\left| \ \tfrac{F_n(y) - F_n(x)}{y - x} - f_n(x) \ \right| \le |f_n(c) - f_n(x)| \le |f_n'(d)||c - x| \le B_a|c - x| \le B_a|y - x|,.$$

Now that we got a bound independent of n, we can let $n \to \infty$, and find

$$\left| \ \tfrac{F(y) - F(x)}{y - x} - f(x) \ \right| \le B_a|y - x|,.$$

So

$$\lim_{n \to \infty} \left| \ \tfrac{F(y) - F(x)}{y - x} - f(x) \ \right| = 0\,.$$

So

$$\lim_{n \to \infty} \frac{F(y) - F(x)}{y - x} = f(x)\,.$$

So we have proved that

(*) If $0 < a < 1$, then F is differentiable on $[0, a)$ and $F'x() = f(x)$ for every $x \in [0, a)$.

Now, given any $x \in [0, 1)$, we can choose a such that $x < a < 1$ and apply (*). The result is:

(**) F is differentiable on $[0, 1)$, and $f'(x) = f(x)$ for every $x \in [0, 1)$.

We have not yet proved that F is differentiable at 1, and that $F'(1) = f(1)$.

I will do that next time. (10 minutes.)

Let us prove that F is continuous at 1.

If $0 < x < 1$, we have: $F_n(1) - F_n(x) = f_n(c)(1-x)$ for some $c \in (0, 1)$. Since $|f_n(c)| \leq 1$, we have $|F_n(1) - F_n(x)| \leq |1 - x|$.

Letting $n \to \infty$, we get $|F(1) - F(x)| \leq |1 - x|$. But $\lim_{x \to 1} |1 - x| = 0$. So

$$\lim_{x \to 1} |F(1) - F(x)| = 0\,.$$

And then

$$\lim_{x \to 1} F(x) = F(1)\,.$$

DEFINITION. The number $4F(1)$ is called π ("pi ").

Then $\frac{\pi}{4}$ is given by

$$\frac{\pi}{4} = \lim_{n\to\infty} F_n(1) = \lim_{n\to\infty} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{4n+1},$$

so

$\frac{\pi}{4}$ is the sum of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

Lecture 26

December 9, 2020

In today's lecture:

1. End of the construction of the trigonometric functions.

2. Some general reflections about Real Analysis: what it is about, how it came about, and why it is one of the greatest, most revolutionary things that ever happened in the world.

We have constructed a continuous function $F : \mathbb{R} \mapsto \mathbb{R}$, and a positive real number π, such that

1. F is strictly increasing.
2. $F(x) > 0$ whenever $x > 0$ and $F(x) < 0$ whenever $x < 0$.
3. $F(-x) = -F(x)$ for every $x \in \mathbb{R}$.
4. $F(1) = \frac{\pi}{4}$ and $F(-1) = -\frac{\pi}{4}$.
5. $F(x) + F\left(\frac{1}{x}\right) = \frac{\pi}{2})$ whenever $x > 0$.
6. $F(x) + F\left(\frac{1}{x}\right) = -\frac{\pi}{2}$ whenever $x < 0$.
7. $\lim_{x \to +\infty} F(x) = \frac{\pi}{2}$ and $\lim_{x \to -\infty} F(x) = -\frac{\pi}{2}$.

We have also proved that

(D1) F is differentiable everywhere on the interval $[0, 1)$, and $F'(x) = \frac{1}{1+x^2}$ for $0 \leq x < 1$.

Using the formula

$$F(x) = \frac{\pi}{2} - F\left(\frac{1}{x}\right),$$

we see that F is also differentiable on $(1, +\infty)$. We can compute the derivative using the Chain Rule, we find that $F'(x) = \frac{1}{1+x^2}$ also for $x > 1$.

Then, using the formula

$$F(x) = -F(-x),$$

we see that F is also differentiable on $(-\infty, 0)$, except possibly at $x = -1$. We can compute the derivative using the Chain Rule, and we find that $F'(x) = \frac{1}{1+x^2}$ also for $x < 0$, $x \neq -1$.

So now we know that

(D2) F is differentiable everywhere on $\mathbb{R}$, except possibly at $x = 1$ and $x = -1$, and $F'(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$, $x \neq 1$, $x \neq -1$.

What about the missing points, 1 and -1?

A TECHNICAL LEMMA

LEMMA: Suppose the functions H and h are continuous on an interval I, and H is differentiable everywhere on I except possibly at a point p, and $H'(x) = h(x)$ for all $x \in I - \{p\}$. Then H is differentiable at p and $H'(p) = h(p)$.

Proof. Let ε be an arbitrary positive real number.

Since h is continuous, find a positive real number δ such that

$$|h(u) - h(p)| < \varepsilon \qquad \text{whenever} \quad u \in (p - \delta, p + \delta) \cap I\,.$$

Let $x \in (p - \delta, p + \delta) \cap I$ be such that $x \neq p$.

Then, by the Mean Value Theorem (MVT), there exists a point c lying between x and p such that

$$\frac{H(x) - H(p)}{x - p} = h(c)\,.$$

IMPORTANT REMARK: The MVT does not require that H be differentiable at the endpoints of the interval with endpoints p, x. So the fact that we don't know H to be differentiable at p does not matter.

Since c lies between p and x, c belongs to $(p-\delta, p+\delta) \cap I$. So

$$|h(c) - h(p)| < \varepsilon\,.$$

Therefore

$$\left|\ \frac{H(x)-H(p)}{x-p} - h(p)\ \right| = |h(c) - h(p)| < \varepsilon\,.$$

So

$$\left|\ \frac{H(x)-H(p)}{x-p} - h(p)\ \right| < \varepsilon \qquad \text{whenever } x \in I,\ x \neq p,\ |x-p| < \delta,. \quad (24)$$

So we have shown that for every positive ε there exists a positive δ such that (24) holds.

Therefore

$$\lim_{x \to p, x \in I} \left(\frac{H(x)-H(p)}{x-p} \right) = h(p)\,. \qquad (25)$$

So H is differentiable at p and $H'(p) = h(p)$. **QED**

So now we know that

(D3) F is differentiable everywhere on $\mathbb{R}$, and

$$F'(x) = \frac{1}{1+x^2}$$

for every $x \in \mathbb{R}$.

Since F is strictly increasing, F is one-to-one, so F has an inverse $G : \mathrm{Ran}\, F \mapsto \mathbb{R}$.

Since $\lim_{x\to+\infty} F(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} F(x) = -\frac{\pi}{2}$, the range of F is the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$.

So

$$G : \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \mapsto \mathbb{R}\,.$$

Since F is strictly increasing, G is strictly increasing as well.

Since $\lim_{x\to+\infty} F(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} F(x) = -\frac{\pi}{2}$, we have

$$\lim_{x\to\frac{\pi}{2}} G(x) = +\infty \text{ and } \lim_{x\to-\frac{\pi}{2}} G(x) = -\infty\,.$$

Since $F(0) = 0$, we have $G(0) = 0$.

Therefore $G(x) > 0$ if $0 < x < \frac{\pi}{2}$ and $G(x) < 0$ if $0 > x > -\frac{\pi}{2}$

Finally, since F is differentiable everywhere and $F'(y) = \frac{1}{1+y^2}$ for every y, the Inverse Function Theorem tells us that G is differentiable everywhere on $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$ and, if $y = G(x)$, then

$$G'(x) = \frac{1}{F'(y)} = \frac{1}{\frac{1}{1+y^2}} = 1 + y^2 = 1 + G(x)^2$$

so

$$G'(x) = 1 + G(x)^2 \, .$$

So

The function

$$G : \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \mapsto \mathbb{R}$$

is everywhere differentiable, and satisfies

$$G' = 1 + G^2 , \tag{26}$$

$$G(0) = 0 . \tag{27}$$

FINAL EXAM PROBLEM (WITH HINTS): PROVE THAT THE SOLUTION OF THE DIFFERENTIAL EQUATION WITH INITIAL CONDITION IS UNIQUE.

The function F is called "arc tan" ("arc tangent"), and the function G is called "tan" ("tangent").

Then

1. $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \mapsto \mathbb{R}$,
2. tan is strictly increasing.
3. $\tan x > 0$ whenever $x > 0$ and $\tan x < 0$ whenever $x < 0$.
4. $\tan(-x) = -\tan x$ for every $x \in \mathbb{R}$.
5. $\tan 1 = \frac{\pi}{4}$ and $\tan(-1) = -\frac{\pi}{4}$.
6. $\lim_{x \to +\frac{\pi}{2}} \tan x = +\infty$ and $\lim_{x \to -\frac{\pi}{2}} \tan x = -\infty$.
7. tan is everywhere differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, and

$$\frac{d}{dx} \tan x = 1 + \tan^2 x \,.$$

It is convenient to extend the tan function to the whole real line, as a periodic function with period π, so that

$$\tan(x+\pi) = \tan x \text{ for all } x.$$

So tan is defined and continuous on the set

$$S - \{\,(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\,\}\,.$$

One can then easily construct all the other trigonometric functions.

We should have

$$\begin{aligned}\frac{\sin x}{\cos x} &= \tan x \\ \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2}\,, \\ \sin^2 x + \cos^2 x &= 1\,.\end{aligned}$$

So

$$\begin{aligned}\frac{1}{\cos^2 x} &= \frac{\sin^2 x}{\cos^2 x} + 1 \\ &= \tan^2 x + 1\,,\end{aligned}$$

so

$$\cos^2 x = \frac{1}{\tan^2 x + 1}\,,$$

and then

$$
\begin{aligned}
\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\
&= 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2} \\
&= 2\tan\frac{x}{2}\cos^2\frac{x}{2} \\
&= \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},
\end{aligned}
$$

so we define

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$

Then sin is a continuous function on $\mathbb{R}$, except possibly at the points of $2S$. (Here $2S$ is the set of odd multiples of π, i.e. the numbers $(2n+1)\pi$, $n \in \mathbb{Z}$.)

But at the points of $2S$ $|\tan\frac{x}{2}|$ goes to $+\infty$, so $\lim_{x\to p}\sin x = 0$ if $p \in 2S$.

So sin is actually a continuous function on all of $\mathbb{R}$, and it is periodic with period 2π, that is

$$\sin x + 2\pi = \sin x \text{ for every } x.$$

One can then define $\cos x$, and prove that cos is actually a continuous function on all of $\mathbb{R}$, and it is periodic with period 2π, that is

$$\cos x + 2\pi = \cos x \text{ for every } x\,.$$

And then one can prove all the usual properties of the sine and cosine:

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= 1\,, \\
\sin x + 2\pi &= \sin x\,, \\
\cos x + 2\pi &= \cos x\,, \\
\sin x + y &= \sin x\, \cos y + \cos x \sin y\,, \\
\cos x + y &= \cos x\, \cos y - \sin x \sin y\,, \\
\sin -x &= -\sin x\,, \\
\cos -x &= \cos x\,, \\
\sin 0 &= 0\,, \\
\cos 0 &= 1\,, \\
\frac{d}{dx} \sin x &= \cos x\,, \\
\frac{d}{dx} \cos x &= -\sin x\,.
\end{aligned}
$$

Summarizing: we have seen how the methods of real analysis (limits and derivatives), allow us to construct all the trigonometric functions without using geometry, by purely analytic arguments.

Furthermore, this method yields results that one can obtain with geometric arguments, such as, for example, the formual for $\frac{\pi}{4}$ that we found earlier.

This is a small example illustrating **the great revolution that took place in the 17th and 18th centuries, when mathematics changed, and became analytic instead of geometric.**

Now I will explain this, and give you a major example, one that has had monumental consequences and changed the world.

At the beginning of the 17th Century, mathematics was considered to be primarily Goemetry.

In 1623, in *The Assayer*, Galileo Galilei (1564-1642) wrote:

> **[The book of nature] is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.**

So, as you can see, mathematics was about "triangles, circles, and other geometrical figures".

During the 17th and 18th Century, thanks to the work of several great scientists, this changed. Mathematics became analytic rather than geometric. It was no longer about

> triangles, circles, and other geometrical figures,

but about

> functions, derivatives, and integrals.

This revolutionary change was complete by the end of the 18th Century.

In 1788, Joseph-Louis Lagrange (1736-1813) published a monumental mathematics book called *Analytical Mechanics*. And the preface of the book, he wrote:

On ne trouvera point de Figures dans cet Ouvrage

that is,

No figures will be found in this work.

So the world of mathematics was no longer Galileo's world of "triangles, circles, and other geometrical figures".

Geometry had been dethroned.

The transformation was complete.

Analysis now ruled.

- How did this revolution come about?
- Who made it?
- Why was it important?
- How did it change the world?

“Analysis” is a method for studying change, and in particular motion. (Motion is a particular kind of change: it is position changing in time.)

The great idea of Analysis, conceived by Isaac Newton (1642-1726) and Gottfried Wilhelm Leibniz (1646-1716) was this:

- Break every change into very small, “infinitesimal” steps.

- Study the structure of each of these tiny steps.

- Then add up the effects of these infinitesimal steps to get the magnitude and direction of the overall change.

For example, to understand how the angle θ changes when its tangent changes, we look at what happens when u changes by a "very small", or "infinitesimal" amount du. We compute the the amount $d\theta$ by which θ changes, and, as we saw, we find that

$$d\theta = \frac{1}{1+u^2} du\,,$$

which we write as

$$\frac{d\theta}{du} = \frac{1}{1+u^2}\,.$$

Finally, we add up these changes to obtain the total amount of change:

$$\begin{aligned} \Delta\theta &= \text{sum of all the } d\theta\,\text{s} \\ &= S d\theta\,, \end{aligned}$$

and, following Leibniz, instead of the S, which stands for "sum", we write a stretched, elongated version:

$$\Delta\theta = \int d\theta\,,$$

and, since $d\theta = \frac{1}{1+u^2}du$, we end up with

$$\Delta\theta = \int \frac{du}{1+u^2}\,.$$

We already saw how this method can be used to analyze the arc tan and tangent functions, and get new insights that tradictional geometry couldn't yield.

But that just a small exmaple. There was a really big one, an example that, as I said, changed the world.

I am talking about how **by analyzing Kepler's laws of planetary motion Newton discovered the law of universal gravitation**.

Since time immemorial, people had been trying to explain the motion of the planets.

They tried all kinds of ideas, and they all failed.

Partly, they failed because they were trying to see the planets as orbiting around the Earth, which was a very bad idea.

Eventually, Copernicus (1473-1543) realized that the planets orbit around the Sun, not around the Earth.

But he thought the planets move in circles, which was wrong.

Johannes Kepler (1571-1630) tried to understand how the planets more, and came up with his three laws.

KEPLER'S LAWS

1. The planets move in ellipses, with the Sun as one of the foci.
2. Each planet sweeps equal areas in equal time.
3. The square of the period is proportional to the cube of the major axis of the ellipse.

While Kepler was discovering his laws, René Descartes (1596-1650) had the briliant idea of drawing two lines (“the x axis and the y axis”) and representing points as pairs (x, y) of numbers.

This had the effect of allowing us to write equations to describe curves. For example, the equation of a circle is

$$x^2 + y^2 = R^2 \,,$$

if you draw the axes so that they cross at the center of the circle.

And the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,,$$

if a, b are the major and minor semiaxes of the ellipse.

Then Newton was born (on the exactly the same year when Galileo died, 1642), and he was lucky to find, available to him,

- a fascinating set of phenomena, obeying some complicated but clear and precise laws (found by Kepler), ready to be “analyzed”;

- the tools for analyzing these phenomena, in the form of a method (invented by Descartes) for translating the laws of Kepler into equations that one could study.

And Newton put the two sets of ideas together, and figured out how to analyze Kepler’s laws.

The equation of an ellipse in polar coordinates, if we draw the axis crossing at a focus, is

$$r = \frac{\alpha}{1 - k\cos\theta},$$

where k is a constant knows as the "eccentricty" of the ellipse.

This is very easy to show. I recommend that you do the calculation.

Then Newton translated Kepler's equal area law into a formula:

As θ changes by an amount $d\theta$, the area dA swept by the moving planet is approximately given by

$$dA = r^2 d\theta\,.$$

So Kepler's equal area law becomes $\frac{dA}{dt} =$ constant, that is,

$$\frac{d\theta}{dt} = \frac{M}{r^2}\,,$$

where M is a constant. (I call it "M" because it is the angular momentum.)

Then Newton analyzed the motion, using the derivative, that he had invented.

Let me do the analysis, using vector language. Write

$$\vec{u} = \cos\theta\vec{i} + \sin\theta\vec{j},, \qquad \vec{v} = -\sin\theta\vec{i} + \cos\theta\vec{j},.$$

Then

$$\vec{u} = \theta'\vec{v} \qquad \text{and} \quad \vec{v} = -\theta'\vec{u}\,.$$

Let $\vec{q}$ be the position of the planet, so

$$\vec{q} = x\vec{i} + y\vec{j} = r\cos\theta\vec{i} + r\sin\theta\vec{j} = r\vec{u}\,.$$

Then the velocity $\vec{q}'$ is given by

$$\vec{q}' = r\vec{u}' + r'\vec{u} = r\theta'\vec{v} + r'\vec{u}\,.$$

This does not say much, so Newton computed the acceleration $\vec{u}''$, and got

$$\begin{aligned}
\vec{q}'' &= r\theta''\vec{v} + r'\theta'\vec{v} + r\theta'\vec{v}' + r''\vec{u} + r'\vec{u}' \\
&= r\theta''\vec{v} + r'\theta'\vec{v} - r(\theta')^2\vec{u} + r''\vec{u} + r'\vec{v}\theta' \\
&= r\theta''\vec{v} + 2r'\theta'\vec{v} + r''\vec{u} - r(\theta')^2\vec{u} \\
&= (r\theta'' + 2r'\theta')\vec{v} + (r'' - r(\theta')^2)\vec{u}\,.
\end{aligned}$$

So the acceleration has a **radial component** $\rho = r'' - r(\theta')^2$, and a **tangential component** $\sigma = r\theta'' + 2r\theta'$.

But: $r^2\theta' = M$, so $2rr'\theta' + r^2\theta'' = M'$, and then $M' = r(2r'\theta' + r\theta'') = r\sigma$. So $\boxed{M' = r\sigma}$.

And here comes our first great discovery: since $r \neq 0$, **$\sigma = 0$ if and only if $M' = 0$**. And Kepler's second law says that $M' = 0$. So Kepler's second law actually says that **the tangential component of the acceleration is zero.**

If we write $\vec{F} = m\vec{a}$, then **the planet is pulled towards the focus by a force pointing towards the focus.**

So **Kepler's second law amounts to saying that the planet is pulled towards the Sun by a force pointing towards the Sun.**

Can we compute ρ?

Yes, we can.

It's an easy computation, using the equation of the ellipse. But it's a little bit long.

The result is $\rho = -\frac{C}{r^2}$, where C is a constant.

So, **Kepler's first and second law amount to saying that the planet is pulled towards the Sun with an acceleration of of a magnitude equal to $\frac{C}{r^2}$, where C is a constant.**

So, **Kepler's first and second law amount to saying that the planet is pulled towards the Sun with a force of a magnitude equal to $\frac{Cm}{r^2}$, where C is a constant, and m is the mass of the planet.**

This is **Newton's inverse square law of gravitaion, for a single planet.**

Is there more? If you compute the constant C, it turns out that it is equal to $\frac{2\pi T^2}{a^3}$, where $T$is the period and a is the major semiaxis. I could be off by a factor here.

So **Kepler's third law amounts to saying that the constant C is the same for all planets.**

So we get **Newton's inverse square law of gravitaion, for all the planets, i.e., for all bodies that orbit around the Sun, with the same constant C for all.**

Is there more? How about other "planetary systems", e.g., the Moon orbiting around the Earth, or the moons of Jupiter?

A simple argument shows that the constant C, which is the same for all the planets, has to be proportional to the mass of the attracting body.

So $C = MG$, where M is the mass of the attracting body, and G is a truly universal constant.

So we get **Newton's universal law of gravitation: any two bodies attract each other with a force with magnitude $\frac{GmM}{r^2}$, where r is the distance betwee the bodies and m, M are their masses.**

So Analysis led to the discovery of the law of gravitation.

It could not have been done with Geometry alone.

But there was more to come.

The main point of Newton's discovery was that **bodies interact with each other by acting on the acceleration**.

This was suprising and counterintuitive.

Our intuition tells us that when A "attracts" B, it does so by making B move towards A, that is, by causing the velocity of B to change. The discoveries of Kepler and Newton, made possible by Descartes, showed that "attraction" consists of an effect on the acceleration.

Once this was understood for gravitational attraction, it became possible to use the same methodology (i.e., studying the acceleration produced by a body acting on another body) to study other forces, attractive as well as repulsive.

And then, in the 19th century, when people started studying electricity and magnetism, they discovered that there are **electromagnetic fields**. Maxwell synthesized the laws of electomegnetism into a system of partial differential equations. (The "Maxwell equations".) And by studying the solutions, he discovered that they have solutions that are waves that move with a speed that he was able to compute.

And this speed turned out to be a speed that was known: the speed of light.

So Maxwell concluded that **light is an electromagnetic field.**

And it became possible to manipulate these fields,. For example, one could take sound, create an electromagnetic field that adequately represented the sound, and tramsit it far away.

This made the modern world possible. Without electromagnetism, there would be no radio, TV, computers, automobiles, airplanes, high-rise buildings, air-conditioning, Internet, Google, Facebook.

So **the modern world owes its existence to Mathematical Analysis.**

And Mathematical Analysis was the creation of a few people: Descartes, Newton, and Leibniz, who in turn achieved what they did thanks to their predecessors: Copernicus, Kepler, and Galileo.

But: the creation of Analysis in the 17th and 18th centuries had a huge flaw. Those "infinitely small" numbers that were essential for the whole thing to work, were never found. It turns out that they don't exist. (We saw this in the course earlier.)

The solution to this problem was found much later, in the late 19th Century, when mathematicians developed the rigorous notion of "limit", and were able to define and study rigorously such things as continuity, derivatives, and integrals.

What we covered in the course was a glimpse into what the creators of Analysis did in the 17th and 18th centuries, combined with the insights on the late 19th century that showed how to do things rigorously.