13 More on the course

13.1 Reading for the period from the beginning of the semester until March 29

I. The book, Chapter 1 (all of it) and Chapter 2 (up to and including 2.5). *NOTE: on induction, all you need to know is the Well-Ordering Principle. As far as I am concerned, you are free to use well-ordering any time the book wants you to use induction or complete induction.*

II. The instructor's notes, up to page 88.

In particular,

a. Please read carefully the chapter of the notes on definitions (pages 56 to 67). You are going to be asked (in the second midterm, and in the final exam) to write definitions.

b. Please pay special attention to

   i. the statement and proof of "Euclid's algorithm," in the book, pages 62, 63,

   ii. the statement and proof of the division algorithm for $\mathbb{N}$, on page 115.

   NOTE: I will post be a set of notes on these two theorems and their consequences. (They will be ready, I hope, by Monday March 13.) ***Please read them carefully, because these theorems and their proofs are very important.***

13.2 Homework assignment No. 6, due on Wednesday, March 8

This is a short assignment, consisting of just one problem:

Prove (using well-ordering, or induction, as you wish) that

$$\sum_{k=1}^{n} k^3 = \Big(\frac{n(n+1)}{2}\Big)^2 \qquad \text{for every natural number } n\,.$$

13.3 Homework assignment No. 7, due on Wednesday, March 22

This is a long assignment, because I have included some challenging problems, so that you will not be bored. If you cannot do all the problems, do as many as you can.

I. *The following problems all depend on induction or well-ordering. You can do each one of them by whichever method you prefer: induction, or complete induction, or well-ordering,* ***even when the book tells you to use a specific method****. (My own preference is well-ordering. This method* ***always*** *works whenever one of the other two methods works, so it is quite safe, besides being simple. In a few cases, a proof by induction might be a little bit easier or shorter, so you may be slightly better off using induction.)*

1. Pages 106-107, Problem 8, Parts (b), (c), (d), (f), (g), (h), (i), (j), (l), (m), (n), (p), (q), (t),
2. Pages 107-108, Problem 9, Parts (b), (d), (f).
3. Page 109, Problem 14.

II. (*This is a truly challenging problem!*) On pages 96, 97, the book gives us a list of "axioms" for the natural numbers, and says that "these axioms are sufficient to derive all the familiar properties of the natural numbers." I am asking you to ***prove that the book is wrong***[18], by proving the following: ***using the axioms in the book, it is impossible to prove that*** $1.1 = 1$. *Here is a hint: suppose you take "natural number" to mean "even natural number," rather than "ordinary natural number." (This is sort of similar to things we did in the course, where we discussed what would happen if "giraffe" meant "rabbit", "cow" meant "unicorn", and "sheep" meant "elephant".) Also, take "1" to mean "2". (Then, of course, the "successor"* $x + 1$ *of a number is now* $x + 2$*.) With this new interpretation of the meaning of "natural number" and "1", prove that all the 18 axioms listed in the book, pages 96, 97, hold. And yet the assertion that* $1 \cdot 1 = 1$ *is not true,*

[18] Naturally, whether or not the argument I am proposing truly establishes that the book is wrong depends very much on whether you believe that "'$1 \cdot 1 = 1$" is a "familiar property of the natural numbers." In my opinion, it is. What do you think?

because it says, under our new interpretation, that $2 \cdot 2 = 2$, which of course is false.

The following two problems have already been assigned before, as "optional." Very few people did them, and nobody did them right. Now I am asking you to do them again. Remember our discussion of the problems in class: ***any argument you give that would also prove that "every year must have a Friday the 13th" even in a situation where this conclusion can fail to be true (for example, if all the months had 28 days) is necessarily wrong.***

III. Prove that every year must have a Friday the 13th.

IV. Prove that the statement of Problem III remains true even if we change the order of the months (without changing the names of the months or the number of days of each month) in an arbitrary way.

13.4 Solutions to the problems of the first midterm

Problem 1. Prove each of the following. (You will need the definitions of "even" and "odd", so write them down and make sure you use them. You are allowed to use all the basic facts you know about arithmetic, except that you are not allowed to use anything about "even" and "odd" other than the definitions.)

(i) The number 7 is odd.

Proof. The definition of "odd" says that an integer n is ***odd*** if $(\exists k \in \mathbb{Z})n = 2 \cdot k + 1$. Now, $7 = 2 \cdot 3 + 1$, so $(\exists k \in \mathbb{Z})7 = 2 \cdot k + 1$, so 7 is odd.

(ii) The sum of two odd numbers is even.

Proof. The definition of "odd" was given in Part (i). The definition of "even" says that an integer n is ***even*** if $(\exists k \in \mathbb{Z})n = 2 \cdot k$.

Let a, b be arbitrary integers. Suppose that a and b are odd. Then $(\exists k \in \mathbb{Z})a = 2 \cdot k + 1$, since a is odd. Pick a $k \in \mathbb{Z}$ such that $a = 2 \cdot k + 1$, and call it k_1, so $k_1 \in \mathbb{Z}$ and $a = 2 \cdot k_1 + 1$. Also, $(\exists k \in \mathbb{Z})b = 2 \cdot k + 1$, since b is odd. Pick a $k \in \mathbb{Z}$ such that $b = 2 \cdot k + 1$, and call it k_2, so $k_2 \in \mathbb{Z}$ and $b = 2 \cdot k_2 + 1$. Then

$$a + b = (2k_1 + 1) + (2k_2 + 1) = 2k_1 + 2k_2 + 2 = 2(k_1 + k_2 + 1)\,.$$

Since $k_1 + k_2 + 1 \in \mathbb{Z}$, it follows that $(\exists k \in \mathbb{Z})a + b = 2 \cdot k$. Hence $a + b$ is even.

(iii) If the product of two integers is odd, then both integers have to be odd.

Proof. First, we need to show that

(A) *Every integer is even or odd.* That is, in symbolic notation,

$$(\&) \qquad (\forall n \in \mathbb{Z})(n \text{ is even} \vee n \text{ is odd})$$

or, if you do not want to use the predicates "is even" and "is odd":

$$(\forall n \in \mathbb{Z})((\exists k \in \mathbb{Z})n = 2k \vee (\exists k \in \mathbb{Z})n = 2k + 1)\,.$$

Here is the proof. Suppose that (&) was not true. Then there would exist an integer n which is neither even nor odd. Then $n \neq 0$, because 0 is even. Since n is neither even nor odd, it follows that $-n$ is neither even nor odd, because if $-n$ was even then n would be even, and if $-n$ was odd then n would be odd. And one of the two, n or $-n$, is a natural number. So there exists a natural number which is neither even nor odd. By the well-ordering principle, we may pick ν such that $\nu \in \mathbb{N}$, ν is neither even nor odd, and no number $\mu \in \mathbb{N}$ such that μ is $< \nu$ can be neither even nor odd. Then ν cannot be 1, because 1 is odd. So $\nu > 1$. Then $\nu - 1 \in \mathbb{N}$. It follows that $\nu - 1$ is either even or odd. If $\nu - 1$ is even, then ν is odd, so ν is even$\vee\nu$ is odd. If $\nu - 1$ is odd, then ν is even, so ν is even$\vee\nu$ is odd. So in both cases ν is even$\vee\nu$ is odd, contradicting the fact that ν is neither even nor odd. **END OF THE PROOF OF (&)**.

Next we show that

(B) *An integer cannot be both even and odd.* That is, in symbolic notation,

$$(\#) \qquad (\forall n \in \mathbb{Z}) \sim (n \text{ is even} \wedge n \text{ is odd})$$

or, if you do not want to use the predicates "is even" and "is odd":

$$(\forall n \in \mathbb{Z}) \sim ((\exists k \in \mathbb{Z})n = 2k \wedge (\exists k \in \mathbb{Z})n = 2k + 1) .$$

Here is the proof. Suppose that (#) was not true. Then there would exist an integer n which is both even and odd. Pick one and call it ν, so $\nu \in \mathbb{Z}$, ν is even, and ν is odd. Since ν is even $(\exists k \in \mathbb{Z})\nu = 2k$. Pick one such k and call it k_1. Then $k_1 \in \mathbb{Z}$, and $\nu = 2k_1$. Since ν is odd, $(\exists k \in \mathbb{Z})n = 2k + 1)$. Pick one such k and call it k_2. Then $k_2 \in \mathbb{Z}$, and $\nu = 2k_2 + 1$. It follows that $2k_2 + 1 = 2k_1$, so $1 = 2(k_1 - k_2)$. Hence $\frac{1}{2} = k_1 - k_2$, so $\frac{1}{2} \in \mathbb{Z}$. But it is also true that $\sim \frac{1}{2} \in \mathbb{Z}$, because $0 < \frac{1}{2}$, $\frac{1}{2} < 1$, and $\sim (\exists n \in \mathbb{Z})(0 < n \wedge n < 1)$. So $\frac{1}{2} \in \mathbb{Z} \wedge \sim \frac{1}{2} \in \mathbb{Z}$, which is a contradiction. Hence (#) is true. **END OF THE PROOF OF (#)**.

Problem 2. Prove the following statement: If a, b, c are integers, and both a, b are divisible by c, then $a+b$ is divisible by c. (You will need the definition of "divisible," so write it down and make sure you use it. You are allowed to use all the basic facts you know about arithmetic, except that you are not allowed to use anything about the predicate "divisible" other than the definition.)

Proof. The definition of "divisible" says that, if x, y are integers, then x is ***divisible*** by y if $(\exists k \in \mathbb{Z})x = y \cdot k$.

Let a, b, c be arbitrary integers. Suppose a is divisible by c and b is divisible by c. Since a is divisible by c, $(\exists k \in \mathbb{Z})a = c \cdot k$. Pick a $k \in \mathbb{Z}$ such that $a = c \cdot k$, and call it k_1. Then $k_1 \in \mathbb{Z}$ and $a = c \cdot k_1$. Since b is divisible by c, $(\exists k \in \mathbb{Z})b = c \cdot k$. Pick a $k \in \mathbb{Z}$ such that $b = c \cdot k$, and call it k_2. Then $k_2 \in \mathbb{Z}$ and $b = c \cdot k_2$. So $a + b = c \cdot k_1 + c \cdot k_2 = c \cdot (k_1 + k_2)$. Then $(\exists k \in \mathbb{Z})a + b = c \cdot k$. So $a + b$ is divisible by c.

Problem 3. For each of the following three statements:

$$(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R})(\delta > 0 \wedge \delta < \varepsilon)) \ ,$$

$$(\forall \varepsilon \in \mathbb{R})(\varepsilon < 0 \Rightarrow (\exists \delta \in \mathbb{R})(\delta > 0 \wedge \delta < \varepsilon)) \ ,$$

$$(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \wedge (\exists \delta \in \mathbb{R})(\delta > 0 \wedge \delta < \varepsilon)) \ ,$$

(i) translate the statement into plain English, without using letter variables or mathematical symbols,

(ii) indicate whether the statement is true,

(iii) if the statement is true, prove it, and if it is false, prove that it is false.

Answer. First look at $(\forall\varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow (\exists\delta \in \mathbb{R})(\delta > 0 \wedge \delta < \varepsilon))$.

An English translation is "given any positive real number, there exists a smaller positive real number". This is ***true***. Here is a proof: let $\bar{\varepsilon}$ be an arbitrary real number. Assume that $\bar{\varepsilon} > 0$. Let $\bar{\delta} = \frac{\bar{\varepsilon}}{2}$. Then $\bar{\delta} > 0 \wedge \bar{\delta} < \bar{\varepsilon}$. So $(\exists\delta \in \mathbb{R})(\delta > 0 \wedge \delta < \bar{\varepsilon})$. So $\bar{\varepsilon} > 0 \Rightarrow (\exists\delta \in \mathbb{R})(\delta > 0 \wedge \delta < \bar{\varepsilon})$. Since $\bar{\varepsilon}$ was an arbitrary real number, we have proved that $(\forall\varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow (\exists\delta \in \mathbb{R})(\delta > 0 \wedge \delta < \varepsilon))$.

Next, consider $(\forall\varepsilon \in \mathbb{R})(\varepsilon < 0 \Rightarrow (\exists\delta \in \mathbb{R})(\delta > 0 \wedge \delta < \varepsilon))$.

An English translation is "given any negative real number, there exists a smaller positive real number". This is ***false***. Here is a proof: Take $\varepsilon = -1$. Then there cannot exist a $\delta \in \mathbb{R}$ such that $\delta > 0 \wedge \delta < \varepsilon$, because if any such δ existed it would follow that $\varepsilon > 0$, but $\varepsilon = -1$.

Finally, let us look at $(\forall\varepsilon \in \mathbb{R})(\varepsilon > 0 \wedge (\exists\delta \in \mathbb{R})(\delta > 0 \wedge \delta < \varepsilon))$.

An English translation is "given any real number, the number is positive, and there exists a smaller positive real number". This is ***false***. Here is a proof: Just take $\varepsilon = -1$. Then "$\varepsilon > 0 \wedge (\exists\delta \in \mathbb{R})(\delta > 0 \wedge \delta < \varepsilon)$" is false, because "$\varepsilon > 0$" is false.

Problem 4. In this problem, the universe of discourse (i.e., the range of values of the variables) is fixed but unknown to us, and the meaning of the one-variable predicates "is a borogove" and "is mimsy" is also fixed but unknown to us. (In other words, the universe of discourse and the meanings of the two predicates are fixed, and known by our "creator of arbitrary things", but they are unknown to us, and could be anything, as far as we know.)

Prove each of the following. (Informal proofs O.K., but make sure you indicate which logical rules you are using.)

(1) $\Big((\exists x)x \text{ is a borogove} \wedge (\exists x)x \text{ is mimsy}\Big) \Longrightarrow (\exists x)(x \text{ is a borogove} \wedge x \text{ is mimsy})\,;$

(2) $\Big((\forall x)x \text{ is a borogove} \Longrightarrow (\forall x)x \text{ is mimsy}\Big) \Longrightarrow$
$$(\forall x)(x \text{ is a borogove} \Longrightarrow x \text{ is mimsy})\,;$$

(3) $\Big((\forall x)(x \text{ is a borogove} \Longrightarrow x \text{ is mimsy})\Big) \Longrightarrow$
$$\Big((\exists x)x \text{ is a borogove} \Longrightarrow (\exists x)x \text{ is mimsy}\Big)\,.$$

Answer. Statement (1) cannot be proved, because it is not logically valid. To see this, take "is a borogove" to mean "is a cow", and "is mimsy" to mean "is an elephant", and let the universe of discourse be the set of all animals. Then "$(\exists x)x$ is a borogove" says that "there are cows", which is true, and "$(\exists x)x$ is mimsy$\Big)$" says that "there are elephants", which is also true. So the conjunction "$(\exists x)x$ is a borogove $\wedge$ $(\exists x)x$ is mimsy" is true. On the other hand, "$(\exists x)(x$ is a borogove $\wedge$ x is mimsy)" says that there exists an animal that is both a cow and an elephant, and this is clearly false. So (1) is false.

Statement (2) cannot be proved, because it is not logically valid. To see this, we can actually use the same example as for (1). "$(\forall x)x$ is a borogove" says that "all animals are cows", which is false. Hence the implication "$(\forall x)x$ is a borogove $\Longrightarrow$ $(\forall x)x$ is mimsy" is true. On the other hand, "$(\forall x)(x$ is a borogove $\Longrightarrow$ x is mimsy)" says that "every cow is an elephant", which is false. Therefore (2) is false.

Statement (3) is logically valid, and we can prove it. Here is a proof.

1.	Assume $(\forall x)(x$ is a borogove $\Longrightarrow$ x is mimsy)	[Assumption]
2.	Assume $(\exists x)x$ is a borogove	[Assumption]
3.	Pick a such that a is a borogove.	[Rule $\exists_{use}$, from 2]
4.	a is a borogove $\Longrightarrow$ a is mimsy	[Rule $\forall_{use}$, from 1]
5.	a is mimsy	[Rule $\Rightarrow_{use}$, from 3 & 4]
6.	$(\exists x)x$ is mimsy	[Rule $\exists_{get}$, from 5]
7.	$(\exists x)x$ is mimsy	[Rule $\exists_{use}$, from 3 & 6]
8.	$(\exists x)x$ is a borogove $\Longrightarrow$ $(\exists x)x$ is mimsy	[Rule $\Rightarrow_{get}$, from 2 & 7]
9.	$(\forall x)(x$ is a borogove $\Longrightarrow$ x is mimsy) $\Longrightarrow$ $\Big((\exists x)x$ is a borogove $\Longrightarrow$ $(\exists x)x$ is mimsy$\Big)$	[Rule $\Rightarrow_{get}$, from 1 & 8]

END

Problem 5. For each of the following claims and purported proofs (a) indicate if the claim is true, (b) grade the purported proof (using grades A, C, F), (c) if the statement is true but the proof is wrong, give a correct proof. If your grade is not "A", explain why. Please do not use fuzzy, vague, verbose sentences. Be precise. In particular, when a step violates one of the logical rules, indicate which rule is being misapplied or violated, and explain why.

I. *Claim*: The sum of two even integers is divisible by 4. *Proof*: Let x, y be even integers. Then $x = 2k$ and $y = 2k$, so $x + y = 2k + 2k = 4k$, showing that $x + y$ is divisible by 4.

Answer: The grade is **F**. The claim is false. (For example, 2 and 4 are even, but the sum $2+4$ is not divisible by 4. The mistake in the proof is the violation of Rule $\exists_{use}$. The author of the proof is implicitly trying to use this rule, together with the facts that $(\exists k \in \mathbb{Z})x = 2k$ and $(\exists k \in \mathbb{Z})y = 2k$ to pick a k in each case. However, the rule states that each time we pick such a k we have to give it a different name, so it is ***not*** allowed to pick a k for x and another one for y and call them both k.

II. *Claim*: The product of two even integers is divisible by 4. *Proof*: Let x, y be even integers. Then $x = 2k$ and $y = 2k$, so $x \cdot y = 4k^2$, showing that $x \cdot y$ is divisible by 4.

Answer: The grade is **C**. The conclusion is true, but the proof is worng, because of the same mistake in the application of Rule $\exists_{use}$ as in the previous question. *Correct proof*: Let x, y be even integers. Then $(\exists k \in \mathbb{Z})x = 2k$ and $(\exists k \in \mathbb{Z})y = 2k$. Pick a $k \in \mathbb{Z}$ such that $x = 2k$ and call it k_1. Pick a $k \in \mathbb{Z}$ such that $y = 2k$ and call it k_2. Then $x \cdot y = 4k_1k_2$, showing that $(\exists k \in \mathbb{Z})x \cdot y = 2k$, so $x \cdot y$ is divisible by 4. **END**

III. *Claim*: For real numbers x and y, if $x \cdot y = 0$ then $x = 0$ or $y = 0$. *Proof*: We do a proof by cases. **Case 1**: If $x = 0$ then $x \cdot y = 0 \cdot y = 0$. **Case 2**: If $y = 0$ then $x \cdot y = x \cdot 0 = 0$. In either case, $x \cdot y = 0$.

Answer: The grade is **F**. The statement is correct, but the proof is completely wrong, because it begins by assuming the conclusion, that $x = 0$ or $y = 0$, and then proves the hypothesis. *Correct proof*: Let x, y be real numbers such that $x \cdot y = 0$. Assume that $\sim x = 0$. Then $y = x \cdot \frac{y}{x}$. But $x \cdot \frac{y}{x} = \frac{x \cdot y}{x} = \frac{0}{x} = 0$. So $y = 0$. Hence we have proved that $\sim x = 0 \Rightarrow y = 0$, which is equivalent to $x = 0 \vee y = 0$. **END**

IV. *Claim*: For real numbers x and y, if $x \cdot y \geq 0$ then $\sqrt{x^2 + y^2} \leq x + y$. *Proof*: Squaring both sides of $\sqrt{x^2 + y^2} \leq x + y$ we get $x^2 + y^2 \leq (x + y)^2$. But $(x + y)^2 = x^2 + y^2 + 2 \cdot x \cdot y$, so we got $x^2 + y^2 \leq x^2 + y^2 + 2xy$, which is true because $x \cdot y \geq 0$.

Answer: The grade is **F**. The statement is false (for example, take $x = -1$, $y = -1$), and the proof is completely wrong, because it begins by assuming the conclusion, that $\sqrt{x^2 + y^2} \leq x + y$.

Problem 6.

(i) For each of the following four statements: (a) rewrite the statement in plain English, without letter symbols or any mathematical symbol; (b) indicate whether the statement is true or false (no proof necessary).

1. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y < x)$

   Translation. For every integer there exists a strictly smaller integer. TRUE.

2. $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(y \leq x)$

   Translation. There exists a smallest integer. FALSE.

3. $(\exists y \in \mathbb{N})(\forall x \in \mathbb{N})(y \leq x)$

   Translation. There exists a smallest natural number. TRUE.

4. $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\forall z \in \mathbb{Z})(x \cdot z = y \cdot z \Longrightarrow x = y)$

   Translation. If the results of multiplying two integers by a third integer are equal, then the two integers are equal. FALSE. (Take $x = 3$, $y = 21$, $z = 0$.)

(ii) For each of the following four statements: (a) rewrite the statement in formal language, using the basic vocabulary of arithmetic (that is, the parentheses "(" and ")", the logical connectives "$\vee$", "$\wedge$", "$\sim$", "$\Rightarrow'$", "$\Leftrightarrow'$", "$\exists$", and "$\forall$", letter variables such as $n, p, q, x, y, z, a, b,$, etc., the predicates "$\in \mathbb{N}$", "$\in \mathbb{Z}$" and "$\in \mathbb{R}$", the symbols 0, 1, $+$, $-$, $\cdot$, $=$, $<$, $>$, $\leq$, $\geq$), plus the predicate "is prime", and **nothing else**. (b) indicate whether the statement is true or false (no proof necessary).

5. Every real number has a square root.

   Translation: $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y \cdot y = x)$. FALSE.

6. There exists a smallest nonnegative real number.

   Translation: $(\exists x \in \mathbb{R})(x \geq 0 \wedge (\forall y \in \mathbb{R})(y \geq 0 \Rightarrow y \geq x))$. TRUE

7. Every positive integer is the sum of the squares of three integers.

   Translation:

   $(\forall n \in \mathbb{Z})(n > 0 \Rightarrow (\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(p \cdot p + q \cdot q + r \cdot r = n))$. FALSE. (Take $n = 7$.)

8. The product of two prime numbers is not prime.

   Translation:

   $(\forall p \in \mathbb{Z})(\forall q \in \mathbb{Z})((p \text{ is prime} \wedge q \text{ is prime}) \Rightarrow \sim p \cdot q \text{ is prime})$. TRUE.

Problem 7. Prove the following:

For every natural number n, $\sum_{k=1}^{n}(2k-1) = n^2$.

(You may use well-ordering or induction, or even give a direct proof that uses neither, if you remember what was said in class about C. F. Gauss.)

Proof using well-ordering. Call a natural number n "bad" if it is not true that $\sum_{k=1}^{n}(2k-1) = n^2$. We want to prove that there are no bad natural numbers. Suppose there is a bad natural number. Then the well-ordering principle tells us that there exists a smallest bad natural number. Call this number s. Then $s \in \mathbb{N}$ and the equality $\sum_{k=1}^{s}(2k-1) = s^2$ is not true. Furthermore, $\sum_{k=1}^{n}(2k-1) = n^2$ for every $n \in \mathbb{N}$ such that $n < s$. Now, the equality $\sum_{k=1}^{n}(2k-1) = n^2$ is true for $n = 1$, because $\sum_{k=1}^{1}(2k-1) = 1$ and $1^2 = 1$. So 1 is not bad, and then $s \neq 1$. Since $s \in \mathbb{N}$, we have $s > 1$, and then $s-1 \in \mathbb{N}$ and $s-1$ is not bad. Therefore $\sum_{k=1}^{s-1}(2k-1) = (s-1)^2$, and then

$$\sum_{k=1}^{s}(2k-1) = 2s-1+\sum_{k=1}^{s-1}(2k-1) = (s-1)^2+2s-1 = s^2-2s+1+2s-1 = s^2\,.$$

So $\sum_{k=1}^{s}(2k-1) = s^2$, and then s is not bad. But s is bad. So s is not bad and s is bad. This is a contradiction, and we have proved that no bad numbers can exist. **END**

Problem 8.

a. Prove that the product of two rational numbers is rational.

   Proof: Let x, y be arbitrary rational numbers. The definition of "rational number" says that

   $(\forall u \in \mathbb{R})(u \text{ is } \textbf{rational} \Leftrightarrow (\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})(\sim n = 0 \wedge u = \frac{m}{n}))$.

   Since x and y are rational, we may pick integers a, b, c, d such that $\sim b = 0$, $\sim d = 0$, $x = \frac{a}{b}$, and $y = \frac{c}{d}$. Then $xy = \frac{ac}{bd}$, so xy is rational. **END**

b. Prove that $\sqrt{2}$ is irrational.

Proof: Assume $\sqrt{2}$ was rational. Then we may pick integers a, b such that $\sim b = 0$ and $\sqrt{2} = \frac{a}{b}$. After "eliminating all common factors" from a and b, we may assume that a and b have no common factors. In particular, a and b cannot both be even. On the other hand, $2b^2 = a^2$. Hence a^2 is even, so a is even, because if a was odd then a^2 would be odd. So we may pick an integer c such that $a = 2c$. Then $a^2 = 4c^2$. So $2b^2 = 4c^2$, and then $b^2 = 2c^2$, so b^2 is even and then b is even. So we have shown that

a is even and b is even and a and b are not both even.

This is a contradiction, proving that $\sqrt{2}$ is irrational. **END**

c. Prove that the product of two irrational numbers is irrational.

Answer: This cannot be proved because it is false. (Proof that it is false: let $x = \sqrt{2}$, $y = \frac{1}{\sqrt{2}}$. Then x and y are irrational, but the product $x \cdot y$ is equal to 1, which is rational.

d. Prove that $\sqrt{12}$ is irrational.

Proof: Assume $\sqrt{12}$ was rational. Then we may pick integers a, b such that $\sim b = 0$ and $\sqrt{12} = \frac{a}{b}$. After "eliminating all common factors" from a and b, we may assume that a and b have no common factors. In particular, a and b cannot both be divisible by 3. On the other hand, $12b^2 = a^2$. Hence $a^2 = 3 \times (4b^2)$, so a^2 is divisble by 3, so a is divisble by 3, because if a was not divisible by 3 then a^2 would be not be divisble by 3 either. (The general fact we are using is this: if p is prime and a product mn of integers is divisible by p, then m or n must be divisible by p.) So we may pick an integer c such that $a = 3c$. Then $a^2 = 9c^2$. So $12b^2 = 9c^2$, and then $4b^2 = 3c^2$, so $4b^2$ is divisible by 3. Since 4 is not divisible by 3, it follows that b^2 is divisible by 3, and then b is divisible by 3. So we have shown that

a is divisible by 3, b is divisible by 3, and a and b are not both divisible by 3.

This is a contradiction, proving that $\sqrt{12}$ is irrational. **END**

Problem 9.

$P \Longrightarrow (Q \Longrightarrow (R \Longrightarrow (S \Longrightarrow (P \wedge Q \wedge R \wedge S))))$ is a ***tautology***.

$(P \wedge (\sim P)) \Longrightarrow Q$ is a ***tautology***.

$P \Longrightarrow (\sim P)$ is a ***contingency***.

$P \wedge (\sim P)$ is a ***contradiction***.

$P \vee (\sim P)$ is a ***tautology***.

$((\sim P) \wedge (\sim Q)) \wedge (P \vee Q)$ is a ***contradiction***.