13 More on the course

13.1 Reading for the period from the beginning of the semester until March 29

I. The book, Chapter 1 (all of it) and Chapter 2 (up to and including 2.5). *NOTE: on induction, all you need to know is the Well-Ordering Principle. As far as I am concerned, you are free to use well-ordering any time the book wants you to use induction or complete induction.*

II. The instructor's notes, up to page 88.

In particular,

a. Please read carefully the chapter of the notes on definitions (pages 56 to 67). You are going to be asked (in the second midterm, and in the final exam) to write definitions.

b. Please pay special attention to

   i. the statement and proof of "Euclid's algorithm," in the book, pages 62, 63,

   ii. the statement and proof of the division algorithm for $\mathbb{N}$, on page 115.

   NOTE: I will post be a set of notes on these two theorems and their consequences. (They will be ready, I hope, by Monday March 13.) ***Please read them carefully, because these theorems and their proofs are very important.***

13.2 Homework assignment No. 6, due on Wednesday, March 8

This is a short assignment, consisting of just one problem:

Prove (using well-ordering, or induction, as you wish) that

$$\sum_{k=1}^{n} k^3 = \Big(\frac{n(n+1)}{2}\Big)^2 \qquad \text{for every natural number } n\,.$$

13.3 Homework assignment No. 7, due on Wednesday, March 22

This is a long assignment, because I have included some challenging problems, so that you will not be bored. If you cannot do all the problems, do as many as you can.

I. *The following problems all depend on induction or well-ordering. You can do each one of them by whichever method you prefer: induction, or complete induction, or well-ordering,* ***even when the book tells you to use a specific method****. (My own preference is well-ordering. This method* ***always*** *works whenever one of the other two methods works, so it is quite safe, besides being simple. In a few cases, a proof by induction might be a little bit easier or shorter, so you may be slightly better off using induction.)*

1. Pages 106-107, Problem 8, Parts (b), (c), (d), (f), (g), (h), (i), (j), (l), (m), (n), (p), (q), (t),
2. Pages 107-108, Problem 9, Parts (b), (d), (f).
3. Page 109, Problem 14.

II. (*This is a truly challenging problem!*) On pages 96, 97, the book gives us a list of "axioms" for the natural numbers, and says that "these axioms are sufficient to derive all the familiar properties of the natural numbers." I am asking you to ***prove that the book is wrong***[18], by proving the following: ***using the axioms in the book, it is impossible to prove that*** $1.1 = 1$. *Here is a hint: suppose you take "natural number" to mean "even natural number," rather than "ordinary natural number." (This is sort of similar to things we did in the course, where we discussed what would happen if "giraffe" meant "rabbit", "cow" meant "unicorn", and "sheep" meant "elephant".) Also, take "1" to mean "2". (Then, of course, the "successor"* $x + 1$ *of a number is now* $x + 2$*.) With this new interpretation of the meaning of "natural number" and "1", prove that all the 18 axioms listed in the book, pages 96, 97, hold. And yet the assertion that* $1 \cdot 1 = 1$ *is not true,*

[18]Naturally, whether or not the argument I am proposing truly establishes that the book is wrong depends very much on whether you believe that "$1 \cdot 1 = 1$" is a "familiar property of the natural numbers." In my opinion, it is. What do you think?

> *because it says, under our new interpretation, that* $2 \cdot 2 = 2$*, which of course is false.*

The following two problems have already been assigned before, as "optional." Very few people did them, and nobody did them right. Now I am asking you to do them again. Remember our discussion of the problems in class: ***any argument you give that would also prove that "every year must have a Friday the 13th" even in a situation where this conclusion can fail to be true (for example, if all the months had 28 days) is necessarily wrong.***

III. Prove that every year must have a Friday the 13th.

IV. Prove that the statement of Problem III remains true even if we change the order of the months (without changing the names of the months or the number of days of each month) in an arbitrary way.