## 13 More on the course

# 13.1 Reading for the period from the beginning of the semester until March 29

- I. The book, Chapter 1 (all of it) and Chapter 2 (up to and including 2.5). NOTE: on induction, all you need to know is the Well-Ordering Principle. As far as I am concerned, you are free to use well-ordering any time the book wants you to use induction or complete induction.
- II. The instructor's notes, up to page 88.

In particular,

- a. Please read carefully the chapter of the notes on definitions (pages 56 to 67). You are going to be asked (in the second midterm, and in the final exam) to write definitions.
- b. Please pay special attention to
  - i. the statement and proof of "Euclid's algorithm," in the book, pages 62, 63,
  - ii. the statement and proof of the division algorithm for  $\mathbb N,$  on page 115.

NOTE: I will post be a set of notes on these two theorems and their consequences. (They will be ready, I hope, by Monday March 13.) Please read them carefully, because these theorems and their proofs are very important.

## 13.2 Homework assignment No. 6, due on Wednesday, March 8

This is a short assignment, consisting of just one problem:

Prove (using well-ordering, or induction, as you wish) that

$$\sum_{k=1}^{n} k^{3} = \left(\frac{n(n+1)}{2}\right)^{2} \quad \text{for every natural number } n \,.$$

## 13.3 Homework assignment No. 7, due on Wednesday, March 22

This is a long assignment, because I have included some challenging problems, so that you will not be bored. If you cannot do all the problems, do as many as you can.

- I. The following problems all depend on induction or well-ordering. You can do each one of them by whichever method you prefer: induction, or complete induction, or well-ordering, even when the book tells you to use a specific method. (My own preference is well-ordering. This method always works whenever one of the other two methods works, so it is quite safe, besides being simple. In a few cases, a proof by induction might be a little bit easier or shorter, so you may be slightly better off using induction.)
  - Pages 106-107, Problem 8, Parts (b), (c), (d), (f), (g), (h), (i), (j), (l), (m), (n), (p), (q), (t),
  - 2. Pages 107-108, Problem 9, Parts (b), (d), (f).
  - 3. Page 109, Problem 14.
- II. (This is a truly challenging problem!) On pages 96, 97, the book gives us a list of "axioms" for the natural numbers, and says that "these axioms are sufficient to derive all the familiar properties of the natural numbers." I am asking you to **prove that the book is wrong**<sup>18</sup>, by proving the following: **using the axioms in the book, it is impossible to prove that** 1.1 = 1. Here is a hint: suppose you take "natural number" to mean "even natural number," rather than "ordinary natural number." (This is sort of similar to things we did in the course, where we discussed what would happen if "giraffe" meant "rabbit", "cow" meant "unicorn", and "sheep" meant "elephant".) Also, take "1" to mean "2". (Then, of course, the "successor" x + 1 of a number is now x + 2.) With this new interpretation of the meaning of "natural number" and "1", prove that all the 18 axioms listed in the book, pages 96, 97, hold. And yet the assertion that  $1 \cdot 1 = 1$  is not true,

<sup>&</sup>lt;sup>18</sup>Naturally, whether or not the argument I am proposing truly establishes that the book is wrong depends very much on whether you believe that " $1 \cdot 1 = 1$ " is a "familiar property of the natural numbers." In my opinion, it is. What do you think?

because it says, under our new interpretation, that  $2 \cdot 2 = 2$ , which of course is false.

The following two problems have already been assigned before, as "optional." Very few people did them, and nobody did them right. Now I am asking you to do them again. Remember our discussion of the problems in class: any argument you give that would also prove that "every year must have a Friday the 13th" even in a situation where this conclusion can fail to be true (for example, if all the months had 28 days) is necessarily wrong.

- III. Prove that every year must have a Friday the 13th.
- IV. Prove that the statement of Problem III remains true even if we change the order of the months (without changing the names of the months or the number of days of each month) in an arbitrary way.

### 13.4 Solutions to the problems of the first midterm

**Problem 1.** Prove each of the following. (You will need the definitions of "even" and "odd", so write them down and make sure you use them. You are allowed to use all the basic facts you know about arithmetic, except that you are not allowed to use anything about "even" and "odd" other than the definitions.)

(i) The number 7 is odd.

**Proof.** The definition of "odd" says that an integer n is **odd** if  $(\exists k \in \mathbb{Z})n = 2 \cdot k + 1$ . Now,  $7 = 2 \cdot 3 + 1$ , so  $(\exists k \in \mathbb{Z})7 = 2 \cdot k + 1$ , so 7 is odd.

(ii) The sum of two odd numbers is even.

**Proof.** The definition of "odd" was given in Part (i). The definition of "even" says that an integer n is **even** if  $(\exists k \in \mathbb{Z})n = 2 \cdot k$ .

Let a, b be arbitrary integers. Suppose that a and b are odd. Then  $(\exists k \in \mathbb{Z})a = 2 \cdot k + 1$ , since a is odd. Pick a  $k \in \mathbb{Z}$  such that  $a = 2 \cdot k + 1$ , and call it  $k_1$ , so  $k_1 \in \mathbb{Z}$  and  $a = 2 \cdot k_1 + 1$ . Also,  $(\exists k \in \mathbb{Z})b = 2 \cdot k + 1$ ,

since b is odd. Pick a  $k \in \mathbb{Z}$  such that  $b = 2 \cdot k + 1$ , and call it  $k_2$ , so  $k_2 \in \mathbb{Z}$  and  $b = 2 \cdot k_2 + 1$ . Then

$$a + b = (2k_1 + 1) + (2k_2 + 1) = 2k_1 + 2k_2 + 2 = 2(k_1 + k_2 + 1).$$

Since  $k_1 + k_2 + 1 \in \mathbb{Z}$ , it follows that  $(\exists k \in \mathbb{Z})a + b = 2 \cdot k$ . Hence a + b is even.

(iii) If the product of two integers is odd, then both integers have to be odd.

**Proof.** First, we need to show that

(A) Every integer is even or odd. That is, in symbolic notation,

(&) 
$$(\forall n \in \mathbb{Z})(n \text{ is even } \lor n \text{ is odd})$$

or, if you do not want to use the predicates "is even" and "is odd":

$$(\forall n \in \mathbb{Z})((\exists k \in \mathbb{Z})n = 2k \lor (\exists k \in \mathbb{Z})n = 2k + 1).$$

Here is the proof. Suppose that (&) was not true. Then there would exist an integer n which is neither even nor odd. Then  $n \neq 0$ , because 0 is even. Since n is neither even nor odd, it follows that -n is neither even nor odd, because if -n was even then n would be even, and if -n was odd then n would be odd. And one of the two, n or -n, is a natural number. So there exists a natural number which is neither even nor odd. By the wellordering principle, we may pick  $\nu$  such that  $\nu \in \mathbb{N}$ ,  $\nu$  is neither even nor odd, and no number  $\mu \in \mathbb{N}$  such that  $\mu$  is  $< \nu$  can be neither even nor odd. Then  $\nu$  cannot be 1, because 1 is odd. So  $\nu > 1$ . Then  $\nu - 1 \in \mathbb{N}$ . It follows that  $\nu - 1$  is either even or odd. If  $\nu - 1$  is even, then  $\nu$  is odd, so  $\nu$  is even  $\lor \nu$  is odd. If  $\nu - 1$ is odd, then  $\nu$  is even, so  $\nu$  is even  $\lor \nu$  is odd. So in both cases  $\nu$ is even  $\lor \nu$  is odd, contradicting the fact that  $\nu$  is neither even nor odd. **END OF THE PROOF OF (**&).

Next we show that

(B) An integer cannot be both even and odd. That is, in symbolic notation,

$$(\#) \qquad (\forall n \in \mathbb{Z}) \sim (n \text{ is even } \land n \text{ is odd})$$

or, if you do not want to use the predicates "is even" and "is odd":

$$(\forall n \in \mathbb{Z}) \sim ((\exists k \in \mathbb{Z})n = 2k \land (\exists k \in \mathbb{Z})n = 2k + 1).$$

Here is the proof. Suppose that (#) was not true. Then there would exist an integer n which is both even and odd. Pick one and call it  $\nu$ , so  $\nu \in \mathbb{Z}$ ,  $\nu$  is even, and  $\nu$  is odd. Since  $\nu$  is even  $(\exists k \in \mathbb{Z})\nu = 2k$ . Pick one such k and call it  $k_1$ . Then  $k_1 \in \mathbb{Z}$ , and  $\nu = 2k_1$ . Since  $\nu$  is odd,  $(\exists k \in \mathbb{Z})n = 2k + 1)$ . Pick one such k and call it  $k_2$ . Then  $k_2 \in \mathbb{Z}$ , and  $\nu = 2k_2 + 1$ . It follows that  $2k_2 + 1 = 2k_1$ , so  $1 = 2(k_1 - k_2)$ . Hence  $\frac{1}{2} = k_1 - k_2$ , so  $\frac{1}{2} \in \mathbb{Z}$ . But it is also true that  $\sim \frac{1}{2} \in \mathbb{Z}$ , because  $0 < \frac{1}{2}$ ,  $\frac{1}{2} < 1$ , and  $\sim (\exists n \in \mathbb{Z})(0 < n \land n < 1)$ . So  $\frac{1}{2} \in \mathbb{Z} \land \sim \frac{1}{2} \in \mathbb{Z}$ , which is a contradiction. Hence (#) is true. **END OF THE PROOF OF (#)**.

**Problem 2.** Prove the following statement: If a, b, c are integers, and both a, b are divisible by c, then a+b is divisible by c. (You will need the definition of "divisible," so write it down and make sure you use it. You are allowed to use all the basic facts you know about arithmetic, except that you are not allowed to use anything about the predicate "divisible" other than the definition.)

**Proof.** The definition of "divisible" says that, if x, y are integers, then x is *divisible* by y if  $(\exists k \in \mathbb{Z})x = y \cdot k$ .

Let a, b, c be arbitrary integers. Suppose a is divisible by c and b is divisible by c. Since a is divisible by c,  $(\exists k \in \mathbb{Z})a = c \cdot k$ . Pick a  $k \in \mathbb{Z}$  such that  $a = c \cdot k$ , and call it  $k_1$ . Then  $k_1 \in \mathbb{Z}$  and  $a = c \cdot k_1$ . Since b is divisible by c,  $(\exists k \in \mathbb{Z})b = c \cdot k$ . Pick a  $k \in \mathbb{Z}$  such that  $b = c \cdot k$ , and call it  $k_2$ . Then  $k_2 \in \mathbb{Z}$  and  $b = c \cdot k_2$ . So  $a + b = c \cdot k_1 + c \cdot k_2 = c \cdot (k_1 + k_2)$ . Then  $(\exists k \in \mathbb{Z})a + b = c \cdot k$ . So a + b is divisible by c.

**Problem 3.** For each of the following three statements:

 $(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R})(\delta > 0 \land \delta < \varepsilon)),$  $(\forall \varepsilon \in \mathbb{R})(\varepsilon < 0 \Rightarrow (\exists \delta \in \mathbb{R})(\delta > 0 \land \delta < \varepsilon)),$  $(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \land (\exists \delta \in \mathbb{R})(\delta > 0 \land \delta < \varepsilon)),$ 

- (i) translate the statement into plain English, without using letter variables or mathematical symbols,
- (ii) indicate whether the statement is true,
- (iii) if the statement is true, prove it, and if it is false, prove that it is false.

**Answer.** First look at  $(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R})(\delta > 0 \land \delta < \varepsilon))$ .

An English translation is "given any positive real number, there exists a smaller positive real number". This is **true**. Here is a proof: let  $\bar{\varepsilon}$  be an arbitrary real number. Assume that  $\bar{\varepsilon} > 0$ . Let  $\bar{\delta} = \frac{\bar{\varepsilon}}{2}$ . Then  $\bar{\delta} > 0 \land \bar{\delta} < \bar{\varepsilon}$ . So  $(\exists \delta \in \mathbb{R}) (\delta > 0 \land \delta < \bar{\varepsilon})$ . So  $\bar{\varepsilon} > 0 \Rightarrow (\exists \delta \in \mathbb{R}) (\delta > 0 \land \delta < \bar{\varepsilon})$ . Since  $\bar{\varepsilon}$  was an arbitrary real number, we have proved that  $(\forall \varepsilon \in \mathbb{R}) (\varepsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R}) (\varepsilon > 0)$ .

Next, consider  $(\forall \varepsilon \in \mathbb{R}) (\varepsilon < 0 \Rightarrow (\exists \delta \in \mathbb{R}) (\delta > 0 \land \delta < \varepsilon))$ .

An English translation is "given any negative real number, there exists a smaller positive real number". This is **false**. Here is a proof: Take  $\varepsilon = -1$ . Then there cannot exist a  $\delta \in \mathbb{R}$  such that  $\delta > 0 \land \delta < \varepsilon$ , because if any such  $\delta$  existed it would follow that  $\varepsilon > 0$ , but  $\varepsilon = -1$ .

Finally, let us look at  $(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \land (\exists \delta \in \mathbb{R})(\delta > 0 \land \delta < \varepsilon))$ .

An English translation is "given any real number, the number is positive, and there exists a smaller positive real number". This is **false**. Here is a proof: Just take  $\varepsilon = -1$ . Then " $\varepsilon > 0 \land (\exists \delta \in \mathbb{R}) (\delta > 0 \land \delta < \varepsilon)$ " is false, because " $\varepsilon > 0$ " is false.

**Problem 4.** In this problem, the universe of discourse (i.e., the range of values of the variables) is fixed but unknown to us, and the meaning of the one-variable predicates "is a borogove" and "is mimsy" is also fixed but unknown to us. (In other words, the universe of discourse and the meanings of the two predicates are fixed, and known by our "creator of arbitrary things", but they are unknown to us, and could be anything, as far as we know.)

Prove each of the following. (Informal proofs O.K., but make sure you indicate which logical rules you are using.)

- (1)  $((\exists x)x \text{ is a borogove } \land (\exists x)x \text{ is mimsy}) \Longrightarrow (\exists x)(x \text{ is a borogove } \land x \text{ is mimsy});$
- (2)  $((\forall x)x \text{ is a borogove} \Longrightarrow (\forall x)x \text{ is mimsy}) \Longrightarrow$

 $(\forall x)(x \text{ is a borogove} \Longrightarrow x \text{ is mimsy});$ 

- (3)  $((\forall x)(x \text{ is a borogove} \Longrightarrow x \text{ is mimsy})) \Longrightarrow$ 
  - $((\exists x)x \text{ is a borogove} \Longrightarrow (\exists x)x \text{ is mimsy}).$

**Answer.** Statement (1) cannot be proved, because it is not logically valid. To see this, take "is a borogove" to mean "is a cow", and "is mimsy" to mean "is an elephant", and let the universe of discourse be the set of all animals. Then " $(\exists x)x$  is a borogove" says that "there are cows", which is true, and " $(\exists x)x$  is mimsy)" says that "there are elephants", which is also true. So the conjunction " $(\exists x)x$  is a borogove  $\land$  ( $\exists x)x$  is mimsy" is true. On the other hand, " $(\exists x)(x \text{ is a borogove } \land x \text{ is mimsy})$ " says that there exists an animal that is both a cow and an elephant, and this is clearly false. So (1) is false.

Statement (2) cannot be proved, because it is not logically valid. To see this, we can actually use the same example as for (1). " $(\forall x)x$  is a borogove" says that "all animals are cows", which is false. Hence the implication " $(\forall x)x$  is a borogove  $\implies (\forall x)x$  is mimsy" is true. On the other hand, " $(\forall x)(x \text{ is a borogove} \implies x \text{ is mimsy})$ " says that "every cow is an elephant", which is false. Therefore (2) is false.

Statement (3) is logically valid, and we can prove it. Here is a proof.

1.	Assume $(\forall x)(x \text{ is a borogove} \Longrightarrow x \text{ is mimsy})$	[Assumption]
2.	Assume $(\exists x)x$ is a borogove	[Assumption]
3.	Pick $a$ such that $a$ is a borogove.	[Rule $\exists_{use}$ , from 2]
4.	$a \text{ is a borogove} \Longrightarrow a \text{ is mimsy}$	[Rule $\forall_{use}$ , from 1]
5.	a is mimsy	[Rule $\Rightarrow_{use}$ , from 3 & 4]
6.	$(\exists x)x$ is mimsy	[Rule $\exists_{get}$ , from 5]
7.	$(\exists x)x$ is mimsy	[Rule $\exists_{use}$ , from 3 & 6]
8.	$(\exists x)x$ is a borogove $\Longrightarrow (\exists x)x$ is mimsy	[Rule $\Rightarrow_{get}$ , from 2 & 7]
9.	$(\forall x)(x \text{ is a borogove} \Longrightarrow x \text{ is mimsy}) \Longrightarrow$	
	$((\exists x)x \text{ is a borogove} \Longrightarrow (\exists x)x \text{ is mimsy})$	$[\text{Rule} \Rightarrow_{get}, \text{from 1 \& 8}]$
	`````	$\mathbf{END}$

**Problem 5.** For each of the following claims and purported proofs (a) indicate if the claim is true, (b) grade the purported proof (using grades A, C, F), (c) if the statement is true but the proof is wrong, give a correct proof. If your grade is not "A", explain why. Please do not use fuzzy, vague, verbose sentences. Be precise. In particular, when a step violates one of the logical rules, indicate which rule is being misapplied or violated, and explain why.

**I.** Claim: The sum of two even integers is divisible by 4. Proof: Let x, y be even integers. Then x = 2k and y = 2k, so x + y = 2k + 2k = 4k, showing that x + y is divisible by 4.

**Answer:** The grade is  $\mathbf{F}$ . The claim is false. (For example, 2 and 4 are even, but the sum 2 + 4 is not divisible by 4. The mistake in the proof is the violation of Rule  $\exists_{use}$ . The author of the proof is implicitly trying to use this rule, together with the facts that  $(\exists k \in \mathbb{Z})x = 2k$  and  $(\exists k \in \mathbb{Z})y = 2k$  to pick a k in each case. However, the rule states that each time we pick such a k we have to give it a different name, so it is **not** allowed to pick a k for x and another one for y and call them both k.

**II.** Claim: The product of two even integers is divisible by 4. Proof: Let x, y be even integers. Then x = 2k and y = 2k, so  $x \cdot y = 4k^2$ , showing that  $x \cdot y$  is divisible by 4.

**Answer:** The grade is **C**. The conclusion is true, but the proof is worng, because of the same mistake in the application of Rule  $\exists_{use}$  as in the previous question. *Correct proof*: Let x, y be even integers. Then  $(\exists k \in \mathbb{Z})x = 2k$  and  $(\exists k \in \mathbb{Z})y = 2k$ . Pick a  $k \in \mathbb{Z}$  such that x = 2k and call it  $k_1$ . Pick a  $k \in \mathbb{Z}$  such that y = 2k and call it  $k_2$ . Then  $x \cdot y = 4k_1k_2$ , showing that  $(\exists k \in \mathbb{Z})x \cdot y = 2k$ , so  $x \cdot y$  is divisible by 4. **END** 

**III.** Claim: For real numbers x and y, if  $x \cdot y = 0$  then x = 0 or y = 0. Proof: We do a proof by cases. **Case 1**: If x = 0 then  $x \cdot y = 0 \cdot y = 0$ . **Case 2**: If y = 0 then  $x \cdot y = x \cdot 0 = 0$ . In either case,  $x \cdot y = 0$ .

**Answer:** The grade is **F**. The statement is correct, but the proof is completely wrong, because it begins by assuming the conclusion, that x = 0 or y = 0, and then proves the hypothesis. *Correct proof*: Let x, y be real numbers such that  $x \cdot y = 0$ . Assume that  $\sim x = 0$ . Then  $y = x \cdot \frac{y}{x}$ . But  $x \cdot \frac{y}{x} = \frac{x \cdot y}{x} = \frac{0}{x} = 0$ . So y = 0. Hence we have proved that  $\sim x = 0 \Rightarrow y = 0$ , which is equivalent to  $x = 0 \lor y = 0$ . **END** 

**IV.** Claim: For real numbers x and y, if  $x \cdot y \ge 0$  then  $\sqrt{x^2 + y^2} \le x + y$ . Proof: Squaring both sides of  $\sqrt{x^2 + y^2} \le x + y$  we get  $x^2 + y^2 \le (x + y)^2$ . But  $(x + y)^2 = x^2 + y^2 + 2 \cdot x \cdot y$ , so we got  $x^2 + y^2 \le x^2 + y^2 + 2xy$ , which is true because  $x \cdot y \ge 0$ .

Answer: The grade is  $\mathbf{F}$ . The statement is false (for example, take x = -1, y = -1), and the proof is completely wrong, because it begins by assuming the conclusion, that  $\sqrt{x^2 + y^2} \le x + y$ .

#### Problem 6.

- (i) For each of the following four statements: (a) rewrite the statement in plain English, without letter symbols or any mathematical symbol;(b) indicate whether the statement is true or false (no proof necessary).
  - 1.  $(\forall x \in \mathbb{Z}) (\exists y \in \mathbb{Z}) (y < x)$

**Translation.** For every integer there exists a strictly smaller integer. TRUE.

2.  $(\exists y \in \mathbb{Z}) (\forall x \in \mathbb{Z}) (y \le x)$ 

*Translation.* There exists a smallest integer. FALSE.

3.  $(\exists y \in \mathbb{N}) (\forall x \in \mathbb{N}) (y \le x)$ 

**Translation.** There exists a smallest natural number. TRUE.

4.  $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\forall z \in \mathbb{Z})(x \cdot z = y \cdot z \Longrightarrow x = y)$ 

**Translation.** If the results of multiplying two integers by a third integer are equal, then the two integers are equal. FALSE. (Take x = 3, y = 21, z = 0.)

- (ii) For each of the following four statements: (a) rewrite the statement in formal language, using the basic vocabulary of arithmetic (that is, the parentheses "(" and ")", the logical connectives "∨", "∧", "~", "⇒", "⇔", "⇔", "∃", and "∀", letter variables such as n, p, q, x, y, z, a, b,, etc., the predicates "∈ N", "∈ Z" and "∈ ℝ", the symbols 0, 1, +, -, ·, =, <, >, ≤, ≥), plus the predicate "is prime", and **nothing else**. (b) indicate whether the statement is true or false (no proof necessary).
  - 5. Every real number has a square root.

**Translation:**  $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y \cdot y = x)$ . FALSE.

6. There exists a smallest nonnegative real number.

**Translation:**  $(\exists x \in \mathbb{R})(x \ge 0 \land (\forall y \in \mathbb{R})(y \ge 0 \Rightarrow y \ge x)).$ TRUE

7. Every positive integer is the sum of the squares of three integers. *Translation:* 

 $(\forall n \in \mathbb{Z})(n > 0 \Rightarrow (\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(p \cdot p + q \cdot q + r \cdot r = n)).$ FALSE. (Take n = 7.) 8. The product of two prime numbers is not prime.

#### Translation:

 $(\forall p \in \mathbb{Z})(\forall q \in \mathbb{Z})((p \text{ is prime} \land q \text{ is prime}) \Rightarrow \sim p \cdot q \text{ is prime}).$ TRUE.

**Problem 7.** Prove the following:

For every natural number n,  $\sum_{k=1}^{n} (2k-1) = n^2$ .

(You may use well-ordering or induction, or even give a direct proof that uses neither, if you remember what was said in class about C. F. Gauss.)

**Proof using well-ordering.** Call a natural number n "bad" if it is not true that  $\sum_{k=1}^{n} (2k-1) = n^2$ . We want to prove that there are no bad natural numbers. Suppose there is a bad natural number. Then the well-ordering principle tells us that there exists a smallest bad natural number. Call this number s. Then  $s \in \mathbb{N}$  and the equality  $\sum_{k=1}^{s} (2k-1) = s^2$  is not true. Furthermore,  $\sum_{k=1}^{n} (2k-1) = n^2$  for every  $n \in \mathbb{N}$  such that n < s. Now, the equality  $\sum_{k=1}^{n} (2k-1) = n^2$  is true for n = 1, because  $\sum_{k=1}^{1} (2k-1) = 1$  and  $1^2 = 1$ . So 1 is not bad, and then  $s \neq 1$ . Since  $s \in \mathbb{N}$ , we have s > 1, and then  $s - 1 \in \mathbb{N}$  and s - 1 is not bad. Therefore  $\sum_{k=1}^{s-1} (2k-1) = (s-1)^2$ , and then

$$\sum_{k=1}^{s} (2k-1) = 2s - 1 + \sum_{k=1}^{s-1} (2k-1) = (s-1)^2 + 2s - 1 = s^2 - 2s + 1 + 2s - 1 = s^2.$$

So  $\sum_{k=1}^{s} (2k-1) = s^2$ , and then s is not bad. But s is bad. So s is not bad and s is bad. This is a contradiction, and we have proved that no bad numbers can exist. **END** 

#### Problem 8.

a. Prove that the product of two rational numbers is rational.

**Proof:** Let x, y be arbitrary rational numbers. The definition of "rational number" says that

$$(\forall u \in \mathbb{R})(u \text{ is rational} \Leftrightarrow (\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})(\sim n = 0 \land u = \frac{m}{n})).$$

Since x and y are rational, we may pick integers a, b, c, d such that  $\sim b = 0, \ \sim d = 0, \ x = \frac{a}{b}$ , and  $y = \frac{c}{d}$ . Then  $xy = \frac{ac}{bd}$ , so xy is rational. **END** 

b. Prove that  $\sqrt{2}$  is irrational.

**Proof:** Assume  $\sqrt{2}$  was rational. Then we may pick integers a, b such that  $\sim b = 0$  and  $\sqrt{2} = \frac{a}{b}$ . After "eliminating all common factors" from a and b, we may assume that a and b have no common factors. In particular, a and b cannot both be even. On the other hand,  $2b^2 = a^2$ . Hence  $a^2$  is even, so a is even, because if a was odd then  $a^2$  would be odd. So we may pick an integer c such that a = 2c. Then  $a^2 = 4c^2$ . So  $2b^2 = 4c^2$ , and then  $b^2 = 2c^2$ , so  $b^2$  is even and then b is even. So we have shown that

a is even and b is even and a and b are not both even.

This is a contradiction, proving that  $\sqrt{2}$  is irrational. **END** 

c. Prove that the product of two irrational numbers is irrational.

**Answer:** This cannot be proved because it is false. (Proof that it is false: let  $x = \sqrt{2}$ ,  $y = \frac{1}{\sqrt{2}}$ . Then x and y are irrational, but the product  $x \cdot y$  is equal to 1, which is rational.

d. Prove that  $\sqrt{12}$  is irrational.

**Proof:** Assume  $\sqrt{12}$  was rational. Then we may pick integers a, b such that  $\sim b = 0$  and  $\sqrt{12} = \frac{a}{b}$ . After "eliminating all common factors" from a and b, we may assume that a and b have no common factors. In particular, a and b cannot both be divisible by 3. On the other hand,  $12b^2 = a^2$ . Hence  $a^2 = 3 \times (4b^2)$ , so  $a^2$  is divisible by 3, so a is divisible by 3, because if a was not divisible by 3 then  $a^2$  would be not be divisible by 3 either. (The general fact we are using is this: if p is prime and a product mn of integers is divisible by p, then m or n must be divisible by p.) So we may pick an integer c such that a = 3c. Then  $a^2 = 9c^2$ . So  $12b^2 = 9c^2$ , and then  $4b^2 = 3c^2$ , so  $4b^2$  is divisible by 3. Since 4 is not divisible by 3, it follows that  $b^2$  is divisible by 3, and then b is divisible by 3. So we have shown that

a is divisible by 3, b is divisible by 3, and a and b are not both divisible by 3.

This is a contradiction, proving that  $\sqrt{12}$  is irrational. END

Problem 9.

 $\begin{array}{l} P \Longrightarrow (Q \Longrightarrow (R \Longrightarrow (S \Longrightarrow (P \land Q \land R \land S)))) \text{ is a tautology.} \\ (P \land (\sim P)) \Longrightarrow Q \text{ is a tautology.} \\ P \Longrightarrow (\sim P) \text{ is a contingency.} \\ P \land (\sim P) \text{ is a contradiction.} \\ P \lor (\sim P) \text{ is a tautology.} \\ ((\sim P) \land (\sim Q)) \land (P \lor Q) \text{ is a contradiction.} \end{array}$ 

## 14 Integer arithmetic

So far, we have discussed in a scattered way some facts about integers (for example, we have defined "even" and "odd" integers, and we have proved some properties of these concepts).

Now we want to study the integers in a more systematic way, using our proof techniques. We will start with the familiar fact that you can always *divide* an integer a by a nonzero natural number b, and get a *quotient* q and a *remainder* r. The precise meaning of this is that there exist integers q, r such that  $a = q \cdot b + r$ , and  $0 \le r < b$ .

# 14.1 The division theorem (a.k.a. the quotient and remainder theorem)

We are now going to prove something that you already know, and we are going to be careful and give a correct proof using well-ordering. You know that if you take two natural numbers then you can "divide the first number by the second number and find the quotient and the remainder." For example, "103 divided by 19 is 5 with a remainder of 8." What this means, precisely, is that  $103 = 19 \times 5 + 8$ . In this case, 5 is the **quotient** and 8 is the **remainder**. Here is a second example: suppose we want to divide 28 by 7. Of course, the answer is  $28 = 7 \times 4$ , that is,  $28 = 7 \times 4 + 0$ . So the quotient is 4, an the remainder is 0. And here we see that there is a small problem. In our second example, the remainder is **not** a natural number, because it is 0, and in these notes (following theb book) 0 is not a natural number. So we will have to allow for a remainder that need not be a natural number, because it could be zero. We are going to do this by working in  $\mathbb{Z}_+$ , the set consisting of all the natural numbers and the number 0. So  $\mathbb{Z}_+$  is "almost the same" as  $\mathbb{N}$ , except only that  $0 \in \mathbb{Z}_+$  but  $\sim 0 \in \mathbb{N}$ .

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The general situation we will deal with is as follows: suppose we have an integer a and a natural number b, and we want to "divide a by b with a quotient q and a remainder r." What does this mean? Well, it means that we want to find a integers q (the "quotient") and r (the "remainder") such that  $a = b \cdot q + r$ , and  $0 \le r < b$ . For example:

- if a = 103 and b = 19, then q = 5 and r = 8;
- if a = 28 and b = 7, then q = 4 and r = 0;
- if a = -33 and b = 7, then q = -5 and r = 2;
- if a = -105 and b = 10, then q = -11 and r = 5.

Here is the theorem that tells us that q and r always exist.

**Theorem 6.** Let a be an integer and let b be a natural number. Then there exist integers q, r such that  $a = b \cdot q + r$  and  $0 \le r < b$ . (If you prefer a more formal statement, here it is:  $(\forall a \in \mathbb{Z})(\forall b \in \mathbb{N})(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})$  $(a = b \cdot q + r \land (0 \le r \land r < b))$ .)

*Proof.* We fix a (an arbitrary integer) and b (an arbitrary natural number) and set out to prove that  $(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = b \cdot q + r \land (0 \le r \land r < b))$ .

COMMENT (not part of the proof): To prove our theorem, we will look at all the numbers r that can appear in an equality  $a = b \cdot q + r$ , for some integer q, and try to make r as small as possible, while keeping it nonnegative. For example, suppose we want to find q and r if a = 103 and b = 19. Then one obvious possible choice of q, r is q = 1, r = 84, because in that case  $b \cdot q + r = 19 \cdot 1 + 84 = 19 + 84 = 103 = a$ . These q, r are not, however, good enough for us, because r is too big. In fact,  $r \ge b$ , and this very fact tells us how to make r smaller. We can subtract 19 from r, and add 19 to the  $b \cdot q$  term. To add 19 to the  $b \cdot q$  term, we add 1 to q, so we take q = 2 instead of q = 1. On the other hand, when we subtract 19 from r, the number we get is still nonnegative, because r was  $\geq 19$ . In fact, our new value of r is 84-19, i.e., 65. And we now have  $b \cdot q + r = 19 \cdot 2 + 65 = 38 + 65 = 103 = a$ . This is better than before (because r is smaller) but is not yet good enough, because r is still too big, in fact  $r \geq b$ . So we make r smaller again: we subtract 19 from r, and add 19 to the  $b \cdot q$  term. To add 19 to the  $b \cdot q$  term, we add 1 to q, so we take q = 3 instead of q = 2. On the other hand, when we subtract 19 from r, the number we get is still nonnegative, because r was  $\geq 19$ . In fact, our new value of r is 65 - 19, i.e., 46. And we now have  $b \cdot q + r = 19 \cdot 3 + 46 = 57 + 46 = 103 = a$ . This is, again, better than before, because r is smaller, but not yet what we want, since  $r \geq b$ . So we subtract 19 from r again, and add 19 to the  $b \cdot q$  term. To add 19 to the  $b \cdot q$  term, we add 1 to q, so we now take q = 4instead of q = 3. When we subtract 19 from r, our new value of r is 46-19, i.e., 27. And we now have  $b \cdot q + r = 19 \cdot 4 + 27 = 76 + 27 = 103 = a$ . This is even better than before, because r is smaller, but we are not there yet, since r is still  $\geq b$ . So we subtract 19 from r one more time, and add 19 to the  $b \cdot q$  term. To add 19 to the  $b \cdot q$  term, we add 1 to q, so we now take q = 5 instead of q = 4. When we subtract 19 from r, our new value of r is 27 - 19, i.e., 8. And we now have  $b \cdot q + r = 19 \cdot 5 + 8 = 95 + 8 = 103 = a$ . This is even better than before, because r is smaller. Furthemore, now we got where we wanted, because  $0 \le r < b$ . So at this point we can stop our process of succesive subtractions.

Our general strategy is going to be the same as in the special example above. We will start with **some** nonnegative integer r such that  $r = a - b \cdot q$  for some integer q. If r < b, then  $0 \le r \land r < b$ , so got what we wanted. If  $r \ge b$ , then we can produce a new r (and a new q, of course) by subtracting b from r and adding 1 to q. This new r will still be  $\ge 0$ , because our old r was  $\ge b$ , but it may or may not be ,b. If it is < b, we repeat the procedure. We go on like this until we get an r such that  $0 \le r < b$ . This must happen eventually, because the successive subtractions cannot go on for ever, since the r's are all nonnegative integers, and at each step r gets smaller.

On the other hand, this talk about the procedure that "cannot go on for ever" is not rigorous and precise as it is. To make it rigorous, we will use well-ordering. Instead of talking about "repeating the same step over and over again until we have to stop," we will simply take the smallest r such that  $r = a - b \cdot q$  for some q.

There remain, however, two difficulties to be overcome. The first one is sort of artificial, caused by the fact that, according to the book, 0 is not a natural number, while, on the other hand, the number r that we are trying to find could be zero. So our r could fail to be a natural number. We will take care of this problem by writing  $\rho = r + 1$  and trying to find  $\rho$  rather than r. Since  $\rho$  will always be a natural number, even when r = 0, this should work. So (1) we are going to work with the predicate P(u) defined by letting P(u)stand for " $u \in \mathbb{N} \land (\exists q \in \mathbb{Z})(a = b \cdot q + u - 1)$ ", (2) we will take  $\rho$  to be the smallest usuch that P(u), and (3) we will take  $r = \rho - 1$ .

The second difficulty has to do with what the well-ordering principle allows us to do, precisely. In order to guarantee that there exists a **smallest** u, we need to know that there exists at **least one** u. How can we know that? If we knew that  $a \ge 0$ , this would be easy, because  $a = b \cdot 0 + a$ , so if we take  $u_* = a + 1$  and  $q_* = 0$ , then  $u_* \in \mathbb{N} \land a = b \cdot q_* + u_* - 1$ , so  $(\exists q \in \mathbb{Z})(u_* \in \mathbb{N} \land a = b \cdot q + u_* - 1)$ , and then  $P(u_*)$ , so  $(\exists u)P(u)$ . Take, for example, a = -321, b = 23. What we can do in this case is add 23 to a many times, until we get a positive number. We could, for instance, add  $20 \times 230$ , i.e., 460, to a, so that  $-321 + 20 \times 23 = -321 + 20 \times 23$ , and then, if we let q = -20, we have  $a - bq = -321 - (-20 \times 23) = -321 + 460 = 139$ . So, if we take q = -20, r = 139, then  $a = b \cdot q + r$ ,  $q \in \mathbb{Z}$ , and  $r \in \mathbb{Z}_+$ . We can then let u = 140—that is, u = r + 1, and we have a natural number u and an integer q such that  $a = b \cdot q + u - 1$ . END OF COMMENT

Let P(u) be the predicate " $u \in \mathbb{N} \land (\exists q \in \mathbb{Z})(a = b \cdot q + u - 1)$ ". (Recall that a and b are fixed, so this is a one-variable predicate, the free variable being u.)

We want to apply well-ordering and find a smallest  $u \in \mathbb{N}$  such that P(u) is true. For this purpose, we need to know that  $(\exists u \in \mathbb{N})P(u)$ . So we prove this first.

**Lemma 1.**<sup>19</sup> Let  $x \in \mathbb{Z}$ ,  $y \in \mathbb{N}$ . Then there exists an integer q such that  $x - y \cdot q \ge 0$ . (That is,  $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{N})(\exists q \in \mathbb{Z})x - y \cdot q \ge 0$ .)

Proof. Take  $\bar{q} = -|x|$ . Then  $\bar{q} \in \mathbb{Z}$ , and  $x - y \cdot \bar{q} = a - b \cdot (-|a|) = x + y \cdot |a|$ . But we know that  $-x \leq |x|$ , and on the other hand  $|x| \leq y \cdot |x|$  (because  $|x| \geq 0$ and  $y \geq 1$ , so  $|x| = 1 \cdot |x| \leq y|x|$ ). Hence  $-x \leq y \cdot |x|$ , so  $-x - y \cdot |x| \leq 0$ , and then  $x + y \cdot |x| \geq 0$ , so  $x - y \cdot (-|x|) \geq 0$ , and then, finally,  $x - y \cdot \bar{q} \geq 0$ . So  $(\exists q \in \mathbb{Z})x - y \cdot q \geq 0$ . **END OF THE PROOF OF THE LEMMA** 

We now return to the proof of our theorem. Apply the lemma with x = a, y = b. Then there exists a  $q \in \mathbb{Z}$  such that  $a - b \cdot q \ge 0$ . Pick one such q and call it  $q_*$ . Then  $q_* \in \mathbb{Z}$  and  $a - b \cdot q_* \ge 0$ . Let  $u_* = 1 + a - b \cdot q_*$ . Then  $u_* \in \mathbb{Z}$ , and  $u_* \ge 1$ , because  $a - b \cdot q_* \ge 0$ . So  $u_* \in \mathbb{N}$ . Furthermore,  $a - b \cdot q_* = u_* - 1$ , so  $a = b \cdot q_* + u_* - 1$ . Hence  $(\exists q \in \mathbb{Z})a = b \cdot q + u_* - 1$ . Therefore  $u_* \in \mathbb{N} \land (\exists q \in \mathbb{Z})a = b \cdot q + u_* - 1$ . So  $P(u_*)$  is true. Hence  $(\exists u \in \mathbb{N})P(u)$ .

Since  $(\exists u \in \mathbb{N})P(u)$ , we can apply the well-ordering principle (that is, Axiom NZ12) and conclude that there exists a smallest  $u \in \mathbb{N}$  such that P(u). Pick one such u and call it  $\bar{u}$ . Then  $\bar{u} \in \mathbb{N}$ ,  $P(\bar{u})$  is true, and in addition  $\bar{u}$  is the smallest  $u \in \mathbb{N}$  such that P(u), which means that  $(\forall u \in \mathbb{N})(P(u) \Rightarrow \bar{u} \leq u)$ .

Since  $P(\bar{u})$  is true, we can conclude that  $\bar{u} \in \mathbb{N} \land (\exists q \in \mathbb{Z})a = b \cdot q + \bar{u} - 1$ . In particular,  $(\exists q \in \mathbb{Z})a = b \cdot q + \bar{u} - 1$ . So we may pick one such q and call it  $\bar{q}$ . Then  $\bar{q} \in \mathbb{Z} \land a = b \cdot \bar{q} + \bar{u} - 1$ .

Let  $\bar{r} = \bar{u} - 1$ . Then  $\bar{r} \in \mathbb{Z}$ , because  $\bar{u} \in \mathbb{Z}$ . Furthermore,  $\bar{r} \ge 0$ , because  $\bar{u} \ge 1$ . And, finally,  $a = b \cdot \bar{q} + \bar{r}$ .

<sup>&</sup>lt;sup>19</sup>A *lemma* is a statement we proof as a premilinary towards proving some more important result that truly interests us. For example, in the situation that we find ourselves in, we need to know that there exists  $q \in \mathbb{Z}$  such that  $a - b \cdot q \ge 0$ , so we prove that first. The fact that q exists is not of interest to us, except only that we need it to prove our theorem. So we prove this fact as a lemma, we use it in the proof of the theorem, and then we forget about it.

COMMENT: We are now going to prove that  $\bar{r} < b$ . We will do this by contradiction. That is, we will assume that  $\sim \bar{r} < b$  and try to get a contradiction.

Suppose that  $\sim \bar{r} < b$ . Then  $\bar{r} \geq b$ .

Let  $\tilde{r} = \bar{r} - b$ , and let  $\tilde{q} = \bar{q} + 1$ . Then  $\tilde{r} \ge 0$  (because  $\bar{r} \ge b$ , so  $\bar{r} - b \ge 0$ ),  $\tilde{q} \in \mathbb{Z}$ , and

$$a = b \cdot \bar{q} + \bar{r} = b \cdot (\tilde{q} - 1) + \tilde{r} + b = b \cdot \tilde{q} - b + \tilde{r} + b = b \cdot \tilde{q} + \tilde{r}.$$

So  $a = b \cdot \tilde{q} + \tilde{r}$ . Let  $\tilde{u} = \tilde{r} + 1$ . Then  $\tilde{u} \in \mathbb{Z}$ ,  $\tilde{u} \ge 1$ , and  $a = b \cdot \tilde{q} + \tilde{u} - 1$ . Since  $\tilde{q} \in \mathbb{Z}$ , we have  $\tilde{q} \in \mathbb{Z} \land a = b \cdot \tilde{q} + \tilde{u} - 1$ , so  $(\exists q \in \mathbb{Z})a = b \cdot \tilde{q} + \tilde{u} - 1$ . Also,  $\tilde{u} \in \mathbb{N}$  (because  $\tilde{u} \in \mathbb{Z}$  and  $\tilde{u} \ge 1$ ), so  $P(\tilde{u})$  is true.

Hence  $a = b \cdot \tilde{q} + \tilde{u} - 1$ . Since  $\tilde{q} \in \mathbb{Z}$ , we can conclude that

 $(\exists q \in \mathbb{Z})a = b \cdot q + \tilde{u} - 1.$ 

Since  $\tilde{u} \in \mathbb{N}$ , it follows that  $P(\tilde{u})$  is true.

On the other hand,  $\tilde{u} = \tilde{r} + 1 = (\bar{r} - b) + 1 = (\bar{r} + 1) - b = \bar{u} - b$ , so  $\tilde{u} < \bar{u}$ . Since  $\tilde{u} \in \mathbb{N}$ ,  $\tilde{u} < \bar{u}$ , and  $P(\tilde{u})$ , we have shown that  $\bar{u}$  is not the smallest u such that P(u). But this contradicts the fact that  $\bar{u}$  is the smallest u such that P(u). This contradiction was derived by assuming that  $\sim \bar{r} < b$ . Hence  $\bar{r} < b$ .

We already know that  $a = b \cdot \bar{q} + \bar{r}, \bar{r} \ge 0, \bar{r} < b$  and  $\bar{r} \in \mathbb{Z}$ . Hence

$$(\exists r \in \mathbb{Z})(a = b \cdot \bar{q} + r \land (0 \le r \land r < b)).$$

Since we also know that  $\bar{q} \in \mathbb{Z}$ , we can conclude that

$$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = b \cdot q + r \land (0 \le r \land r < b)),$$

which is exactly our desired conclusion. END OF THE PROOF

### 14.2 Divisibility

**Definition 3.** Let x, y be integers. We say that x divides y, or that y is divisible by x, if there exists an integer k such that  $y = x \cdot k$ .

We can say the same thing in formal language:

$$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x \text{ divides } y \Leftrightarrow (\exists k \in \mathbb{Z})(y = x \cdot k)).$$

or

$$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(y \text{ is divisible by } x \Leftrightarrow (\exists k \in \mathbb{Z})(y = x \cdot k)).$$

Instructor's Notes, March 13, 2006

We can even go one step further and introduce a symbolic notation for the two-variable predicate "x divides y". We agree to use "x|y" to stand for "x divides y". Then we can write

$$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x | y \Leftrightarrow (\exists k \in \mathbb{Z})(y = x \cdot k)).$$

**Warning.** If I ask you to define the symbol "|", then your answer should be " $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x|y \Leftrightarrow (\exists k \in \mathbb{Z})(y = x \cdot k))$ ", or, if you prefer:

Let x, y be integers. Then "x|y" means " $(\exists k \in \mathbb{Z})(y = x \cdot k)$ ".

You can also say

Let x, y be integers. Then "x|y" means "there exists an integer k such that  $y = x \cdot k$ .

Or you could say

Let x, y be integers. Then "x|y" means " $y=x \cdot k$  for some integer k".

You can even say the following, if you do not want to use letter variables other than x and y:

Let x, y be integers. Then "x|y" means "y is equal to x times some integer".

On the other hand, if I ask you to define the word "divides", you can say something like

 $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x \text{ divides } y \Leftrightarrow (\exists k \in \mathbb{Z})(y = x \cdot k)),$ 

or

If x, y are integers, we say that x **divides** y if  $(\exists k \in \mathbb{Z})(y = x \cdot k))$ ,

or, perhaps,

If x, y are integers, we say that x *divides* y if  $y = x \cdot k$  for some integer k.

But it would **not** be correct to answer the question "Define 'divides'" by saying

(\*) Let x, y be integers. Then "x|y" means " $y = x \cdot k$  for some integer k".

Why not? (*Think* before you read the answer!)

The answer is quite simple. If I ask you to define "divides" I want you to tell me what "divides" means, so the word "divides" has to appear in your definition. If your answer is (\*), then you are telling me what "|" means, not what "divides" means. Naturally, you will argue that "|" and "divides" mean the same thing. But, how on Earth am I suppose to know that if you do not tell me?

Often, what people try to achieve when they give a definition, is to tell the reader two things, namely: (a) what a new word of phrase means, and (b) how this word or phrase is abbreviated in symbolic language (if such an abbreviation exists). My advice is: **you should do the same**. That is, when I ask you the question was "Define 'divides'", I recommend that you answer

If x, y are integers, we say that x divides y if  $y = x \cdot k$  for some integer k. The mathematical notation for "x divides y" is "x|y".

And when I ask you to define the symbol | you can say exactly the same thing, because what we have just written tells the reader both what the word "divides" means and what the symbol | means. You could also say it in lots of other ways, for example:

If x, y are integers, we write "x|y" if  $y = x \cdot k$  for some integer k. We read "x|y" as "x divides y", or "y is divisible by x".

**Another warning.** Students sometimes confuse the expression "x|y" with the fraction  $\frac{x}{y}$ . These two things are **totally different**. Indeed,

"x|y" is a (two-variable) predicate. When you plug in particular numbers for x and y, "x|y" becomes a proposition, or statement, which can be true or false. (For example, "3|6" is a true statement, and "3|7" is a false statement.)

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•  $\frac{x}{y}$  is a **term** that stands for a **number**. When you plug in particular real numbers a, b for x and y,  $\frac{x}{y}$  becomes a **number** (if  $\sim b = 0$ ). In particular, if a and b are integers, and  $b \neq 0$ , then  $\frac{a}{b}$  is a **rational number**. (You probably learned to call those things "fractions". That is O.K., as long as you make it clear that by "fraction" you mean "a real number of the form  $\frac{a}{b}$ , where  $a \in \mathbb{Z}, b \in \mathbb{Z}$ , and  $b \neq 0$ . But if by "fraction" you also means something like  $\frac{3}{\sqrt{2}}$ , or  $\frac{\pi}{25}$ , then your notion of "fraction" is different, and when you mean "rational number" you have to say "rational number".

So, you see, "x|y" and " $\frac{x}{y}$ " couldn't be more totally different.

A third warning. I know that what I am going to discuss now is a mistake that you would never make, but some students have made it in the past, incredible as this may seem, so I am bringing it to your attention. Please do not be offended if you think that this mistake is too ridiculous for me to even think that you could make it. I **don't** think you can possibly make such a mistake, but you wouldn't believe the kinds of incredible things that people write sometimes, so I want you to be aware of this.

Once, a long time ago, a student actually wrote this: 6|3 + 5 = 7. This is of course absurd, because "6|3" is a statement (which happens to be true, but that's irrelevant), not a number, so it does not make any sense to take the sentence "6|3" and add to it the number 5. (That would be like taking the sum of the sentence "My uncle Jimmy is very smart" and the number 38. The result is the following idiotic, unintelligible string of symbols: "My uncle Jimmy is very smart+38". You obviously realize that this is totally meaningless, don't you?)