OPTIMAL CONTROL+

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§1. Introduction .

Optimal Control Theory (OCT), in its modern sense, began in earnest in the 1950's, with the formulation and proof of the Pontryagin Maximum Principle (PMP), cf. [P], [Be], [LM]. The PMP is a far-reaching generalization of the Euler-Lagrange equations from the classical Calculus of Variations, and the development of OCT has proceeded, to a large extent, as an outgrowth of the Calculus of Variations, from which it has borrowed many of its themes, problems and methods. More recently, techniques based on ideas from "Geometric Control Theory" have been applied, leading to new and exciting results. This paper is an attempt to survey some of the recent developments in OCT, with a special emphasis on the contribution made by these new techniques.

We will deal exclusively with *finite-dimensional, deterministic* optimal control. In §2 we describe some of the basic problems of OCT. In §3 we discuss results on properties of optimal trajectories, including results on necessary conditions for optimality, and the important particular case of Local Controllability. In §4 we discuss results on optimal synthesis and the value function.

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§2. The basic problems of optimal control theory .

Throughout this paper, we deal with control systems

$$\dot{x} = f(x,u) \ , \ x \in \Omega \ , \ u \in U \tag{2.1}$$

where Ω is an open subset of $\mathbb{R}^n$, U is a set, and $f : \Omega \times U \to \mathbb{R}^n$ is of class C^1 in x for each fixed u. A *control* for (2.1) is a function $\eta : I \to U$ whose domain $I = \mathrm{Dom}(\eta)$ is an interval of the real line. The control η is *f-admissible* if the time-varying vector field $f_\eta : \Omega \times I \to \mathbb{R}^n$ given by $f_\eta(x,t) = f(x,\eta(t))$ satisfies the conditions of the Carathéodory existence and uniqueness theorem of ordinary differential equations, that is: (i) f_η is jointly measurable in x and t, and (ii) for each compact interval $J \subseteq I$ and each compact $K \subseteq \Omega$ there exists a Lebesgue integrable function $\varphi : J \to \mathbb{R}$ such that

$$\|f_\eta(x,t)\| + \|D_x f_\eta(x,t)\| \le \varphi(t) \ \text{ for all } \ x \in K \, , \, t \in J \ . \tag{2.2}$$

A *trajectory* for a control $\eta : I \to U$ is an absolutely continuous curve $\gamma : I \to \Omega$ such that $\dot{\gamma}(t) = f(\gamma(t),\eta(t))$ for almost all $t \in I$. The Carathéodory theorem (cf., e.g., [CLe]), implies the local existence and global uniqueness of trajectories, that is: if η is f-admissible, then (i) given $\bar{t} \in I$, $\bar{x} \in \Omega$, there exists $\varepsilon > 0$ with the property that there is a trajectory γ for the restriction of η to the interval $(\bar{t}-\varepsilon, \bar{t}+\varepsilon) \cap I$, such that $\gamma(\bar{t}) = \bar{x}$, and (ii) if γ and δ are trajectories for η such that $\gamma(\bar{t}) = \delta(\bar{t})$ for some $\bar{t}$ in I, then $\gamma \equiv \delta$. If η is f-admissible and γ is a trajectory for η, then (γ,η) is an *f-admissible pair.*

It follows from our definitions that *every piecewise constant control is admissible.* Moreover, the following important *Piecewise Constant Approximation Theorem* (PCAT) holds: if (γ,η) is an admissible pair, $\mathrm{Dom}(\eta)$ is compact, $\bar{t} \in \mathrm{Dom}(\eta)$, $\bar{x} = \gamma(\bar{t})$, $x_j \in \Omega$, $x_j \to \bar{x}$ as $j \to +\infty$, then there exist piecewise constant controls $\eta_j : \mathrm{Dom}(\eta) \to U$ and corresponding trajectories γ_j such that $\gamma_j(\bar{t}) = x_j$ and $\gamma_j \to \gamma$ uniformly as $j \to \infty$. (For a proof, cf. [Su14].)

A control η *starts* at time t_0 if $t_0 \in \mathrm{Dom}(\eta)$ and $\mathrm{Dom}(\eta) \subseteq [t_0,\infty)$. In that case, if γ is a trajectory for η, then the point $x_0 = \gamma(t_0)$ is the *initial point* of γ. The definition of what it means for a control to *end* at a time t_1, and for a point x_1 to be the *terminal point* of a trajectory, is completely analogous. If x_0 and x_1 are the initial and terminal points of a trajectory γ that corresponds to a control η, then we say that γ *goes from* x_0 to x_1, or that η *steers* x_0 to x_1. We want to consider more general terminal conditions, such as the requirement that $\gamma(t) \to 0$ as $t \to \infty$. So we define a *terminal condition on trajectories* (TCT) to be, simply, a set τ of Ω-valued curves with the property that, if two curves γ, δ coincide from a certain time on, then $\gamma \in \tau$ iff $\delta \in \tau$. If τ is a TCT and $\gamma \in \tau$, then we say that γ *satisfies* τ. A particular example of TCT is the *point terminal condition* τ_x, i.e. the set of all curves that end at x. A *target* is a set $\mathcal{T}$ of pairwise disjoint TCT's. A curve γ *reaches the target* $\mathcal{T}$ (or *hits* $\mathcal{T}$, or *ends at* $\mathcal{T}$, or *gets to* $\mathcal{T}$) if γ satisfies some $\tau \in \mathcal{T}$.

We consider an optimal control problem $\mathcal{P} = (\Omega, U, f, L, \mathcal{T}, \varphi)$, given by the specification of (i) a control system (2.1), (ii) a function $L : \Omega \times U \to \mathbb{R}$, called the *Lagrangian*, (iii) a target $\mathcal{T}$, and (iv) a function $\varphi : \mathcal{T} \to \mathbb{R}$. We require that L be of class C^1 in x

for each fixed u. If $\eta : I \to U$ is an admissible control, and $\gamma : I \to \Omega$ is a trajectory for η that satisfies the TCT $\tau \in \mathcal{T}$, then we would like to define the *cost* of (γ, η) by

$$J(\gamma,\eta) = \int_I L(\gamma(t),\eta(t))\,dt + \varphi(\tau) \ . \tag{2.3}$$

However, the function $t \to L(\gamma(t),\eta(t))$ may fail to be integrable (or even measurable) unless some extra technical hypotheses hold. So we need to restrict further the set $\mathrm{Dom}(J)$ of those pairs (γ,η) for which $J(\gamma,\eta)$ is defined. For instance, we may take $\mathrm{Dom}(J)$ to be the set of those f-admissible (γ,η) such that $\gamma \in \tau$ for some $\tau \in \mathcal{T}$ and the integrand $h(t)$ of (2.3) has a well defined (but possibly infinite) Lebesgue integral (i.e. h is measurable and at least one of the integrals $\int_I h_+$, $\int_I h_-$ is finite, where $h_+ = \max(h,0)$ and $h_- = h_+ - h$). If, as is often the case, the Lagrangian is nonnegative, and the f-admissibility of (γ,η) suffices to imply that the function h is measurable, then this amounts to letting $\mathrm{Dom}(J)$ be the set of *all* f-admissible pairs (γ,η). A second possibility is to require η to be (f,L)-*admissible*, i.e. admissible for the *augmented control system* obtained from (2.1) by adding a new variable y with equation $\dot{y} = L(x,u)$. In this case, the integrability of h is guaranteed if I is finite (e.g. if the target $\mathcal{T}$ consists entirely of point TCT's and η has a starting time). If I is allowed to be infinite, then we restrict ourselves even further and require (γ,η) to be such that the integral exists. We shall refer to the above two possibilities as "Formulations I and II of the Optimal Control Problem $\mathcal{P}$."

For $x \in \Omega$ we define $V(x)$ to be the infimum of $J(\gamma,\eta)$ taken over all $(\gamma,\eta) \in \mathrm{Dom}(J)$ such that γ starts at x and ends at the target $\mathcal{T}$. (If no such γ exists then of course $V(x) = +\infty$.) The function $V : \Omega \to \mathbb{R} \cup \{-\infty,+\infty\}$ is the *value function*, or *Bellman function*, of the problem $\mathcal{P}$. An *optimal pair* (often simply referred to as an "optimal trajectory") is a pair (γ,η) that has a starting point x and is such that $J(\gamma,\eta) = V(x)$.

We remark that *the choice of formulation* (i.e. the choice of $\mathrm{Dom}(J)$ as explained above) *may make a fundamental difference.* This is most dramatically illustrated by the "Lavrentieff phenomenon" (cf., e.g., [BM]). One can exhibit a nonnegative polynomial $L : \mathbb{R}^3 \to \mathbb{R}$ and real numbers x_0, x_1 such that, for the problem of minimizing $\int_0^1 L(x(t),\dot{x}(t),t)\,dt$ among all "curves" $t \to x(t)$ such that $x(0) = x_0$ and $x(1) = x_1$, the solution exists if by "curve" we mean "absolutely continuous function," and also if we mean "Lipschitz function." However, the values V_1, V_2 that correspond to these two interpretations satisfy $V_2 > V_1$. In control terms, this corresponds to the control system $\dot{x} = u$, $\dot{y} = 1$, with an obvious choice of Lagrangian. Requiring $x(\cdot)$ to be Lipschitz amounts to demanding that the control be measurable and bounded, in which case it is clearly (f,L)-admissible. Moreover, it follows easily from the PCAT (applied to the augmented system) that every trajectory $x(\cdot)$ for an (f,L)-admissible control can be approximated by Lipschitz trajectories $x_j(\cdot)$ *in such a way that the costs of the* x_j *converge to that of* x. Therefore V_2 is precisely the value for Formulation II. On the other hand, the condition that $x(\cdot)$ be absolutely continuous is precisely the requirement that the control be f-admissible, so V_1 is the value for Formulation I.

The choice of formulation is important, in particular, for the problem of *necessary conditions for optimality.* Indeed, the most basic of such conditions is the PMP, and the PMP is valid for Formulation II but may fail for Formulation I. For this reason, *from* *now on it will always be understood that the optimal control problem $\mathcal{P}$ is interpreted in the sense of Formulation II.*

For a control system (2.1) and a target $\mathcal{T}$, define a *presynthesis* to be a family $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ of admissible pairs such that, for each $x \in S$, γ_x starts at x and ends at the target. The set S is the *domain* of Γ. Call a presynthesis Γ *memoryless* if, whenever $x \in S$ and $t \in \mathrm{Dom}(\eta_x)$, it follows that $y = \gamma_x(t) \in \mathrm{Dom}(\Gamma)$ and η_y is precisely the restriction of η_x to the interval $\mathrm{Dom}(\eta_x) \cap [t, +\infty)$. A memoryless presynthesis Γ will be referred to as a *partial synthesis.* If the domain of Γ is the largest possible (i.e. the set of *all* points x such that there exists a trajectory that goes from x to the target), then Γ will be called a *synthesis*. If each pair (γ_x, η_x) is optimal for the control problem $\mathcal{P}$, then we call Γ an *optimal synthesis* for $\mathcal{P}$.

The following problems are of interest in Optimal Control Theory:

1. existence of optimal trajectories,
2. characterization of optimal trajectories,
3. properties of optimal trajectories,
4. existence of an optimal synthesis,
5. chracterization of optimal syntheses,
6. properties of optimal syntheses,
7. chracterization and properties of the value function.

§3. Optimal Trajectories .

The problem of the *existence* of optimal trajectories is very classical, and has been studied in great detail in the 1950's and 1960's, so we choose not to discuss it here, and refer the reader instead to the book [Be] by L.D. Berkovitz for a rather complete account of the results.

Regarding the *characterization* of optimal trajectories, it has been clear for a long time that, except for very special problems such as linear systems with a convex Lagrangian, the question of *sufficient conditions* is essentially hopeless. On the other hand, the search for *necessary conditions for optimality* (NCO's) has been vigorously and fruitfully pursued since the early days of the PMP. Since the PMP is, roughly, a "first order condition," it was natural to look for "high order conditions." Some classical results of this endeavor are described in the papers [GK], [Go], [JS], the book [BJ], and especially H.W. Knobloch's book [Kn], which gives a detailed and mathematically rigorous treatment based on a formalism of asymptotic expansions. More recently, several new techniques have been introduced, many of which are based on the systematic use of *Lie brackets* and *Lie algebras of vector fields.* In many cases, the purpose was to solve some of the other problems in our list. In particular, the *synthesis problem* has led —as explained below— to an interest in specific *properties of optimal trajectories*, and new NCO's had to be developed in order to establish these properties.

To illustrate why Lie brackets come into the picture, let us remark that, in order to obtain necessary conditions for a pair (γ, η) (with $\mathrm{Dom}(\eta)$ assumed to be a compact interval $[a, b]$) to be optimal, the time-honored method is to make a perturbation of η —called a *control variation*— and embed η in a one-parameter family $\{\eta_\varepsilon : 0 \leq \varepsilon \leq \bar{\varepsilon}\}$ of controls with $\mathrm{Dom}(\eta_\varepsilon)) = [a, b_\varepsilon]$. These η_ε will then give rise to trajectories $\gamma_\varepsilon : [a, b_\varepsilon] \to \Omega$ (with $\gamma_\varepsilon(a) = \gamma(a)$). If the variation is smooth in some appropriate sense, then the optimality of (γ,η) will imply that the derivative of $J(\gamma_\varepsilon,\eta_\varepsilon)$ with respect to ε at $\varepsilon=0$ has to be nonnegative. (We are ignoring here the extra difficulty that the perturbed trajectories must also satisfy the desired target conditions.) If we assume that we are already dealing with the augmented problem, as explained above, then the requirement becomes the condition that $v\cdot\nabla\psi(p)\geq 0$, where $p=\gamma(b)$, ψ is a function on the state space Ω, and v is the *variational vector*

$$v=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\gamma_\varepsilon(b_\varepsilon)\ . \tag{3.1}$$

So the vector v must belong to a particular subset of $\mathbb{R}^n$. When one applies this condition to a reasonably large class of variations, one obtains an NCO. In particular, the PMP is obtained by taking variations of a specially simple kind (cf., e.g. [P]), and more powerful NCO's arise by considering more sophisticated variations.

One is thus naturally led to the study of the set of all possible variational vectors. And it can be seen in a number of ways that *these vectors can be obtained as leading terms of asymptotic expansions that involve Lie brackets.* (This idea, as well as its use for high order NCO's, was proposed by [Bro], [AG2] and [Kr].) For a simple illustration of this situation let us consider a system of the form $\dot{x}=f(x)+ug(x)$, where f and g are C^∞ vector fields and the scalar control u takes values in $U=[-1,1]$. (This is called a *two vector field system*, because the possible directions of motion at a point p are $X(p)$, $Y(p)$, and their convex combinations, where $X=f-g$ and $Y=f+g$.) Assume moreover that a point p is an *equilibrium*, i.e. that $f(p)=0$. Then the constant trajectory $\gamma(t)\equiv p$ corresponds to the control $\eta(t)\equiv 0$. One can prove that, if $\eta:[0,T]\to[-1,1]$ is a general control, and γ_η is the corresponding trajectory with initial condition $\gamma_\eta(0)=p$, then the asymptotic expansion

$$\gamma_\eta(T)\sim\cdots e^{\alpha_3(\eta)b_3}e^{\alpha_2(\eta)b_2}e^{\alpha_1(\eta)b_1}p \tag{3.2}$$

holds, where (i) the b_i are vector fields obtained from certain formal Lie brackets B_i in the two indeterminates F and G by plugging in f for F and g for G, (ii) the B_i are a *P. Hall basis* (cf. [Se], [Vi]) of the free Lie algebra in the indeterminates F and G (for instance, one can take $B_1=F$, $B_2=G$, $B_3=[F,G]$, $B_4=[F,[F,G]]$, $B_5=[G,[F,G]]$, $\ldots$), (iii) the $\alpha_i(\eta)$ are *iterated integrals* calculated from η by explicit formulae (for instance, if the P. Hall basis is as indicated above, then $\alpha_1(\eta)=T$, $\alpha_2(\eta)=\int_0^T\eta(t)\,dt$, $\alpha_3(\eta)=\int_0^T t\,\eta(t)\,dt$, $\alpha_4(\eta)=\frac{1}{2}\int_0^T t^2\eta(t)\,dt$, $\alpha_5(\eta)=\int_0^T\int_0^t t\,\eta(t)\,\eta(s)\,ds\,dt$), (iv) the exponential notation stands for the *flow*, i.e. $t\to e^{tV}p$ is the integral curve of the vector field V that goes through p at time $t=0$, (iv) "$\sim$" stands for "asymptotic equality," in the sense that, if P_k denotes the right-hand side of (3.2) with all the B_i of degree $>k$ omitted, then $\gamma_\eta(T)-P_k=\mathrm{O}(T^{k+1})$ as $T\to 0$, uniformly in η. (The right-hand side of (3.2) is the *Chen series*. Cf. [Su2] for the asymptotic formula, and [Su11] for the fact that the Chen series is given as an infinite product of exponentials.) If one now manages to choose $\eta_\varepsilon:[0,T_\varepsilon]\to[-1,1]$ so that all the integrals $\alpha_k(\eta_\varepsilon)$ other than a particular $\alpha_m(\eta_\varepsilon)$ are $\mathrm{o}(\varepsilon^{\delta(m)})$ (where $\delta(m)$ is the *degree* of the integral α_m, i.e. the degree of the formal Lie bracket B_m) whereas $\alpha_m(\eta_\varepsilon)$ itself is $\sim c\varepsilon^{\delta(m)}$, with $c\neq 0$, then we will have exhibited a control variation in the direction of $\pm b_m(p)$, the sign being that of c. It is natural to choose $T_\varepsilon=\varepsilon$. The iterated integrals associated to Lie monomials of degree $> \delta(m)$ are then automatically $\mathrm{O}(\varepsilon^{\delta(m)+1})$, so the only real difficulty is to make sure that the integrals of degree $i \leq \delta(m)$, whose natural asymptotic behavior is $\sim \varepsilon^i$, are in fact much smaller. As an example, let us exhibit a variation in the direction of the vector $-[g,[f,g]](p)$. We begin by picking a function $\eta : [0,1] \to [-1,1]$ which is L^2-orthogonal to the functions 1, t and t^2 but not $\equiv 0$. (One can take η to be a cubic polynomial.) Then define $\eta_\varepsilon(t) = \eta(\frac{t}{\varepsilon})$, with domain $[0,\varepsilon]$. An easy calculation shows that $\alpha_1(\eta_\varepsilon) = \varepsilon$, $\alpha_2(\eta_\varepsilon) = \alpha_3(\eta_\varepsilon) = \alpha_4(\eta_\varepsilon) = 0$, and $\alpha_5(\eta_\varepsilon) = -c\varepsilon^3$, where $c = \frac{1}{2}\int_0^1(\int_0^t \eta(s)\,ds)^2\,dt$, so that $c > 0$. It is clear that the brackets B_i of degree ≤ 3 are precisely B_1, B_2, B_3, B_4 and B_5. Since $\mathrm{e}^{\alpha_1(\eta)b_1}p = p$ for all η (because $b_1 = f$ and $f(p) = 0$), and $b_5 = [g,[f,g]]$, we see that $\gamma_{\eta_\varepsilon}(\varepsilon) = p - c\varepsilon^3[g,[f,g]](p) + \mathrm{o}(\varepsilon^3)$. So $-[g,[f,g]](p)$is a variational vector.

The preceding example shows how a particular direction can be exhibited as a variational direction by "neutralizing" those directions that in principle might have given rise to larger terms in the asymptotic expansion. Moreover, the example makes it clear that there are "semialgebraic" properties of the iterated integrals that make it possible to neutralize certain vectors but not others. For instance, the fact that $\alpha_5(\eta)$ is negative follows from the equalities $\alpha_2(\eta) = \alpha_3(\eta) = \alpha_4(\eta) = 0$. Therefore the vector $+[g,[f,g]](p)$ cannot be exhibited as a variational vector by this method. On the other hand, it may happen for a particular system that $[g,[f,g]](p)$ is equal to some other vector (e.g. $g(p)$) which is a variational vector for other reasons. In that case, $[g,[f,g]](p)$ will be a variational vector after all. Therefore, in order to understand the variational vectors, one has to understand the semialgebriac properties of iterated integrals that cause certain vectors such as $-[g,[f,g]](p)$ (but not $+[g,[f,g]](p)$) to show up automatically as variational vectors, as well as the interplay between this algebra and the special Lie-algebraic properties of particular systems that permit more general vectors such as $[g,[f,g]](p)$ to be variational in certain cases even though they are not variational in general.

The line of thought outlined above has been actively pursued in recent years, leading to a series of papers on *small-time locally controllable* (SLTC) systems. A system Σ is SLTC from a point p if for arbitrarily small $T > 0$ the set $\mathcal{R}(\leq T, p)$ (i.e. the reachable set from p in time $\leq T$) contains p in its interior. The study of the STLC condition is important because it is a very interesting special case of the more general question of NCO's. Indeed, a system fails to be SLTC from p iff the constant trajectory $\gamma(t) \equiv p$ —which for an equilibrium point of a two vector field system corresponds to the control $\eta(t) \equiv 0$— is a *boundary trajectory*, in the sense that $\gamma(t) \in \partial\mathcal{R}(\leq t, p)$ for all sufficiently small small $t > 0$. So a sufficient condition for the SLTC property immediately yields a necessary condition for a particular γ to be a boundary trajectory, i.e. an NCO of a special kind.

The search for sufficient conditions for a system to be STLC has been guided by the above ideas. It is at least intuitively clear that, if a suitable definition of "variational vector" is given, then the set $\mathcal{V}(p)$ of all variational vectors at p should be a convex cone, and a system should be SLTC from p if $\mathcal{V}(p)$ is the whole space $\mathbb{R}^n$. To make these considerations rigorous one may proceed as in the classical theory of the PMP (cf. [P]), or one may follow the approach proposed by H. Frankowska (cf. [Fra]) and use open mapping theorems for set-valued maps. In either case, one is led to the question of sufficient conditions for $\mathcal{V}(p)$ to be the whole space, and this requires the algebraic analysis outlined above. As expected, Lie brackets arise most naturally. The development of the theory was initiated by H. Hermes in a series of papers ([He1], [He2],[He3]). For the particular case of two vector field systems, Hermes pointed out that it was crucial to distinguish between those brackets with an even number of g's and those with an odd number. Let us assume that we are considering *real analytic systems.* Then we may assume that the "accessibility property" (AP) holds (i.e. that the Lie algebra Λ generated by f and g satisfies $\Lambda(p) = \mathbb{R}^n$), since this is known in any case to be necessary for the reachable set from p to have a nonvoid interior (cf. [SJ]). Hermes singled out the property that $S_k(f,g)(p) = S_{k-1}(f,g)(p)$ for all *even* k and conjectured that, together with the AP, this condition was sufficient for STLC. (Here $S_k(f,g)$ is the linear span of all the brackets of f's and g's with no more than k g's. Notice that the Hermes condition holds if p is an equilibrium and $S_1(f,g)$ already equals $\mathbb{R}^n$. This is precisely the case when the linearization at p is controllable, so the Hermes condition is weaker than the classical condition on the linearization, and the corresponding sufficiency theorem stronger.) In [Su3] it was proved that the Hermes condition is indeed sufficient for the STLC property. This was done by introducing a new technique for handling the algebraic part of the argument, namely, the use of groups of symmetries. Using this technique in a more refined way, a much stronger sufficiency theorem was established in [Su7]. Since then, new and more powerful sufficient conditions have been found by K. Wagner, R.M. Bianchini and G. Stefani, and M. Kawski (cf. [Wa], [BS1], [Ka1], [St1], [St2], and especially the survey article [Ka2] by Kawski). Moreover, the work on the SLTC condition has been extended to the study of local controllability about more general —i.e. not necessary constant— reference trajectories, that is to the general NCO problem that had originally provided the motivation for studying this particular case. Work by A.A. Agrachev and R.V. Gamkrelidze ([AG2]) developed a general formalism for expressing, in terms of Lie brackets, variational vectors arising from perturbations of general controls (cf. also [Su4]). In various other papers (e.g. [AG1], [AG3]), these authors applied their formalism to the derivation of new NCO's. P. Crouch and F. Lamnabhi-Lagarrigue ([CLa]) have pursued the development of the algebra needed to understand the perturbation expansions. Using their work, new NCO's have been obtained by Lamnabhi-Lagarrigue, Bianchini and Stefani (cf. [LLa], [BS2], [LLS], [St3]).

Another set of NCO's were obtained by A. Bressan in [Br1] using methods from nonlinear functional analysis (the Mountain Pass Lemma), and then applied by Bressan himself in his work on synthesis (cf. [Br2]). A third approach to obtaining NCO's has been used by H. Schättler in his synthesis work (cf. [Sc1], [Sc2] and [SSu]). The method here is to use the Campbell-Hausdorff series to compare two controls defined on small time intervals. Suppose, for instance, that we are dealing with the minimum time problem for a two-vector field system as above. Then we may want to study bang-bang controls of the form $\eta^{\tau_1,\ldots,\tau_k}_{t_1,\ldots,t_k}$, i.e. controls obtained by letting $u = -1$ during a time τ_1, then $u = 1$ during a time t_1, then $u = 1$ during a time τ_2, and so on. Such a control steers a point p to a point

$$q = \mathrm{e}^{t_k Y}\mathrm{e}^{\tau_k X}\cdots\mathrm{e}^{t_2 Y}\mathrm{e}^{\tau_2 X}\mathrm{e}^{t_1 Y}\mathrm{e}^{\tau_1 X}p \ . \tag{3.3}$$

If we have two different controls of the above type that steer p to the same point q, then we can use the asymptotic expansion of the right-hand side of (3.3) to compare the total time $\sum t_i + \sum \tau_i$ for both controls, and we may be able to conclude that one the two cannot be optimal.

Finally, we give an example of an NCO which is of a different type, in that it generalizes the theory of envelopes from the classical Calculus of Variations. Define a *one-parameter field of extremals* (1PFE) to be a family $\{(\gamma_\varepsilon, \eta_\varepsilon, \lambda_\varepsilon) : 0 \le \varepsilon \le \bar{\varepsilon}\}$ of triples such that the $(\gamma_\varepsilon, \eta_\varepsilon)$ are extremals, defined on intervals $[a_\varepsilon, b_\varepsilon]$, the $\lambda_\varepsilon : [a_\varepsilon, b_\varepsilon] \to \mathbb{R}^n$ are Pontryagin adjoint variables for the $(\gamma_\varepsilon, \eta_\varepsilon)$, and the a_ε, b_ε, γ_ε and λ_ε "depend smoothly on ε." (Naturally, to make this precise we need a precise definition of smoothness. In [Su12], the particular case of bang-bang extremals is considered, and "smoothness" simply means that the switching times of the controls and the initial conditions for the γ_ε and the λ_ε are continously differentiable functions of ε. A paper developing the general theory under minimal smoothness assumptions is in preparation.) If the a_ε and the $\gamma_\varepsilon(a_\varepsilon)$ do not depend on ε, then we shall say that we have a *fixed initial condition* (FIC) field. Now suppose that we have a Pontryagin extremal (γ, η) defined on an interval $[a, b]$. Define an *envelope* for (γ, η) to be a curve $\delta : [-\bar{\varepsilon}, 0] \to \Omega$, with $\bar{\varepsilon} > 0$ with the property that there is a FIC 1PFE $\{(\gamma_\varepsilon, \eta_\varepsilon, \lambda_\varepsilon) : 0 \le \varepsilon \le \bar{\varepsilon}\}$, defined on intervals $[a, b_\varepsilon]$, such that $\delta(-\varepsilon) = \gamma(b_\varepsilon)$ for $0 \le \varepsilon \le \bar{\varepsilon}$, and δ is itself a trajectory, corresponding to some control $\theta : [-\bar{\varepsilon}, 0] \to U$, in such a way that $\langle \lambda_\varepsilon(b_\varepsilon), f(\delta(-\varepsilon), \theta(\varepsilon)) \rangle = 0$ for all ε. One then proves that the trajectory obtained by concatenating $\gamma_{\bar{\varepsilon}}$ with δ (with an obvious definition of the controls) has exactly the same cost as (γ, η). In particular, if (γ, η) is optimal then it follows that (δ, θ) is optimal as well. This can be used in many cases to prove that (γ, η) cannot be optimal. For instance, suppose that the terminal point q of γ lies in a region R such that every optimal trajectory in R is bang-bang. Suppose also that we manage to produce an envelope δ which is not bang-bang. Then it follows that (γ, η) is not optimal. This method has been successfully applied in [Su12], [Sc1], [Sc2] and [SSu] to the study of synthesis problems in low dimensions.

We now turn to the problem of *properties* of optimal trajectories. Naturally, there are no sharp boundaries separating this question from that of NCO's since, strictly speaking, an NCO is exactly the same as a statement that an optimal trajectory must have some special property. However, the question that will interest us now is that of proving theorems that say that optimal trajectories must have some interesting "regularity properties." For instance, if it was true that every optimal control is piecewise smooth (assuming now that we are dealing with problems with $U \subseteq \mathbb{R}^m$ and f reasonably smooth in u as well as in x), then this would be an example of the kind of theorem we want. Unfortunately, no such theorem is true in full generality, and two natural questions arise. First, one wishes to know whether some weaker result (e.g. that optimal controls necessarily have countably many discontinuities, or that the set of points of discontinuity has measure zero) is true *in general.* Second, one wishes to know whether strong theorems are true for *interesting special classes* of systems. Two classical examples of the latter situation are (i) the theorem that, for a classical Calculus of Variations problem whose Lagrangian $L(x, \dot{x})$ is of class C^∞ and has an everywhere nonsingular Hessian with respect to $\dot{x}$, all optimal trajectories are of C^∞, and (ii) the bang-bang theorem from Linear System Theory, according to which, for a linear system $\dot{x} = Ax + Bu$ with a polyhedral control constraint $u \in K = \mathrm{co}(u_1, \ldots, u_k)$, whenever there is a trajectory from a point p to another point q, then there is a time-optimal trajectory from p to q which is bang-bang with a finite number N of switchings that can actually be estimated to be $\leq C + DT$, where C, D are constants and T is the optimal time for going from p to q. Notice that the second result does not say that a time-optimal trajectory necessarily has to be bang-bang, since the system may be such that, for instance, one of the state coordinates —say x_1— satisfies $\dot{x}_1 = 1$, in which case *every* trajectory is time-optimal. What the result does say is that "bang-bang trajectories suffice for time-optimality." More generally, let us say that a class $\mathbf{C}$ of trajectories *suffices* for an optimal control problem $\mathcal{P}$ if, whenever there is an optimal trajectory from a point p to the target $\mathcal{T}$, it follows that there is an optimal trajectory from p to the target which belongs to $\mathbf{C}$, and let us say that $\mathbf{C}$ suffices for a class of problems if it suffices for each problem in the class. (So, for instance, bang-bang trajectories suffice for linear time-optimal control problems with polyhedral constraints.) Notice that this formulation includes the situation when the cost is identically zero and the targets are points, in which case our question is just that of finding classes that are *sufficient for reachability*, i.e. such that whenever it is possible to go from a point p to another point q by means of *some* trajectory, then this can be done by means of a trajectory in $\mathbf{C}$.

The problem of finding interesting sufficient classes of trajectories is essentially handled by the use of NCO's, i.e. by assuming that one has an optimal pair (γ, η) and trying to prove —using all available NCO's— that (γ, η) is "special" in some sense. There is, however, a significant twist that gives this line of research a slightly different technical flavor: we are now no longer restricted to working with given (γ, η); we are allowed to modify (γ, η) in order to make it "nicer," as long as we do not change the initial and terminal conditions.

The work on sufficient classes has so far proceeded by a combination of analytic arguments based on the use of the PMP and other NCO's, and geometric reasoning. In many cases (such as the work by Schättler and Bressan mentioned above) it was the need to prove sufficiency theorems that led to the discovery of new NCO's. In most of this work, Lie brackets have played an essential role.

The simplest of the new sufficiency theorems is the result of [Su1], according to which bang-bang trajectories are also sufficient for certain classes of two vector field systems. The specific condition on the systems can easily be understood as a particular consequence of linearity: a two vector field system is linear iff all Lie brackets involving two g's vanish identically, whereas the property of [Su1] says that all brackets with two g's must be expressible in a special way as linear combinations of brackets with only one g (cf. [Su1] for the precise details). It then turns out that this weaker property is all that is required for the bang-bang theorem.

The result of [Su1] only makes use of the PMP. In [Su6], [Su8], [Su9], [Su10], [Su13], [Sc1], [Sc2], [Br2]) a number of other results on sufficient classes were proved for various classes of systems. Remarkably, the result of [Su8], [Su9] and [Su10] shows that, for real analytic two vector field systems in two dimensions, trajectories of piecewise analytic controls (with a *finite* number of points of nonanalyticity), are sufficient for time-optimality. Unfortunately, this result fails, even in two dimensions, as soon as more complicated cost functionals are considered. For instance, for the problem of reaching the origin with minimum cost, subject to $\dot{x} = y$, $\dot{y} = u$, and $|u| \leq 1$, the cost functional being $\int x^2$, it can be shown that the optimal control from any initial condition $p \neq (0,0)$ is bang-bang *with infinitely many switchings.* (This is "Fuller's problem;" cf. [Ma] for a classical treatment, and [Ku] for a an analysis showing that Fuller-like situations appear in most optimal control problems, in sufficiently high dimensions.) One can easily modify Fuller's problem to obtain a *minimum time* problem in $\mathbb{R}^3$ for which similar "pathology" occurs.

One may then ask whether there are any results at all that can be expected to hold in full generality. The answer turns out to be closely related to *real analyticity*. Indeed, it was shown in [Su14] that, if an *arbitrary* measurable function $\eta : [0,T] \rightarrow [-1,1]$ is given, then one can construct a two vector field system of class C^∞ in $\mathbb{R}^3$, and points p, q, such that η steers p to q and no other control does. So no class of controls strictly smaller than that of all controls can be sufficient for all C^∞ problems. On the other hand, it turns out that the above construction is not possible if one wants the vector fields to be analytic. As shown in [Su13], the class **RAD** of controls η with the property that η is real analytic on an open dense subset of $\mathrm{Dom}(\eta)$ is sufficient for reachability and for time optimality for all real analytic two vector field problems.

The problem of the general structure and regularity properties of optimal controls, even for the case of two vector field systems (and even in $\mathbb{R}^3$) appears at the moment to be beyond the reach of currently existing methods. (However, substantial progress has been made, in three dimensions, in the understanding of generic cases, cf [Sc1], [Sc2], [Br2].) It is still a mystery where exactly, somewhere in between the very large class **RAD** and the much smaller class of piecewise analytic controls with at most countably many points of nonanalyticty, lies the frontier that characterizes the maximum possible pathology of optimal controls.

§4. Optimal Synthesis .

The *existence problem* for optimal synthesis is closely related to the question of existence of optimal trajectories. Once one knows that an optimal trajectory exists for every initial point p, then an optimal presynthesis is obtained by just choosing one such trajectory for each p, which is certainly possible if one accepts the Axiom of Choice. However, to get a synthesis one has to make the presynthesis *memoryless*. Whether this is possible in general appears at the moment to be an open question.

Regarding the *characterization* of optimal synthesis, the situation is quite remarkably different from what happens for optimal trajectories. Nice *sufficient conditions* can now be proved. Indeed, the question of optimal synthesis is intimately related to the study of the solutions V of the *Hamilton-Jacobi-Bellman (HJB) equation*

$$H(\nabla V(x), x) = 0 \ . \tag{4.1}$$

The function H is the *normal minimized Hamiltonian*, given by $H(\lambda, x) = \tilde{H}(\lambda, 1, x)$, where

$$\tilde{H}(\lambda, \lambda_0, x) = \inf\left\{\mathcal{H}(\lambda, \lambda_0, x, u) : u \in U\right\} \ , \tag{4.2}$$

and

$$\mathcal{H}(\lambda, \lambda_0, x, u) = \langle \lambda, f(x,u)\rangle + \lambda_0 L(x,u) \ . \tag{4.3}$$

(Here $\lambda \in \mathbb{R}^n$ and $\lambda_0 \in \mathbb{R}$.)

Now, let Γ be a synthesis and let V_Γ be the corresponding value function, i.e. $V_\Gamma(x) = J(\gamma_x, \eta_x)$. Let V be the value function of the problem under consideration. Then of course Γ is optimal iff $V_\Gamma \equiv V$. So the problem of deciding whether Γ is optimal is precisely that of checking whether $V_\Gamma \equiv V$. Moreover, for Γ to be optimal it is clearly necessary that each trajectory (γ_x, η_x) be a Pontryagin extremal. (Recall that a *Pontryagin extremal* is an (f, L) admissible pair (γ, η) with the property that that there exist (i) an absolutely continuous function $t \to \lambda(t)$, defined for $t \in \mathrm{Dom}(\eta)$, and (ii) a constant $\lambda_0 \geq 0$, such that the equalities

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(\lambda(t), \lambda_0, \gamma(t), \eta(t)) \tag{4.4}$$

and

$$\tilde{H}(\lambda(t), \lambda_0, \gamma(t)) = \mathcal{H}(\lambda(t), \lambda_0, \gamma(t), \eta(t)) = 0 \tag{4.5}$$

hold for almost all $t \in \mathrm{Dom}(\eta)$, and $(\lambda(t), \lambda_0) \neq (0, 0)$ for some —and hence every— value of t. The PMP says that an optimal trajectory must be a Pontryagin extremal.) So an obvious necessary condition for Γ to be optimal is that it be an *extremal synthesis*, i.e. a synthesis all whose trajectories are Pontryagin extremals. We sall use t_x to denote the starting time of the control η_x.

It turns out that, "modulo some technical conditions," the value function of the optimal control problem can be characterized as the unique solution of the HJB equation that satisfies certain boundary conditions. So what we need to know is whether V_Γ is a solution of the HJB equation and whether it satisfies the required boundary conditions. If one compares the HJB equation with the statement of the PMP given above, it is easy to see that the value function V_Γ will satisfy the HJB equation *provided we show that the adjoint variable $\lambda_x(\cdot)$ whose existence is asserted by the PMP can be chosen so that $\nabla V_\Gamma(x) = \lambda(t_x)$, and the corresponding λ_0 can be taken to be equal to* 1. It turns out that —again "modulo technical conditions"— this is indeed possible. So an extremal synthesis is indeed optimal if suitable technical conditions hold.

The preceding paragraph is intendedly vague, because of the repeated reference to unspecified "technical conditions." As written, the paragraph constitutes an accurate reflection of the current state of our knowledge. It is "known" that in some sense the statements are true in general, but no satisfactory way has yet been found to translate them into theorems that are both precise and sufficiently general. We shall refer to the problem of finding such theorems as *the Fundamental Problem of Synthesis Theory (FPST).* Naturally, the main question is that of finding the appropriate technical conditions that make all the desired results true.

The FPST splits into three parts. *First*, there is the problem of rigorously defining what is meant for a function to be a solution of the HJB equation. The difficulty here is that the value function typically is not everywhere differentiable, so one needs some concept of "weak solution." To illustrate the difficulties involved, consider the simple example of the system $\dot{x} = u$ on the interval $[-1, 1]$, with control constraint $|u| \leq 1$. Assume that we want to reach the boundary of the interval in minimum time. It is easy to see that the value function V is just $V(x) = 1 - |x|$, the HJB equation is $|V'(x)| = 1$, and the boundary condition is $V(-1) = V(1) = 0$. Requiring that a solution of the HJB equation be differentiable everywhere would exclude the function V. If we only require differentiability almost everywhere, or even everywhere except at just one point, then we obtain the desired function V as well as many undesirable functions, such as $-V$.

Second, there is the question of passing from the fact that V_Γ satisfies the HJB equation and the boundary conditions to the conclusion that V_Γ is the value function. It is clear that the value function V satisfies the *Dynamic Programming Inequality (DPI)*

$$V(x) \le V(y) + \int L(\gamma(t),\eta(t))\,dt \ , \tag{4.6}$$

whenever (γ,η) is a trajectory that goes from x to y and that, conversely, any function that satisfies this inequality plus the boundary conditions must be bounded above by the value function. In particular, if V_Γ satisfies the DPI and the boundary conditions then V_Γ is the value function. So the technical problem here is the passage from the HJB equation to the DPI. Formally, this is done by observing that, the HJB equation implies that $\langle \nabla V(\gamma(t)),\dot\gamma(t)\rangle + L(\gamma(t),\eta(t)) \ge 0$, from which the DPI follows by integration. The technical problem is to handle the case when ∇V does not exist everywhere, and therefore the proof by integration is not justified. Actually, all that is used in the formal derivation is the inequality $H(\nabla V(x),x) \ge 0$. Let us call a function that satisfies this inequality in some sense a *subsolution* of HJB. Then the real problem is to find the adequate technical definition of "subsolution" so that it becomes a true theorem that a function is a subsolution iff it satisfies the DPI.

Third, there is the derivation of the HJB equation from the property that Γ is extremal. In this case, there is a formal proof which is just based on differentiating V_Γ at a point x in a direction v. Asssuming, for simplicity, that the target is just a single point, one writes $V_\Gamma(x+hv)$ as an integral, and differentiates with respect to h at $h=0$. The result is precisely $\langle\lambda(t_x),v\rangle$ and, since v is an arbitrary direction, the desired conclusion follows. One is then left with the technical problem of justifying the differentiation under the integral sign.

V.G. Boltyansky (cf. [Bo]) developed a theory of "regular synthesis" in order to tackle the above problems. Roughly speaking, a regular synthesis in Boltyansky's sense is a synthesis in which the trajectories and the controls depend on x in a "piecewise smooth" way. (The precise definition is quite technical, and we shall not repeat it here.) Boltyansky's result is that a regular synthesis in his sense is indeed optimal. However, it is not hard to give examples of optimal syntheses that are not regular in Boltyansky's sense but are good enough so that Boltyansky's arguments work, possibly after some trivial modifications. So Boltyansky's theory has to be extendedand generalized. Examples of such extensions have been proposed in [Bru2], [Su3] and [Su5], but none of these can be regarded as definitive.

The main reason why we do not have a good theory of "regular synthesis" is that we lack a good theory of how "nice" the optimal synthesis has to be for reasonably large classes of problems. Indeed, suppose we could identify a property **P** such that one could prove, for every optimal control problem in a sufficiently large class, that (a) an optimal synthesis with Property **P** exists, and (b) if an extremal synthesis has Property **P** then it is optimal. Then we could regard the FPST as solved: one would consider **P** to be *the* natural property to be required of a synthesis, in that an existence theorem holds, and there is a simple necessary and sufficient condition for optimality, namely, extremality.

So we are naturally led to the question of *properties of optimal syntheses.* The study of this problem began with the work of P. Brunovsky [Bru1], [Bru2], who showed, for certain classes of problems (linear time optimal with a polyhedral control constraint) that a "regular synthesis" (in a sense slightly different from Boltyansky's) exists. Although the class of problems originally studied by Brunovsky was quite limited, his work made a far-reaching contribution, in that it introduced the idea of using the theory of *subanalytic sets* (cf. [Ha], [Hi], [Su15]) in optimal control. As was subsequently noticed (e.g. by Brunovsky himself in [Bru2], and also in [Su3]), the crucial point of his approach is that, thanks to the use of subanalytic sets, the problem of proving existence of a nice synthesis can be reduced to that of proving that certain classes of trajectories are sufficient for optimality. This has led to a number of results on existence of regular synthesis (cf. Bressan [Br2], Schättler [Sc1], [Sc2], Sussmann [Su10]).

The problem of the *characterization of the value function* is closely related to the FPST, except that now we are just given a function V_g and we ask for conditions under which this function will be the value function V. For the synthesis problem the function V_g already comes from a synthesis, so it automatically satisfies $V_g \leq V$. Here, on the other hand, we need conditions that will imply both inequalities $V_g \leq V$ and $V_g \geq V$. An elegant characterization of the value function along these lines is given by the theory of *viscosity solutions* due to M.G. Crandall and P.L. Lions (cf. [CLi], [LS]). A continuous function V is a *viscosity subsolution* of the HJB equation if, whenever φ is a C^1 function defined on a neighborhood of a point x, and such that $V - \varphi$ has a local maximum at x, it follows that $H(\nabla\varphi(x), x) \geq 0$. It is then easy to prove that, if V is continuous, then V is a viscosity subsolution if and only if it satisfies the DPI. Replacing "maximum" by "minimum" and "$H \geq 0$" by "$H \leq 0$" one obtains the definition of viscosity supersolution. Naturally, a viscosity solution is then defined to be a function which is both a subsolution and a supersolution. It then turns out that for large classes of problems the value function can be characterized as the unique viscosity solution of the HJB equation that satisfies the appropriate boundary conditions (cf.[LS]).

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