TRANSVERSALITY, SET SEPARATION, AND THE PROOF OF THE MAXIMUM PRINCIPLE, PART II

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This is the continuationd of the notes of January 3. In particular, the page and section numbering continue those of the previous set of notes.

CONTENTS

4. END OF THE PROOF OF THE MAIN TECHNICAL LEMMA

We have to show that

$$\omega(\vec{\varepsilon}) = o(\|\vec{\varepsilon}\|) \quad \text{as} \quad \vec{\varepsilon} \to 0\,, \tag{4.1}$$

where

$$\omega(\vec{\varepsilon}) \stackrel{\text{def}}{=} \sup\{\|R^{\vec{\varepsilon}}(s)\| : a \le s \le b\}\,,$$

$$\begin{aligned} R^{\vec{\varepsilon}}(t) \quad &\stackrel{\text{def}}{=} \quad \int_a^t \Big(\Delta_1^{\vec{\varepsilon}}(s) - A(s)\cdot(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s) - \xi_*(s))\Big)\, ds \\ &\qquad + \int_{[a,t]\cap E(\vec{\varepsilon})} \Big(\Delta_2^{\vec{\varepsilon}}(s) - \theta^{\vec{\varepsilon}}(s)\Big)\, ds\,, \end{aligned}$$

$$\Delta_1^{\vec{\varepsilon}}(t) \stackrel{\text{def}}{=} f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t), \eta_*(t), t) - f(\xi_*(t), \eta_*(t), t)\,,$$

$$\Delta_2^{\vec{\varepsilon}}(t) \stackrel{\text{def}}{=} f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t), \eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t), t) - f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t), \eta_*(t), t)\,,$$

$$A(t) = \frac{\partial f}{\partial x}(\xi_*(t), \eta_*(t), t)\,,$$

and

$$\theta^{\vec{\varepsilon}}(t) = \begin{cases} 0 & \text{if} \quad t \notin E(\vec{\varepsilon})\,, \\ f(x_j, u_j, \tau_j) - f(x_j, u_{*,j}, \tau_j) & \text{if} \quad t \in [\tau_j, \tau_j + \varepsilon_j]\,. \end{cases}$$

First, we have

$$\Delta_1^{\vec{\varepsilon}}(t) = \int_0^1 \frac{\partial f}{\partial x}\Big(\xi_*(t) + \nu(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t)), \eta_*(t), t\Big)\cdot\Big(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t)\Big)\, d\nu\,,$$

so that

$$\Delta_1^{\vec{\varepsilon}}(t) - A(t)\cdot\Big(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t)\Big) = \int_0^1 B^{\vec{\varepsilon}}(\nu, t)\cdot\Big(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t)\Big)\, d\nu\,,$$

where

$$B^{\vec{\varepsilon}}(\nu, t) = \frac{\partial f}{\partial x}\Big(\xi_*(t) + \nu(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t)), \eta_*(t), t\Big) - \frac{\partial f}{\partial x}\Big(\xi_*(t), \eta_*(t), t\Big)\,.$$

Since the function

$$Q \times U \times [a,b] \ni (x, u, t) \mapsto \frac{\partial f}{\partial x}(x, u, t)$$

is continuous, the quantity

$$\lambda(r) \stackrel{\text{def}}{=}$$
$$\sup\Big\{\Big\|\frac{\partial f}{\partial x}(x+v, u, t) - \frac{\partial f}{\partial x}(x, u, t)\Big\| : (x,u,t)\in K, (x+v,u,t)\in K, \|v\|\le r\Big\}$$

satisfies

$$\lim_{r\downarrow 0} \lambda(r) = 0\,.$$

Using the bound

$$\|\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t)\| \le 2\kappa\|\vec{\varepsilon}\| e^{\kappa(b-a)} \tag{4.2}$$

(cf. equation (7.4)) we get

$$\|B^{\vec{\varepsilon}}(\nu,t)\| \le \lambda\Big(2\kappa\|\vec{\varepsilon}\|e^{\kappa(b-a)}\Big) \quad \text{whenever} \quad t\in[a,b], 0\le\nu\le 1\,.$$

Then

$$\|\Delta_1^{\vec{\varepsilon}}(t) - A(t)\cdot\Big(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t)-\xi_*(t)\Big)\| \le \lambda\Big(2\kappa\|\vec{\varepsilon}\|e^{\kappa(b-a)}\Big)\cdot 2\kappa\|\vec{\varepsilon}\|e^{\kappa(b-a)}$$

whenever $t\in[a,b]$. Therefore

$$\begin{aligned}(4.3)\qquad & \Big\|\int_a^t \Big(\Delta_1^{\vec{\varepsilon}}(s) - A(s)\cdot(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s)-\xi_*(s))\Big)\,ds\Big\| \\ & \le \lambda\Big(2\kappa\|\vec{\varepsilon}\|e^{\kappa(b-a)}\Big)\cdot 2(b-a)\kappa\|\vec{\varepsilon}\|e^{\kappa(b-a)}\end{aligned}$$

whenever $t\in[a,b]$. This provides the desired estimate for the first of the two terms in the definition of $R^{\vec{\varepsilon}}(t)$.

To estimate the second term, we fix j, and then $t\in[\tau_j,\tau_j+\varepsilon_j]$, and write

$$\Delta_2^{\vec{\varepsilon}}(t) = f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t),u_j,t) - f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t),\eta_*(t),t)\,.$$

Then

$$\Delta_2^{\vec{\varepsilon}}(t) - \theta^{\vec{\varepsilon}}(t) = \sigma_1^{\vec{\varepsilon}}(t) - \sigma_2^{\vec{\varepsilon}}(t)\,,$$

where

$$\begin{aligned}\sigma_1^{\vec{\varepsilon}}(t) &\stackrel{\text{def}}{=} f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t),u_j,t) - f(\xi_*(\tau_j),u_j,\tau_j)\,,\\ \sigma_2^{\vec{\varepsilon}}(t) &\stackrel{\text{def}}{=} f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t),\eta_*(t),t) - f(\xi_*(\tau_j),\eta_*(\tau_j),\tau_j)\,.\end{aligned}$$

Let

$$\hat{\lambda}(r) \stackrel{\text{def}}{=} \sup\{\|F(s_1)-F(s_2)\| : s_1\in[a,b], s_2\in[a,b], |s_1-s_2|\le r\}\,,$$

where

$$F(s) \stackrel{\text{def}}{=} f(\xi_*(s),\eta_*(s),s)\,.$$

Then

$$(4.4)\qquad \lim_{r\downarrow 0}\hat{\lambda}(r) = 0\,,$$

because F is continuous, since f, ξ_* and η_* are continuous.

Then

$$\begin{aligned}(4.5)\quad \|f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t),\eta_*(t),t) - f(\xi_*(t),\eta_*(t),t)\| &\le \kappa\|\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t)-\xi_*(t)\|\\ &\le 2\kappa^2\|\vec{\varepsilon}\|e^{\kappa(b-a)}\,,\end{aligned}$$

and

$$(4.6)\qquad \|f(\xi_*(t),\eta_*(t),t) - f(\xi_*(\tau_j),\eta_*(\tau_j),\tau_j)\| \le \hat{\lambda}(\|\vec{\varepsilon}\|)\,.$$

Therefore

$$\sigma_2^{\vec{\varepsilon}}(t) \le 2\kappa^2\|\vec{\varepsilon}\|e^{\kappa(b-a)} + \hat{\lambda}(\|\vec{\varepsilon}\|)\,.$$

To estimate $\sigma_1^{\vec{\varepsilon}}(t)$, we define

$$\tilde{\lambda}(r) \stackrel{\text{def}}{=} \sup\{\|F_j(s_1) - F_j(s_2)\| : s_1 \in [a,b], s_2 \in [a,b], |s_1 - s_2| \le r, j = 1,\ldots,k\}\,,$$

where

$$F_j(s) \stackrel{\text{def}}{=} f(\xi_*(s), u_j, s)\,.$$

Then

$$\lim_{r\downarrow 0} \tilde{\lambda}(r) = 0\,,$$

because the F_j are continuous, since f and ξ_* are continuous.

It follows that

$$\begin{aligned} \|f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t), u_j, t) - f(\xi_*(t), u_j, t)\| &\le \kappa\|\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t)\| \\ &\le 2\kappa^2\|\vec{\varepsilon}\|e^{\kappa(b-a)}\,, \end{aligned}$$

and

$$\|f(\xi_*(t), u_j, t) - f(\xi_*(\tau_j), u_j, \tau_j)\| \le \tilde{\lambda}(\|\vec{\varepsilon}\|)\,.$$

Therefore

$$\sigma_1^{\vec{\varepsilon}}(t) \le 2\kappa^2\|\vec{\varepsilon}\|e^{\kappa(b-a)} + \tilde{\lambda}(\|\vec{\varepsilon}\|)\,.$$

Hence

$$\|\Delta_2^{\vec{\varepsilon}}(t) - \theta^{\vec{\varepsilon}}(t)\| \le 4\kappa^2\|\vec{\varepsilon}\|e^{\kappa(b-a)} + \hat{\lambda}(\|\vec{\varepsilon}\|) + \tilde{\lambda}(\|\vec{\varepsilon}\|)\,.$$

Then

$$\begin{aligned} &\|\int_{[a,t]\cap E(\vec{\varepsilon})} \Big(\Delta_2^{\vec{\varepsilon}}(s) - \theta^{\vec{\varepsilon}}(s)\Big)\,ds\| \\ &\quad\le \Big(4\kappa^2\|\vec{\varepsilon}\|e^{\kappa(b-a)} + \hat{\lambda}(\|\vec{\varepsilon}\|) + \tilde{\lambda}(\|\vec{\varepsilon}\|)\Big)\cdot\|\vec{\varepsilon}\|\,. \end{aligned} \tag{4.7}$$

If we combine (4.3) and (4.7), we get the bound

$$\|R^{\vec{\varepsilon}}(t)\| \le \mu(\vec{\varepsilon})\cdot\|\vec{\varepsilon}\|\,,$$

where

$$\mu(\vec{\varepsilon}) \stackrel{\text{def}}{=} \lambda\Big(2\kappa\|\vec{\varepsilon}\|e^{\kappa(b-a)}\Big)\cdot 2(b-a)\kappa e^{\kappa(b-a)} + 4\kappa^2\|\vec{\varepsilon}\|e^{\kappa(b-a)} + \hat{\lambda}(\|\vec{\varepsilon}\|) + \tilde{\lambda}(\|\vec{\varepsilon}\|)\,.$$

It follows that

$$\|\omega(\vec{\varepsilon})\| \le \mu(\vec{\varepsilon})\cdot\|\vec{\varepsilon}\|\,.$$

Since $\mu(\vec{\varepsilon}) \to 0$ as $\vec{\varepsilon} \to 0$, (4.1) is proved. $\diamondsuit$

5. TRANSVERSALITY OF CONES AND SET SEPARATION

This section presents in detail the "topological argument" that plays a crucial role in the proof of the maximum principle. The purpose of the argument is to extract information from the facts that

1. the reachable set $\mathcal{R}_{[a,b]}(Q,U,f;x)$ and the set S are locally separated at $\hat{x}$,
2. S has a Boltyanskii tangent cone C,
3. $\mathcal{R}_{[a,b]}(Q,U,f;x)$ contains the image of a neighborhood of 0 relative to the positive orthant $\mathbb{R}^k_+$ of $\mathbb{R}^k$ under the endpoint map $\mathcal{E}^{\vec{\tau},\vec{u}}$,
4. the map $\mathcal{E}^{\vec{\tau},\vec{u}}$ is continuous near 0 and differentiable at 0.

Facts 2, 3 and 4 tell us that we have

a. two subsets S_1 and S_2 (given by $S_1 = \mathcal{R}_{[a,b]}(Q,U,f;x)$ and $S_2 = S$) of a finite-dimensional real linear space $Y = \mathbb{R}^n$,
b. two convex cones C_1 and C_2 (given by $C_1 = \mathbb{R}^k_+$ and $C_2 = C$), in linear spaces X_1, X_2 (where $X_1 = \mathbb{R}^k$ and $X_2 = \mathbb{R}^n$),
c. neighborhoods U_1, U_2 of the origin in X_1, X_2, respectively,
d. continuous maps $F_1 : U_1 \cap C_1 \mapsto S_1$ and $F_2 : U_2 \cap C_2 \mapsto S_2$ (given by $F_1 = \mathcal{E}^{\vec{\tau},\vec{u}}$ and $F_2 = \varphi$, where φ is the approximating map that occurs in the definition of a Boltyanskii cone),
e. linear maps $L_1 : X_1 \mapsto Y$, $L_2 : X_2 \mapsto Y$ such that $F_1(x) = L_1 x + o(\|x\|)$ as $x \to 0$, $x \in C_1$, and $F_2(x) = L_2 x + o(\|x\|)$ as $x \to 0$, $x \in C_2$.

Our goal is then to conclude that "the separation of the sets S_1 and S_2 implies that the linear approximations L_1C_1, L_2C_2 to these sets are separated in the linear sense." Here "linear separation" of two cones K_1, K_2 in Y means that "there exists a nontrivial linear functional $\lambda : Y \mapsto \mathbb{R}$ such that $\lambda(v) \geq 0$ for $v \in K_1$ and $\lambda(v) \leq 0$ for $v \in K_2$."

Unfortunately, *separation of two sets does not imply linear separation of their approximating cones.* This can be see most easily by considering the following trivial example. Let $Y = \mathbb{R}^2$, and take S_1, S_2 to be the x axis and the y axis, respectively. Then S_1 and S_2 are separated at the origin. On the other hand, it is clear that S_1 and S_2 are their own linear approximations at 0. Yet S_1 and S_2 are *not* linearly separated.

The true corespondence between separation of two sets and linear separation of their approximating cones is given by a property which is slightly *weaker* than linear separation of the approximating cones. Precisely:

i. linear separation of two cones is equivalent to the property that the cones are not "transversal,"
ii. there is another property, called "strong transversality," which is slightly stronger than transversality,

iii. therefore the property of *not being strongly transversal* is slightly *weaker* than non-transversality, i.e., slightly weaker than linear separation,
iv. strong transversality of the approximating cones implies that the sets are not separated, that is *separation of the sets implies that the approximating cones are not strongly transversal.*

The notion of "transversality" of cones is a natural extension of the well known notion of transversality of linear subspaces. One says that two linear subspaces A_1, A_2 of a finite-dimensional real linear space Y are *transversal* if the sum A_1+A_2 (that is, the set of all sums a_1+a_2, $a_1 \in A_1$, $a_2 \in A_2$) is the whole space Y. Naturally, when the A_i are subspaces we could equally well have used the *difference* $A_1 - A_2$ (that is, the set of all differences $a_1 - a_2$, $a_1 \in A_1$, $a_2 \in A_2$). It turns out that, *once we use set difference rather than set sum, the resulting notion of "transversality" is the one that works for cones as well.*

The general philosophy of transversality theory is that, if two objects B_1 and B_2 have linear approximations A_1, A_2 at a point $\bar{x}$, then $B_1 \cap B_2$ looks, near $\bar{x}$, like $A_1 \cap A_2$, *if A_1 and A_2 are transversal.* For example, suppose B_1, B_2 are smooth submanifolds of $\mathbb{R}^n$ of dimensions n_1, n_2, $\bar{x} \in B_1 \cap B_2$, and A_1, A_2 are the tangent spaces to B_1, B_2 at $\bar{x}$. Then, if A_1 and A_2 are transversal, it follows that $n_1 + n_2 \geq n$, and the intersection $A_1 \cap A_2$ is a subspace of dimension $\nu = n_1 + n_2 - n$. The implicit function theorem then implies that, near $\bar{x}$, $B_1 \cap B_2$ is a ν-dimensional submanifold of $\mathbb{R}^n$. (Transversality is essential here! For example, if we take $n = 2$, and let B_1 be the x axis, and B_2 be the parabola $\{(x, y) : y = x^2\}$, we see that B_1 and B_2 intersect at the origin, and their tangent spaces A_1, A_2 at $(0, 0)$ both coincide with x-axis. So $A_1 \cap A_2$ is one-dimensional, but $B_1 \cap B_2$ consists of a single point, so $B_1 \cap B_2$ does not look at all like $A_1 \cap A_2$. This shows that the principle that "$B_1 \cap B_2$ looks near $\bar{x}$ like $A_1 \cap A_2$" can fail if A_1 and A_2 fail to be transversal.)

In our case, the "general philosophy" suggest the following possibility: if the approximating cones C_1 and C_2 are transversal, then $S_1 \cap S_2$ will contain a nontrivial set if $C_1 \cap C_2$ does. Therefore, to guarantee that $S_1 \cap S_2$ contains a nontrivial set, we have to require

(a) that C_1 and C_2 be transversal,

and

(b) that $C_1 \cap C_2$ be nontrivial, that is, that $C_1 \cap C_2 \neq \{0\}$.

The conjunction of these two conditions is precisely what we are going to call "strong transversality." It will then be a rigorous theorem that

(#a) *strong tranversality of the approximating cones implies non-separation of the sets,* that is, that

($\#'$) *separation of the sets implies that the approximating cones are not strongly transversal.*

Using ($\#'$), we will conclude, in the proof of the maximum principle, that the cones $D\mathcal{E}^{\vec{\tau},\vec{u}}(0)(\mathbb{R}^k_+)$ and C are not strongly transversal. We will then want to draw the stronger conclusion that the cones are not transversal. This will follow from the observation (proved in Lemma 5.2.1) that, as long as the cones under consideration are not both linear subspaces, transversality of two cones is in fact equivalent to strong transversality. That is, "transversality and strong transversality are essentially the same thing," except only in the case when both cones are linear subspaces.

5.1. Transversality and strong transversality of cones. If S_1, S_2 are subsets of a real linear space X, and $S_3 \subseteq \mathbb{R}$, we write

$$\begin{aligned} S_1 + S_2 &\stackrel{\text{def}}{=} \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}\,, \\ S_1 - S_2 &\stackrel{\text{def}}{=} \{s_1 - s_2 : s_1 \in S_1, s_2 \in S_2\}\,, \\ S_3 \cdot S_1 &\stackrel{\text{def}}{=} \{s_3 \cdot s_1 : s_1 \in S_1, s_3 \in S_3\}\,. \end{aligned}$$

When one of the sets consists of a single point x, we will write "x" rather than "$\{x\}$" in the above formulae. Thus, for example, if $S \subseteq X$ and $x \in X$, then $x + S$ is the translate of S by x, i.e., the set $\{x + s : s \in S\}$. Similarly, if $S \subseteq X$ and $r \in \mathbb{R}$, then $r \cdot S$ is the set $\{r \cdot s : s \in S\}$. In particular, if B is the closed unit ball of X centered at 0 (with respect to some norm on X) and $x \in X$, $r \in \mathbb{R}$, $r > 0$, then $x + r \cdot B$ is the closed ball of radius r centered at x.

Definition 5.1.1. Let C_1, C_2 be cones in a finite-dimensional real linear space X. We say that C_1 and C_2 are *transversal*—and write $C_1 \mathrel{\overline{\pitchfork}} C_2$—if

$$C_1 - C_2 = X\,. \qquad \diamondsuit$$

Remark 5.1.2. *If C_1 and C_2 are convex, then $C_1 \mathrel{\overline{\pitchfork}} C_2$ if and only if there does not exist a nonzero linear functional $\lambda : X \mapsto \mathbb{R}$ such that*

$$\begin{aligned} \lambda(c) &\geq 0 \quad \text{whenever} \quad c \in C_1\,, \\ \lambda(c) &\leq 0 \quad \text{whenever} \quad c \in C_2\,. \end{aligned}$$

Equivalently,

$$C_1 \mathrel{\overline{\pitchfork}} C_2 \Longleftrightarrow (-C_1)^\perp \cap (C_2)^\perp = \{0\}\,. \tag{5.1}$$

To see that (5.1) holds, assume first that $C_1 \mathrel{\overline{\pitchfork}} C_2$. Let $\lambda \in (-C_1)^\perp \cap (C_2)^\perp$. Let $x \in X$. Write $x = c_1 - c_2$, $c_1 \in C_1$, $c_2 \in C_2$. Then $\lambda(c_1) \geq 0$, because $\lambda \in (-C_1)^\perp$, and then $\lambda(c_2) \leq 0$, because $\lambda \in (C_2)^\perp$. Therefore $\lambda(c_1 - c_2) \geq 0$. So $\lambda(x) \geq 0$. Since this inequality is true for all $x \in X$, we can take an $x \in X$ and apply the inequality to $-x$, thereby concluding that $\lambda(-x) \geq 0$, i.e., that $\lambda(x) \leq 0$. So $\lambda(x) = 0$ for all $x \in X$, i.e., $\lambda = 0$.

To prove the converse, we assume that $(-C_1)^\perp \cap (C_2)^\perp = \{0\}$ and try to prove that $C_1 \pitchfork C_2$. Suppose it is not true that $C_1 \pitchfork C_2$. Let $D = C_1 - C_2$. Then D is a convex cone in X, and $D \neq X$. So D is a proper convex cone in X. Let E be the closure of D in X. Then E is a closed convex cone in X, and $E \neq X$. (Here we are using the fact that X is finite-dimensional, to conclude that $E \neq X$ from the fact that $D \neq X$. In infinite dimensions this can fail, because there are proper convex cones that are dense.) So by the Hahn-Banach Theorem there exists a nonzero linear functional $\lambda : X \mapsto \mathbb{R}$ such that $\lambda(x) \geq 0$ for all $x \in E$. In particular, if $c \in C_1$ then $c \in E$, so $\lambda(c) \geq 0$. Therefore $\lambda \in (-C_1)^\perp$. Also, if $c \in C_2$ then $-c \in E$, so $\lambda(-c) \geq 0$, and then $\lambda(c) \leq 0$. Therefore $\lambda \in (C_2)^\perp$. So $\lambda \in (-C_1)^\perp \cap (C_2)^\perp = \{0\}$. Since $\lambda \neq 0$, this contradicts the assumption that $(-C_1)^\perp \cap (C_2)^\perp = \{0\}$. Therefore $C_1 \pitchfork C_2$. ♢

Definition 5.1.3. Let C_1, C_2 be cones in a finite-dimensional real linear space X. We say that C_1 and C_2 are *strongly transversal*—and write $C_1 \pitchfork\!\!\!\!\pitchfork C_2$—if $C_1 \pitchfork C_2$ and $C_1 \cap C_2 \neq \{0\}$. ♢

5.2. Transversality vs. strong transversality.

Lemma 5.2.1. *Let C_1, C_2 be two convex cones in a finite-dimensional real linear space X. Then the following two conditions are equivalent:*

1. $C_1 \pitchfork C_2$,
2. *either*
   - a. $C_1 \pitchfork\!\!\!\!\pitchfork C_2$

   or

   - b. C_1 *and* C_2 *are both linear subspaces and* $X = C_1 \oplus C_2$.

Proof. It is clear that $2 \Longrightarrow 1$. Let us show that $1 \Longrightarrow 2$. Assume that $C_1 \pitchfork C_2$ but C_1 is not strongly transversal to C_2. We have to show that Condition b holds. Clearly, our assumptions imply that $C_1 \cap C_2 = \{0\}$.

First, we show that C_2 is a linear subspace. Let $c \in C_2$. Since $C_1 \pitchfork C_2$, we can write $c = c_1 - c_2$, $c_1 \in C_1$, $c_2 \in C_2$. Then $c + c_2 = c_1$, so $c_1 \in C_1 \cap C_2$. Therefore $c_1 = 0$, so $c + c_2 = 0$. It follows that $-c = c_2$, so $-c \in C_2$. We have thus shown that

$$(\forall c)(c \in C_2 \Longrightarrow -c \in C_2)\,. \tag{5.2}$$

Since C_2 is a convex cone, (5.2) implies that it is a linear subspace.

Second, a similar argument shows that C_1 is a linear subspace as well.

It then follows that $C_1 + C_2 = X$, because $C_1 - C_2 = X$, since $C_1 \pitchfork C_2$. Moreover, we know that $C_1 \cap C_2 = \{0\}$. So the sum is direct, that is, $X = C_1 \oplus C_2$. ♢ □

5.3. The main topological lemma.

LEMMA 5.3.1. *Let $r \in \mathbb{R}$, $r > 0$, and let B be the ball of radius r in a finite-dimensional normed space Y. Let ρ be such that $0 < \rho < r$, and let $F : B \to Y$ be a continuous map such that*

$$\|F(y) - y\| \leq \rho \quad \text{whenever} \quad y \in B\,.$$

Then

$$(r - \rho)B \subseteq F(B)\,.$$

PROOF. Fix $z \in (r - \rho)B$. We want to find $y \in B$ such that $F(y) = z$. Now, the equation $F(y) = z$ is equivalent to $y = y - F(y) + z$. Let

$$G(y) = y - F(y) + z\,,$$

for $y \in B$. If $y \in B$, then $\|y - F(y)\| \leq \rho$. Since $\|z\| \leq r - \rho$, we can conclude that $\|G(y)\| \leq r$. So G is a continuous map from B to B. By the Brouwer fixed point theorem, G has a fixed point. That is, there exists $y \in B$ such that $G(y) = y$. But then this y satisfies $F(y) = z$, and our proof is complete. $\square$

5.4. The transversal intersection theorem.

THEOREM 5.4.1. *Assume that the following conditions hold.*

1. *X_1, X_2, Y are finite-dimensional real linear spaces.*
2. *C_1, C_2 are convex cones in X_1, X_2.*
3. *U_1, U_2 are neighborhoods of 0 in X_1, X_2.*
4. *$F_1 : U_1 \cap C_1 \mapsto Y$, $F_2 : U_2 \cap C_2 \mapsto Y$, are continuous maps such that $F_1(0) = F_2(0) = 0$.*
5. *F_1 and F_2 are differentiable at 0 along C_1, C_2 with differentials L_1, L_2. (That is, for $i = 1, 2$, L_i is a linear map from X_i to Y such that $F_i(x) = L_i x + o(\|x\|)$ as $x \to 0$ via values in C_i.)*
6. *$L_1(C_1) \pitchfork L_2(C_2)$.*

Then the sets $F_1(U_1 \cap C_1)$ and $F_2(U_2 \cap C_2)$ are not locally separated at 0. That is, $F_1(U_1 \cap C_1) \cap F_2(U_2 \cap C_2) \cap V \neq \{0\}$ for every neighborhood V of 0 in Y.

REMARK 5.4.2. The statement of the theorem involves (in Item 5) norms on the spaces X_1, X_2, Y, but it is easy to see that the validity of the condition on Item 5 does not depend on the choice of the norms. $\diamondsuit$

PROOF. We fix once and for all norms on X_1, X_2, Y.

Let $(e_1, \ldots, e_n)$ be a basis of Y. Write $e_0 = -(e_1 + \ldots + e_n)$. Then every $y \in Y$ can be written uniquely as an affine combination of $e_0, e_1, \ldots, e_n$. (Recall that an *affine combination* of vectors $v_1, \ldots, v_m$ is a linear combination $a_1 v_1 + \ldots + a_m v_m$ with scalar coefficients a_j such that $a_1 + \ldots + a_m = 1$. If $y \in Y$, then we can write $y = a_1 e_1 + \ldots + a_n e_n$ and, using $0 = e_0 + e_1 + \ldots + e_n$, we can pick an arbitrary scalar r and write $y = b_0 e_0 + b_1 e_1 + \ldots + b_n e_n$, with

$b_0 = r$ and $b_j = a_j + r$ for $j = 1, \ldots, n$. Then $b_0 + b_1 + \ldots + b_n = a + (n+1)r$, where $a = a_1 + \ldots + a_m$. By choosing $r = \frac{1-a}{n+1}$, we get $b_0 + b_1 + \ldots + b_n = 1$. This yields an expression of the desired form for y. To prove uniqueness, assume $y = b_0e_0 + b_1e_1 + \ldots + b_ne_n = c_0e_0 + c_1e_1 + \ldots + c_ne_n$, with $b_0 + b_1 + \ldots + b_n = c_0 + c_1 + \ldots + c_n = 1$. Then $(b_0 - c_0)e_0 + (b_1 - c_1)e_1 + \ldots + (b_n - c_n)e_n = 0$, so $(b_1 - c_1 - (b_0 - c_0))e_1 + \ldots + (b_n - c_n - (b_0 - c_0))e_n = 0$. Since the vectors $e_1, \ldots, e_n$ are linearly independent, we find $b_1 - c_1 - (b_0 - c_0) = \ldots = b_n - c_n - (b_0 - c_0) = 0$. Therefore $b_0 - c_0 = b_1 - c_1 = \ldots = b_n - c_n$. But $\sum_{j=0}^n b_j = \sum_{j=0}^n c_j = 1$, and then $\sum_{j=0}^n (b_j - c_j) = 0$. So the $n+1$ numbers $b_j - c_j$ are equal and add up to zero. Therefore $b_j - c_j = 0$ for $j = 0, 1, \ldots, n$.) Use $a_j(y)$ to denote the coefficients of this affine combination, so

$$y \in Y \Longrightarrow \Big(y = \sum_{j=0}^n a_j(y)e_j \ \text{ and } \ \sum_{j=0}^n a_j(y) = 1 \Big).$$

It is easy to see that the functions $Y \ni y \mapsto a_j(y) \in \mathbb{R}$ are continuous. Since

$$0 = e_0 + e_1 + \ldots + e_n = \frac{1}{n+1} \cdot e_0 + \frac{1}{n+1} \cdot e_1 + \ldots + \frac{1}{n+1} \cdot e_n \,,$$

we have

$$a_j(0) = \frac{1}{n+1} \ \text{ for } \ j = 0, 1, \ldots, n\,.$$

Since the functions a_j are continuous, we can fix a positive number δ such that

$$\Big(y \in Y \ \text{and} \ \|y\| \le \delta \Big) \Longrightarrow \Big(a_j(y) \ge 0 \ \text{for} \ j = 0, 1, \ldots, n \Big).$$

Let B be the closed ball $\{y \in Y : \|y\| \le \delta\}$. Then every member y of B is a convex combination $\sum_{j=0}^n a_j(y)e_j$ of $e_0, e_1, \ldots, e_n$.

The assumption that the cones $L_1(C_1)$ and $L_2(C_2)$ are transversal tells us that we can find vectors $\tilde{e}_{1,0}, \tilde{e}_{1,1}, \ldots, \tilde{e}_{1,n}$ in C_1, $\tilde{e}_{2,0}, \tilde{e}_{2,1}, \ldots, \tilde{e}_{2,n}$ in C_2, such that

$$e_j = L_1\tilde{e}_{1,j} - L_2\tilde{e}_{2,j} \quad \text{for} \quad j = 0, 1, \ldots .n\,.$$

The assumption that $L_1(C_1)$ and $L_2(C_2)$ are strongly transversal implies that there exists a nonzero vector $\bar{y} \in Y$ such that $\bar{y} \in L_1(C_1) \cap L_2(C_2)$. Write

$$\bar{y} = L_1v_1 = L_2v_2\,, \ v_1 \in C_1\,, \ v_2 \in C_2\,.$$

Then, if $r \in \mathbb{R}$ is arbitrary, we have

$$e_j = L_1(\tilde{e}_{1,j} + rv_1) - L_2(\tilde{e}_{2,j} + rv_2) \quad \text{for} \quad j = 0, 1, \ldots .n\,,$$

because $L_1v_1 - L_2v_2 = 0$.

We will choose r in a special way, and then define

$$e_{1,j} = \tilde{e}_{1,j} + rv_1\,, \ e_{2,j} = \tilde{e}_{2,j} + rv_2 \quad \text{for} \quad j = 0, 1, \ldots .n\,.$$

The choice of r is made by first fixing a linear functional $\mu : Y \mapsto \mathbb{R}$ such that $\mu(\bar{y}) = 1$, and observing that the resulting vectors $e_{i,j}$ will then satisfy

$$\mu(L_1e_{1,j}) = \mu(L_1\tilde{e}_{1,j}) + r\,, \quad \mu(L_2e_{2,j}) = \mu(L_1\tilde{e}_{2,j}) + r\,.$$

We choose r such that $r>0$ and all the numbers $\mu(L_1\tilde{e}_{1,j})+r$, $\mu(L_1\tilde{e}_{2,j})+r$ are ≥ 1.

With this choice of r, the vectors $e_{1,j}$ belong to C_1, and the $e_{2,j}$ belong to C_2. So we now have

$$\left.\begin{array}{rrcl} & e_j & = & L_1e_{1,j}-L_2e_{2,j}\,, \\ & e_{1,j} & \in & C_1\,, \\ & e_{2,j} & \in & C_2\,, \\ & \mu(L_1e_{1,j}) & \geq & 1\,, \\ \text{and} & \mu(L_2e_{2,j}) & \geq & 1 \end{array}\right\} \text{for} \qquad j=0,1,\ldots.n\,.$$

We now define a positive number $\bar{r}$ and a map

$$]0,\bar{r}]\times B\ni(r,y)\mapsto H(r,y)\in Y$$

by letting

$$H(r,y)=\frac{1}{r}\left(F_1\Big(r\theta_1(y)\Big)-F_2\Big(r\theta_2(y)\Big)\right)$$

whenever $0<r\leq\bar{r}$ and $y\in B$. Here

$$\begin{aligned}\theta_1(y) &\stackrel{\text{def}}{=} a_0(y)e_{1,0}+a_1(y)e_{1,1}+\ldots+a_n(y)e_{1,n}\,,\\ \theta_2(y) &\stackrel{\text{def}}{=} a_0(y)e_{2,0}+a_1(y)e_{2,1}+\ldots+a_n(y)e_{2,n}\,.\end{aligned}$$

The number $\bar{r}$ is chosen so that

$$\bar{r}(\|e_{i,0}\|+\|e_{i,1}\|+\ldots+\|e_{i,n}\|)\leq\delta_i \quad \text{for} \quad i=1,2\,,$$

where the numbers δ_i are such that

$$\delta_i>0 \quad \text{and} \quad (\forall x)\Bigg(\Big(x\in X_i \text{ and } \|x\|\leq\delta_i\Big)\Longrightarrow x\in U_i\Bigg)\,.$$

It follows from our choice of $\bar{r}$ that $\|r\theta_i(y)\|\leq\delta_i$ whenever $i=1,2$, $y\in B$, and $0<r\leq\bar{r}$. (Recall that the coefficients $a_j(y)$ satisfy $0\leq a_j(y)\leq 1$ whenever $y\in B$.) Therefore H is well defined on the set $]0,\bar{r}]\times B$. It is then clear that H is continuous.

Now use the assumption that

$$F_i(x)=L_ix+o(\|x\|) \quad \text{as} \quad x\to 0\,,\ x\in C_i\,,$$

to write

$$F_i\Big(r\theta_i(y)\Big)=L_i\Big(r\theta_i(y)\Big)+o(r) \ \text{ as } \ r\downarrow 0\,,$$

since $\|\theta_i(y)\|$ is bounded by a fixed constant. Then

$$F_i\Big(r\theta_i(y)\Big)=rL_i\Big(\theta_i(y)\Big)+o(r) \ \text{ as } \ r\downarrow 0\,,$$

since L_i is linear. So

$$H(r,y)=L_1\Big(\theta_1(y)\Big)-L_2\Big(\theta_2(y)\Big)+o(1) \ \text{ as } \ r\downarrow 0\,.$$

But

$$
\begin{aligned}
L_1\Big(\theta_1(y)\Big) - L_2\Big(\theta_2(y)\Big) &= L_1\Big(\sum_{j=0}^{n} a_j(y)e_{1,j}\Big) - L_2\Big(\sum_{j=0}^{n} a_j(y)e_{2,j}\Big) \\
&= \sum_{j=0}^{n} a_j(y)(L_1e_{1,j} - L_2e_{2,j}) \\
&= \sum_{j=0}^{n} a_j(y)e_j \\
&= y\,.
\end{aligned}
$$

Therefore

$$H(r,y) = y + o(1) \ \text{ as } \ r \downarrow 0\,.$$

In other words, if we define $H_r(y) = H(r,y)$, we find

$$\lim_{r\downarrow 0} H_r(y) = y\,, \ \text{ uniformly with respect to } y \in B\,.$$

Pick any α such that $0 < \alpha < \delta$. Then choose $\hat{r}$ such that $0 < \hat{r} \leq \bar{r}$ having the property that $\|H_r(y) - y\| \leq \alpha$ whenever $0 < r \leq \hat{r}$. Then Lemma 5.3.1 implies that

$$(\delta - \alpha)B \subseteq H_r(B) \quad \text{whenever} \quad 0 < r \leq \hat{r}\,.$$

In particular,

$$0 \in H_r(B) \quad \text{whenever} \quad 0 < r \leq \hat{r}\,.$$

This means that, for each $r \in]0,\hat{r}]$, we can find a point $y_r \in B$ such that

$$H_r(y_r) = 0\,.$$

It then follows from the definition of H_r that

$$F_1(r\theta_1(y_r)) = F_2(r\theta_2(y_r))\,.$$

So, if we let $w_r \stackrel{\text{def}}{=} F_1(r\theta_1(y_r)) = F_2(r\theta_2(y_r))$, we have shown that

$$w_r \in F_1(U_1 \cap C_1) \cap F_2(U_2 \cap C_2) \quad \text{whenever} \quad 0 < r \leq \hat{r}\,.$$

Moreover, it is clear that

$$\lim_{r\downarrow 0} w_r = 0\,,$$

since $r\theta_1(y_r) \to 0$ and F_1 is continuous. It follows that, to conclude our proof, it suffices to show that

$$w_r \neq 0 \ \text{ if } \ r \ \text{ is small enough}\,.$$

To see this, we estimate $\mu(w_r)$. We have

$$\begin{aligned} w_r &= F_1(r\theta_1(y_r)) \\ &= L_1(r\theta_1(y_r)) + o(r) \\ &= L_1\Big(r\sum_{j=0}^{n} a_j(y_r)e_{1,j}\Big) + o(r) \\ &= \Big(r\sum_{j=0}^{n} a_j(y_r)L_1e_{1,j}\Big) + o(r)\,, \end{aligned}$$

and then

$$\begin{aligned} \mu(w_r) &= \mu\Big(r\sum_{j=0}^{n} a_j(y_r)L_1e_{1,j}\Big) + o(r) \\ &= r\sum_{j=0}^{n} a_j(y_r)\mu(L_1e_{1,j}) + o(r) \\ &\geq r\sum_{j=0}^{n} a_j(y_r) + o(r) \\ &= r + o(r) \\ &> 0 \text{ if } r \text{ is small enough}\,, \end{aligned}$$

since $\mu(L_1e_{1,j}) \geq 1$. This concludes our proof. ☐

REMARK 5.4.3. The preceding proof is somewhat unusual, in that we actually end up proving more than is really needed. All we need to conclude that the sets $F_i(U_i \cap C_i)$ are not separated at 0 is to find a nonzero point in $F_1(U_1 \cap C_1) \cap F_2(U_2 \cap C_2)$. To establish the stronger conclusion that the $F_i(U_i \cap C_i)$ are not locally separated at 0 it suffices to find nonzero points of $F_1(U_1 \cap C_1) \cap F_2(U_2 \cap C_2)$ arbitrarily close to zero. Yet, our proof actually yields a whole "continuous" one-parameter family $\{w_r\}_{r>0,r \text{ small}}$ of such points! This is clearly an anomalous situation, suggesting that in fact the "true conclusion" of the theorem ought to be stronger, saying not only that these points exist, but that that there is a whole continuum of them, perhaps a continuous curve.

It turns out that the correct answer is *not* that the set $F_1(U_1 \cap C_1) \cap F_2(U_2 \cap C_2)$ contains a whole continuous curve $r \mapsto w_r$ of nonzero points. Such a strong conclusion can fail to be true, but the weaker conclusion that $F_1(U_1 \cap C_1) \cap F_2(U_2 \cap C_2)$ contains a *nontrivial connected set containing* 0 is true, even though *this set may fail to be path-connected.* The proof of this stronger conclusion is based on a theorem of Leray-Schauder on connected sets of zeros of a homotopy. The issue is discussed in detail in the paper

Sussmann, H. J., “Transversality conditions and a strong maximum principle for systems of differential inclusions.” In *Proceedings of the 37th IEEE Conference on Decision and Control, Tampa, FL, Dec. 1998*. IEEE publications, New York, 1998, pp. 1-6.

A Postscript version of the paper (compressed with gzip) can be downloaded from the author’s Web page:

http://www.math.rutgers.edu/~sussmann ◊

6. COMPLETION OF THE PROOF

6.1. Application of the transversal intersection theorem. Fix k, $\vec{\tau}$, $\vec{u}$ as above.

Using the fact that C is a Boltyanskii tangent cone to S at $\hat{x}$, we pick a neighborhood V of the origin in $\mathbb{R}^n$ and a continuous map $\varphi : V \cap C \mapsto S$ such that $\varphi(v) = \hat{x} + v + o(\|v\|)$ as $v \to 0$, $v \in C$.

Since the image of $\mathbb{R}^k_+$ under the endpoint map $\mathcal{E}^{\vec{\tau},\vec{u}}$ is contained in the reachable set $\mathcal{R}_{[a,b]}(Q,U,f;\bar{x})$, the image of C under φ is contained in$S$, and the sets $\mathcal{R}_{[a,b]}(Q,U,f;\bar{x})$ and S are locally separated at $\hat{x}$, it follows that the sets $\mathcal{E}^{\vec{\tau},\vec{u}}(\mathbb{R}^k_+)$ and $\varphi(C)$ are locally separated at $\hat{x}$. The transversal intersection theorem then implies that the cones $D\mathcal{E}^{\vec{\tau},\vec{u}}(0)(\mathbb{R}^k_+)$ and C are not strongly transversal. Since C is not a linear subspace, we can conclude from Lemma 5.2.1 that the cones $D\mathcal{E}^{\vec{\tau},\vec{u}}(0)(\mathbb{R}^k_+)$ and C are not transversal. Therefore there exists a nonzero $\bar{\pi} \in \mathbb{R}_n$ such that

$$\bar{\pi} \cdot v \leq 0 \quad \text{whenever} \quad v \in D\mathcal{E}^{\vec{\tau},\vec{u}}(0)(\mathbb{R}^k_+)$$

and

$$\bar{\pi} \cdot v \geq 0 \quad \text{whenever} \quad v \in C\,. \tag{6.1}$$

The formula for $D\mathcal{E}^{\vec{\tau},\vec{u}}(0)(\mathbb{R}^k_+)$ given by the main technical lemma implies that

$$M(b,\tau_j)\cdot\Big(f(x_j,u_j,\tau_j)-f(x_j,u_{*,j},\tau_j)\Big) \in D\mathcal{E}^{\vec{\tau},\vec{u}}(0)(\mathbb{R}^k_+) \quad \text{for} \quad j=1,\ldots,k\,.$$

Therefore

$$\bar{\pi}\cdot M(b,\tau_j)\cdot\Big(f(x_j,u_j,\tau_j)-f(x_j,u_{*,j},\tau_j)\Big) \leq 0 \quad \text{for} \quad j=1,\ldots,k\,,$$

and then

$$\begin{aligned} \bar{\pi}\cdot M(b,\tau_j)\cdot f(x_j,u_j,\tau_j) \quad &\leq \quad \bar{\pi}\cdot M(b,\tau_j)\cdot f(x_j,u_{*,j},\tau_j) \\ &\qquad \text{for} \quad j=1,\ldots,k\,. \end{aligned} \tag{6.2}$$

Clearly, we can also assume that

$$\|\bar{\pi}\| = 1\,. \tag{6.3}$$

6.2. The compactness argument. In the previous subsection, we proved that for every choice of k, $\vec{\tau}$, $\vec{u}$, there exists a $\bar{\pi}$ for which (6.1), (6.2), (6.3) hold.

Given any subset W of $[a,b]\times U$, let $\Pi(W)$ be the set of all $\bar{\pi}\in\mathbb{R}_n$ that satisfy (6.1), (6.3), and

$$\bar{\pi}\cdot M(b,\tau)\cdot f(\xi_*(\tau),u,\tau) \;\le\; \bar{\pi}\cdot M(b,\tau)\cdot f(\xi_*(\tau),\eta_*(\tau),\tau) \quad \text{for all} \quad (\tau,u)\in W\,. \tag{6.4}$$

We have established that $\Pi(W)$ is nonempty whenever W is a finite set $\{(\tau_1,u_1),\ldots,(\tau_k,u_k)\}$ such that $a\le\tau_1<\tau_2<\ldots<\tau_k<b$. Now suppose that W is any finite subset of $[a,b[\times U$. Let

$$W=\{(\tau_1,u_1),\ldots,(\tau_k,u_k)\}\,,\;\; a\le\tau_1\le\tau_2\le\ldots\le\tau_k<b\,.$$

Let W_ℓ be the set

$$\{(\tau_1,u_1),(\tau_2+\frac{1}{\ell},u_2),(\tau_3+\frac{2}{\ell},u_3),\ldots,(\tau_k+\frac{k-1}{\ell},u_k)\,.$$

Then $\Pi(W_\ell)\neq\emptyset$. Pick $\bar{\pi}_\ell\in\Pi(W_\ell)$. Then the $\bar{\pi}_\ell$ belong to the unit sphere of $\mathbb{R}^n$, which is compact. So there is a subsequence $\{\bar{\pi}_{\ell(\nu)}\}_{\nu\in\mathbb{N}}$ that converges to a limit $\bar{\pi}$. Then $\bar{\pi}\in\Pi(W)$, so $\Pi(W)\neq\emptyset$.

A similar limiting argument then shows that $\Pi(W)\neq\emptyset$ if W is any finite subset of $[a,b]\times U$.

Now, it is clear that the set $\Pi(W)$ is compact for every W. Moreover,

$$\Pi(W_1)\cap\ldots\cap\Pi(W_s)=\Pi(W_1\cup\ldots\cup W_s)\neq\emptyset$$

if $W_1,\ldots,W_s$ are finite subsets of $[a,b]\times U$.

Let $\mathcal{W}$ be the set of all finite subsets of $[a,b]\times U$. Then

$$\Big\{\Pi(W)\Big\}_{W\in\mathcal{W}}$$

is a family of compact subsets of the unit sphere of $\mathbb{R}^n$ having the property that every finite intersection of members of the family is nonempty. It follows that

$$\bigcap\Big\{\,\Pi(W):W\in\mathcal{W}\Big\}\neq\emptyset\,.$$

Therefore

$$\Pi([a,b]\times U)\neq\emptyset\,.$$

This means that there exists a $\bar{\pi}\in\mathbb{R}_n$ that satisfies (6.1), (6.3), and is such that

$$\bar{\pi}\cdot M(b,\tau)\cdot f(\xi_*(\tau),u,\tau) \;\le\; \bar{\pi}\cdot M(b,\tau)\cdot f(\xi_*(\tau),\eta_*(\tau),\tau) \quad \text{for all} \quad (\tau,u)\in[a,b]\times U\,. \tag{6.5}$$

6.3. The momentum. Let $\bar\pi$ be such that (6.1), (6.3), and (6.5) hold. Define

$$\pi(t) = \bar\pi \cdot M(b,t)\,.$$

Then π satisfies the adjoint equation

$$\dot\pi(t) = -\pi(t) \cdot A(t)\,.$$

(*Proof.* We know that

$$\frac{\partial M}{\partial t}(t,s) = A(t) \cdot M(t,s)\,.$$

Moreover,

$$M(t,s) \cdot M(s,t) = 1_n\,.$$

If we differentiate this with respect to s, we get

$$\frac{\partial M}{\partial s}(t,s) \cdot M(s,t) + M(t,s) \cdot A(s) \cdot M(s,t) = 0\,.$$

Therefore

$$\frac{\partial M}{\partial s}(t,s) + M(t,s) \cdot A(s) = 0\,.$$

So

$$\frac{\partial M}{\partial s}(t,s) = -M(t,s) \cdot A(s)\,.$$

Then

$$\dot\pi(s) = \bar\pi \cdot \frac{\partial M}{\partial s}(b,s) = -\bar\pi \cdot M(b,s) \cdot A(s) = -\pi(s) \cdot A(s)\,,$$

as desired.)

Condition (6.1) says that

$$-\pi(b) \in C^\perp\,.$$

Condition (6.3) implies that π is nonzero.

Finally, Condition (6.5) says that

$$\begin{aligned} \pi(\tau) \cdot f(\xi_*(\tau), u, \tau) \;\; &\le \;\; \pi(\tau) \cdot f(\xi_*(\tau), \eta_*(\tau), \tau) \\ &\quad\;\; \text{for all} \quad (\tau, u) \in [a,b] \times U\,. \end{aligned} \tag{6.6}$$

Therefore

$$\begin{aligned} H(\xi_*(\tau), u, \pi(\tau), \tau) \;\; &\le \;\; H(\xi_*(\tau), \eta_*(\tau), \pi(\tau), \tau) \\ &\quad\;\; \text{for all} \quad (\tau, u) \in [a,b] \times U\,. \end{aligned}$$

It then follows that

$$H(\xi_*(\tau), \eta_*(\tau), \pi(\tau), \tau) = \max\{H(\xi_*(\tau), u, \pi(\tau), \tau) : u \in U\}$$

for all $\tau \in [a,b]$.

We have thus shown that π satisfies all the desired conditions. Our proof is therefore complete, *under the extra assumption that η_* is continuous.*

6.4. Elimination of the continuity assumption on the reference control. The assumption that η_* is continuous was made to simplify some steps of the proof, but is not really necessary. We now explain how to avoid this hypothesis, and assume only that η_* is a bounded, measurable U-valued function on $[a, b]$.

The continuity of η_* was used in the proof exactly four times. We will now show how, in each case, the hypothesis can be avoided.

1. In page 11, we defined the matrix-valued function $[a, b] \ni t \mapsto A(t)$, and asserted that this function was continuous. If η_* is only bounded and measurable, then the set $\{(\xi_*(t), \eta_*(t), t) : a \leq t \leq b\}$ is contained in a compact subset of $Q \times U \times [a, b]$, so A turns out to be bounded and measurable, though not necessarily continuous. This, however, is good enough, and the properties of the fundamental solution M are the same.
2. In page 13, we defined the set U_0 and asserted that U_0 is compact. If η_* is not continuous, the conclusion that U_0 is compact no longer follows. On the other hand, we can define U_0 in this case to be the *closure* of the set of page 11, and this new set is compact. Using the new U_0 instead of the original one, all the arguments where U_0 occurs are still valid.
3. In page 19, we used the fact that the function F is continuous, which depended very strongly on the continuity of η_*. This problem is much more serious than the previous ones, and to solve it we need some basic facts from the theory of functions of a real variable.
   Recall that, if $[a, b] \ni t \mapsto \psi(t) \in \mathbb{R}^n$ is an integrable function, a *Lebesgue point* of ψ is a $\tau \in]a, b[$ such that
   $$\lim_{h \downarrow 0} \frac{1}{h} \int_{\tau-h}^{\tau+h} \|\psi(t) - \psi(\tau)\| \, dt = 0$$
   It is then a theorem that *almost every point* of $[a, b]$ is a Lebesgue point of ψ.
   To avoid invoking the fact that F is continuous, we make the further restriction that *the times τ_j where our needle variations are made are Lebesgue points of the function F.*
   The argument is then modified as follows. Estimate (4.5) is still valid and useful, but (4.6)—which is still valid—is no longer useful, because there is no reason now for (4.4) to hold.
   On the other hand, if we let

$$\zeta_0(h) \stackrel{\text{def}}{=}$$
$$\frac{1}{h} \max \Big\{ \int_{\tau_j-h}^{\tau_j+h} \|f(\xi_*(t), \eta_*(t), t) - f(\xi_*(\tau_j), \eta_*(\tau_j), \tau_j)\| \, dt : j = 1 \ldots, k \Big\},$$
$$\zeta(h) \stackrel{\text{def}}{=} \sup\{\zeta_0(h') : 0 < h' \leq h\}$$

then the condition that the τ_j are Lebesgue points says precisely that

$$\lim_{h\downarrow 0}\zeta(h)=0\,. \tag{6.7}$$

We then get the integral estimate

$$\Big\|\int_{[a,t]\cap E(\vec{\varepsilon})}\Big(\Delta_2^{\vec{\varepsilon}}(s)-\theta^{\vec{\varepsilon}}(s)\Big)\,ds\Big\|\le$$
$$\Big(4\kappa^2\|\vec{\varepsilon}\|e^{\kappa(b-a)}+k\zeta(\|\vec{\varepsilon}\|)+\tilde{\lambda}(\|\vec{\varepsilon}\|)\Big)\cdot\|\vec{\varepsilon}\|\,,$$

which replaces (4.7). The rest of proof of the main technical lemma remains unchanged.

4. In the compactness argument in page 31, we approximated the τ_j, that could in principle be equal, by points $\tau_{j,\ell}$ that are all different. In our new situation this argument has to be refined, because the approximating $\tau_{j,\ell}$ have to be Lebesgue points of F, and the passage to the limit in (6.4) requires that

$$\lim_{\ell\to\infty}f(\xi_*(\tau_{j,\ell}),\eta_*(\tau_{j,\ell}),\tau_{j,\ell})=f(\xi_*(\tau_j),\eta_*(\tau_j),\tau_j)\,.$$

To take care of this we use *Lusin's theorem*, which guarantees that for every positive β there exists a compact subset J_β of $[a,b]$ such that $\mathrm{meas}([a,b]\backslash J_\beta)\le\beta$ having the property that the restriction of F to J_β is continuous.

We pick sets J_{β_ν} as above, corresponding to a sequence $\{\beta_\nu\}$ that converges to zero and then, for each ν, we let $\tilde{G}_\nu$ be the set of all points of J_{β_ν} that are also Lebesgue points of F, and let G_ν be the set of all points of density of G_ν. (Recall that a *point of density* of a measurable set E is a point of E which is a Lebesgue point of the indicator function of E.) We let $G=\bigcup_{\nu\in\mathbb{N}}G_\nu$. Then G is a subset of full measure of $[a,b]$, and it is easy to see that the limiting argument works when the τ_j belong to G.

The proof of the maximum principle, thus modified, works exactly as before, provided that the needle variations are made using points $\tau_j\in G$. The result now is the same as before, except that the Hamiltonian maximization condition is only proved for $t\in G$. This explains why Condition E4 of Theorem 2.1.1 says that the maximization holds “almost everywhere” rather than “everywhere.”