TRANSVERSALITY, SET SEPARATION, AND THE PROOF OF THE MAXIMUM PRINCIPLE

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CONTENTS

1. THE TRANSVERSALITY CONDITIONS

We now move one step further, and present two even more general versions of the Pontryagin maximum principle. The new versions are still rather simple as far as technical hypotheses go, but contain a new ingredient, namely, the "transversality condition." This shows up as an additional necessary condition for optimality for a problem in which the terminal constraint, instead of being of the form "$\xi(b) = \hat{x}$" considered so far, is of the more general form

$$\xi(b) \in S\,,$$

where S is some given subset of Q.

1.1. Cones. If X is a real linear space, a *cone* in X is a subset C of X such that

1. $C \neq \emptyset$,
2. C is a union of rays (that is, whenever $r \in \mathbb{R}$, $r \geq 0$, and $c \in C$, it follows that $r \cdot c \in C$).

If C is a cone in X then necessarily $0 \in C$. In general, a cone need not be convex. A cone C is convex if and only if it is closed under addition, that is, $c_1 + c_2 \in C$ whenever $c_1, c_2 \in C$.

If X is endowed with a topology, then we can talk about *closed cones.* We will only be interested in cones in *finite-dimensional* real linear spaces X. In that case, X has a canonical topology, so the concept of a closed cone is well defined.

The (algebraic) *polar* of a cone C in a real linear space X is the set

$$C^{\perp} \stackrel{\text{def}}{=} \{w \in X^{\dagger} : \langle w, c\rangle \leq 0 \text{ for all } c \in C\}\,,$$

where $X^{\dagger}$ denotes the (algebraic) dual of X, that is, the set of all linear functionals $w : X \mapsto \mathbb{R}$. (When X is a topological linear space one wants to consider the *topological dual*, consisting of the continuous linear functionals, and the corresponding *topological polar* of a cone. But for us, since we are only interested in the finite-dimensional case, algebraic duals and polars are sufficient.)

It is easy to see that *the polar of any cone C in X is a convex cone.* Moreover, if X is finite-dimensional and C is a cone in X, then $C^{\perp}$ is closed in $X^{\dagger}$, and $C^{\perp\perp}$—which is a subset of $X^{\dagger\dagger}$, a space which is canonically identified with X—is the closed convex hull of C. In particular, $C^{\perp\perp} = C$ if and only if C is closed and convex.

1.2. Boltyanskii tangent cones to a set at a point. Let S be a subset of a finite-dimensional real linear space X, and let $\hat{x}$ be a point of S. A *Boltyanskii tangent cone* (or "approximating cone") to S at $\hat{x}$ is a closed convex cone C in X such that there exist a neighborhood U of 0 in X and a continuous map $\varphi : U \cap C \mapsto S$ having the property that

$$\varphi(v) = \hat{x} + v + o(\|v\|) \qquad \text{as} \qquad v \to 0,\ v \in C\,.$$

1.3. The maximum principle with transversality conditions for fixed time interval problems. We assume that the following conditions are satisfied:

C1. n, m are nonnegative integers;
C2. Q is an open subset of $\mathbb{R}^n$;
C3. U is a closed subset of $\mathbb{R}^m$;
C4. a, b are real numbers such that $a \leq b$;
C5. $Q \times U \times [a,b] \ni (x,u,t) \mapsto f(x,u,t) = (f^1(x,u,t), \dots, f^n(x,u,t)) \in \mathbb{R}^n$ and $Q \times U \times [a,b] \ni (x,u,t) \mapsto L(x,u,t) \in \mathbb{R}$ are continuous maps;
C6. for each $(u,t) \in U \times [a,b]$ the maps

$$Q \ni x \mapsto f(x,u,t) = \Big(f^1(x,u,t), \dots, f^n(x,u,t)\Big) \in \mathbb{R}^n$$

and

$$Q \ni x \mapsto L(x,u,t) \in \mathbb{R}$$

are continuously differentiable, and their partial derivatives with respect to the x coordinates are continuous functions of (x,u,t);
C7. $\bar{x}$ is a given point of Q, and S is a subset of Q;
C8. $TCP_{[a,b]}(Q,U,f)$ (the set of all "trajectory-control pairs defined on $[a,b]$ for the data Q, U, f") is the set of all pairs (ξ,η) such that:
  a. $[a,b] \ni t \mapsto \eta(t) \in U$ is a measurable bounded map,
  b. $[a,b] \ni t \mapsto \xi(t) \in Q$ is an absolutely continuous map,
  c. $\dot{\xi}(t) = f(\xi(t), \eta(t), t)$ for almost every $t \in [a,b]$;.
C9. $TCP_{[a,b]}(Q,U,f) \ni (\xi,\eta) \mapsto J(\xi,\eta) \in \mathbb{R}$ is the functional given by

$$J(\xi,\eta) \stackrel{\text{def}}{=} \int_a^b L(\xi(t), \eta(t), t)\, dt\,;$$

C10. $\gamma_* = (\xi_*, \eta_*)$ (the "reference TCP") is such that
  a. $\gamma_* \in TCP_{[a,b]}(Q,U,f)$,
  b. $\xi_*(a) = \bar{x}$ and $\xi_*(b) \in S$,
  c. $J(\xi_*, \eta_*) \leq J(\xi,\eta)$ for all $(\xi,\eta) \in TCP_{[a,b]}(Q,U,f)$ such that $\xi(a) = \bar{x}$ and $\xi(b) \in S$,
C11. C is a Boltyanskii tangent cone to S at the point $\hat{x} = \xi_*(b)$.

THEOREM 1.3.1. *Assume that the data n, m, Q, U, a, b, f, L, $\bar{x}$, S, satisfy conditions C1-C7, $TCP_{[a,b]}(Q,U,f)$ and J are defined by C8-C9, $\gamma_* = (\xi_*, \eta_*)$ satisfies C10, $\hat{x}$ is defined by C11 and C satisfies C11. Define the Hamiltonian H to be the function*

$$Q \times U \times \mathbb{R}^n \times \mathbb{R} \times [a,b] \ni (x,u,p,p_0,t) \mapsto H(x,u,p,p_0,t) \in \mathbb{R}$$

given by

$$H(x,u,p,p_0,t) \stackrel{\text{def}}{=} \langle p, f(x,u,t)\rangle - p_0 L(x,u,t)\,.$$

Then there exists a pair (π,π_0) *such that*

E1. $[a,b] \ni t \mapsto \pi(t) \in \mathbb{R}^n$ *is an absolutely continuous map;*
E2. $\pi_0 \in \mathbb{R}$ *and* $\pi_0 \geq 0$*;*
E3. $(\pi(t),\pi_0) \neq (0,0)$ *for every* $t \in [a,b]$*;*
E4. *the "adjoint equation" holds, that is,*

$$\dot{\pi}(t) = -\frac{\partial H}{\partial x}(\xi_*(t),\eta_*(t),\pi(t),\pi_0,t)$$

for almost every $t \in [a,b]$*;*

E5. *the "Hamiltonian maximization condition" holds, that is,*

$$H(\xi_*(t),\eta_*(t),\pi(t),\pi_0,t) = \max\Big\{H(\xi_*(t),u,\pi(t),\pi_0,t) : u \in U\Big\}$$

for almost every $t \in [a,b]$,

E6. *the "transversality condition" holds, that is,*

$$-\pi(b) \in C^{\perp}\,.$$

1.4. The maximum principle with transversality conditions for variable time interval problems. We assume that the following conditions are satisfied:

C1. n, m are nonnegative integers;
C2. Q is an open subset of $\mathbb{R}^n$;
C3. U is a closed subset of $\mathbb{R}^m$;
C4. the maps $Q \times U \ni (x,u) \mapsto f(x,u) = (f^1(x,u),\ldots,f^n(x,u)) \in \mathbb{R}^n$ and $Q \times U \ni (x,u) \mapsto L(x,u) \in \mathbb{R}$ are continuous;
C5. for each $u \in U$ the maps

$$Q \ni x \mapsto f(x,u) = \Big(f^1(x,u),\ldots,f^n(x,u)\Big) \in \mathbb{R}^n$$

and $$Q \ni x \mapsto L(x,u) \in \mathbb{R}$$

are continuously differentiable, and their partial derivatives with respect to the x coordinates are continuous functions of (x,u);

C6. $\bar{x}$ is a given point and S is a given subset of Q;
C7. $TCP(Q,U,f)$ (the set of all "trajectory-control pairs for the data Q,U,f") is the set given by

$$TCP(Q,U,f) \stackrel{\text{def}}{=} \bigcup \Big\{TCP_{[a,b]}(Q,U,f) : a,b \in \mathbb{R}, a \leq b\Big\}\,,$$

where, for $a,b \in \mathbb{R}$ such that $a \leq b$, $TCP_{[a,b]}(Q,U,f)$ is the set of all pairs (ξ,η) such that:

a. $[a,b] \ni t \mapsto \eta(t) \in U$ is a measurable bounded map;
b. $[a,b] \ni t \mapsto \xi(t) \in Q$ is an absolutely continuous map;
c. $\dot{\xi}(t) = f(\xi(t),\eta(t))$ for almost every $t \in [a,b]$.

C8. $TCP(Q,U,f) \ni (\xi,\eta) \mapsto J(\xi,\eta) \in \mathbb{R}$ is the functional given by

$$J(\xi,\eta) \stackrel{\text{def}}{=} \int_a^b L(\xi(t),\eta(t))\,dt \text{ for } (\xi,\eta) \in TCP_{[a,b]}(Q,U,f),\ a,b \in \mathbb{R},\ a \leq b.$$

C9. $\gamma_* = (\xi_*,\eta_*)$ (the "reference TCP") and a_*, b_* are such that
   a. $a_* \in \mathbb{R}$, $b_* \in \mathbb{R}$, and $\gamma_* \in TCP_{[a_*,b_*]}(Q,U,f)$;
   b. $\xi_*(a_*) = \bar{x}$ and $\xi_*(b_*) \in S$;
   c. $J(\xi_*,\eta_*) \leq J(\xi,\eta)$ for all $a,b \in \mathbb{R}$ such that $a \leq b$ and all $(\xi,\eta) \in TCP_{[a,b]}(Q,U,f)$ such that $\xi(a) = \bar{x}$ and $\xi(b) \in S$.

C10. C is a Boltyanskii tangent cone to S at the point $\hat{x} = \xi_*(b_*)$.

THEOREM 1.4.1. *Assume that the data n, m, Q, U, f, L, $\bar{x}$, S satisfy conditions C1-C6, $TCP(Q,U,f)$ and J are defined by C7-C8, and a_*, b_*, $\gamma_* = (\xi_*,\eta_*)$, $\hat{x}$, C, satisfy C9. Define the Hamiltonian H to be the function*

$$Q \times U \times \mathbb{R}^n \times \mathbb{R} \ni (x,u,p,p_0) \mapsto H(x,u,p,p_0) \in \mathbb{R}$$

given by

$$H(x,u,p,p_0) \stackrel{\text{def}}{=} \langle p, f(x,u)\rangle - p_0 L(x,u)\,.$$

Then there exists a pair (π,π_0) such that

E1. $[a_*,b_*] \ni t \mapsto \pi(t) \in \mathbb{R}^n$ *is an absolutely continuous map;*

E2. $\pi_0 \in \mathbb{R}$ *and* $\pi_0 \geq 0$*;*

E3. $(\pi(t),\pi_0) \neq (0,0)$ *for every* $t \in [a_*,b_*]$*;*

E4. *the "adjoint equation" holds, that is,*

$$\dot{\pi}(t) = -\frac{\partial H}{\partial x}(\xi_*(t),\eta_*(t),\pi(t),\pi_0)$$

for almost every $t \in [a_*,b_*]$*;*

E5. *the "Hamiltonian maximization condition with zero value" holds, that is,*

$$0 = H(\xi_*(t),\eta_*(t),\pi(t),\pi_0) = \max\Big\{H(\xi_*(t),u,\pi(t),\pi_0) : u \in U\Big\}$$

for almost every $t \in [a,b]$,

E6. *the "transversality condition" holds, that is,*

$$-\pi(b_*) \in C^{\perp}\,.$$

1.5. An example. We consider the "one-dimensional landing problem" in which it is desired to find, for a given initial point $(\alpha,\beta) \in \mathbb{R}^2$, a trajectory-control pair (ξ,η) of the system

$$\dot{x} = y\,, \qquad \dot{y} = u\,, \qquad -1 \leq u \leq 1\,,$$

such that ξ goes from (α,β) to some point in the y axis in minimum time. This is exactly the same as the "soft landing problem" discussed before, except that now we are not asking the landing to be soft, that is, our terminal condition is that the terminal position x be zero, but we are not imposing any condition on the terminal velocity.

The configuration space Q for our problem is $\mathbb{R}^2$, the control set U is the compact interval $[-1,1]$, the dynamical law f is given by

$$f(x,y,u) = \left[\begin{array}{c} y \\ u \end{array}\right],$$

and the Lagrangian is identically equal to 1. The terminal constraint is $\xi(b) \in S$, where

$$S = \{(x,y) \in \mathbb{R}^2 : x = 0\}.$$

The Hamiltonian H is given by

$$H(x,y,u,p_x,p_y,p_0) = p_x y + p_y u - p_0,$$

where we are using p_x, p_y to denote the two components of the momentum variable p.

Assume that (ξ_*, η_*) is a solution of our minimum time problem, and that $(\xi_*, \eta_*) \in TCP_{[a_*,b_*]}(Q,U,f)$.

Then Theorem 1.4.1 tells us that there exists a pair (π, π_0) satisfying all the conditions of the conclusion. Write $\pi(t) = (\pi_x(t), \pi_y(t))$. The adjoint equation then implies

$$\begin{array}{rcl} \dot{\pi}_x(t) & = & 0, \\ \dot{\pi}_y(t) & = & -\pi_x(t). \end{array}$$

Therefore the function π_x is constant. Let $A \in \mathbb{R}$ be such that $\pi_x(t) = A$ for all $t \in [a_*, b_*]$. Then there must exist a constant B such that

$$\pi_y(t) = B - At \quad \text{for} \quad t \in [a_*, b_*].$$

Now, if A and B were both equal to zero, the function π_y would vanish identically and then the Hamiltonian maximization condition would say that the function

$$[-1,1] \ni u \mapsto 0$$

is maximized by taking $u = \eta_*(t)$, a fact that would give us no information whatsoever about η_*.

Fortunately, the conditions of Theorem 1.4.1 imply that *A and B cannot both vanish.* To see this, observe that if $A = B = 0$ then it follows that $\pi_x(t) \equiv \pi_y(t) \equiv 0$. But then the nontriviality condition tells us that $\pi_0 \neq 0$. So the value $H(\xi_*(t), \eta_*(t), \pi(t), \pi_0)$ would be equal to $-\pi_0$, which is not equal to zero. This contradicts the fact that, for our time-varying problem, the Hamiltonian is supposed to vanish.

It is clear that the set S itself is a Boltyanskii tangent cone to S at the terminal point $\xi_*(b_*)$. Therefore the transversality condition says that $\pi(b_*) \in S^\perp$, i.e., that

$$\pi_y(b_*) = 0.$$

So $\mathbb{R} \ni t \mapsto \pi_y(t) = B - At$ is a linear function which is not identically zero (because A and B do not both vanish) but vanishes at the endpoint b_* of the interval $[a_*, b_*]$. Therefore $\pi_y(t)$ *never vanishes on* $[a_*, b_*[$. It follows that *the optimal control* η_* *is either constantly equal to* 1 *or constantly equal to* -1. We have thus proved that *all optimal controls are bang-bang and constant.*

1.6. A remark on more general endpoint conditions. Suppose we wanted to consider endpoint conditions of the form

$$(\xi(a), \xi(b)) \in S\,,$$

where S is a subset of $Q \times Q$. (For example, we could require that $\xi(a) \in S_1$ and $\xi(b) \in S_2$, in which case we would take $S = S_1 \times S_2$. Or we could require "periodic endpoint conditions" $\xi(a) = \xi(b)$, in which case we would take S to be the diagonal of Q, that is, the set $\{(x, x) : x \in Q\}$.)

These problems can easily be reduced to the ones considered in §1.3 and §1.4, by means of the following trick: work with a new system in $Q \times Q$, with state evolution equations $\dot{x} = f(x, u, t)$, $\dot{x}' = 0$. (This has the effect of keeping track of the initial state so that it is still there as part of the state at any later time.) One needs an aditional transformation because now we do not want to have to initialize the state at a prespecified point $(\bar{x}, \bar{x}')$ of $Q \times Q$. For this reason, we add an extra piece of dynamics prior to time a_*, by means of the dynamical law $\dot{x} = v$, $\dot{x}' = v$, on the time interval $[a_*-1, a_*]$, where v is a new control with values in $\mathbb{R}^n$. This has the effect of allowing us to fix a point $(\bar{x}, \bar{x}')$ of $Q \times Q$ to be the initial condition at time $a_* - 1$, while letting the state at time a_* be of the form (x, x) with x arbitrary.

2. THE SEPARATION THEOREM

We now proceed to "geometrize" the problem by completely eliminating the optimization aspect, and working instead with a question about *separation of two sets.*

Precisely, let us say that two subsets S_1, S_2 of a set A are *separated at a point* $\hat{x} \in S_1 \cap S_2$ if

$$S_1 \cap S_2 = \{\hat{x}\}\,,$$

that is, if $S_1 \cap S_2$ contains no points other than $\hat{x}$.

If A is a topological space, we say that S_1 and S_2 are *locally separated at* $\hat{x}$ if there exists a neighborhood U of $\hat{x}$ such that

$$U \cap S_1 \cap S_2 = \{\hat{x}\}\,.$$

2.1. The maximum principle as a set separation condition for fixed time interval problems. We assume that the following conditions are satisfied:

C1. n, m are nonnegative integers;
C2. Q is an open subset of $\mathbb{R}^n$;
C3. U is a closed subset of $\mathbb{R}^m$;
C4. a, b are real numbers such that $a \leq b$;

C5. $Q\times U\times[a,b]\ni(x,u,t)\mapsto f(x,u,t)=(f^1(x,u,t),\ldots,f^n(x,u,t))\in\mathbb{R}^n$ is a continuous map;

C6. for each $(u,t)\in U\times[a,b]$ the map

$$Q\ni x\mapsto f(x,u,t)=\Big(f^1(x,u,t),\ldots,f^n(x,u,t)\Big)\in\mathbb{R}^n$$

is continuously differentiable, and its partial derivatives with respect to the x coordinates are continuous functions of (x,u,t);

C7. $\bar{x}$ is a given point of Q, and S is a subset of Q;

C8. $TCP_{[a,b]}(Q,U,f)$ (the set of all "trajectory-control pairs defined on $[a,b]$ for the data Q,U,f") is the set of all pairs (ξ,η) such that:
 a. $[a,b]\ni t\mapsto\eta(t)\in U$ is a measurable bounded map,
 b. $[a,b]\ni t\mapsto\xi(t)\in Q$ is an absolutely continuous map,
 c. $\dot{\xi}(t)=f(\xi(t),\eta(t),t)$ for almost every $t\in[a,b]$;.

C9. $\mathcal{R}_{[a,b]}(Q,U,f;x)$ (the "reachable set from x for the data Q,U,f over the interval $[a,b]$") is defined by

$$\mathcal{R}_{[a,b]}(Q,U,f;x)\stackrel{\text{def}}{=}$$
$$\Big\{y:(\exists(\xi,\eta)\in TCP_{[a,b]}(Q,U,f))(\xi(a)=x\wedge\xi(b)=y)\Big\}.$$

C10. $\gamma_*=(\xi_*,\eta_*)$ (the "reference TCP") is such that
 a. $\gamma_*\in TCP_{[a,b]}(Q,U,f)$,
 b. $\xi_*(a)=\bar{x}$ and $\xi_*(b)\in S$,
 c. $\mathcal{R}_{[a,b]}(Q,U,f;\bar{x})$ and S are blocally separated at the point $\hat{x}=\xi_*(b)$;

C11. C is a Boltyanskii tangent cone to S at $\hat{x}$.

THEOREM 2.1.1. *Assume that the data n, m, Q, U, a, b, f, $\bar{x}$, S, satisfy conditions C1-C7, $TCP_{[a,b]}(Q,U,f)$ and $\mathcal{R}_{[a,b]}(Q,U,f;x)$ are defined by C8-C9, $\gamma_*=(\xi_*,\eta_*)$ satisfies C10, $\hat{x}$ is defined by C10, and C satisfies C11.*

Assume, moreover, that C is not a linear subspace of $\mathbb{R}^n$.

Define the Hamiltonian H to be the function

$$Q\times U\times\mathbb{R}^n\times[a,b]\ni(x,u,p,t)\mapsto H(x,u,p,t)\in\mathbb{R}$$

given by

$$H(x,u,p,t)\stackrel{\text{def}}{=}\langle p,f(x,u,t)\rangle\,.$$

Then there exists a π such that

E1. $[a,b]\ni t\mapsto\pi(t)\in\mathbb{R}^n$ *is an absolutely continuous map;*

E2. $\pi(t)\neq 0$ *for every* $t\in[a,b]$*;*

E3. *the "adjoint equation" holds, that is,*

$$\dot{\pi}(t)=-\frac{\partial H}{\partial x}(\xi_*(t),\eta_*(t),\pi(t),t)$$

for almost every $t\in[a,b]$*;*

E4. *the "Hamiltonian maximization condition" holds, that is,*

$$H(\xi_*(t),\eta_*(t),\pi(t),t)=\max\Big\{H(\xi_*(t),u,\pi(t),t):u\in U\Big\}$$

for almost every $t\in[a,b]$,

E5 *the "transversality condition" holds, that is,*

$$-\pi(b)\in C^{\perp}\,.$$

2.2. The maximum principle as a set separation condition for variable time interval problems. We assume that the following conditions are satisfied:

C1. n, m are nonnegative integers;
C2. Q is an open subset of $\mathbb{R}^n$;
C3. U is a closed subset of $\mathbb{R}^m$;
C4. the map $Q\times U\ni(x,u)\mapsto f(x,u)=(f^1(x,u),\ldots,f^n(x,u))\in\mathbb{R}^n$ is continuous;
C5. for each $u\in U$ the map

$$Q\ni x\mapsto f(x,u)=\Big(f^1(x,u),\ldots,f^n(x,u)\Big)\in\mathbb{R}^n$$

is continuously differentiable, and its partial derivatives with respect to the x coordinates are continuous functions of (x,u);
C6. $\bar{x}$ is a given point and S is a given subset of Q;
C7. $TCP(Q,U,f)$ (the set of all "trajectory-control pairs for the data Q,U,f") is the set given by

$$TCP(Q,U,f)\stackrel{\text{def}}{=}\bigcup\Big\{TCP_{[a,b]}(Q,U,f):a,b\in\mathbb{R},a\le b\Big\}\,,$$

where, for $a,b\in\mathbb{R}$ such that $a\le b$, $TCP_{[a,b]}(Q,U,f)$ is the set of all pairs (ξ,η) such that:
  a. $[a,b]\ni t\mapsto\eta(t)\in U$ is a measurable bounded map;
  b. $[a,b]\ni t\mapsto\xi(t)\in Q$ is an absolutely continuous map;
  c. $\dot{\xi}(t)=f(\xi(t),\eta(t))$ for almost every $t\in[a,b]$.

C8. $\mathcal{R}(Q,U,f;x)$ (the "reachable set from x for the data Q,U,f") is defined by

$$\mathcal{R}(Q,U,f;x)\stackrel{\text{def}}{=}\bigcup_{-\infty<a\le b<+\infty}\mathcal{R}_{[a,b]}(Q,U,f;x)$$

where

$$\mathcal{R}_{[a,b]}(Q,U,f;x)\stackrel{\text{def}}{=}\Big\{y:(\exists(\xi,\eta)\in TCP_{[a,b]}(Q,U,f))(\xi(a)=x\wedge\xi(b)=y)\Big\}\,.$$

C9. $\gamma_*=(\xi_*,\eta_*)$ (the "reference TCP") and a_*, b_* are such that
  a. $a_*\in\mathbb{R}$, $b_*\in\mathbb{R}$, and $\gamma_*\in TCP_{[a_*,b_*]}(Q,U,f)$;
  b. $\xi_*(a_*)=\bar{x}$ and $\xi_*(b_*)\in S$;
  c. $\mathcal{R}(Q,U,f;\bar{x})$ and S are locally separated at the point $\hat{x}=\xi_*(b_*)$.

C10. C is a Boltyanskii tangent cone to S at $\hat{x}$.

THEOREM 2.2.1. *Assume that the data n, m, Q, U, f, $\bar{x}$, S satisfy conditions C1-C6, $TCP(Q,U,f)$ and $\mathcal{R}(Q,U,f;\bar{x})$ are defined by C7-C8, and a_*, b_*, $\gamma_* = (\xi_*, \eta_*)$, $\hat{x}$, C, satisfy C9 and C10.*

Assume, moreover, that C is not a linear subspace of $\mathbb{R}^n$.

Define the Hamiltonian H to be the function

$$Q \times U \times \mathbb{R}^n \ni (x, u, p) \mapsto H(x, u, p) \in \mathbb{R}$$

given by

$$H(x, u, p) \stackrel{\text{def}}{=} \langle p, f(x, u) \rangle \,.$$

Then there exists a π such that

E1. $[a_*, b_*] \ni t \mapsto \pi(t) \in \mathbb{R}^n$ *is an absolutely continuous map;*
E2. $\pi(t) \neq 0$ *for every* $t \in [a_*, b_*]$*;*
E3. *the "adjoint equation" holds, that is,*

$$\dot{\pi}(t) = -\frac{\partial H}{\partial x}(\xi_*(t), \eta_*(t), \pi(t))$$

for almost every $t \in [a_*, b_*]$*;*

E4. *the "Hamiltonian maximization condition with zero value" holds, that is,*

$$0 = H(\xi_*(t), \eta_*(t), \pi(t)) = \max\Big\{H(\xi_*(t), u, \pi(t)) : u \in U\Big\}$$

for almost every $t \in [a, b]$,

E5. *the "transversality condition" holds, that is,*

$$-\pi(b_*) \in C^\perp \,.$$

3. THE PROOF OF THE MAXIMUM PRINCIPLE

We now present a proof of Theorem 2.1.1. The basic strategy is to make "needle variations," combine these variations into "packets of needle variations," "propagate" the effects of these variations to the terminal point of the reference trajectory, and construct an "approximating cone" to the reachable set. Then a topological argument will be used to derive, from the separation assumption for the reachable set and the set, the existence of a hyperplane separating the corresponding tangent cones. By propagating backwards via the adjoint equation the linear functional that defines this hyperplane, we will get the desired momentum vector π.

In order to avoid technical complications, we will first do the proof under the following additional simplifying assumption:

(SA) the reference control η_* is continuous.

Assumption (SA) is not needed, but the proof without it is a little bit more technical, so we will first assume (SA), and then explain how to get rid of this condition.

So from now on *we assume that the data n, m, Q, U, a, b, f, $\bar{x}$, S, satisfy conditions C1-C7 of §2.1, $TCP_{[a,b]}(Q,U,f)$ and $\mathcal{R}_{[a,b]}(Q,U,f;x)$ are defined by C8-C9 of §2.1, $\gamma_* = (\xi_*, \eta_*)$ satisfies C10 of §2.1, $\hat{x}$ is defined by C10 of §2.1, C satisfies C11 of §2.1, and (SA) holds.*

3.1. The variational equation. Define

$$A(t) = \frac{\partial f}{\partial x}(\xi_*(t), \eta_*(t), t)\,.$$

Then, under our hypotheses—including the additional condition (SA)—the map $[a,b] \ni t \mapsto A(t) \in \mathbb{R}^{n\times n}$ is continuous.

The linear time-varying ordinary differential equation

$$\dot{v}(t) = A(t)\cdot v(t) \tag{3.1}$$

is the *variational equation along the reference TCP* (ξ_*, η_*).

We use $M(t,s)$ to denote the fundamental matrix solution of (3.1). That is, for each $s \in [a,b]$ the map

$$[a,b] \ni t \mapsto M(t,s) \in \mathbb{R}^{n\times n}$$

is the solution of the matrix-valued initial value problem

$$\begin{cases} \dot{X}(t) = A(t)\cdot X(t)\,, \\ X(s) = \mathbf{1}_n\,, \end{cases} \tag{3.2}$$

where $\mathbf{1}_n$ is the identity matrix of $\mathbb{R}^n$.

It then follows that the map

$$[a,b]\times[a,b] \ni (t,s) \mapsto M(t,s) \in \mathbb{R}^{n\times n}$$

is continuous and satisfies the integral equation

$$M(t,s) = \mathbf{1}_n + \int_s^t A(r)\cdot M(r,s)\,dr \quad \text{for} \quad t,s\in[a,b]\,.$$

3.2. Packets of needle variations. Fix

NV1. a positive integer k

NV2. a k-tuple $\vec{\tau} = (\tau_1, \dots, \tau_k)$ (the "variation times") such that

$$a \le \tau_1 < \tau_2 < \cdots < \tau_k < b\,,$$

NV3. a k-tuple $\vec{u} = (u_1, \dots, u_k)$ (the "variation controls") of members of the control set U.

Use $MB([a,b],U)$ to denote the set of all controls defined on the interval $[a,b]$. (That is, the members of $MB([a,b],U)$ are the bounded measurable maps $\eta : [a,b] \mapsto U$.)

For a positive real number α, define

$$\mathbb{R}^k_+(\alpha) \stackrel{\text{def}}{=} \{(x_1, \dots, x_k) \in \mathbb{R}^k : 0 \le x_j \le \alpha \text{ for } j = 1, \dots, k\}\,.$$

Let

$$\bar{\varepsilon} = \min(\tau_2 - \tau_1, \tau_3 - \tau_2, \dots, \tau_k - \tau_{k-1}, b - \tau_k)\,.$$

We then define the *packet of needle variations of the reference control* η_* *associated to* $\vec{\tau}, \vec{u}$ to be the map

$$\mathbb{R}^k_+(\bar{\varepsilon}) \ni \vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k) \mapsto \eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}} \in MB([a,b], U)$$

that associates to every $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k) \in \mathbb{R}^k_+(\bar{\varepsilon})$ a control $\eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}$ defined as follows:

$$\eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) = \begin{cases} \eta_*(t) & \text{if} \quad t \in [a,b] \backslash \bigcup_{j=1}^k [\tau_j, \tau_j + \varepsilon_j]\,, \\ u_j & \text{if} \quad t \in [\tau_j, \tau_j + \varepsilon_j],\ j \in \{1, \dots, k\}\,. \end{cases}$$

3.3. The endpoint map. Given $k, \vec{\tau}, \vec{u}$ as above, we let $\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}$ be, for $\vec{\varepsilon} \in \mathbb{R}^k_+(\bar{\varepsilon})$, the unique solution $t \mapsto \xi(t)$ of the equation

$$\dot{\xi}(t) = f(\xi(t), \eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t), t)$$

such that $\xi(a) = \bar{x}$.

Uniqueness follows because the map

$$Q \times [a,b] \ni (x,t) \mapsto f(x, \eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t), t)$$

is Lipschitz with respect to x with a bounded Lipschitz constant as long as (x,t) belongs to a compact subset of $Q \times [a,b]$.

In principle, $\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}$ need not exist on the whole interval $[a,b]$. The *endpoint map*

$$\vec{\varepsilon} \mapsto \mathcal{E}^{\vec{\tau},\vec{u}}(\vec{\varepsilon})$$

associated to $\vec{\tau}, \vec{u}$ is the map that assigns to each $\vec{\varepsilon} \in \mathbb{R}^k_+(\bar{\varepsilon})$ the terminal point

$$\mathcal{E}^{\vec{\tau},\vec{u}}(\vec{\varepsilon}) \stackrel{\text{def}}{=} \xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(b) \tag{3.3}$$

of the trajectory $\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}$. Precisely, the *domain* $\mathcal{D}^{\vec{\tau},\vec{u}}$ of $\mathcal{E}^{\vec{\tau},\vec{u}}$ is the set of those $\vec{\varepsilon} \in \mathbb{R}^k_+(\bar{\varepsilon})$ such that $\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}$ exists on the whole interval $[a,b]$ and, for $\vec{\varepsilon} \in \mathcal{D}^{\vec{\tau},\vec{u}}$, $\mathcal{E}^{\vec{\tau},\vec{u}}(\vec{\varepsilon})$ is defined by (3.3).

3.4. The main technical lemma.

LEMMA 3.4.1. *Given* $k, \vec{\tau}, \vec{u}$ *as above, write*

$$x_j = \xi_*(\tau_j)\,, \quad u_{*,j} = \eta_*(\tau_j)\,.$$

Then there exists an $\tilde{\varepsilon}$ *such that:*

1. $0 < \tilde{\varepsilon} \le \bar{\varepsilon}$,
2. $\mathbb{R}^k_+(\tilde{\varepsilon}) \subseteq \mathcal{D}^{\vec{\tau},\vec{u}}$,
3. *the endpoint map* $\mathcal{E}^{\vec{\tau},\vec{u}}$ *is continuous on* $\mathbb{R}^k_+(\tilde{\varepsilon})$,

4. *$\mathcal{E}^{\vec{\tau},\vec{u}}$ is differentiable at $0 \in \mathbb{R}^k$, and the differential $D\mathcal{E}^{\vec{\tau},\vec{u}}(0)$ of $\mathcal{E}^{\vec{\tau},\vec{u}}$ at 0 is the linear map*

$$\mathbb{R}^k \ni \vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \mapsto \Lambda(\vec{\alpha}) \in \mathbb{R}^n$$

given by

$$\Lambda(\vec{\alpha}) \stackrel{\text{def}}{=} \sum_{j=1}^{k} \alpha_j \cdot M(b, \tau_j) \cdot \Big(f(x_j, u_j, \tau_j) - f(x_j, u_{*,j}, \tau_j) \Big) .$$

Conclusion 4 says that

$$\mathcal{E}^{\vec{\tau},\vec{u}}(\vec{\varepsilon}) = \xi_*(b) + \sum_{j=1}^{k} \varepsilon_j \cdot M(b, \tau_j) \cdot \Big(f(x_j, u_j, \tau_j) - f(x_j, u_{*,j}, \tau_j) \Big) + o(\|\vec{\varepsilon}\|)$$

as $\vec{\varepsilon} \to 0$ via values in $\mathbb{R}^k_+$.

Proof. We fix a positive number δ such that the set

$$\tilde{K} = \{(x, t) \in \mathbb{R}^n \times [a, b] : \|x - \xi_*(t)\| \leq \delta\}$$

is entirely contained in $Q \times [a, b]$.

We let U_0 be the union of the sets $\{\eta_*(t) : a \leq t \leq b\}$ and $\{u_1, \ldots, u_k\}$. Then U_0 is compact. We define $K = \{(x, u, t) : (x, t) \in \tilde{K} \wedge u \in U_0\}$, so K is a compact subset of $Q \times U \times [a, b]$.

Since f and $\frac{\partial f}{\partial x}$ are continuous on $Q \times U \times [a, b]$, we can pick a constant κ such that

$$\|f(x, u, t)\| + \left\| \frac{\partial f}{\partial x}(x, u, t) \right\| \leq \kappa \quad \text{whenever} \quad (x, u, t) \in Q \times U \times [a, b] .$$

For $\vec{\varepsilon} \in \mathbb{R}^k_+(\bar{\varepsilon})$, let $\beta(\vec{\varepsilon})$ be the maximum of all real numbers β such that (i) $a \leq \beta \leq b$, (ii) $\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}$ exists on $[a, \beta]$, (iii) $(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t), t) \in \tilde{K}$ whenever $t \in [a, \beta]$. (It is easy to prove that such a maximum exists.)

Fix $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{R}^k_+(\bar{\varepsilon})$. If $t \in [a, \beta(\vec{\varepsilon})]$, then

$$\begin{aligned}
\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t) &= \int_a^t \Big(f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), s) - f(\xi_*(s), \eta_*(s), s) \Big)\, ds \\
&= \int_a^t \Big(f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*(s), s) - f(\xi_*(s), \eta_*(s), s) \Big)\, ds \\
&\quad + \int_{[a,t] \cap E(\vec{\varepsilon})} \Big(f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), s) - f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*(s), s) \Big)\, ds ,
\end{aligned}$$

where

$$E(\vec{\varepsilon}) = \bigcup_{j=1}^{k} [\tau_j, \tau_j + \varepsilon_j] .$$

Then

$$\begin{aligned}\|\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(t)-\xi_*(t)\| &\le \int_a^t \left\|f(\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(s),\eta_*(s),s)-f(\xi_*(s),\eta_*(s),s)\right\|ds\\ &\quad+\int_{[a,t]\cap E(\vec\varepsilon)}\left\|f(\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(s),\eta_*^{\vec\tau,\vec u,\vec\varepsilon}(s),s)-f(\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(s),\eta_*(s),s)\right\|ds\\ &\le \kappa\int_a^t\left\|\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(s)-\xi_*(s)\right\|ds+2\kappa\|\vec\varepsilon\|\,,\end{aligned}$$

where

$$\|\vec\varepsilon\|=\varepsilon_1+\ldots+\varepsilon_k\,.$$

Gronwall's inequality then implies

$$\|\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(t)-\xi_*(t)\|\le 2\kappa\|\vec\varepsilon\|e^{\kappa(b-a)}\,. \tag{3.4}$$

Choose $\tilde\varepsilon$ such that

$$0<\tilde\varepsilon\le\bar\varepsilon \tag{3.5}$$

and

$$4k\kappa\tilde\varepsilon e^{\kappa(b-a)}\le\delta\,.$$

Then

$$\vec\varepsilon\in\mathbb{R}_+^k(\tilde\varepsilon)\Longrightarrow\left(\|\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(t)-\xi_*(t)\|\le\frac{\delta}{2}\quad\text{whenever}\quad t\in[a,\beta(\vec\varepsilon)]\right).$$

It then follows that

$$\vec\varepsilon\in\mathbb{R}_+^k(\tilde\varepsilon)\Longrightarrow\beta(\vec\varepsilon)=b\,,$$

so that

$$\mathbb{R}_+^k(\tilde\varepsilon)\subseteq\mathcal{D}^{\vec\tau,\vec u} \tag{3.6}$$

and

$$\left(\vec\varepsilon\in\mathbb{R}_+^k(\tilde\varepsilon)\wedge t\in[a,b]\right)\Longrightarrow\left(\|\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(t)-\xi_*(t)\|\le2\kappa\|\vec\varepsilon\|e^{\kappa(b-a)}\le\frac{\delta}{2}\right).$$

A similar Gronwall inequality argument establishes that

$$\left(\vec\varepsilon\in\mathbb{R}_+^k(\tilde\varepsilon)\wedge\vec\varepsilon\,'\in\mathbb{R}_+^k(\tilde\varepsilon)\wedge t\in[a,b]\right)\Longrightarrow$$
$$\left(\|\xi_*^{\vec\tau,\vec u,\vec\varepsilon}(t)-\xi_*^{\vec\tau,\vec u,\vec\varepsilon\,'}(t)\|\le2\kappa\Big(\sum_{j=1}^k|\varepsilon_j-\varepsilon_j'|\Big)e^{\kappa(b-a)}\le\frac{\delta}{2}\right).$$

It follows, in particular, that

$$\left(\vec\varepsilon\in\mathbb{R}_+^k(\tilde\varepsilon)\wedge\vec\varepsilon\,'\in\mathbb{R}_+^k(\tilde\varepsilon)\right)\Rightarrow\|\mathcal{E}^{\vec\tau,\vec u}(\vec\varepsilon)-\mathcal{E}^{\vec\tau,\vec u}(\vec\varepsilon\,')\|\le2\kappa\Big(\sum_{j=1}^k|\varepsilon_j-\varepsilon_j'|\Big)e^{\kappa(b-a)}\,,$$

so the endpoint map $\mathcal{E}^{\vec\tau,\vec u}$ is continuous on $\mathbb{R}_+^k(\tilde\varepsilon)$.

Finally, we have to prove that $\mathcal{E}^{\vec{\tau},\vec{u}}$ is differentiable at 0 and that the differential is given by the formula of our statement. For this purpose, we go back to the formula

$$\begin{aligned}\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t) \quad &= \quad \int_a^t \Big(f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*(s), s) - f(\xi_*(s), \eta_*(s), s)\Big)\, ds \\ &\quad + \int_{[a,t]\cap E(\vec{\varepsilon})} \Big(f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), s) - f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*(s), s)\Big)\, ds\,,\end{aligned}$$

that we rewrite as

$$\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t) = \int_a^t \Delta_1(s)\, ds + \int_{[a,t]\cap E(\vec{\varepsilon})} \Delta_2(s)\, ds\,. \tag{3.7}$$

We observe that

1. the difference
$$\Delta_1(s) \stackrel{\text{def}}{=} f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*(s), s) - f(\xi_*(s), \eta_*(s), s)$$
is "approximately equal" to $\frac{\partial f}{\partial x}(\xi_*(s), \eta_*(s), s) \cdot (\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s) - \xi_*(s))$, i.e., to $A(s) \cdot (\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s) - \xi_*(s))$, whenever $t \in [a, b]$.
2. the difference
$$\Delta_2(s) \stackrel{\text{def}}{=} f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), s) - f(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s), \eta_*(s), s)$$
is "approximately equal" to
$$w_j \stackrel{\text{def}}{=} f(x_j, u_j, \tau_j) - f(x_j, u_{*,j}, \tau_j)$$
whenever $s \in [\tau_j, \tau_j + \varepsilon_j]$,

To make this precise, we let $\theta : [a, b] \to \mathbb{R}^n$ be the function such that

$$\theta(t) = \begin{cases} 0 & \text{if} \quad t \notin E(\vec{\varepsilon})\,, \\ f(x_j, u_j, \tau_j) - f(x_j, u_{*,j}, \tau_j) & \text{if} \quad t \in [\tau_j, \tau_j + \varepsilon_j]\,, \end{cases}$$

and we let $\mu : [a, b] \to \mathbb{R}^n$ be the solution of

$$\dot{\mu}(t) = A(t) \cdot \mu(t) + \theta(t)\,, \qquad \mu(a) = 0\,.$$

Then

$$\mu(t) = \int_a^t (A(s) \cdot \mu(s) + \theta(s))\, ds\,. \tag{3.8}$$

If we subtract (3.8) from (3.7), we get

$$\begin{aligned}&\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t) - \mu(t) \\ &= \int_a^t \Big(\Delta_1(s) - A(s) \cdot \mu(s)\Big)\, ds + \int_{[a,t]\cap E(\vec{\varepsilon})} \Big(\Delta_2(s) - \theta(s)\Big)\, ds \\ &= \int_a^t \Big(A(s) \cdot (\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s) - \xi_*(s) - \mu(s)\Big)\, ds + R(t)\,,\end{aligned}$$

where the remainder $R(t)$ is given by

$$R(t) \stackrel{\text{def}}{=} \int_a^t \Big(\Delta_1(s) - A(s)\cdot(\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(s) - \xi_*(s))\Big)\, ds + \int_{[a,t]\cap E(\vec{\varepsilon})} \Big(\Delta_2(s) - \theta(s)\Big)\, ds\,.$$

Then Gronwall's inequality yields

$$\|\xi_*^{\vec{\tau},\vec{u},\vec{\varepsilon}}(t) - \xi_*(t) - \mu(t)\| \le e^{\kappa(b-a)} \sup\{\|R(s)\| : a \le s \le b\}\,. \tag{3.9}$$

The functions θ and μ depend on $\vec{\varepsilon}$. The variations of constants formula yields

$$\mu^{\vec{\varepsilon}}(b) = \int_a^b M(b,s)\cdot\theta^{\vec{\varepsilon}}(s)\, ds\,,$$

where we have now made the $\vec{\varepsilon}$-dependence explicit. So

$$\begin{aligned} \mu^{\vec{\varepsilon}}(b) &= \sum_{j=1}^k \int_{\tau_j}^{\tau_j+\varepsilon_j} M(b,s)\cdot w_j\, ds \\ &= \sum_{j=1}^k \varepsilon_j \cdot M(b,\tau_j)\cdot w_j \\ &\quad + \sum_{j=1}^k \int_{\tau_j}^{\tau_j+\varepsilon_j} \Big(M(b,s) - M(b,\tau_j)\Big)\cdot w_j\, ds \\ &= \sum_{j=1}^k \varepsilon_j \cdot M(b,\tau_j)\cdot w_j + o(\|\vec{\varepsilon}\|)\,. \end{aligned} \tag{3.10}$$

If we apply (3.9) with $t = b$, and use (3.10), we get

$$\|\mathcal{E}^{\vec{\tau},\vec{u}}(\vec{\varepsilon}) - \xi_*(b) - \sum_{j=1}^k \varepsilon_j \cdot M(b,\tau_j)\cdot w_j\| \le e^{\kappa(b-a)}\omega(\vec{\varepsilon}) + o(\|\vec{\varepsilon}\|)\,, \tag{3.11}$$

where

$$\omega(\vec{\varepsilon}) \stackrel{\text{def}}{=} \sup\{\|R^{\vec{\varepsilon}}(s)\| : a \le s \le b\}\,.$$

So our desired conclusion will follow if we prove that

$$\omega(\vec{\varepsilon}) = o(\|\vec{\varepsilon}\|) \quad \text{as} \quad \varepsilon \to 0\,.$$

The proof of this is a fairly routine and somewhat tedious calculation, that will be done later.

□