18 EXAMPLES OF CALCULUS OF VARIATIONS AND OPTIMAL CONTROL PROBLEMS

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Here is a list of examples of calculus of variations and/or optimal control problems. Some are easy, others hard. Three of them are still unsolved. Some can be solved directly by elementary arguments, others cannot be solved unless one uses the machinery of the calculus of variations or optimal control. Some statements may look a little bit too technical at this point, but will become clearer later in the course. In all the problems, when we talk about "all paths" we mean "all appropriately smooth paths." The precise meaning will depend on the problem, and usually this question—of choosing the correct space of functions or paths in which to work—is regarded nowadays as part of the problem. On the other hand, the founders of the subject—people like the Bernoulli brothers, Newton, Leibniz, Euler, Lagrange, to name a few—were perfectly happy talking about "all paths," without ever being troubled by the fact that not every function has a derivative and not every function can be integrated. For the purposes of this particular discussion, you can follow our illustrious predecessors and just forget about the choice of the space of paths.

1. (*The oldest of all calculus of variations problems; at least 3000 years old, for $n = 2$ or $n = 3$, of course.*) Of all paths $\xi : [0, 1] \mapsto \mathbb{R}^n$ such that $\xi(0) = A$, $\xi(1) = B$, find the paths or paths that minimize the integral

$$J(\xi) = \int_0^1 \|\dot{\xi}(t)\| \, dt \,,$$

where $\| \cdot \|$ is Euclidean norm, that is, $\|(x_1, \ldots, x_n)\| = \sqrt{\sum_{i=1}^n x_i^2}$, and A, B are two given points in $\mathbb{R}^n$.

2. (*The rope-stretching problem.*) Given the length of a string, find the configurations that *maximize* the distance between the endpoints. (NOTE: this is easily proved to be equivalent to Problem 1. In ancient Egypt and India the art of *rope-stretching* was practiced in order to produce segments and other geometrical figures, showing knowledge of the fact that a segment solves the rope-stretching problem. In Egypt, according to Democritus, geometric constructions were carried out with the help of specialized workers—whom Democritus describes as experts in "composing lines," and calls by the Greek word "harpenodaptai," which means "rope-stretchers"—by means of pegs and cords. Their skills were used to build altars, temples and pyramids, where it was deemed necessary to produce certain geometric shapes that would obey very precise specifications, such as being made of perfectly straight segments or perfect circles. These demands often originated with the Gods, who insisted that an altar or temple be built so as to meet very strict requirements, failing which their wrath would be unleashed upon the builders and their communities. Rope-stretching is a good analog device for making long segments *precisely because the solution of the rope-stretching problem is a segment.* This is one of the oldest examples I know of mathematics applied to engineering.)

3. (*A more modern version of Problem 1.*) Same question as in Problem 1, but $\| \cdot \|$ is now a general norm on $\mathbb{R}^n$, not necessarily Euclidean or smooth or strictly convex. (A good example to keep in mind is the "ℓ^1 norm" given by $\|x\| = |x_1| + \ldots + |x_n|$.)

4. (*The isoperimetric problem, also known as "Dido's" problem, inspired by the story told by Virgil (70-19 BCE) in the* **Aeneid** *about the foundation of Carthage, ca. 850 BCE; more than 2100 years old; essentially solved by* **Zenodorus**, *ca. 200-140 BCE, who showed that, (i) among all polygons with a given perimeter P and given number N of sides, the regular polgyon has the largest area, and (ii) a circle had a larger area than any regular polygon of the same perimeter.*) Of all simple closed rectifiable curves $\xi : [0, 1] \mapsto \mathbb{R}^2$ of a given length L, find the ones that maximize the enclosed area.

5. (*The light reflection problem, mathematically trivial but historically important; about 1900 years old.*) Of all paths in the closed upper half plane $\{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ that go from a given point A to another point B and go through at least one point in the x axis, find the ones of minimum length. (NOTE: Hero of Alexandria—believed to have been born in 10 CE and to have died in 75 CE—pointed out in his *Catoptrics* that when a light ray emitted by an object is reflected by a mirror it follows a path from the object to the eye which is the shortest of all possible such paths. *This is the first example in history of an explanation of a physical phenomenon on the basis of a "curve minimization principle."*)

6. (*The light refraction problem, easily doable today as a Freshman Calculus exercise, but important for people such as Leibniz when they were first able to solve it; about 350 years old. In his 1684 Nova Methodus paper, Leibniz wrote, referring to his solution of the light refraction and other related problems, that "Other very learned men have sought in many devious ways what someone versed in this calculus can accomplish in these lines as by magic. ... This is the beginning of much more sublime Geometry, pertaining to even the most difficult and most beautiful problems of applied mathematics, which without our differential calculus or something similar noone could attack with any such ease."*) Assuming that the speed of light is equal to a constant c_+ when $y > 0$ and to a different constant c_- when $y < 0$, find the light rays (i. e., minimum time paths) from a given point A in the open upper half plane to another given point B in the open lower half plane.

7. (*The catenary; solved incorrectly by Galileo in the 1630s; solved correctly by Johann Bernoulli in 1691.*) Find the shape of a chain (or rope) of uniform mass density and given length L, held fixed at its endpoints, and otherwise hanging freely. Mathematically, this becomes: given two points A, B in the two-dimensional plane $\mathbb{R}^2$, and a length L (which should be $\geq \text{dist}(A,B)$), then, among all the paths $[0,1] \ni t \mapsto \xi(t) = (x(t), y(t))$ such that $\xi(0) = A$, $\xi(1) = B$, and $\int_0^1 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt = L$, find the one that minimizes the integral

$$J(\xi) = \int_0^1 y(t)\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt \,.$$

(Note: $J(\xi)$ is equal to L times the ordinate of the center of mass of the hanging chain. So the problem amounts to finding the path whose center of mass is as low as possible.)

8. (*The brachistochrone; solved incorrectly by Galileo in 1638; solved correctly by Johann Bernoulli and others in 1696-7.*) Suppose we drop a particle at a point A so that it will fall freely along a curve ξ going from A to another point B. For given A, B, how should ξ be chosen so that the falling time is minimized?

The above is, roughly, Johann Bernoulli's formulation of the problem. A standard translation into mathematics, found in most textbooks, is as follows. The energy E is constant, and $E = K + P$, where K is kinetic energy and P is potential energy. Since $K = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ (assuming the particle mass to be equal to 1) and $P = gy$, we have

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy = E \,.$$

If we write $A = (a, \alpha)$, $B = (b, \beta)$, and assume further that our curve is actually given by $y = \varphi(x)$, $a \leq x \leq b$, we have

$$\frac{1}{2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) + gy = E \,,$$

so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2E - 2gy \,,$$

$$(dx)^2 + (dy)^2 = (2E - 2gy)(dt)^2 \,,$$

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right)(dx)^2 = (2E - 2gy)(dt)^2 \,,$$

$$dt = \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2E - 2gy}}\, dx \,,$$

and then our problem is to minimize the integral

$$J(\varphi) = \int_a^b \frac{\sqrt{1+\varphi'(x)^2}}{\sqrt{2E - 2g\varphi(x)}}\,dx\,,$$

among "all functions" $\varphi : [a,b] \to \mathbb{R}$ such that $\varphi(a) = \alpha$, $\varphi(b) = \beta$, and $g\varphi(x) \leq E$ for all x.
If we factor out the $\sqrt{2}$ and just omit it, take $g = 1$ (which is possible by a suitable choice of units) and relabel $E - \varphi$ as our new φ, then the integral becomes

$$J(\varphi) = \int_a^b \frac{\sqrt{1+\varphi'(x)^2}}{\sqrt{\varphi(x)}}\,dx\,,$$

and the constraint on φ is just $\varphi(x) \geq 0$ for all x.
Later in the course we will discuss whether the above formulation is a satisfactory translation into mathematics of Johann Bernoulli's original statement. It will be argued that it is a very imperfect translation, and that a much better one is possible and yields nicer results.

9. (*The "reflected brachistochrone."*) Assuming that the speed of light $c(x,y)$ is equal to $\sqrt{|y|}$, find the light rays (i. e., minimum time paths) from a given point A to another given point $B \in \mathbb{R}^2$.

10. (*One-dimensional soft landing.*) For the system

$$\dot{x} = y\,, \qquad \dot{y} = u\,, \qquad |u| \leq 1\,,$$

given a starting position x_- and velocity y_-, find the path of minimum time that ends at $x = 0$, $y = 0$.

11. (n*-dimensional soft landing.*) Same as in Problem 10, but now x, y, u are n-dimensional vectors, and the constraint on u is $\|u\| \leq 1$, $\|\cdot\|$ being Euclidean norm.

12. (*Fuller's problem; a very important problem, famous because of the surprising nature of the solutions.*) For the system

$$\dot{x} = y\,, \qquad \dot{y} = u\,, \qquad |u| \leq 1\,,$$

given a starting position x_- and velocity y_-, find the path ending at $(0,0)$ and defined on an interval $[0,T]$ for some $T > 0$ (to be found) that minimizes the integral $\int_0^T x(t)^2\,dt$.

13. Given $a \in \mathbb{R}$, $b \in \mathbb{R}$, $L > 0$, find, of all functions $\xi : [0,L] \mapsto \mathbb{R}$ such that $\xi(0) = a$, $\xi(L) = b$, and $|\dot{\xi}(t)\| \leq 1$ for all t, the ones that minimize the integral

$$J(\xi) = \int_0^L \xi(t)^2\,dt\,.$$

14. (*The "Markov-Dubins problem," going back to A. A. Markov, 1887. Solved by L. Dubins in 1957 for* $n = 2$*, and by H. Sussmann in 1992 for* $n = 3$*. D. Mittenhuber later proved that the* $n > 3$ *case reduces to the* $n = 3$ *case.*) Of "all paths" ξ in $\mathbb{R}^n$ that are parametrized by arc-length (that is, $\|\dot{\xi}(t)\| = 1$ for all t), satisfy the *curvature bound* $\|\ddot{\xi}(t)\| \leq 1$, and go from a given initial point x_- and initial direction v_- to a given terminal point x_+ and terminal direction v_+, find the ones of minimum length.

15. (*The "Markov-Dubins-Reeds-Shepp problem," a variant of Problem 9, solved by J. Reeds and L. Shepp in 1990 for* $n = 2$*, and by Sussmann in 1992 for* $n = 3$*. Mittenhuber later proved that the* $n > 3$ *case reduces to the* $n = 3$ *case.*) Of "all paths" ξ in $\mathbb{R}^n$ that satisfy $\dot{\xi}(t) = \varepsilon(t)v(t)$, $\varepsilon(t) \in \{-1, 1\}$, $\varepsilon(\cdot)$ measurable, $v(\cdot)$ Lipschitz, $\|\dot{v}(t)\| \leq 1$, that go from a given initial point x_- and initial direction v_- to a given terminal point x_+ and terminal direction v_+, find the ones of minimum length. (NOTE: think of a "vehicle" that is moving in such a way that at each time t it points to some direction $v(t)$—so $v(t)$ is a unit vector—and the motion takes place with speed 1, while at the same time the vehicle can turn—that is, $v(t)$ can change—but with a bound $\|\dot{v}(t)\|$ on how fast this can happen. Then the Dubins case corresponds to a vehicle that is not allowed to back up—that is, $\varepsilon = 1$—whereas the Reed-Shepp case corresponds to allowing backups, that is $\varepsilon = \pm 1$.)

16. (*The "Markov-Dubins problem with angular acceleration control," a variant of Problem 9. Still open, although some partial—and rather counterintuitive—results are known.*) For the system

$$\dot{x} = \cos\theta\,, \qquad \dot{y} = \sin\theta\,, \qquad \dot{\theta} = \omega\,, \qquad \dot{\omega} = u\,, \qquad \|u\| \le 1\,, \tag{1}$$

of "all paths" $t \mapsto (x(t), y(t), \theta(t), \omega(t)) \in \mathbb{R} \times \mathbb{R} \times S^1 \times \mathbb{R}$ that go from a given initial condition $(x_-, y_-, \theta_-, \omega_-)$ to a given terminal condition $(x_+, y_+, \theta_+, \omega_+)$, find the ones of minimum length.

17. (*General regularity of optimal controls. Still open, although some answers are known, and the answers that are known are somewhat surprising, as will be explained later in the course.*) For two smooth (i. e., C^∞) vector fields f and g in $\mathbb{R}^n$, and two points A, B in $\mathbb{R}^n$, let $P(f, g, A, B)$ be the following *minimum time problem*: find a $T \ge 0$ and a measurable function $[0, T] \ni t \mapsto u(t) \in [-1, 1]$ such that the solution $[0, T] \ni t \mapsto \xi(t) \in \mathbb{R}^n$ of the O.D.E. initial value problem

$$\dot{\xi}(t) = f(\xi(t)) + u(t)g(\xi(t))\,, \quad \xi(0) = A\,,$$

exists on $[0, T]$, is such that $\xi(T) = B$, and T is as small as possible.
Which of the following statements are true?

a. Whenever f, g are C^∞ vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most finitely many points of discontinuity.
b. Whenever f, g are real-analytic vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most finitely many points of discontinuity.
c. Whenever f, g are real-analytic vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most finitely many points of non-analyticity.
d. Whenever f, g are C^∞ vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most countably many points of discontinuity.
e. Whenever f, g are real-analytic vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most countably many points of discontinuity.
f. Whenever f, g are real-analytic vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most countably many points of non-analyticity.
g. Whenever f, g are C^∞ vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most a set of measure zero of points of discontinuity.
h. Whenever f, g are real-analytic vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most a set of measure zero of points of discontinuity.
i. Whenever f, g are real-analytic vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ has at most a set of measure zero of points of non-analyticity.
j. Whenever f, g are C^∞ vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ is continuous on some dense open set of its interval of definition.
k. Whenever f, g are real-analytic vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ is continuous on some dense open set of its interval of definition.
ℓ. Whenever f, g are real-analytic vector fields and $P(f, g, A, B)$ has a unique solution $(T, u(\cdot))$, then $u(\cdot)$ is real-analytic on some dense open set of its interval of definition.

18. (*General regularity of subriemannian minimizers. Still open, although several erroneous proofs of an affirmative answer have appeared in the literature, e. g. one by R. Strichartz in J. Diff. Geometry* **24**, *1983, pp. 221-263, later retracted in the same journal,* **30**, *1989, pp. 595-596.*) A *subriemannian metric* is like a Riemannian metric except that the metric is only defined (and smooth) on a smooth subbundle E of the tangent bundle, and only curves ξ whose direction $\dot{\xi}(t)$ at each time t belongs to the corresponding space $E(\xi(t))$ are permitted. Are all length-minimizers smooth (if parametrized by arc-length)?