FINAL EXAM

Héctor J. Sussmann—The Weizmann Institute—January 2001

This is a take-home exam, consisting of FOUR problems. You can send your solutions to me by e-mail, to *sussmann@math.rutgers.edu*. Send either a Plain TeX or LateX file **without using any special Weizmann-Institute macros**, or a Postscript file. When you e-mail your file, **do not use any "MIME-encoding" or Microsoft-dependent stuff.**

PROBLEM 1: Do Problem 3 of the list in the first handout, dated November 1, 2000. (Recall: a *norm* on $\mathbb{R}^n$ is a function

$$\mathbb{R}^n \ni x \mapsto \|x\| \in \mathbb{R}$$

such that (i) $\|x\| > 0$ for all $x \in \mathbb{R}^n$ such that $x \neq 0$, (ii) $\|x + y\| \leq \|x\| + \|y\|$ for all x, y, (iii) $\|r \cdot x\| = |r| \cdot \|x\|$ whenever $r \in \mathbb{R}$ and $x \in \mathbb{R}^n$.)

PROBLEM 2: Do Problem 11 of the list in the first handout, dated November 1, 2000.

PROBLEM 3: For Problem 16 of the list in the first handout, dated November 1, 2000, prove the following:

(a) For every optimal trajectory-control pair (ξ, η) with a continuous control, the control η is is fact constant and equal to $+1$, -1 or 0; moreover, the zero control case can only occur where $\omega = 0$.
(b) For an optimal trajectory-control pair (ξ, η), it is **not** possible for the control η to have a jump discontinuity from 0 to 1 or -1 or from 1 or -1 to zero. (In other words, there are no optimal "jump junctions" between bang and singular arcs.)

NOTE: Result (b) is fairly counterintuitive. One would imagine in view of (a) that, if you want to control the vehicle from a certain initial position $(x_-, y_-, \theta_-, \omega_-)$ to a terminal position $(x_+, y_+, \theta_+, \omega_+)$ which is very far away, then the right thing to do is to follow a straight line, if possible, and if this cannot be done then you follow a straight line most of the time, except that at the beginning and at the end you use some bang-bang manoeuvers to get into the line. Now, motion along a straight line corresponds to $\theta = \text{constant}$, so $\omega = 0$ and $u = 0$. Result (b) says that the "bang-bang manoeuvers" cannot be finite concatenations of $+1$ and -1 controls.

OPTIONAL PROBLEM: Give a complete description of the optimal trajectories for Problem 3. NOTE: this is a **hard open problem**. If you solve it correctly then this will be a major mathematical achievement that will make you famous.

PROBLEM 4: Fill in the details (for the fixed time interval case) for the result on "more general endpoint conditions" outlined in the remark in Page 7 of the January 3 handout. In particular: (a) give a complete statement of the theorem, (b) give a complete proof that the statement follows from Theorem 1.3.1 of the handout.