Underwater vehicles: The minimum time problem

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Abstract

We consider the minimum time problem for a class of underwater vehicles. We focus on the situation of initial and final configurations at rest satisfying $x_0 \neq x_f, z_0 = z_f, \theta_0 = \theta_f = 0$. We supplement our theory with numerical simulations and include a discussion on a possible Fuller-like phenomenon.

I. Introduction

In this paper we pursue the analysis of the minimum time problem for a special class of underwater vehicles, see [5], [6] for previous works. More precisely, the fully actuated controlled mechanical system we consider here describes the equations of motion for an ellipsoidal, neutrally buoyant, uniformly distributed vehicle submerged in an infinitely large volume of incompressible, irrotational and inviscid fluid at rest at infinity. Moreover, we restrict the vehicle to move in the plane (x, z) and we assume it is fully actuated by the use of thrusters. Boundedness on the power of each thruster is assumed.

The primary goal is to understand the structure of the time optimal trajectories. Our computations are based on the Pontryagin maximum principle. Many questions have to be answered: local and global bounds on the number of switchings, optimality of singular arcs, etc. In [6] we study the 2-singular extremals while their optimality is discussed in [5]. We focus here our attention on the following special situation of initial and final configurations both at rest: $x_0 \neq x_f, z_0 = z_f, \theta_0 = \theta_f = 0$.

We conjecture that in this case an optimal trajectory must contain totally bang arcs as well as u_3-singular arcs.

The theoretical conjectures are supported by a numerical study of optimal solutions for different values of the moment of inertia I. For higher values of I we obtain a purely bang-bang control in all three components with a large total number of switching points. Specifically, for $I = 2$ we get a total number of 9 switching points. The structure of the bang-bang control is determined by applying nonlinear programming techniques to the discretized control problem using a fine grid. We then proceed to a refinement of the solution by directly optimizing the switching points. This approach enables us to perform a numerical test of second-order sufficient conditions [1], [2], [10], [11] which shows that the computed solution is indeed a strong local minimum. For a smaller and more realistic value of I (here $I = 0.12$), the nonlinear programming approach with 1000 grid points yields a rather complicated combination of bang-bang and singular arcs that support our hypothesis of a Fuller-like phenomenon. However, optimality of the computed bang-singular solution cannot be shown since applicable sufficient optimality conditions are not available in the literature.

Clearly our assumptions in the model are idealized to a high degree and we cannot draw any practical conclusion from our results. However, our simplified model raises challenging theoretical and numerical questions and could be a good starting point to study a more complete model taking into account viscous effects.

II. The model

For more details on the model of underwater vehicles we consider here, see for instance [5]. We restrict our underwater vehicle to motions in a plane. The variables x and z denotes respectively the absolute horizontal and vertical positions of the vehicle, while θ represents its orientation. The horizontal and vertical velocities of the vehicle in the body frame coordinates are denoted v_1, v_3. Ω is the scalar angular rate in the plane. We assume the body-fluid mass terms in the body horizontal and vertical directions m_1, m_3 not to be equal, while I is the body-fluid moment of inertia in the plane. Under few assumptions on the fluid and on the vehicle and neglecting viscous effects, the equations of motion for the conservative system are given by

$$\begin{pmatrix} \dot{x} \\ \dot{z} \\ \dot{\theta} \\ \dot{v}_1 \\ \dot{v}_3 \\ \dot{\Omega} \end{pmatrix} = \begin{pmatrix} \cos\theta v_1 + \sin\theta v_3 \\ \cos\theta v_3 - \sin\theta v_1 \\ \Omega \\ -v_3\Omega\frac{m_3}{m_1} \\ v_1\Omega\frac{m_1}{m_3} \\ v_1 v_3 \frac{m_3-m_1}{I} \end{pmatrix}$$

We assume the vehicle actuated with thrusters. More precisely, we consider the following inputs: u_1 is a force in the body 1-axis, u_2 is a force in the body 2-axis and u_3 is a pure torque in the plane. It follows that the equations of motion of our fully actuated underwater vehicle are described by an affine control system: $\dot{w} = f(w) + \sum_{i=1}^{3} g_i(w)u_i$, where the drift f is given by the equations of motion of the conservative system described above, the g_i are the following constant vector fields: $g_1 = (0,0,0,\frac{1}{m_1},0,0)$, $g_2 = (0,0,0,0,\frac{1}{m_3},0)$, $g_3 = (0,0,0,0,0,\frac{1}{I})$ and the controls u_i are measurable bounded functions. Moreover, to reflect the fact that the thrusters have limited power, we assume the following constraints on the inputs: $\{-1 \le u_i \le 1\}$, $i = 1,2,3$. Our system belongs to the class of controlled mechanical systems for which a study on the minimum time problem with bounded controls was initiated ! in [7].

III. Conjecture

Our previous results are based on the maximum principle. For our class of underwater vehicles the Hamiltonian function takes the form $H(w,\lambda,u) = \lambda^t f(w) + \lambda_4\frac{1}{m_1}u_1 + \lambda_5\frac{1}{m_3}u_2 + \lambda_6\frac{1}{I}u_3$ where the λ_i's are the **adjoint variables** which are solutions of $\dot{\lambda} = -\frac{\partial H}{\partial w}$:

$$\dot{\lambda}_1 = 0 \qquad (1)$$

$$\dot{\lambda}_2 = 0 \qquad (2)$$

$$\dot{\lambda}_3 = \lambda_1(v_1\sin\theta - v_3\cos\theta) + \lambda_2(v_1\cos\theta + v_3\sin\theta) \qquad (3)$$

$$\dot{\lambda}_4 = -\lambda_1\cos\theta + \lambda_2\sin\theta - \lambda_5\Omega\frac{m_1}{m_3} - \lambda_6 v_3\alpha \qquad (4)$$

$$\dot{\lambda}_5 = -\lambda_1\sin\theta - \lambda_2\cos\theta + \lambda_4\Omega\frac{m_3}{m_1} - \lambda_6 v_1\alpha \qquad (5)$$

$$\dot{\lambda}_6 = -\lambda_3 + \lambda_4 v_3\frac{m_3}{m_1} - \lambda_5 v_1\frac{m_1}{m_3} \qquad (6)$$

where $\alpha = \frac{m_3-m_1}{I}$ is a nonzero constant. A solution of the maximum principle is called an **extremal**. The maximization condition implies that $u_{i-3}(t) = sign(\lambda_i(t))$ if $\lambda_i(t) \neq 0$, $i = 4,5,6$. In other words, the structure of the time optimal path is governed by the zeroes of the adjoint variables $\lambda_{i+3} = \lambda^t g_i(w)$. We call $\lambda_4, \lambda_5, \lambda_6$ the **switching functions**. An extremal is said to be u_i**-bang** if the control u_i takes it value in $\{-1,+1\}$ for almost every time t and is said to be **totally bang** if it is u_i-bang for all controls. A consequence is that an extremal is u_i-bang if the corresponding switching function does not vanish almost everywhere. If there exists a nonempty interval such that a given switching function is identically zero, no information on the control u_i is directly provided by the maximum principle. We then say that the extremal is $u!_i$**-singular** on that interval. An extremal is **totally singular** if it is u_i-singular for all controls at the same time. A time t_s such that u_i is not almost everywhere constant on any interval of the form $]t_s-\varepsilon, t_s+\varepsilon[$, $\varepsilon > 0$ is called a **switching time** for u_i.

It is well known that for every fully actuated controlled mechanical system there is no totally singular extremal, see [12]. In [7] the 2-singular extremals are analyzed (i.e. extremals such that 2 controls are singular at the same time) and it is proved that along such extremals there is at most one switching for the nonsingular control. The 2-singular extremals such that the singular controls are identically zero correspond to motions that are horizontal ($u_2 = u_3 \equiv 0$) or vertical ($u_1 = u_3 \equiv 0$) translations in body frame coordinates or pure rotations ($u_1 = u_2 \equiv 0$). Notice that if there is a switching time along a 2-extremal, then the 2 nonsingular controls have to be identically zero. Optimality of these trajectories is discussed in [5]. It is proved that the horizontal translation in body frame coordinates between two configurations at rest with a u_1-switching at half time is not time optimal. Generalization of! this result to vertical translations in body frame coordinates and to pure rotations, both cases for initial and final configurations at rest and a half time switching on the nonsingular controls, uses similar arguments. Notice that the trajectories described above are the fastest between the set of 2-singular extremals joining two positions at rest (indeed it implies that we need at least one switching time). Then, a time optimal trajectory between two positions at rest cannot be a 2-singular extremal. It is an easy verification that the arguments used in [5] still hold without the rest condition on the initial and final configurations. In other words, we have the following.

Proposition *3.1:* Horizontal and vertical translations in body frame coordinates as well as pure rotations are not time optimal trajectories. In particular, 2-singular extremals with one switching point on the nonsingular control are not time optimal.

Remark *3.2:* Since we study the problem of joining two positions at rest a control has to be either identically zero or has to contain at least one switching. It follows that an optimal trajectory between two positions at rest cannot be a 2-singular extremal or be formed by concatenations of 2-singular extremals. The simulations show an optimal trajectory that does not contain any 2-singular arc.

Let us now restrict to the situation when the initial and final configurations are respectively $w_0 = (0,1,0,0,0,0)$ and $w_f = (x_f,1,0,0,0,0)$, see Fig. 1. Numerical solutions are presented in Section IV. From Remark 3.2 we know that an

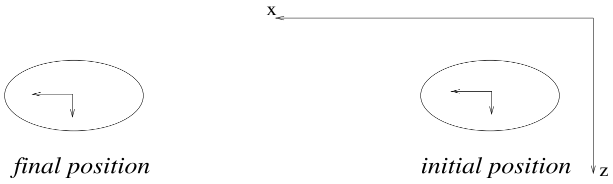

Fig. 1. Initial and Final configurations

optimal trajectory for this choice of configurations is not a concatenation of 2-singular arcs. We then have to find a way to go faster. Let us assume for the next computations that the bounds on the control u_3 are very big, in other words assume we can apply a pure torque almost as big as we want. Now, instead of restricting ourself to an horizontal motion in body fixed frame using only full power on the force in the 1-axis direction we assume the vehicle to be rotate in such angle so that we use both forces in the 1 and 2 -axis direction. Let us compute this angle and show that we go faster. Since we do not want any vertical motion along our path, we must satisfy $\dot{z} \equiv 0$ which is, in other words:

$$\cos\theta v_3(t) - \sin\theta v_1(t) = 0 \qquad (7)$$

for each t. If we take the maximum value for the two forces u_1, u_2, we get $v_1(t) = \frac{t}{m_1}$ and $v_3(t) = \frac{t}{m_3}$ (we focus only on the first half of the trajectory since it is symmetric). It follows that equation (7) becomes $\frac{\cos\theta}{m_3} + \frac{\sin\theta}{m_1} = 0$ which can be written as

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{m_1}{m_3}$$

(it is obvious that the optimal solution will not be $\theta = \frac{\pi}{2}$). Using now the fact that the previous equation leads to $\sin\theta = \frac{m_1}{\sqrt{m_1^2+m_3^2}}$ and that $\dot{x} = \cos\theta v_1 + \sin\theta v_3 = t\sin\theta(\frac{m_3}{m_1^2} + \frac{1}{m_3})$ we have:

$$\dot{x}(t) = t\frac{m_1}{\sqrt{m_1^2+m_3^2}}(\frac{m_3}{m_1^2} + \frac{1}{m_3}). \qquad (8)$$

We claim that the coefficient multiplying t in equation (8) is strictly bigger than 1. Indeed, this coefficient is $\frac{m_1}{\sqrt{m_1^2+m_3^2}}(\frac{m_3^2+m_1^2}{m_1^2 m_3}) = \frac{\sqrt{m_3^2+m_1^2}}{m_1 m_3} = \frac{1}{m_1}\sqrt{1+\frac{m_1^2}{m_3^2}} \geq \frac{1}{m_1}$. In other words, if θ is determined such that equation (8) is verified, then we have

$$\dot{x}(t) > \frac{t}{m_1}$$

. This is the angle at which we can move the fastest along the straight line since it uses the maximum power on both thrusters u_1, u_2. Notice that due to the constraint $\dot{\Omega} = 0$ it has to be a strict u_3-singular arc. It seems reasonable to believe that an optimal trajectory has to be symmetric. Then, since our computations on the previously described u_3-singular arc hold if θ is replaced by $\pi - \theta$ a plausible conjecture for an optimal trajectory would be that it is a concatenation of totally bang and strict u_3-singular arcs as represented in Fig. **??**. We begin and end with totally bang arcs (to move the vehicle to the right orientation), while a third totally bang arc would connect the two u_3-singular pieces. The numerical computations, see Section IV, show that for our choice of values of I, m_1, m_3 when the bounds on the control are taken to be one then an optimal trajectory is still formed by totally bang arcs as well as u_3-singular arcs while if the bounds are very small then there is no longer a u_3-singular arc.

IV. Numerical bang-bang and singular solutions

For numerical computation we choose the values $m_1 = 13.2$ and $m_3 = 25.6$ and the boundary conditions $x(0) = 0$, $x(t_f) = 2$ and $z(0) = z(t_f) = 1$. We study the behavior of the optimal controls w.r.t. the parameter I. For higher values of I, the optimal controls are bang-bang in all components, whereas for smaller values of I, the control component u_3 contains singular arcs while u_1, u_2 are bang-bang. The computations for a wide range of values I show that the optimal control u_1 is bang-bang with one switching time $t_f/2$ and the control sequence $u(t) = 1 \,|\, -1$. Results for the following two cases were obtained by discretizing the optimal control problem and then using the AMPL/LOQO programming and optimization environment [8], [13].

Case $I = 2$: Using 10^4 grid points and Heun's integration method for ODEs, AMPL/LOQO gives the purely bang–bang controls u_2, u_3 and the corresponding switching functions λ_5, λ_6 shown in Fig. 1.

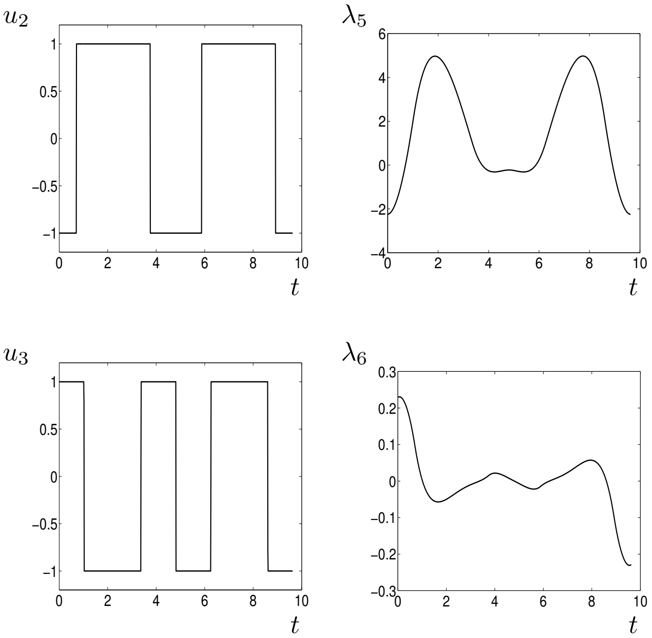

Fig. 2. $I = 2$: optimal controls $u_2(t), u_3(t)$ and switching functions $\lambda_5(t), \lambda_6(t)$.

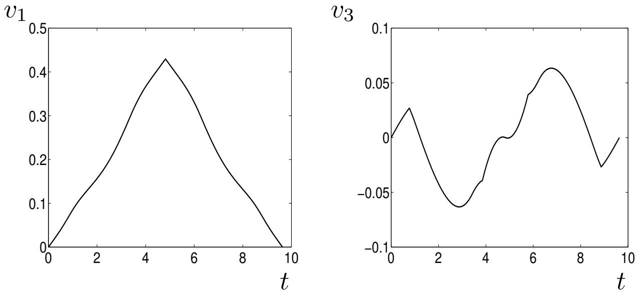

Fig. 3. Velocities $v_1(t)$ and $v_3(t)$.

Fig. 1 shows that there is a total of 9 switching points for all three control components. To compute the switching points $0 =: t_0 < t_1 < ... < t_9 < t_{10} := t_f$ and the optimal final time $t_{10} = t_f$ with higher precision than the grid size, we use the control package NUDOCCCS of Büskens [3], [4] in combination with an arc–parametrization method described in Kaya, Noakes [9]. In addition, this approach will allow us to verify the second–order sufficient conditions (SSC) for bang–bang controls which have recently been developed in [1], [10], [11]. The optimal control problem is reformulated as a finite-dim. optimization problem using the optimization vector

$$z := (\xi_1, \ldots, \xi_{10}), \quad \xi_k := t_k - t_{k-1}, \ k = 1, \ldots, 10,$$

where ξ_k denotes the arc duration of the k-th bang-bang arc. The resulting scaled problem (cf. problem formulation (PM2) in [9], p. 81) is given by

$$\begin{array}{ll} \min & t_f = \sum_{k=1}^{10} \xi_k \\ \text{s.t.} & \dot{w} = \zeta(f(w) + g_1(w)u_1 + g_2(w)u_2 + g_3(w)u_3), \\ & \zeta(t) := 10\xi_k \text{ for } t \in I_k := \left[\frac{k-1}{10}, \frac{k}{10}\right], \ k = 1, .., 10, \\ & w(0) = (0,1,0,0,0,0), \quad w(1) = (2,1,0,0,0,0). \end{array}$$

The values of $u(t)$ in I_k, the arc durations ξ_k and the switching points t_k are listed in table 1, Fig. 4.

k	ξ_k	t_k	$u_1\vert_{I_k}$	$u_2\vert_{I_k}$	$u_3\vert_{I_k}$
1	0.71011826	0.71011826	1	1	-1
2	0.31894166	1.02905992	1	-1	-1
3	2.34499990	3.37405982	1	-1	1
4	0.37987736	3.75393718	1	-1	-1
5	1.06396020	4.81789738	1	1	-1
6	1.06638530	5.88428268	-1	1	1
7	0.37894378	6.26322646	-1	-1	1
8	2.34425230	8.60747876	-1	-1	-1
9	0.31793702	8.92541578	-1	-1	1
10	0.71080287	9.63621866	-1	1	1

Fig. 4. Table1 - Switching points and optimal control structure

The computed initial values for the adjoint variables are

$$\begin{array}{ll} \lambda_1(0) = 2.293213\,, & \lambda_2(0) = -0.000011\,, \\ \lambda_3(0) = 0.052879\,, & \lambda_4(0) = 10.51066\,, \\ \lambda_5(0) = 2.243136\,, & \lambda_6(0) = -0.229126\,. \end{array}$$

The strict bang-bang property in [1] holds in all switching points t_k.

The test for SSC in [1], [2], [10], [11] requires to compute the Hessian of the Lagrangian associated with the above optimization problem. Then one has to show the Hessian to be positive definite on the kernel of the Jacobian of the terminal constraints. Here the Hessian is a 10×10–matrix while the Jacobian is 6×10–matrix. The projected Hessian is a (4×4)-matrix that has smallest eigenvalue 0.065 and, hence, is positive definite. Thus the solution in Table 1 provides a strong local minimum in the L^1–norm. Moreover, a sensitivity analysis of optimal solutions and switching points w.r.t. variations of system parameters can be performed along the lines of the approach in [10]. Detailed numerical results for the parametric sensitivity derivatives will be given elsewhere.

Case $I = 0.12$: Here, we use a very fine grid with 10^5 grid points and again Heun‘s method for AMPL/LOQO. The optimal time is computed as 9.254699 while the optimal control candidates $u_2(t), u_3(t)$ and the corresponding switching functions $\lambda_5(t), \lambda_6(t)$ are depicted in Fig. 5. A zoom on Fig. 5 shows that the control u_3 contains in fact five singular arcs. It is interesting to remark that along these arcs, either there is no switching for u_1, u_2 or both switching functions λ_4, λ_5 have to vanish at the same time. For instance, on the third u_3-singular arc, there is a u_1-switching at $t_f/2$ while the switching function λ_5 vanishes (but does not change sign). It can be explained with the adjoint equations. Indeed, it is a consequence of the fact that in the equation for $\dot{\lambda}_6$ we have the two terms $\lambda_4 v_3 \frac{m_3}{m_1} - \lambda_5 v_1 \frac{m_1}{m_3}$ while λ_3 is absolutely continuous. A more general result can be stated.

PROPOSITION *4.1:* Let i, j, k be all distincts. If t^* is a switching time for u_j, then the switching function λ_{k+3} corresponding to u_k has to vanish at t_0.

Another interesting point is to notice from the pictures is that if Ω is constant then it is along a u_3-singular arc. This could not have happened along a totally bang arc. Indeed if we assume the extremal to be u_1 and u_2-bang, then the condition $\dot{\Omega} \equiv 0$ implies $v_1 v_3(m_3 - m_1) = -u_3$. If u_3 is ± 1, then we must have $v_1 v_3$ constant. Assume $\Omega \equiv 0$. Then, since u_1, u_2 are bang we have $\dot{v}_1$ and $\dot{v}_3$ constants which implies that v_1 and v_3 are linear. For two linear functions such that their product is constant have to be constant. In our case it would mean that u_1 and u_2 are zero which is a contradiction. If Ω is a constant that not zero we can write the equation for $\dot{v}_1$ and replace all terms with v_1 in terms of v_3. This gives us a polynomial of order 4 for v_3: $-v_3^4 \Omega \frac{m_3}{m_1} + v_3^3 \frac{u_1}{m_1} - v_3 \frac{u_2}{m_3} + \frac{\Omega}{v_3} \frac{m_1}{m_3}$ which implies th! at v_3 has to be constant if u_1 and u_2 are.

The simulations raise an other interesting question. It concerns the junction between bang and singular arcs. Fig. 5 suggests that the control u_3 is chattering at these junctions. This would not be surprising since the order of the singular arc is two which mean that along a strict u_3-singular arc we need to compute the fourth derivative of λ_6 to derive the singular control. If it can be proved, it would provide us with another interesting example for Fuller-like phenomenon.

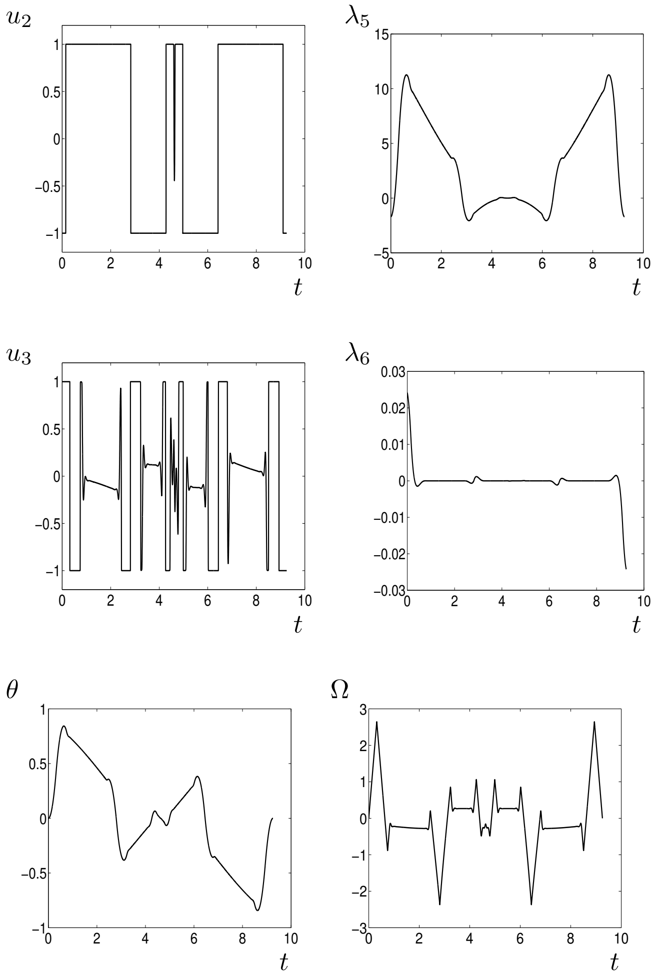

Fig. 5. $I = 0.12$: optimal controls $u_2(t), u_3(t)$, their switching functions $\lambda_5(t), \lambda_6(t)$ and θ, Ω.

REMARK *4.2:* The simulations show that for each choice of bounds $\alpha_3 < 0 < \beta_3$ for the control u_3: $\alpha_3 \leq u_3 \leq \beta_3$, there exists a value I^* at which we have the transition between totally bang-bang solutions and bang-singular solutions, see Fig. 6. Equivalently, if we fix I and varies the bounds on the control there exists $\alpha_3 < 0 < \beta_3$ at which the transition occurs. For instance for the value $I = 0.12$ the solution is totally bang-bang if we consider the bounds $|u_3| \leq 0.08$ instead of 1, the transition value being 0.107. It means that the power we allow on the torque influence the structure of the time-optimal trajectories for our choice of initial and final configurations at rest. Based on the pictures for θ and Ω a possible explanation of this phenomenon could be that along a u_3-singular extremal Ω needs to be almost constant. This means that u_3 has to balance the term $v_1 v_3(m_3 - m_1)$ and if there is no! enough power on the torque it is impossible.

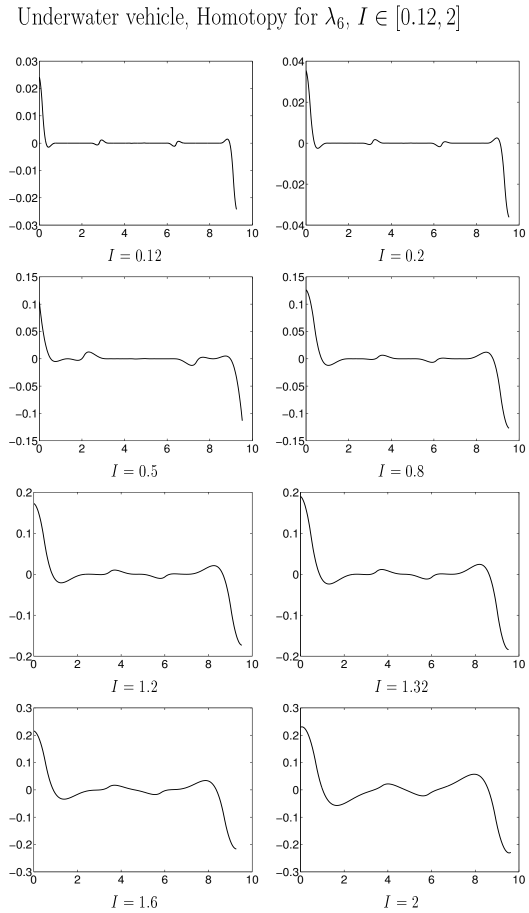

Fig. 6. Homotopy with respect to the value I

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