SYNTHESIS, PRESYNTHESIS, SUFFICIENT CONDITIONS FOR OPTIMALITY AND SUBANALYTIC SETS

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§1. Introduction.

The purpose of this paper is twofold. First, we want to explain the basic concepts of *optimal presynthesis and synthesis*, and show how the theory of subanalytic sets can be used to prove existence theorems. Second, we want to establish the link with several other papers in this volume, by showing how the results on synthesis lead to the search for theorems on the structure of optimal trajectories.

The most basic result of Optimal Control Theory (OCT) is the Pontryagin Maximum Principle (PMP), stated below in §3. Moreover, it is well known that, in some sense, if one manages to cover the whole space by a set Γ of trajectories that satisfy the necessary conditions given by the PMP, then all the trajectories in Γ are optimal, provided Γ is "sufficiently smooth." The problem that will concern us here is that of turning the previous sentence into a theorem which is both rigorous and useful. To have a rigorous theorem one needs to give a precise definition of the terms involved, and then give a

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rigorous proof. For the theorem to be useful, the hypotheses on Γ have to be such that a Γ that satisfies them exists for large classes of problems. We will present one such set of conditions in §3, following a method based on ideas of Boltyansky, cf.[Bo]. Our approach differs from Boltyansky's in that (i) we do not require the cost function V_Γ associated to Γ to be continuous, but make instead the weaker –and much more natural– hypothesis that V_Γ satisfies a set of "weak continuity conditions," (ii) we dispense with Boltyansky's partition into "type I and type II cells," and retain only those properties that are needed to justify the identification of the gradient of V_Γ with the adjoint variable of the PMP, and (iii) we do not require Γ to be memoryless, so that our Γ is really a "presynthesis" rather than a true synthesis.

In §4 we show how the existence of a Γ with the desired properties follows if the optimal control problem under consideration admits a proper subanalytic sufficient family of trajectories. Finally, in §5 we explain how the theorem of §4 reduces the proof of the existence of Γ to a problem of trajectory analysis.

§2. Optimal Control Theory.

Throughout this paper, we deal with control systems

$$\dot{x} = f(x,u),\ x \in \Omega,\ u \in U \tag{2.1}$$

where Ω is an open subject of $\mathbb{R}^n$, U is a set, and $f : \Omega \times U \mapsto \mathbb{R}^n$ is of class C^1 in x for each fixed u. A *control* for (2.1) is a function $\eta : I \mapsto U$ whose domain I=Dom(η) is an interval of the real line. The control η is *f-admissible* if the time-varying vector field $f_\eta : \Omega \times I \mapsto \mathbb{R}^n$ given by $f_\eta(x,t) = f(x,\eta(t))$ satisfies the conditions of the Carathéodory existence and uniqueness theorem of ordinary differential equations, that is: (i) f_η is jointly measurable in x and t, and (ii) for each compact interval $J \subseteq I$ and each compact $K \subseteq \Omega$ there exist a Lebesgue integrable function $\varphi : J \mapsto \mathbb{R}$ such that

$$\|f_\eta(x,t)\| + \|D_x f_\eta(x,t)\| \leq \varphi(t) \ \text{ for all } \ x \in K, t \in J. \tag{2.2}$$

A *trajectory* for a control $\eta : I \mapsto U$ is an absolutely continuous curve $\gamma : I \mapsto \Omega$ such that $\dot{\gamma}(t) = f(\gamma(t),\eta(t))$ for almost all $t \in I$. The Carathéodory theorem (cf., e.g., [CLe]), implies the local existence and global uniqueness

of trajectories, that is : if η is f-admissible, then (i) given $\overline{t} \in I, \overline{x} \in \Omega$, there exists $\varepsilon > 0$ with the property that there is a trajectory γ for the restriction of η to the interval $(\overline{t} - \varepsilon, \overline{t} + \varepsilon) \cap I$ such that $\gamma(\overline{t}) = \overline{x}$, and (ii) if γ and δ are trajectories for η such that $\gamma(\overline{t}) = \delta(\overline{t})$ for some $\overline{t}$ in I, then $\gamma \equiv \delta$. If η is f-admissible and γ is a trajectory for η, then (γ, η) is an *f-admissible pair*.

It follows from our definitions that *every piecewise constant control is admissible*. Moreover, the following important *Piecewise Constant Approximation Theorem* (PCAT) holds: if (γ, η) is an admissible pair, $\mathrm{Dom}(\eta)$ is compact, $\overline{t} \in \mathrm{Dom}(\eta)$, $\overline{x} = \gamma(\overline{t})$, $x_j \in \Omega$, $x_j \to \overline{x}$ as $j \to +\infty$, then there exist piecewise constant controls $\eta_j : \mathrm{Dom}(\eta) \mapsto U$ and corresponding trajectories γ_j such that $\gamma_j(\overline{t}) = x_j$ and $\gamma_j \to \gamma$ uniformly as $j \to \infty$. (For a proof, cf. [Su10].)

A control η *starts* at time t_0 if $t_0 \in \mathrm{Dom}(\eta)$ and $\mathrm{Dom}(\eta) \subseteq [t_0, \infty)$. In that case, if γ is a trajectory for η, then the point $x_0 = \gamma(t_0)$ is the *initial point* of γ. The definition of what it means for a control to *end* at a time t_1, and for a point x_1 to be the *terminal point* of a trajectory, is completely analogous. If x_0 and x_1 are the initial and terminal points of a trajectory γ that corresponds to a control η, then we say that γ *goes from* x_0 to x_1, or that η *steers* x_0 to x_1. In that case, we write $\gamma^+ = x_1$ and $\gamma^- = x_0$. A *target* is a nonempty subset $\mathcal{T}$ of Ω. A curve γ *ends at* $\mathcal{T}$ if γ^+ belongs to $\mathcal{T}$.

We consider an optimal control problem $\mathcal{P} = (\Omega, U, f, L, \mathcal{T}, \varphi)$, given by the specification of (i) a control system (2.1), (ii) a function $L : \Omega \times U \mapsto \mathbb{R}$, called the *Lagrangian*, (iii) a target $\mathcal{T}$, and (iv) a function $\varphi : \mathcal{T} \mapsto \mathbb{R}$. We require that L be of class C^1 in x for each fixed u . If $\eta : I \mapsto U$ is an admissible control, and $\gamma : I \mapsto \Omega$ is a trajectory for η that ends at $\mathcal{T}$, then we would like to define the *cost* of (γ, η) by

$$J(\gamma, \eta) = \int_I L(\gamma(t), \eta(t))dt + \varphi(\gamma^+). \tag{2.3}$$

However, the function $t \mapsto L(\gamma(t), \eta(t))$ may fail to be integrable (or even measurable) unless some extra technical hypotheses hold. So we need to restrict further the set $\mathrm{Dom}(J)$ of those pairs $(\gamma, \eta\ )$ for which $J(\gamma, \eta)$ is defined. For instance, we may take $\mathrm{Dom}(J)$ to be the set of those f-admissible (γ, η) such that γ ends at $\mathcal{T}$ and the integrand $h(t)$ of (2.3) has a well defined (but possible infinite) Lebesgue integral (i.e. h is measurable and at least one of the integrals $\int_I h_+, \int_I h_-$is finite, where $h_+ = \max(h, 0)$ and $h_- = h_+ - h$). If, as is often the case, the Lagrangian is nonnegative, and the f-

admissibility of (γ, η) suffices to imply that the function h is measurable, then this amounts to letting $\mathrm{Dom}(J)$ be the set of *all* f-admissible pairs (γ, η). A second possibility is to require η to be *(f,L)-admissible*, i.e. admissible for the augmented control system obtained from (2.1) by adding a new variable φ with equation $\dot{y} = L(x, u)$. In this case, the integrability of h is guaranteed if I is finite (i.e. if η has a starting time). We shall refer to the above two possibilities as "Formulations I and II of the Optimal Control Problem $\mathcal{P}$."

For $x \in \Omega$ we define $V(x)$ to be the infimum of $J(\gamma, \eta)$ taken over all $(\gamma, \eta) \in \mathrm{Dom}(J)$ such that γ starts at x and ends at the target $\mathcal{T}$. (If no such γ exists then of course $V(x) = +\infty$.) The function $V : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ is the *value function*, or *Bellman function*, of the problem $\mathcal{P}$. An *optimal pair* (often simply referred to as an "optimal trajectory") is a pair (γ, η) that has a starting point x and is such that $J(\gamma, \eta) = V(x)$.

We remark that *the choice of formulation* (i.e. the choice of $\mathrm{Dom}(J)$ as explained above) *may make a fundamental difference.* This is most dramatically illustrated by the "Lavrentieff phenomenon"(cf., e.g., [BM]). One can exhibit a nonnegative polynomial $L : \mathbb{R}^3 \to \mathbb{R}$ and real numbers x_0, x_1 such that, for the problem of minimizing $\int_0^1 L(x(t), \dot{x}(t), t)dt$ among all "curves" $t \to x(t)$ such that $x(0) = x_0$ and $x(1) = x_1$, the solution exists if by "curve" we mean "absolutely continuous function," and also if we mean "Lipschitz function." However, the values V_1, V_2 that correspond to these two interpretations satisfy $V_2 > V_1$. In control terms, this corresponds to the control system $\dot{x} = u,\ \dot{y} = 1$ with an obvious choice of Lagrangian. Requiring $x(\cdot)$ to be Lipschitz amounts to demanding that the control be measurable and bounded, in which case it is clearly $(f, L)-$admissible. Moreover, it follows easily from the PCAT (applied to the augmented system) that every trajectory $x(\cdot)$ for an $(f, L)-$admissible control can be approximated by Lipschitz trajectories $x_j(\cdot)$ *in such a way that the costs of the* x_j *converge to that of* x. Therefore V_2 is precisely the value for Formulation II . On the other hand, the condition that $x(\cdot)$ be absolutely continuous is precisely the requirement that the control be f-admissible, so V_1 is the value for Formulation I.

The choice of formulation is important, in particular, for the problem of *necessary conditions for optimality.* Indeed, the most basic of such conditions is the PMP, and the PMP is valid for Formulation II but may fail for Formulation I. For this reason, *from now on it will always be understood that the optimal control problem* $\mathcal{P}$ *is interpreted in the sense of Formulation II.*

For a control system (2.1) and a target $\mathcal{T}$, define a *presynthesis* to be a family $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ of admissible pairs such that, for each $x \in S$, γ_x starts at x and ends at the target. The set S is the *domain* of Γ. Call a presynthesis Γ *memoryless* if, whenever $x \in S$ and $t \in \mathrm{Dom}(\eta_x)$, it follows that $y = \gamma_x(t) \in \mathrm{Dom}(\Gamma)$ and η_y is precisely the restriction of $\eta_x$to the interval $\mathrm{Dom}(\eta_x) \cap [t, +\infty)$. A memoryless presynthesis Γ will be referred to as a *synthesis*. If the domain of Γ is the largest possible (i.e. the set of *all* points x such that there exists a trajectory that goes from x to the target), then Γ will be said to be *total*. If each pair (γ_x,η_x) is optimal for the control problem $\mathcal{P}$, then we call Γ *optimal* for $\mathcal{P}$.

§3. Optimal Presynthesis: Sufficient Conditions.

The problem of the *existence* of optimal trajectories is very classical, and has been studied in great detail in the 1950's and 1960's, so we choose not to discuss it here, and refer the reader instead to the book [Be] by L.D. Berkovitz for a rather complete account of the results. On the other hand, the problem of getting a good *characterization* of optimal trajectories is much more difficult. The PMP gives a neccessary condition for optimality, and many other conditions have been found, but it has been clear for a long time that, except for very special problems such as linear systems with a convex Lagrangian, the question of *sufficient conditions* is essentially hopeless.

The *existence problem for optimal presynthesis and synthesis* is closely related to the question of existence of optimal trajectories. Once one knows that an optimal trajectory exists for every initial point p, then an optimal presynthesis is obtained by just choosing one such trajectory for each p, which is certainly possible if one accepts the Axiom of Choice. However, to get a synthesis one has to make the presynthesis *memoryless.* Whether this is possible in general appears at the moment to be a open question.

Regarding the *characterization* of optimal presyntheses and syntheses, the situation is quite remarkably different from what happens for optimal trajectories. Nice *sufficient conditions* can now be proved. Indeed, the question of the optimality of a presynthesis is intimately related to the study of the solutions V of the *Hamilton-Jacobi-Bellman (HJB) equation*

$$H(\nabla V(x), x) = 0. \tag{3.1}$$

The function H is the *normal minimized Hamiltonian*, given by $H(\lambda, x) = \tilde{H}(\lambda, 1, x)$, where

$$\tilde{H}(\lambda, \lambda_0, x) = \inf\{\mathcal{H}(\lambda, \lambda_0, x, u) : u \in U\}, \tag{3.2}$$

and

$$\mathcal{H}(\lambda, \lambda_0, x, u) =< \lambda, f(x, u) > +\lambda_0 L(x, u). \tag{3.3}$$

(Here $\lambda \in \mathbb{R}^n$ and $\lambda_0 \in \mathbb{R}$.)

Now, let Γ be a presynthesis and let V_Γ be the corresponding value function, i.e. $V_\Gamma(x) = J(\gamma_x, \eta_x)$. Let V be the value function of the problem under consideration. Then of course Γ is optimal iff $V_\Gamma \equiv V$. So the problem of deciding whether Γ is optimal is precisely that of checking whether $V_\Gamma \equiv V$. Moreover, for Γ to be optimal it is clearly necessary that each trajectory (γ_x,η_x) be a *Pontryagin extremal*. (Recall that a Pontryagin extremal is an (f, L) admissible pair (γ, η) with the property that that there exist (i) an absolutely continuous function $t \to \lambda(t)$, defined for $t \in \mathrm{Dom}(\eta)$, and (ii) a constant $\lambda_0 \geq 0$, such that the equalities

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(\lambda(t), \lambda_0, \gamma(t), \eta(t)) \tag{3.4}$$

and

$$\tilde{H}(\lambda(t), \lambda_0, \gamma(t)) = \mathcal{H}(\lambda(t), \lambda_0, \gamma(t), \eta(t)) = 0 \tag{3.5}$$

hold for almost all $t \in \mathrm{Dom}(\eta)$, and $(\lambda(t), \lambda_0) \neq (0, 0)$ for some —and hence every— value of t.) The PMP says that an optimal trajectory must be a Pontryagin extremal. So an obvious necessary condition for γ to be optimal is that it must be an *extremal presynthesis*. i.e. a presynthesis all whose trajectories are Pontryagin extremals.

If in addition the target $\mathcal{T}$ is a C^1 submanifold of Ω, and the function φ is differentiable at γ^+, then a further necessary condition for optimality of (γ, η) is that it be possible to choose $\lambda(\cdot)$ and λ_0 so that, in addition to the above properties, the equality $< \lambda(t^+), v >= \lambda_0 < d\varphi(\gamma^+), v >$ holds for all vectors v that are tangent to $\mathcal{T}$ at γ^+. (Here t^+ is the terminal time of η). An extremal for which such a choice is possible will be called a *t-extremal*. (The "t" stands for "transversality"). A presynthesis that consists entirely of t-extremals will be called a *t-extremal presynthesis*.

It turns out that, "modulo some technical conditions," the value function of the optimal control problem $\mathcal{P}$ can be characterized as the unique solution

—or as a special solution— V of the HJB equation that satisfies certain boundary conditions. (For instance, if the value function V is of class C^1, and satisfies the boundary condition

$$V(x) = \varphi(x) \text{ for all } x \in \mathcal{T}, \tag{3.6}$$

then it is the unique solution of HJB for which (3.6) holds. Even if the value function does not satisfy (3.6) but only the weaker condition

$$V(x) \leq \varphi(x) \text{ for all } x \in \mathcal{T}, \tag{3.7}$$

it can still be characterized as the largest of all solutions of HJB for which (3.7) is true, cf. [Go].) In the case that interests us, i.e. that of a function V_Γ arising from a presynthesis, so that V_Γ is known *a priori* to satisfy $V_\Gamma \geq V$, it turns out that all we need to know to conclude that $V_\Gamma \equiv V$ is that V_Γ is a solution of HJB that satisfies (3.7). If one compares the HJB equation with the statement of the PMP given above, it is easy to see that the value function will satisfy the HJB equation *provided we show that the adjoint variable* $\lambda_x(\cdot)$ *whose existence is asserted by the PMP can be chosen so that* $\nabla V_\Gamma(x) = \lambda(t_x^-)$, *and the corresponding* λ_0 *can be taken to be equal to 1.* (Here we use t_x^- to denote the starting time of the control η_x.) It turns out that—again "modulo technical conditions"—this is indeed possible. So an extremal synthesis is indeed optimal if suitable technical conditions hold.

The preceding paragraph is intendedly vague, because of the repeated reference to unspecified "technical conditions." As written, the paragraph constitutes an accurate reflection of the current state of our knowledge. It is "known" that in some sense the statements are true in general, but no satisfactory way has yet been found to translate them into theorems that are both precise and sufficiently general. We shall refer to the problem of finding such theorems as *the Fundamental Problem of Synthesis Theory (FPST).* Naturally, the main question is that of finding the appropriate technical conditions that make all the desired results true.

The FPST splits into three parts. *First*, there is the problem of rigorously defining what is meant for a function to be a solution of the HJB equation. The difficulty here is that the value function typically is not everywhere differentiable, so one needs some concept of "weak solution." To illustrate the difficulties involved, consider the simple example of the system $\dot{x} = u$ on the interval $[-1, +1]$, with control constraint $|u| \leq 1$. Assume that we want

to reach the boundary of the interval in minimum time. It is easy to see that the value function V is just $V(x) = 1 - |x|$, the HJB equation is $|V'(x)| = 1$, and the boundary condition is $V(-1) = V(1) = 0$. Requiring that a solution of the HJB equation be differentiable everywhere would exclude the function V. If we only require differentiability almost everywhere, or even everywhere except at just one point, then we obtain the desired function V as well as many undesirable ones, such as $-V$.

Second, there is the question of passing from the fact that V_Γ satisfies the HJB equation and the boundary conditions to the conclusion that V_Γ is the value function. It is clear that the value function V satisfies the *Dynamic Programming Inequality* (DPI)

$$V(x) \leq V(y) + \int L(\gamma(t), \eta(t))dt \tag{3.6}$$

whenever (γ, η) is a trajectory that goes from x to y and, conversely, any function that satisfies this inequality plus the boundary condition (3.7) must be bounded above by the value function. In particular, if V_Γ satisfies the DPI and (3.7) then it is the value function. So the technical problem here is the passage from the HJB equation to the DPI. Formally, this is done by observing that the HJB equation implies that $< \nabla V(\gamma(t)), \dot{\gamma}(t) > + L(\gamma(t), \eta(t)) \geq 0$, from which the DPI follows by integration. The technical problem is to handle the case when ∇V does not exist everywhere, and therefore the proof by integration is not justified. Actually, all that is used in the formal derivation is the inequality $H(\nabla V(x), x) \geq 0$. Let us call a function that satisfies this inequality in some sense *a subsolution* of HJB. Then the real problem is to find the adequate technical definition of "subsolution" so that it becomes a true theorem that a function is a subsolution iff it satisfies the DPI.

Third, there is the derivation of the HJB equation from the property that Γ is extremal. In this case, there is a formal proof which is just based on differentiating $\lambda_0 V_\Gamma$ at a point x in a direction v. Assuming, for simplicity, that the target is just a single point, one writes $\lambda_0 V_\Gamma(x + hv)$ as an integral, and differentiates with respect to h at $h = 0$. The result is precisely $< \lambda(t_x^-), v >$ and, since v is an arbitrary direction, the desired conclusion follows. (Indeed, if $\lambda_0 = 0$, then it would follow that $< \lambda(t_x^-), v >= 0$ for all v, so that $\lambda(t_x^-) = 0$, and therefore the nontriviality condition of the PMP is violated. So $\lambda_0 \neq 0$, and this means that we could have assumed that $\lambda_0 = 1$, so that $\nabla V_\Gamma(x) = \lambda(t_x^-)$. One is then left with the technical problem

of justifying the differentiation under the integral sign.

V.G. Boltyansky (cf. [Bo]) developed a theory of "regular synthesis" in order to tackle the above problems. Roughly speaking, a regular synthesis in Boltyansky's sense is a synthesis in which the trajectories and the controls depend on x in a "piecewise smooth" way. (The precise definition is quite technical, and we shall not repeat it here.) Boltyansky's result is that a regular synthesis in his sense is indeed optimal. However, it is not hard to give examples of optimal syntheses that are not regular in Boltyansky's sense but are good enough so that Boltyansky's arguments still work, possibly after some trivial modifications. So Boltyansky's theory has to be extended and generalized. Examples of such extensions have been proposed in [Bru2], [Su2] and [Su3], but none of these can be regarded as definitive.

Here we shall present a set of conditions similar to those of [Su2]. First, let us call a function $V : \Omega \to \mathbb{R} \cup \{+\infty, -\infty\}$ *weakly upper semicontinuous* at a point x if there exists sequences $\{x_j\}, \{\varepsilon_j\}, \{\delta_j\}$ such that $x_j \in \Omega, x_j \to x, \varepsilon_j > 0, \delta_j > 0, \delta_j \to 0$, with the property that $V(y) < V(x) + \delta_j$ whenever $||y - x_j|| < \varepsilon_j$. If X is a locally Lipschitz vector field on Ω, we say that V has the *NDJ* ("no downwards jumps") property along X if, whenever $t \to \gamma(t)$ is an integral curve of X, defined on an interval $[a, b]$, and $\{t_j\}, \{t'_j\}$ are sequences of points in $[a, b]$ such that $\lim_j t_j = \lim_j t'_j = t,\ t_j \leq t \leq t'_j$ and $\lim [V(t'_j) - V(t_j)] = L$, it follows that $L \geq 0$.

For a control system (2.1), we will say that V has the *NDJ property* if it has the NDJ property along every associated vector field $x \to f(x, u)$. Finally, we will say that V satisfies the *weak continuity conditions* for the optimal control problem $\mathcal{P}$ if V is lower semicontinuous and weakly upper semicontinuous, and has the NDJ property.

We now define what it means for a presynthesis $\Gamma = \{(\gamma_x, \eta_x\} : x \in S\}$ to satisfy the *differentiability conditions* at a point $\overline{x}$. We let $\tilde{f}$ be the $\mathbb{R}^{n+1}$-valued map $(x, u) \to (f(x, u), L(x, u))$, and require the following:

(DC) there exist an open set W, containing the set $\{\gamma_{\overline{x}}(t) : t \in Dom(\eta_{\overline{x}})\}$, a neighborhood N of $\overline{x}$ in Ω, and an interval $[a, b]$ such that $\mathrm{Dom}(\eta_x) \subseteq [a, b]$ for all $x \in N$, with the property that

(DC1) there exist, for sufficiently small $\varepsilon > 0$, integrable functions $\psi_\varepsilon : [a, b] \to \mathbb{R}$ such that $\lim_{\varepsilon \to 0} \int_a^b \psi_\varepsilon(t)dt = 0$, and the inequalities

$$||\tilde{f}(y, \eta_x(t)) - \tilde{f}(y, \eta_{\overline{x}}(t))|| \leq \psi_\varepsilon(t) \tag{3.9}$$

$$||D_y\tilde{f}(y,\eta_x(t)) - D_y\tilde{f}(y,\eta_{\overline{x}}(t))|| \leq \psi_\varepsilon(t) \tag{3.10}$$

hold whenever $y \in W$, $x \in N$, $|||x - \overline{x}|| \leq \varepsilon$.

(DC2) the map $v \to \rho_v$, where ρ_v is the integrable vector-valued function on $[a,b]$ given by

$$\rho_v(t) = \tilde{f}(\gamma_{\overline{x}}(t), \eta_{\overline{x}+v}(t)) - \tilde{f}(\gamma_{\overline{x}}(t), \eta_{\overline{x}}(t)) \tag{3.11}$$

is weakly* differentiable at $v = 0$, regarded as a map into the space of vector-valued Borel measures on $[a,b]$.

In the above conditions, it is always understood that $\tilde{f}(y,\eta_x(t)) = 0$ if $t \notin \mathrm{Dom}(\eta_x)$. The weak* differentiability in (DC2) means that, if $\xi : [a,b] \to \mathbb{R}$ is an arbitrary continuous function, then the map $v \to \int_a^b \xi(t)\rho_v(t)dt$ is differentiable in the usual sense.

If (DC) holds, we will call Γ (f,L)-*differentiable* at $\overline{x}$.

Call a subset A of Ω *thin* if $A = A_1 \cup A_2$ where

(1) A_1 is a finite or countable union of connected C^1 submanifolds of positive codimension, and

(2) A_2 has Hausdorff dimension strictly less than $n-1$.

We then have

THEOREM 3.1. *Let $\mathcal{P}$ be an optimal control problem whose target $\mathcal{T}$ is a C^1 submanifold of Ω, such that the function φ: $\mathcal{T} \to \mathbb{R}$ is of class C^1. Let $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ be a total t-extremal presynthesis for $\mathcal{P}$. Assume that*

(1) the value function V_Γ satisfies the weak continuity conditions for $\mathcal{P}$,

(2) $V_\Gamma \leq \varphi$ on $\mathcal{T}$,

(3) Γ is $(f,L)-$differentiable at all points x in the complement of a thin subset A of S.

Then Γ is optimal.

The proof of this theorem is a modification of the argument given by Boltyansky in [Bo]. The first step is to pick a point $x \in S-A$, choose $(\lambda(\cdot), \lambda_0)$ for the t-extremal (γ_x, η_x), and compute the derivative of $h \to \lambda_0 V_\Gamma(x+hv)$ at $h = 0$, v being an arbitrary vector. A formal calculation shows that this derivative is equal to $< \lambda(t_x^-), v >$, and the (f,L)-differentiability of Γ at x is precisely what is needed for a rigorous justification. As explained above,

this implies in particular that $\lambda_0 \neq 0$. So λ_0 could have been chosen to be equal to 1 to begin with, and with this choice we see that the derivative of $V_\Gamma(x+hv)$ at $h=0$ is equal to $<\lambda(t_x^-), v>$. Using the (f,L)-differentiability we conclude that V_Γ is differentiable at x and $dV_\Gamma(x) = \lambda(t_x^-)$. On the other hand, there exists $t > t_x^-$ arbitrarily close to t_x^- such that

$$< \lambda(t), f(\gamma(t), u) > + L(\gamma(t), u) \geq 0 \tag{3.12}$$

for all $u \in U$. So we can let $t \to t_x^-$ and conclude that the above inequality holds for $t = t_x^-$, so that

$$< dV_\Gamma(x), f(x, u) > + L(x, u) \geq 0 \tag{3.13}$$

for all $u \in U$. We now pick an arbitrary trajectory (γ, η) that goes from a point $x \in \Omega$ to the target $\mathcal{T}$. Let $\mathrm{Dom}(\eta) = [a,b]$. We use the PCAT to find (γ_j, η_j), with domain $[a,b]$, η_j piecewise constant, and $\gamma_j(b) = \gamma(b)$, such that $\gamma_j \to \gamma$ uniformly, and $J(\gamma_j, \eta_j) \to J(\gamma, \eta)$. Let $x_j = \gamma_j(a)$. If we show that $V_\Gamma(x_j) \leq J(\gamma_j, \eta_j)$, then the lower semicontinuity of V_Γ will imply that $V_\Gamma(x) \leq J(\gamma, \eta)$. So we need to prove that $V_\Gamma(x) \leq J(\gamma, \eta)$ assuming that η is piecewise constant. We pick (using the weak upper semicontinuity) $y_j \to \gamma(b)$, $\varepsilon_j > 0$, $\delta_j > 0$, $\delta_j \to 0$, such that $V_\Gamma(z) \leq V_\Gamma(\gamma(b)) + \delta_j$ whenever $||z - y_j|| < \delta_j$. We then let, for each z, γ^z be the trajectory for η which ends at z. Suppose that we find, for each j, a z_j such that $||z_j - y_j|| < \varepsilon_j$ and

$$V_\Gamma(x_j) \leq \int_a^b L(\gamma^{z_j}(t), \eta(t))dt + V_\Gamma(z_j), \tag{3.14}$$

where $x_j = \gamma^{z_j}(a)$. Then

$$V_\Gamma(x_j) \leq \int_a^b L(\gamma^{z_j}(t), \eta(t))dt + V_\Gamma(\gamma(b)) + \delta_j \ . \tag{3.15}$$

If we take limits and use the lower semicontinuity of V_Γ and the boundary condition $V_\Gamma(\gamma(b)) \leq \varphi(\gamma(b))$, we get $V_\Gamma(x) \leq J(\gamma, \eta)$. Hence it suffices to show, for a fixed piecewise constant control $\eta : [a,b] \to U$, that every nonvoid open set W contains a z such that $V_\Gamma(\gamma^z(a)) \leq \int_a^b L(\gamma^z(t), \eta(t))dt + V_\Gamma(z)$. To prove this, we use a transversality argument similar to Boltyansky's, and find a $z \in W$ such that $\gamma^z(t)$ only belongs to A for a finite or countable set

B of "bad" values of t. By adding finitely many points to B, we may assume that B contains all the switching times of η. Now let

$$\psi(t) = V_\Gamma(\gamma^z(t)) + \int_a^t L(\gamma^z(s), \eta(s))ds \quad . \tag{3.16}$$

Then ψ is differentiable at every $t \notin B$, and $\psi'(t) \geq 0$. Moreover, ψ has no downwards jumps as a function of t. From this it follows easily that $\psi(a) \leq \psi(b)$, and the proof is complete.

We remark that the conditions imposed on V_Γ are not too restrictive. Indeed, lower semicontinuity of the value function typically follows whenever one can apply the standard methods to prove existence of optimal controls. The NDJ property always holds for the value function, and the weak upper semicontinuity follows if the control system (2.1) has the accessibility property (cf. [SJ]).

The boundary condition that $V \leq \varphi$ on $\mathcal{T}$ is clearly satisfied by the value function. So the only "unnatural" condition is the differentiability requirement. In the next section, we outline a method for proving, for some classes of optimal control problems, the existence of an optimal presynthesis that satisfies this requirement.

§4. Subanalytic sets and optimal presynthesis.

We will make systematic use of theory of *subanalytic sets and stratifications* (cf. [Ha], [Hi], [Su11]).

We remark that, throughout this section and §5, the word "manifold" has the usual meaning (Hausdorff, countable at infinity) *except for the fact that a manifold is allowed to have connected components of different dimensions.* (In particular, it can contain components of arbitrarily high dimension.)

A *parametrized family of controls* (PFC) is a triple $\mathcal{F} = (N, S, \{\eta_\sigma\}_{\sigma \in S})$ such that N is a set, S is a subset of N and, for each $\sigma \in S$, η_σ is a control. If $\mathcal{U}$ is any class of controls, we will say that that $\mathcal{F}$ is a PFC in $\mathcal{U}$ if each η_σ belongs to $\mathcal{U}$. The definition of a *parametrized family of trajectory-control pairs* (PFTCP) $\mathcal{F} = (N, S, \{(\gamma_\sigma, \eta_\sigma)\}_{\sigma \in S})$ is similar.

If $\mathcal{F}$ is a PFC as above, the *graph* of $\mathcal{F}$ is the set

$$G(\mathcal{F}) = \{(\sigma, t, u) : \sigma \in S, t \in \mathrm{Dom}(\eta_\sigma), u = \eta_\sigma(t)\}.$$

Similarly, for a PFCTP $\mathcal{F}$ as above, the graph $G(\mathcal{F})$ is given by

$$G(\mathcal{F}) = \{(\sigma, t, x, u) : \sigma \in S, t \in \mathrm{Dom}(\eta_\sigma), x = \gamma_\sigma(t), u = \eta_\sigma(t)\}.$$

If the control space U is a subset of some real-analytic "ambient manifold" U_{amb}, then a PFC $\mathcal{F}$ is *subanalytic* if N is a real-analytic manifold and the graph $G(\mathcal{F})$ is a subanalytic subset of $N \times \mathbb{R} \times U_{amb}$. Similarly, if $\mathcal{F}$ is a PFTCP, then $\mathcal{F}$ will be called subanalytic if $G(\mathcal{F})$ is subanalytic in $N \times \mathbb{R} \times \Omega \times U_{amb}$.

From now on we always assume that $U \subseteq U_{amb}$ as above. We are interested in the class $\mathcal{U}_0$ of controls η such that (i) $\mathrm{Dom}(\eta)$ is a compact interval, and (ii) η is bounded (i.e. the set of values of η is relatively compact in U_{amb}).

A PFC $\mathcal{F}$ is *proper* if the projection map $(\sigma, t, u) \to \sigma$ is proper on the closure of $G(\mathcal{F})$, i.e. if for every compact subset K of N there exists a compact subset K' of $\mathbb{R} \times U_{amb}$ such that $G(\mathcal{F}) \cap (K \times \mathbb{R} \times U) \subseteq K \times K'$. The definition of a proper PFTCP is similar. We then have the following *structure theorem* for proper, subanalytic PFC's in $\mathcal{U}_0$:

THEOREM 4.1. *Let U_{amb} be a real-analytic manifold, and let $U \subseteq U_{amb}$. Let $\mathcal{F} = (N, S, \{\eta_\sigma\}_{\sigma \in S})$ be a proper, subanalytic PFC in $\mathcal{U}_0$. Then there exists a real-analytic stratification $\mathcal{S}$ of N with subanalytic strata, such that (i) $\mathcal{S}$ is a union of strata of $\mathcal{S}$ and (ii) for every $\Sigma \in \mathcal{S}$ such that $\Sigma \subseteq S$ there exist an integer $\mu(\Sigma) \geq 0$, and real-valued real-analytic functions $\tau_0, \cdots, \tau_{\mu(\Sigma)}$ on Σ, with the following properties:*

(1) for each $\sigma \in \Sigma$, the inequalities

$$\tau_0(\sigma) < \tau_1(\sigma) < \cdots < \tau_{\mu(\Sigma)}(\sigma)$$

hold, and $\mathrm{Dom}(\eta_\sigma) = [\tau_0(\sigma), \tau_{\mu(\Sigma)}(\sigma)]$,

(2) for each $i \in \{1, \cdots, \mu(\Sigma)\}$ the function $\eta^{i,\Sigma} : (\sigma, t) \to \eta_\sigma(t)$ is real-analytic on

$$G^{i,\Sigma} = \{(\sigma, t) : \sigma \in \Sigma, \tau_{i-1}(\sigma) < t < \tau_i(\sigma)\},$$

and there exist positive integers $\nu_{i,\Sigma,-}$, $\nu_{i,\Sigma,+}$, such that the functions

$$(\sigma, s) \to \eta^{i,\Sigma}(\sigma, \tau_{i-1}(\sigma) + s^{\nu_{i,\Sigma,-}})$$

and

$$(\sigma, s) \to \eta^{i,\Sigma}(\sigma, \tau_i(\sigma) - s^{\nu_{i,\Sigma,+}})$$

(which are defined for $(\sigma, s) \in \Sigma \times \mathbb{R}$, $s > 0$, s *sufficiently small) extend to real-analytic functions on some open neighborhood of* $\Sigma \times \{0\}$ *in* $\Sigma \times \mathbb{R}$.

A proof of Theorem 4.1 using desingularization is given in [Su4].

The structure theorem can be used to prove that certain extremal presyntheses are optimal. Let $\mathcal{P}$ be an optimal control problem as above. Call $\mathcal{P}$ *analytic* if (i) the target $\mathcal{T}$ is an analytic submanifold of Ω, (ii) $U \subseteq U_{amb}$, where U_{amb} is a real-analytic manifold, and (iii) the functions $f : \Omega \times U \to \mathbb{R}^n$, $L : \Omega \times U \to \mathbb{R}$, $\varphi : \mathcal{T} \to \mathbb{R}$ are real-analytic. (This means, for f and L, that they have analytic extensions to open neighborhoods of the closure of $\Omega \times U$ in $\Omega \times U_{amb}$.) Let $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ be a presynthesis for $\mathcal{P}$. Let $\Gamma_0 = (\Omega, S, \{\eta_x\}_{x \in S})$. Let us call Γ *control-subanalytic* (resp. *control-proper*) if Γ_0 is subanalytic (resp. proper). More generally, let us call Γ *countably control-proper control-subanalytic* (CCPCS) if we can write S as a countable union $S = \cup_{i=1}^{\infty} S^i$ of sets, in such a way that each $\Gamma^i = \{(\gamma_x, \eta_x) : x \in S^i\}$ is control-proper and control-subanalytic.

We then have:

THEOREM 4.2. *Let* $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ *be a CCPCS total t-extremal presynthesis for an analytic optimal control problem* $\mathcal{P}$ *, such that each control* η_x *occurring in* Γ *takes values in a compact subset of* U_{amb}. *Assume that the cost function* V_Γ *satisfies the weak continuity conditions and the boundary condition (3.7) associated with* $\mathcal{P}$. *Then* Γ *is optimal.*

PROOF. In view of Theorem 3.1, it suffices to show that Γ is (f, L)-differentiable at all the points of the complement of a thin subset S_0 of S. Write $S = \cup_{i=1}^{\infty} S^i$, where the S^i are such that $\Gamma^i = \{(\gamma_x, \eta_x) : x \in S^i\}$ is control-subanalytic and control-proper. Then Γ_0^i is subanalytic proper. Pick a stratification $\mathcal{S}^i$ with the properties of the structure theorem. Let S_0 be the union of all the strata Σ of positive codimension of all the $\mathcal{S}^i$ such that $\Sigma \subseteq S$. Then S_0 satisfies the desired condition. □

We now turn to the question of the *existence* of an optimal subanalytic

presynthesis. Let $\mathcal{F} = (N, S, \{(\gamma_\sigma, \eta_\sigma)_{\sigma \in S}\})$ be a PFTCP for a problem $\mathcal{P}$. We call $\mathcal{F}$ *sufficient for* $\mathcal{P}$ if for every $x \in \Omega$ that can be steered to the target there exists a $\sigma \in S$ such that $(\gamma_\sigma, \eta_\sigma)$ goes from x to the target and is optimal. That is, $\mathcal{F}$ is large enough to contain an optimal trajectory from x for every x that can be steered to the target, but this trajectory need not be unique, and $\mathcal{F}$ may contain nonoptimal trajectories as well, and also trajectories that do not end at the target.

Define S_{opt} to be the set of those $\sigma \in S$ such that $(\gamma_\sigma, \eta_\sigma)$ ends at the target and is optimal. Define $\mathcal{F}_{opt}$ to be the PFTCP

$$\mathcal{F}_{opt} = (N, S_{opt}, \{(\gamma_\sigma, \eta_\sigma)\}_{\sigma \in S_{opt}}).$$

Clearly, $G(\mathcal{F}_{opt}) = G(\mathcal{F}) \cap (S_{opt} \times \mathbb{R} \times \Omega \times U_{amb})$. So $\mathcal{F}_{opt}$ will be subanalytic if $\mathcal{F}$ is, provided that S_{opt} is subanalytic in N.

Let $\pi^-_{\mathcal{F}}, \pi^+_{\mathcal{F}}$ be the *initial condition* and *terminal condition maps* associated with $\mathcal{F}$, so that γ_σ goes from $\pi^-_{\mathcal{F}}(\sigma)$ to $\pi^+_{\mathcal{F}}(\sigma)$. Let $B = S - S_{opt}$. Then $B = B_1 \cup B_2 \cup B_3$, where $B_1 = N - S$, $B_2 = \{\sigma \in S : \pi^+_{\mathcal{F}}(\sigma) \notin \mathcal{T}\}$, and B_3 is the set of those $\sigma \in S$ such that $\pi^+_{\mathcal{F}}(\sigma) \in \mathcal{T}$ but $(\gamma_\sigma, \eta_\sigma)$ is not optimal. If $\mathcal{F}$ is subanalytic and proper, then S is subanalytic in N , because it is the image of $G(\mathcal{F})$ under the map $(\sigma, t, x, u) \to \sigma$. So B_1 is subanalytic as well. The set B_2 is the image of $(S \times (\Omega - \mathcal{T})) \cap G(\pi^+_{\mathcal{F}})$ under the map $\mu : (\sigma, x) \to \sigma$. (Here $G(\pi^+_{\mathcal{F}})$ is the graph of $\pi^+_{\mathcal{F}}$.) So B_2 will be subanalytic if $\mathcal{T}$ is subanalytic, $\pi^+_{\mathcal{F}}$ is a subanalytic map, and μ is proper on the closure of $G(\pi^+_{\mathcal{F}})$. The latter condition is the requirement that $\pi^+_{\mathcal{F}}$ be *locally bounded*, i.e. that, if $K \subseteq N$ is compact, then $\pi^+_{\mathcal{F}}(K \cap S)$ be relatively compact in Ω. Finally, the set B_3 is the image under the map $\nu : (\sigma, \sigma', x, y, z, a, b) \to \sigma$ of the set A of those 7-tuples $(\sigma, \sigma', x, y, z, a, b) \in N \times N \times \Omega \times \Omega \times \Omega \times \mathbb{R} \times \mathbb{R}$ such that the following hold:
(i) $(\sigma, x) \in G(\pi^-_{\mathcal{F}})$, **(ii)** $(\sigma', x) \in G(\pi^-_{\mathcal{F}})$, **(iii)** $(\sigma, y) \in G(\pi^+_{\mathcal{F}})$,
(iv) $(\sigma', z) \in G(\pi^+_{\mathcal{F}})$, **(v)** $y \in \mathcal{T}$, **(vi)** $z \in \mathcal{T}$, **(vii)** $(\sigma, a) \in G(J_{\mathcal{F}})$,
(viii) $(\sigma', b) \in G(J_{\mathcal{F}})$, and **(ix)** $b < a$. (Here $J_{\mathcal{F}}$ is the cost function associated to $\mathcal{F}$, i.e. $J_{\mathcal{F}}(\sigma) = J(\gamma_\sigma, \eta_\sigma)$.)

It follows that B_3 is subanalytic provided that $G(\pi^+_{\mathcal{F}})$, $G(\pi^-_{\mathcal{F}})$, $\mathcal{T}$ and $G(J_{\mathcal{F}})$ are subanalytic sets, and ν is proper on the closure of A. To ensure the properness of ν, it suffices to have $\pi^+_{\mathcal{F}}$, $\pi^-_{\mathcal{F}}$, $J_{\mathcal{F}}$ locally bounded and $\pi^-_{\mathcal{F}}$ proper. If we observe that the properness and subanalyticity of $\mathcal{F}$ imply that $\pi^+_{\mathcal{F}}$ and $\pi^-_{\mathcal{F}}$ are locally bounded, we conclude that S_{opt} (and hence $\mathcal{F}_{opt}$) is subanalytic if

(I) $\mathcal{F}$ is subanalytic and proper,

(II) $\pi^-_{\mathcal{F}}$ is proper on the closure of S,

(III) $\mathcal{T}$ is subanalytic,

(IV) $J_{\mathcal{F}}$ is subanalytic and locally bounded.

Notice also that the only purpose of having $\pi^-_{\mathcal{F}}$ proper is to guarantee that, as σ takes values in a relatively compact set, so does σ'. If $J_{\mathcal{F}}$ is bounded below, this conclusion will follow under the weaker hypothesis:

(II′) the map $\sigma \to (\pi^-_{\mathcal{F}}(\sigma), J_{\mathcal{F}}(\sigma))$ is proper on the closure of S.

So we have shown that, if $\mathcal{F}$ is a proper subanalytic family and conditions (I), (III), (IV) hold, and either (II) holds or $J_{\mathcal{F}}$ is bounded below and (II′) holds, then $\mathcal{F}_{opt}$ is also subanalytic. It is clear that $\mathcal{F}_{opt}$ is proper as well. To construct a presynthesis, we must select one value of σ for each $x \in \pi^-_{\mathcal{F}}(\mathcal{F}_{opt})$. For this purpose we use the *subanalytic selection theorem*, according to which, if M_1, M_2 are real-analytic manifolds, $A \subseteq M_1$ is subanalytic, and $\rho : M_1 \to M_2$ is real-analytic, then there exists a "subanalytic single-valued selection of ρ^{-1}," that is, a subanalytic subset B of M_1 such that $B \subseteq A$, $\rho(B) = \rho(A)$, and ρ is one-to-one on B. We apply this with $M_1 = N \times \Omega$, $M_2 = \Omega$, $A = G(\pi^-_{\mathcal{F}_{opt}})$, and we take ρ to be the projection $(\sigma, x) \to x$. We then obtain a set $B \subseteq N_1 \times \Omega$ such that $B \subseteq G(\pi^-_{\mathcal{F}_{opt}})$, B is subanalytic in $N_1 \times \Omega$, ρ is one-to-one on B, and $\rho(B) = \pi^-_{\mathcal{F}_{opt}}(S_{opt})$. Now the family $\tilde{\mathcal{F}} = (N \times \Omega, B, \{(\gamma_\sigma, \eta_\sigma) : (\sigma, x) \in B\})$ is "almost" a presynthesis, the only difference being that the parameter set B is not a subset of the state space Ω but a set that, by means of the map ρ, is in a bijective correspondence with such a subset. Precisely, ρ establishes a bijection between B and $\pi^-_{\mathcal{F}_{opt}}(S_{opt})$. If $\mathcal{F}$ is sufficient for $\mathcal{P}$ then $\pi^-_{\mathcal{F}_{opt}}(S_{opt}) = D$, where D is the set of all x that can be steered to the target. Now define $\hat{\mathcal{F}} = (\Omega, D, \Gamma)$, where $\Gamma = \{(\gamma_{\hat{\sigma}(x)}, \eta_{\hat{\sigma}(x)}) : x \in D\}$, and $\hat{\sigma}(x) = \sigma$ iff $(\sigma, x) \in B$. Then Γ is an optimal presynthesis.

The graph $G(\hat{\mathcal{F}})$ may fail to be subanalytic. However, if we let $Q \subseteq N \times \Omega \times \mathbb{R} \times \Omega \times U_{amb}$ be the set of those (σ, x, t, y, u) such that $(\sigma, x) \in B$ and $(\sigma, t, y, u) \in G(\mathcal{F}_{opt})$, then Q is clearly subanalytic, and $\theta(Q) = G(\hat{\mathcal{F}})$,

where θ is the projection $(\sigma, x, t, y, u) \to (x, t, y, u)$. If θ is proper, then $\hat{\mathcal{F}}$ is subanalytic and proper. (This is the case, in particular, if (II) holds.)

Even if θ is not proper, we can write $B = \cup_{i=1}^{\infty} B^i$, with the B^i relatively compact and subanalytic. Then θ is proper on the closure of each B^i. So, if we let $D^i = \rho(B^i)$, $\hat{\mathcal{F}}^i = (\Omega, D^i, \{\gamma_{\hat{\sigma}(x)}, \eta_{\hat{\sigma}(x)} : x \in D^i\})$, we see that $\hat{\mathcal{F}}^i$ is proper subananlytic. So $\hat{\mathcal{F}}$ is CCPCS and, in fact, CPS (countably proper and subanalytic). So we have proved:

THEOREM 4.3. *If an analytic optimal control problem $\mathcal{P}$ is such that there is a proper, subanalytic sufficient family $\mathcal{F}$ for which (I), (II), (III), (IV) hold, then $\mathcal{P}$ has a proper subanalytic optimal synthesis. If, instead of (II), we assume that $J_{\mathcal{F}}$ is bounded below and (II′) holds, then $\mathcal{P}$ has a countably subanalytic countably proper optimal presynthesis.*

§5. From optimal presynthesis to trajectory analysis.

The simplest example of an application of Theorem 4.3 is provided by the *minimum time problem for linear systems with a polyhedral control set.* This case was studied by Brunovsky in [Br1], and was actually the first instance of an application of subanalytic set theory in optimal control. The key fact is the following *bang-bang theorem with bounds on the number of switchings*: for a linear system $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, U a polyhedron, there exist $\alpha > 0$, $\beta \geq 0$ such that, whenever there is a trajectory from a point x to a point y in time $\leq T$, then there is an optimal trajectory from x to y that corresponds to a bang-bang control with at most $\alpha + \beta T$ switchings. Using this result, we construct a proper subanalytic sufficient family as follows. Let V be the set of vertices of the polyhedron U. A *strategy* is a finite sequence $\zeta = (\zeta_1, \cdots, \zeta_r)$ of members of V. The manifold N has connected components $N_\zeta = \mathbb{R}^n \times \mathbb{R}^r$, for all strategies ζ. The set S is such that $S \cap N_\zeta = \mathbb{R}^n \times (\mathbb{R}^r_+ \cap A_r)$, where $\mathbb{R}^r_+$ is the nonnegative orthant of $\mathbb{R}^r$, and A_r consists of those $(t_1, \cdots, t_r)$ such that $\alpha + \beta(t_1 + \cdots + t_r) \geq r + 1$. If $\sigma = (x, t_1, \cdots, t_r) \in S \cap N_\zeta$ and $\zeta = (\zeta_1, \cdots, \zeta_r)$, then η_σ is the control that takes the value ζ_1 for $0 \leq t < t_1$, then ζ_2 for $t_1 \leq t < t_1 + t_2$, and so on. Then γ_σ is the trajectory for η_σ that starts at x. It is clear that the family so constructed is subanalytic and proper, and sufficient for time-optimality. (The latter conclusion follows from the bang-bang theorm, but does not yet

use the full force of the bound on the number of switchings.) Suppose we choose a target $\mathcal{T}$ which consists of a single point $\overline{x}$. Then conditions (I), (III) and (IV) of the hypothesis of Theorem 4.3 are trivially verified. Also, the cost functional is nonnegative, and therefore bounded below. To verify (II′), we notice that, if T is bounded by a $\overline{T}$, then $S \cap N_\zeta$ will be empty except possibly for those $\zeta = (\zeta_1, \cdots, \zeta_r)$ such that $r \leq \alpha + 1 + \beta\overline{T}$. So there are only finitely many ζ for which $S \cap N_\zeta$ contains points σ with $J(\gamma_\sigma, \eta_\sigma) \leq \overline{T}$. If, in addition, x varies in a compact set K, then the possible values of $\sigma = (x, t_1, \cdots, t_r)$ will satisfy $x \in K$, $t_i \geq 0$, $t_1 + \cdots, t_r \leq \overline{T}$, $r \leq \alpha + 1 + \beta T$. So σ is restricted to a compact set, and (II′) holds. Hence Theorem 4.3 applies.

It is clear that the important property of this particular example is the possibility of finding a family of controls parametrized by a finite-dimensional parameter, which contains enough optimal trajectories. This naturally leads to the question of analyzing the structure of optimal trajectories for general optimal control problems, in order to prove similar results. In a number of cases (cf., e.g., [Su1], [Su5], [Su6], [Su7]), such an analysis has been carried out and has yielded the existence of a nice presynthesis. For general problems, progress has been made in proving nontrivial conclusions about optimal trajectories (cf. [Su9], [Su10]). Although the results thus established are too weak to be applicable to synthesis theory, they suffice to show that optimal trajectories have interesting structure properties that deserve further study. The chapters in this book by Agrachev-Gamkrelidze, Bressan, and Schättler, constitute examples of the new directions now opening up in the search for properties of optimal trajectories.

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