Geometry and Optimal Control*

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Dedicated to Roger W. Brockett on his 60th Birthday

ABSTRACT Optimal control has strongly influenced geometry since the early days of both subjects. In particular, it played a crucial role in the birth of differential geometry in the nineteenth century through the revolutionary ideas of redefining the notion of "straight line" (now renamed "geodesic") by means of a curve minimization problem, and of emphasizing general invariance and covariance conditions. More recently, modern control theory has been heavily influenced by geometry. One aspect of this influence is the geometrization of the necessary conditions for optimality, which are recast as geometric conditions about reachable sets, thus becoming special cases of the broader question of the structure and properties of these sets. Recently, this has led to a new general version of the finite-dimensional maximum principle, stated here in full detail for the first time. A second aspect—in which Roger Brockett's ideas have played a crucial role—is the use in control theory of concepts and techniques from differential geometry. In particular, this leads to regarding a control system as a collection of vector fields, and exploiting the algebraic structure given by the Lie bracket operation. This approach has led to new important developments on various nonlinear control problems. In optimal control, the vector-field view has produced invariant formulations of the maximum principle on manifolds—using either Poisson brackets or connections along curves—which, besides being more general and mathematically natural, actually have real advantages for the solution of concrete problems.

5.1 Introduction

The interplay among control, geometry, and physics goes back to ancient times. The words "control theory" are, of course, of recent origin, but the subject itself is much older, since it contains the classical calculus of variations as a special case, and the first calculus of variations problems go back to classical Greece. (The same is true of the calculus of variations, whose name was introduced by Euler in 1755, although the subject itself is at least as old as the classical isoperimetric problem.)

In this discussion we will describe some aspects of the relationship between optimal control and geometry, beginning with a brief outline of some events prior to the formulation of the Maximum Principle in the 1950s, and

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then focusing on the geometrization of optimal control theory that took place after 1960. The pre-Maximum Principle relationship is primarily one-sided, with optimal control (in the form of its ancestor, the calculus of variations) exercising a decisive influence on geometry, and serving as the source of some of the ideas that revolutionized geometry, such as differential geometry. In the much shorter post-Maximum Principle era, the influence goes mainly in the other direction, with control theory being heavily influenced by geometric ideas. (An important exception is provided by recent developments in subriemannian geometry, discussed by V. Jurdjevic in Chapter 7 of this volume.)

The geometrization of control theory has brought into the field many mathematical concepts and structures borrowed from differential geometry and related areas—such as Lie algebras of vector fields, integral manifolds, systems of differential forms and Cartan prolongations, distributions[1], Carnot-Carathéodory metrics, Lie groups, homogeneous spaces, symplectic structures, Hamiltonian mechanics—together with tools from real analytic geometry such as subanalytic sets, stratifications and desingularization. The combined use of these tools has led to new results on several important control-theoretic questions, such as (i) controllability, observability and minimal realizations of nonlinear systems; (ii) path finding; (iii) feedback stabilization; (iv) system equivalence under state space diffeomorphisms, static feedback, and dynamic feedback; (v) system linearization under state space diffeomorphisms and under feedback; (vi) disturbance decoupling; (vii) tracking; and (viii) optimal control.

Our primary area of interest here will be *optimal control*, and we will not review other aspects of the differential-geometrization of nonlinear control theory since those topics have been extensively discussed elsewhere—for example in the books by Jurdjevic [22], Isidori [20], Nijmeijer and var der Schaft [33], and the survey books [8], [15], [21], [32], [41]—or are covered by other papers in this volume, e.g. in the contributions by A.J. Krener and V. Jurdjevic. One of the aspects that will not be covered here is the relationship with mechanics, which is beautifully described in the paper by Bloch and Crouch (Chapter 8 in this volume).

Even more narrowly, we will focus on the *geometrization of the Maximum Principle* based on two independent ideas: *first*, the "differential geometric" formulation, invariant under arbitrary nonlinear changes of coordinates, and *second*, the reformulation of the Maximum Principle as a geometric property of reachable sets, namely, a condition for a reachable set and some other set to be *separated*. Underlying both ideas lies the emphasis on regarding a control system as a *system of vector fields*[2]. This makes it possible to exploit the algebraic structure of the space of vector fields, given by the usual linear operations and the Lie bracket. As will be explained below, this can be used to do at least two important things, namely, (a) give a coordinate-free user-friendly formulation of the Maximum Princi-

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[1] That is, subbundles of a tangent bundle.

[2] For example, Kaskosz and Łojasiewicz proposed in [24] to refer to a general family of time-varying vector fields as a "generalized control system." We adopt the more radical view that the Kaskosz-Łojasiewicz approach provides the most reasonable way to define the concept of a *control system*.

ple, and (b) generate high-order variations and obtain high-order necessary conditions for optimality. Since the first idea, pertaining to the differential-geometrization of the Maximum Principle, has already been described at length in Sussmann [46], we will pay special attention here to the second idea, and mainly concentrate on one of its by-products, namely, our recent general version of the finite-dimensional Maximum Principle. This result, of which several special cases have already been presented in preliminary announcements in Sussmann [42]-[45], will be stated here in a complete, self-contained form.

A natural alternative to the systems of vector fields point of view is that of *differential inclusions*. For about 30 years it has been widely believed that the differential inclusions approach is strictly more general than the one based on vector field systems, owing to the fact that set-valued maps, even if they are very smooth, do not usually admit continuous single-valued selections. (For example, the map $z \to |z|^N \sqrt{z}$, on the complex plane, has two values at each nonzero point, and is of class C^N, but has no continuous single-valued selections on any neighborhood of the origin.)

We will argue that—at least as far as the finite-dimensional Maximum Principle is concerned—the vector-field point of view is actually sufficient to cover the differential inclusions case. The reason for this is that all the differential inclusions for which there appears to be a version of the Maximum Principle are given by set-valued maps that admit sufficiently many selections that, even though they may fail to be continuous, are nice enough to make it possible to construct needle variations and extend the method of proof of the classical Maximum Principle to a differential-inclusions setting. (These facts are based on ideas due to A. Bressan [5]-[7]. The application to the Maximum Principle is discussed in Sussmann [44].) So the vector-fields point of view suffices, provided that we allow our vector fields to be very nonsmooth (i.e., not necessarily Lipschitz continuous or even continuous).

The key mathematical tools needed to carry out this program turn out to be

I. A concept of "generalized differential" having the right properties, especially the chain rule and suitable open mapping and transversal intersection theorems. Two such concepts—the "semidifferentials" and the "multidifferentials"—were introduced in Sussmann [42]-[45], [47], and will be reviewed here.

II. Separation and transversality theorems, giving necessary conditions for two sets S_1, S_2 to be "separated" at a point q, in the sense that $S_1 \cap S_2 = \{q\}$. These conditions are stated in terms of objects D_j that "linearly approximate" the sets S_j near q. A natural choice is to take the D_j to be multidifferentials—at $(0, q)$—of set-valued maps F_j that map cones C_j into the S_j.

III. An extension to set-valued maps (cf. Theorem 5.7.2 and Remark 5.8.1) of a theorem from homotopy theory, originally formulated by Leray and Schauder [29], and proved rigorously by Browder in [9] (cf. also Rabinowitz [36, 37]), about the existence of connected sets of zeros of certain homotopies. This makes it possible to extend the usual transversality results and include the sufficient condition for local controllability along a reference trajectory as a special case of the maximum principle.

IV. An extension of Bressan's ideas of [5]-[7] to "almost lower semicontinuous maps," with a strengthening of the conclusions of his selection theorems, to obtain selections that are continuous at a given point.

V. An extension to the pseudo-Lipschitz case of the results of Colombo *et al.* [13], and those of Fryszkowski and Rzežuchowski [16], on uniform approximation of relaxed trajectories of a differential inclusion by ordinary trajectories.

As explained in Sections 5.8 and 5.9, this theory makes it possible to give a geometric, intrinsic version of the Maximum Principle that covers the nonsmooth case of families of Lipschitz-continuous vector fields, the differential inclusions case, and many problems with discontinuous vector fields, while at the same time it includes, in the smooth case, the extra results arising from high-order conditions corresponding to complicated variations constructed using Lie brackets.

5.2 From Queen Dido to the Maximum Principle

The first optimal control result ever discovered must have been the statement that

(LS) *The shortest path joining two points is a straight line segment.*

This is of course a rather trivial observation, but it is undoubtedly (a) very old, (b) geometric, and (c) a result in optimal control. Moreover, we shall see later that, in spite of its deceptive obviousness, (LS) is a fact of enormous importance that, properly interpreted or reinterpreted, has led to key discoveries in geometry and theoretical physics.

Next came the ***isoperimetric problem*** (IP)[3], the solution of which was known in classical Greece. The IP is the problem of finding, among all simple closed plane curves of a given length, one that encloses the largest possible area. Once again, this is obviously a geometric question, and it is also an optimal control problem in at least two ways:

1. If one uses the well-known formula $A(\xi) = \frac{1}{2}\int_\xi (x\,dy - y\,dx)$ for the area enclosed by a simple closed curve ξ, oriented counterclockwise, then the IP asks us to maximize the integral $A(\xi)$ among all the curves $\xi : [0,1] \to \mathbb{R}^2$ that satisfy endpoint constraints $\xi(0) = \xi(1) = (0,0)$ and the side constraint $\int_0^1 \|\dot{\xi}(t)\|\,dt = P$, where $P > 0$ is given.

[3] According to the story told by Virgil in the *Aeneid* about the foundation of Carthage (*ca.* 850 B.C.E.), "when Dido landed in North Africa, she persuaded the local chief to sell her as much land as an oxhide could contain. She then cut a hide into very narrow strips, and joined them to make a continuous thread more than two-and-a-half miles long. She was then able to enclose between the thread and the sea the land on which Carthage was built" (Pars [34]).

2. Alternatively, our curves can be thought of as trajectories of a three-dimensional control system $\dot{x} = u$, $\dot{y} = v$, $\dot{z} = vx - uy$, and the IP as the minimum-time optimal control problem for this system, with control constraint $u^2 + v^2 = 1$, as well as endpoint constraints $x_{\text{initial}} = x_{\text{final}} = y_{\text{initial}} = y_{\text{final}} = z_{\text{initial}} = 0$, $z_{\text{final}} = \bar{A}$, where $\bar{A} > 0$ is given.

(The justification for the second formulation is that minimizing the length of the boundary of a region given the area is easily proved to be equivalent to maximizing the area given the length of the boundary.)

If we use Formulation 1, then the IP is a standard fixed-endpoint calculus of variations problem—minimize $I_0 \stackrel{\text{def}}{=} \int_a^b L(\xi(t), \dot{\xi}(t))dt$ among all curves $[a, b] \ni t \to \xi(t)$ such that $\xi(a) = \bar{q}$, $\xi(b) = \hat{q}$—with an extra constraint $I_1 = \alpha$, where I_1 is an integral of the form $\int_a^b L_1(\xi(t), \dot{\xi}(t))dt$. Calculus of variations problems of this form, with one or several constraints $\int_a^b L_i(\xi(t), \dot{\xi}(t))dt = \alpha_i$, are in fact called *isoperimetric problems.* [4]

If we use Formulation 2, then the IP is just a standard fixed-endpoint minimum-time optimal control problem, with no extra qualifications.

The solution of the IP was known to be what the most obvious guess immediately suggests, namely, a circle. (It is almost evident that the solution—modulo translations—has to be rotationally symmetric, since the question, as stated, treats all directions in the plane equally. This, however, is not a rigorous argument, since it might conceivably happen that there is "symmetry breaking," so that the solution modulo translations is not unique, and each solution individually is not rotationally symmetric, even though the *set* of all solutions is. [5] To transform the rotational symmetry argument into a rigorous proof that the solution of the IP is a circle, one would have to establish, for example, uniqueness of the solution modulo translations.)

Knowing that the solution of the IP was a circle, Greek scholars could not help thinking that *this fact had to mean something* about the physical world, even if they couldn't quite tell what this might be. For example, Aristotle reasoned that

> Clearly, the movement of the heavens must be the swiftest of all movements. Now of lines which return upon themselves the line which bounds the circle is the shortest; and that movement is the swiftest which follows the shortest line. Therefore, if the heaven moves in a circle and moves more swiftly than anything else, it must necessarily be spherical.

To the extent that one can make sense of Aristotle's argument, it would appear to be saying that planetary motion is circular because the circle is

[4] Throughout this paper, the letters q, u, and p will denote the usual state, control and momentum variables. Greek letters will be used for *curves* in the spaces of these variables. For example, the letters ξ, η, and ψ will denote curves in q, u, and p space, respectively.

[5] This is exactly what happens, for example, for the problem of *minimizing* rather than maximizing an enclosed area. In this case, the set S of solutions is very large, and consists of *all* loops of length P that enclose a zero area. (For example, S contains all "degenerate rectangles" with two opposite sides of length $P/2$ and the other two of length zero.) None of these solutions is rotationally symmetric, but of course the family of all the solutions is symmetric.

the solution of the IP. For Aristotle's logic to work, even if we ignore minor details such as the failure to distinguish between circles and spheres, one needs to know or assume that, for some reason, planets are trying to move as fast as possible, that each planet wants its orbit to enclose in its motion some fixed—planet-dependent—area, and, finally, that the speed of motion is a fixed—planet-dependent—constant. These assumptions—especially the planets' insistence on wanting to enclose a given area—would be considered somewhat implausible today, even by those who believe in a Creator who for some reason is seeking to minimize some cost functional.

The first two examples of minimization properties of curves that were truly tied to physics in a way that we still find credible today were two minimum principles of geometrical optics, namely:

1. The observation by ***Hero of Alexandria*** (*ca.* 65-*ca.* 125) that the laws of the *reflection* of light can be explained by assuming that light follows a shortest path.

2. The *minimum time principle* of ***Fermat*** (1601-1665), according to which, in a medium of variable velocity of propagation of light, the light rays are precisely the minimum-time paths.

Fermat's principle provides, in particular, an "explanation" for Snell's Law of refraction. The statement of Fermat's principle can be interpreted in at least two ways, namely, (a) as a description of the *physics* of a certain process (light propagation) that takes places in ordinary space, in which the governing principle is *time minimization*, or (b) as asserting that the governing principle is still that of *motion along shortest paths*, except that the geometry of space is now different—the "length" of a curve being, by definition, travel time along the curve—so shortest paths no longer have to be ordinary line segments. The first point of view looks simpler and more natural, but the second one has turned out, in the long run, to be more fruitful, as we shall see below.

In 1686, ***Newton*** studied the problem of characterizing the solid of revolution of least resistance. Mathematically, this means that, for given $a, b, \bar{y}, \hat{y}$, with $a < b$, we have to find the function $y = y(x)$ that minimizes the integral

$$I = \int_a^b \frac{x}{1 + y'(x)^2} dx$$

subject to $y(a) = \bar{y}$, $y(b) = \hat{y}$.

In 1696, ***Johann Bernoulli*** (1667-1748) challenged the mathematicians of his time to solve the "brachystochrone problem," in which it is desired to determine, among all curves ξ going from a point A to a point B in a vertical plane, the one for which a particle falling freely along ξ—subject only to the action of the gravitational force plus whatever "virtual force" is needed to keep the particle on ξ—will reach B from A in minimum time. Mathematically, one can formulate this by observing that, if our curve ξ is given—in terms of a parameter s which is an increasing function of time t—by $s \to (x(s), y(s))$, $0 \leq s \leq 1$, $A = (a, \alpha)$, $B = (b, \beta)$, and t is time, initialized so that $t(0) = 0$, then the total energy

$$E = \frac{1}{2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) + gy$$

is constant (taking the mass to be 1). So

$$dt = \sqrt{\frac{dx^2 + dy^2}{2E - 2gy}}\,.$$

If we assume that $a < b$, and make the Ansatz that our minimizing curve is the graph of a function $x \to y(x)$, then $dt = \sqrt{\frac{1+y'(x)^2}{2E-2gy}}\,dx$, so our problem becomes that of minimizing the integral $\int_a^b L(y(x), y'(x))\,dx$, subject to the constraints $y(a) = \alpha$, $y(b) = \beta$, where

$$L(y, u) \stackrel{\text{def}}{=} \sqrt{\frac{1 + u^2}{2E - 2gy}}\,. \tag{1}$$

On the other hand, without assuming that $a < b$ and without making any *Ansatz* about y being a function of x, we can formulate our problem as that of minimizing the integral

$$I = \int_0^1 \Lambda(x(s), y(s), \dot{x}(s), \dot{y}(s))\,ds\,, \tag{2}$$

subject to $x(0) = a$, $y(0) = \alpha$, $x(1) = b$, $y(1) = \beta$, where

$$\Lambda(x, y, u, v) \stackrel{\text{def}}{=} \sqrt{\frac{u^2 + v^2}{2E - 2gy}}\,. \tag{3}$$

(The curves under consideration are, of course, restricted to lie in the closed half-plane $H_E = \{(x, y) : y \leq \frac{E}{g}\}$.) Finally, we can also use a *minimum-time optimal control* formulation, by writing the equations of motion (after choosing units so that $2g = 1$, and changing y to $2E - y$) as

$$\dot{x} = u\sqrt{y}\,,\ \ \dot{y} = v\sqrt{y}\,,\ \ u^2 + v^2 = 1\,, \tag{4}$$

where x and y are the state variables, u and v are the controls, and y is restricted to satisfy $y \geq 0$. This third formulation is clearly more natural than the first two, since it is a direct mathematical translation of the problem. This is the first of a number of reasons that in our view show that Johann Bernoulli's question properly belongs to optimal control. (Cf. Sussmann and Willems [49, 50] for a more detailed argument).

The brachystochrone problem attracted a lot of interest in 1696-97, leading to the publication, in June 1697, of an issue of *Acta Eruditorum* containing six responses to Johann Bernoulli's challenge, by Johann Bernoulli himself, his brother Jakob, Leibniz, Newton, l'Hôpital, and Tschirnhaus.

What made the problem appealing at the time was that the solution was a curve that everybody knew—a cycloid—but there was no obvious way to guess that this was so, and one had to resort to the new methods of infinitesimal calculus that were being invented at that time in order to figure out the formula for the solution. (This should be contrasted with the case of the isoperimetric problem—for which, as indicated above, it is a fairly natural guess that the solution must be a circle—and that of Newton's 1686 problem, whose solution is a curve of no particular significance.)

What makes the solution of the brachystochrone problem important for us now, 300 years later, is that it marked the beginning of the systematic study—which would be given the name "calculus of variations" by Euler in 1755—of minimization problems of the form

(SCV)

Minimize	$\int_a^b L(\xi(t), \dot{\xi}(t), t)dt$	
subject to	$\xi(a) = \bar{q}\,,$	$\xi(b) = \hat{q}\,.$

(Here we will refer to (SCV) as a *standard* calculus of variations problem, to distinguish this question from other, "nonstandard" ones such as isoperimetric problems and problems with constraints other than fixed endpoints.)

Remark 5.2.1 Johann Bernoulli's problem happens to have a very significant differential-geometric aspect, that Bernoulli himself failed to appreciate, thereby failing to make an important discovery for which he had all the clues in front of his eyes. In fact, (4) describes the curves parametrized by arc-length for a Riemannian metric on the upper half-plane, given by

$$ds^2 = \frac{dx^2 + dy^2}{y}\,. \tag{5}$$

The cycloids that solve the problem are the geodesics of this metric. Johann Bernoulli studied the same problem with other functions instead of $\sqrt{y}$ in (4), and in particular for the function y, which in modern terms corresponds to the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,. \tag{6}$$

In this case, he found that the minimum-time paths were ***half-circles*** rather than cycloids. So Johann Bernoulli had found what we would call today the *Poincaré half-plane*, i.e., the simplest model of a non-Euclidean (hyperbolic) geometry. All that was missing was for him to take the "small step" of calling his travel time "length," i.e. of thinking that an expression such as (5) or (6) could serve as just another way of measuring *length*. Had he hit upon this idea, it would have been natural for him to turn (LS) around and regard it as a *definition* of "segment," for a new "geometry" in which "length" was measured differently. This step—which would eventually be taken by Riemann in 1854—was perfectly within Johann Bernoulli's reach, and would have enabled him to show that his minimum-time arcs—for the Poincaré half-plane—gave rise to a geometry where all of Euclid's axioms held, except for the famous Fifth Postulate.[6]

It is worth noting that Johann Bernoulli lived at a time when the question of Euclid's postulate was at the center of mathematicians' interest. (For example, the book Euclides ab Omni Naevo Vindicato, by G. Saccheri (1667-1733), claiming to give a proof of Euclid's Fifth Postulate in terms of the other ones, was published in 1733.) Moreover, Johann Bernoulli had an obsessive desire for fame and glory,[7] so it is a sad irony that he, of all people, should have come so close to a truly momentous discovery, and missed it. ◇

With ***Euler*** (1707-1783) and ***Lagrange*** (1736-1813), it became clear that a necessary condition for a curve $[a,b] \ni t \to \xi_*(t) \in \mathbb{R}^n$ to be a solution of a problem (SCV) is that it satisfy the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}\,, \tag{7}$$

i.e., if we use u instead of $\dot{q}$:

$$\frac{d}{dt}\frac{\partial L}{\partial u}\Big(\xi_*(t),\dot{\xi}_*(t),t\Big) = \frac{\partial L}{\partial q}\Big(\xi_*(t),\dot{\xi}_*(t),t\Big)\,, \tag{8}$$

or

$$\frac{d}{dt}\frac{\partial L}{\partial u^i}\Big(\xi_*(t),\dot{\xi}_*(t),t\Big) = \frac{\partial L}{\partial q^i}\Big(\xi_*(t),\dot{\xi}_*(t),t\Big)\,,\quad i=1,\ldots,n\,. \tag{9}$$

In addition, Lagrange tried to free mechanics from "geometry" by doing everything analytically. But he unwittingly took a fundamental step leading towards the geometrization of physics and the birth of differential geometry, by proving the ***invariance of the Euler-Lagrange equations under arbitrary, nonlinear coordinate changes.***

Lagrange's invariance theorem was the first example of what we would call today a "proof that some object is differential-geometrically intrinsic." In modern terminology, Lagrange's result says that (7) makes sense intrinsically on manifolds. What Lagrange proved, in his *Mécanique Analytique*, was that **(7) has the same form in arbitrary "curvilinear" or "generalized" coordinates.**

Remark 5.2.2 A more modern formulation of Lagrange's invariance result would be that, for a sufficiently smooth curve ξ_* in a smooth manifold Q, if TQ denotes the tangent bundle of Q, and $L : TQ \times \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function, and we define

$$\mathcal{EL}_{\xi_*}(t) = \frac{d}{dt}\frac{\partial L}{\partial u}(\xi_*(t),\dot{\xi}_*(t),t) - \frac{\partial L}{\partial q}(\xi_*(t),\dot{\xi}_*(t),t)\,, \tag{10}$$

[6] The cycloids that minimize arc-length for the metric (5) do not have this property. For example, the length of these cycloids is finite.

[7] He was involved in many fights about priorities. He was jealous of the scientific achievements of his brother Jakob and his own son Daniel, whom he once expelled from the house for having dared to win a scientific prize for which the father was also a candidate.

then $\mathcal{EL}_{\xi_*}$—the "Euler-Lagrange form along ξ_*"—is a well-defined field of covectors along ξ_*. This implies that the equation $\mathcal{EL}_{\xi_*}(t) = 0$ makes sense intrinsically, but it also implies that other things that can be written using $\mathcal{EL}_{\xi_*}$ make sense intrinsically. For example, if the Lagrangian L governs the motion of a particle that is subject in addition to constraints $\omega_i(q).\dot{q} = 0$, where the ω_i, $i = 1, \ldots, k$, are differential 1-forms on Q, such that $\omega_1(q), \ldots, \omega_k(q)$ are linearly independent at each q, then the statement that

$$\mathcal{EL}_{\xi_*}(t) \in \text{linear span}\Big(\{\omega_1(\xi(t)), \ldots, \omega_k(\xi(t))\}\Big) \quad \text{for all} \quad t \tag{11}$$

also makes sense intrinsically.

Formula (11) is the well-known *D'Alembert principle*, governing constrained motion. If we think of the covector $\mathcal{EL}_{\xi_*}(t)$ as a "virtual force," then (10) says that the "force"—i. e., $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$, the time derivative of the momentum $p = \frac{\partial L}{\partial \dot{q}}$—is equal to $\frac{\partial L}{\partial q}$ plus the virtual force. Then (7) says that for unconstrained motion the virtual force should be zero, and (11) says that that for constrained motion the virtual force a point q must be orthogonal to the set $\{v : \omega_i(q).v = 0\,,\, i = 1, \ldots, k\}$ of feasible velocity vectors at q. ◇

Lagrange was aware of the importance of his invariance proof. He wrote:

> It is perhaps one of the principal advantages of our method that it expresses the equations of every problem in the most simple form relative to each set of variables and that it enables us to see beforehand which variables one should use in order to facilitate the integration as much as possible.

Remark 5.2.3 An obvious illustration of the power of this idea is provided by the calculation showing that the three laws of Kepler follow from Newton's inverse-square law. The calculation is made much simpler by using spherical coordinates, i.e., by writing the equations in terms of those variables that "facilitate the integration as much as possible." ◇

Remark 5.2.4 The era of Euler and Lagrange also saw the birth of the ***least action principle***, that made the link between curve minimization and physics much stronger. According to the principle, the behavior of mechanical systems is such as to minimize an *action functional*

$$S = \int_a^b L(\xi(t), \dot{\xi}(t), t)\, dt\,.$$

Moreover, at least some of the discoverers of this principle felt, like Aristotle, that the fact that Nature obeys a minimization principle had to have a profound meaning. For example, the least action principle was thought to express the fact that the Creator was making the wisest possible use of his power, as Maupertuis wrote in 1759:

> Notre principe, le plus conforme aux idées que nous devons avoir des choses, laisse le monde dans le besoin continuel de la puissance du Créateur, et est une suite nécessaire de l'emploi le plus sage de cette puissance ... Ces lois si belles et si simples sont peut-être les seules que le Créateur et l'Ordonnateur des choses a établies dans la matière pour y opérer tous les phénomènes de ce Monde visible.

A radically different view was expressed by Hamilton in 1833:

> Although the law of least action has thus attained a rank among the highest theorems of physics, yet its pretensions to a cosmological necessity, on the ground of economy of the universe, are now generally rejected. And the rejection appears just, for this, among other reasons, that the quantity pretended to be economized is in fact often lavishly expended .... We cannot, therefore, suppose the economy of this quantity to have been designed in the divine idea of the universe.

The least action principle remains to this day one of the basic principles of physics, classical as well as relativistic and quantum mechanical (cf. Yourgrau and Mandelstam [57]). ◇

In 1834 and 1835, ***W.R. Hamilton*** (1805-1865) published papers on dynamics showing how to rewrite the Euler-Lagrange equations in what we would now call "Hamiltonian form" : for a curve $t \mapsto \xi_*(t)$, (7) is exactly equivalent to the system of equations

$$\begin{aligned} \tfrac{d\xi_*}{dt}(t) &= \tfrac{\partial H}{\partial p}\Big(\xi_*(t), \dot{\xi}_*(t), \psi(t), t\Big)\,, \\ \tfrac{d\psi}{dt}(t) &= -\tfrac{\partial H}{\partial q}\Big(\xi_*(t), \dot{\xi}_*(t), \psi(t), t\Big)\,, \\ 0 &= \tfrac{\partial H}{\partial u}\Big(\xi_*(t), \dot{\xi}_*(t), \psi(t), t\Big)\,, \end{aligned} \tag{12}$$

where the *Hamiltonian* H and the *momentum* ψ are defined by

$$H(q,u,p,t) \stackrel{\text{def}}{=} \langle p, u\rangle - L(q,u,t)\,, \tag{13}$$

$$\psi(t) \stackrel{\text{def}}{=} \frac{\partial L}{\partial u}\Big(\xi_*(t), \dot{\xi}_*(t), t\Big)\,. \tag{14}$$

The system (12)-(14) is not exactly what is commonly known as "Hamilton's equations," although in our view it is how Hamilton's equations should be written. The usual Hamilton equations are

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}\,,\quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}\,, \tag{15}$$

where $(q,p,t) \to \mathcal{H}(q,p,t)$ is a function of p, q and t alone, defined by the formula

$$\mathcal{H}(q,p,t) \stackrel{\text{def}}{=} \langle p, \dot{q}\rangle - L(q,\dot{q},t)\,. \tag{16}$$

This is certainly similar to (12)-(14), but with a crucial difference: in (15) and (16) $\dot{q}$ is supposed to be treated *not* as an independent variable, but as a function of q, p, and t, defined implicitly by the equation

$$p = \frac{\partial L}{\partial \dot{q}}(q,\dot{q},t)\,, \tag{17}$$

whereas in (12)-(14) q, u, p, and t are treated as independent variables.

The fact that Hamilton's equations were written in the form (15,16) rather than (12,13,14) has had important consequences, discussed in detail in Sussmann and Willems [49, 50]. Indeed, it is argued in [49, 50] that the maximum principle of optimal control theory could have been discovered much earlier, perhaps by Weierstrass, if only the "correct" Hamiltonian system (12,13,14) been used.

Hamilton's equations also have the *coordinate invariance property* proved by Lagrange for the Euler-Lagrange equations, provided that the p variables are transformed according to the correct law: for a coordinate change $q^{\text{new}} = F(q)$, i.e. $q_i^{\text{new}} = F_i(q_1, \ldots, q_n)$, the new variables p_i^{new} must be related to the old ones via

$$p_i = \sum_j \frac{\partial F_j}{\partial q_i} p_j^{\text{new}} ,$$

which is the dual of the transformation law

$$\dot{q}_i^{\text{new}} = \sum_j \frac{\partial F_i}{\partial q_j} \dot{q}_j ,$$

and guarantees that

$$\sum_i p_i \dot{q}_i = \sum_i p_i^{\text{new}} \dot{q}_i^{\text{new}} .$$

In other words, using modern terminology, the p_i are the components of a *covector.*

In 1854, ***B. Riemann*** (1826-1866) introduced a number of ideas that today lie at the foundation of differential geometry:

1. He turned Statement (LS) around, and made it into a *definition* of "line segment," now called "geodesic."

2. He introduced more general ways of measuring the length of an arbitrary curve $t \to (q^1(t), \ldots, q^n(t))$ by using the *arc-length* element

$$ds^2 = \sum_{ij} g_{ij}(q)\, dq^i dq^j , \tag{18}$$

   where $(g_{ij})_{i,j=1}^n$ is an $n \times n$ matrix of smooth functions of q, assumed to be positive definite for each q.

3. He introduced the systematic use of "covariant quantities," i. e., quantities that transform under general nonlinear coordinate changes according to a specific law. Two examples of that, namely, "vectors" such as $\dot{q}$, and "covectors" such as p, have already been discussed before. A third example is the Riemannian metric itself, which is a tensor with two covariant indices, meaning that the g_{ij} transform according to

$$g_{ij} = \sum_{k,\ell} \frac{\partial F_k}{\partial q_i} \frac{\partial F_\ell}{\partial q_j} g_{k,\ell}^{\text{new}} .$$

Moreover, in his effort to understand the properties of what we now call "Riemannian metrics," given locally by (18), Riemann introduced the ***curvature tensor***, whose components are given, in coordinates, by

$$R^i_{jk\ell} = \frac{\partial \Gamma^i_{j\ell}}{\partial q^k} - \frac{\partial \Gamma^i_{jk}}{\partial q^\ell} + \sum_s \left(\Gamma^i_{sk}\Gamma^s_{j\ell} - \Gamma^i_{s\ell}\Gamma^s_{jk} \right), \tag{19}$$

where the Γ^i_{jk} are the "Christoffel symbols," given by

$$\Gamma^i_{jk} = \frac{1}{2}\sum_\ell g^{i\ell}\left(\frac{\partial g_{\ell k}}{\partial q^j} + \frac{\partial g_{\ell j}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^\ell} \right), \tag{20}$$

and the g^{ij} are the entries of the inverse of the matrix (g_{ij}), so that

$$\sum_j g^{ij} g_{jk} = \delta^i_k \,.$$

(Here $\delta^i_k = 0$ if $k \neq i$, $\delta^i_i = 1$.)

The crucial points about the curvature tensor R are:

(1) Its ***covariance property***, i.e., the fact that the components $\tilde{R}^\alpha_{\beta\gamma\delta}$ of R relative to a coordinate chart $\tilde{q} = F(q)$ are related by a specific formula to the components $R^i_{jk\ell}$ relative to q. The transformation law is

$$\tilde{R}^\alpha_{\beta\gamma\delta} = \sum_{ijk\ell} R^i_{jk\ell} \cdot \frac{\partial F^\alpha}{\partial q^i} \cdot \frac{\partial G^j}{\partial \tilde{q}^\beta} \cdot \frac{\partial G^k}{\partial \tilde{q}^\gamma} \cdot \frac{\partial G^\ell}{\partial \tilde{q}^\delta} \,,$$

where $\tilde{q} \to G(\tilde{q})$ is the inverse transformation of $q \to F(q)$. This implies, in particular, that *if $R \equiv 0$ in* ***some*** *coordinate system, then $R \equiv 0$ in* ***every*** *coordinate system.* So $R \equiv 0$ is an *invariant property.*

(2) The fact that R locally determines the metric up to isometry; for example, $R \equiv 0$ near a point $\bar{q}$ iff the metric is ***flat*** (i.e., such that, for some coordinate chart, the g_{ij} are constant functions) near $\bar{q}$.[8]

It then follows that flatness can be detected even if the metric is presented to us in completely arbitrary "curvilinear" coordinates, since it suffices to check whether $R \equiv 0$, and this property holds in ***one*** coordinate system iff it holds in ***every*** such system.

Through Riemann's ideas, optimal control exercised a decisive influence on the birth of differential geometry by showing that certain optimal control problems could be used as ***sources of new geometries.*** Moreover,

[8] The general statement of the result that "R determines the metric up to isometry" is more delicate, because, if (M, g) and $(\tilde{M}, \tilde{g})$ are two different Riemannian manifolds, then the corresponding curvature tensors live in different spaces, so it is not clear how to define what it means for them to be the same. One way to do this is given in Do Carmo [14].

the concept of invariance under general nonlinear changes of coordinates, which had first surfaced with Lagrange's proof of the invariance of the Euler-Lagrange system, turned out to play a crucial role in the theory. In fact, the systematic use of ***general covariant quantities*** and the search for ***intrinsic formulations*** became the hallmarks of modern differential geometry. As we shall see below, the ideas of covariant objects and intrinsic formulations have in turn been influential in recent developments in optimal control.

5.3 Invariance, Covariance and Lie Brackets

The evolution of nonlinear control theory exhibits some interesting similarities to that of differential geometry. In both cases, a special class of situations (flat Euclidean spaces, linear control systems) was studied first, and a rich theory was developed. In both cases, there was a mathematical language (Euclidean geometry, linear algebra) adequate for the special theory, but not appropriate for the general case. In both cases, the general theory requires the use of the language of covariant quantities, some of which turn out to play a key role. For differential geometry, one needs the various spaces of tensors and other covariant quantities (densities, spinors, jets, connections, etc.), of which the most important ones for classical (Riemannian) differential geometry are the metric and the curvature tensor. For nonlinear control, one needs many covariant objects, such as vector fields and differential forms, and covariant operations such as the Lie bracket, exterior multiplication, and exterior differentiation.

The analogue of the Riemann curvature tensor in this setting is the *structure of the accessibility Lie algebra of a system.* More precisely, for an autonomous control system $\Sigma : \dot{q} = f^{\Sigma}(q,u)$, $u \in U$, on a smooth manifold Q^{Σ}, define Λ^{Σ} to be the free Lie algebra generated by a family $\{F_u : u \in U\}$ of indeterminates indexed by the set U. Let L^{Σ} be the Lie algebra of vector fields generated by the f_u^{Σ}, $u \in U$, where we let $f_u^{\Sigma}(q) = f^{\Sigma}(q,u)$. Then there is a unique Lie-algebra homomorphism $P^{\Sigma} : \Lambda^{\Sigma} \to L^{\Sigma}$ that sends F_u to f_u^{Σ} for each u. Given a point $q \in Q^{\Sigma}$, we can define a linear map $EV_q^{\Sigma} : \Lambda^{\Sigma} \to T_qQ^{\Sigma}$—where T_qQ^{Σ} is the tangent space of Q^{Σ} at q—by letting $EV_q^{\Sigma}(V) = P^{\Sigma}(V)(q)$.

Definition 5.3.1 The kernel

$$REL_q^{\Sigma} \stackrel{\text{def}}{=} \ker\left(EV_q^{\Sigma}\right)$$

is the *set of Lie-bracket relations at* q of the vector fields of Σ. ◇

Example 5.3.1 Suppose u, v, and w are three members of U. Then the formal expression $[F_u,[F_v,F_u]] + 7[F_u,F_w] + F_v$ belongs to REL_q^{Σ} if and only if $[f_u,[f_v,f_u]](q) + 7[f_u,f_w](q) + f_v(q) = 0$. ◇

The following can be proved by a simple application of E. Cartan's graph method.

Theorem 5.3.1 *For a fixed control set U, if we consider pairs (Σ, q) such that Σ is a real analytic system with control set U, q belongs to the state space Q^Σ of Σ, and Σ has the accessibility property at q, then the space REL_q^Σ is a complete set of local invariants for the germ of Σ at q under diffeomorphisms.* $\diamondsuit$

(Cf. for example Sussmann [48], Theorem 7). The precise meaning of the conclusion of Theorem 5.3.1 is the following:

(CSI) *Given two pairs (Σ_i, q_i), $i = 1, 2$, the following two conditions are equivalent:*

(i) There exists a diffeomorphism Φ from a neighborhood U_1 of q_1 in Q^{Σ_1} onto a neighborhood U_2 of q_2 in Q^{Σ_2} that maps each vector field $f_u^{\Sigma_1}$ to the corresponding vector field $f_u^{\Sigma_2}$ and is such that $\Phi(q_1) = q_2$,

(ii) $REL_{q_1}^{\Sigma_1} = REL_{q_2}^{\Sigma_2}$.

(Recall that the "accessibility property at q," for a system Σ, is the property that $\{X(q) : X \in L^\Sigma\} = T_q Q^\Sigma$.)

Theorem 5.3.1 tells us that—for real-analytic systems—all properties of interest that are invariant under nonlinear coordinate changes are determined by the Lie bracket relations. For systems where U is a linear space and the control enters the equations in an affine-linear way, it can be shown the class of *linear systems* is characterized by the vanishing of certain brackets. (Precisely: for a real-analytic system $\dot q = f(q) + \sum_{i=1}^m u_i g_i(q)$, if $f(\bar q) = 0$ and the system has the accessibility property at $\bar q$, then the system is linear near $\bar q$—up to a nonlinear change of coordinates that sends $\bar q$ to 0—if and only if all the iterated brackets of f and the g_i that involve at least two g_i's vanish at $\bar q$.)

The fact that the Lie bracket relations give a complete set of invariants under diffeomorphism makes it reasonable to expect that the structural, coordinate invariant properties of a nonlinear system should be expressible in terms of Lie brackets, in perfect analogy with the way that the structural, coordinate invariant properties of a Riemannian manifold are expressible in terms of the curvature tensor. For example, the structure of the time-optimal trajectories (e. g., whether the time-optimal controls are bang-bang) should be determined by Lie bracket conditions. Since the main tool for analyzing this structure is the necessary condition for optimality given by the Maximum Principle, it is clear that it would be useful to have a statement of the Maximum Principle where the Lie brackets appear explicitly. We will describe such a formulation in Section 5.10 (outlining the much more detailed exposition of Sussmann [46]), and then show by means of a simple example how this formulation can be used to derive qualitative properties of optimal trajectories.

But first we must proceed to our discussion of the Maximum Principle.

5.4 The Maximum Principle

The ***Pontryagin Maximum Principle*** [4, 35] extends the necessary conditions for optimality of the calculus of variations to the much more general setting of fixed-endpoint optimal control problems with a state space Q which is an open subset of $\mathbb{R}^n$, and a set U of control values:

(FEOCP)		
	Minimize	$\int_a^b L(\xi(t), \dot{\xi}(t), t)dt$
	subject to	$\dot{\xi}(t) = f(\xi(t), \eta(t), t)$ for a.e. t,
		$(\xi(t), \eta(t)) \in Q \times U$ for all t,
	and	$\xi(a) = \bar{q}\,, \qquad \xi(b) = \hat{q}\,,$

or to problems

(GEOCP)		
	Minimize	$\int_a^b L(\xi(t), \dot{\xi}(t), t)dt$
	subject to	$\dot{\xi}(t) = f(\xi(t), \eta(t), t)$ for a.e. t,
		$(\xi(t), \eta(t)) \in Q \times U$ for all t,
	and	$(\xi(b), \xi(a)) \in S\,.$

with a more general endpoint condition.

The result is, modulo technical conditions that will not be discussed at this point, that, for a trajectory-control pair $[a,b] \ni t \to (\xi_*(t), \eta_*(t))$ to be a minimizer, it is necessary that there exist an absolutely continuous curve $[a,b] \ni t \to \psi(t) \in \mathbb{R}_n$ and a constant $\psi_0 \geq 0$ such that the *adjoint equation*

$$-\dot{\psi}(t) = \frac{\partial H}{\partial q}(\xi_*(t), \eta_*(t), \psi(t), \psi_0, t)$$

and the *maximization condition*

$$H(\xi_*(t), \eta_*(t), \psi(t), \psi_0, t) = \max\{H(\xi_*(t), u, \psi(t), \psi_0, t) : u \in U\}$$

hold for a.e. $t \in [a,b]$, as well as the *nontriviality condition*

$$(\psi(t), \psi_0) \neq (0,0) \text{ for some (and hence every) } t\,,$$

and the *transversality condition*

$$(-\psi(b), \psi(a)) \in C^\dagger\,.$$

Here

1. H is the *Hamiltonian*, defined by

$$H(q, u, p, p_0, t) = p.f(q, u, t) - p_0 L(q, u, t)\,,$$

2. C is a "tangent cone" at the point $(\xi_*(b), \xi_*(a))$ to the set S,[9]

3. if K is any subset of a real linear space X, then $K^\dagger$ is the *polar cone* of K, i.e. the set

$$K^\dagger \stackrel{\text{def}}{=} \{v \in X^* : \langle v, w\rangle \leq 0 \text{ for all } w \in K\}\,,$$

where X^* is the dual space of X,

[9]The question of how to give a precise definition of "tangent cone" will be discussed later, cf. Section 5.6.

and

4. the symbol $\mathbb{R}_n$ denotes the set of n-dimensional real *row vectors*, whereas the members of $\mathbb{R}^n$ are *column vectors*.

5.5 The Maximum Principle as a Necessary Condition for Set Separation

It turns out that the maximum principle, as formulated, is really a "geometric" result, giving a necessary condition for a separation property between a reachable set of a control system and some other set.

To see this, we first observe that the Maximum Principle for problems of the form (GEOCP) can easily be reduced to the special case of a problem of the same form but with a more restrictive endpoint condition of the form

$$\xi(a) = \bar{q}\,, \qquad \xi(b) \in S\,, \tag{21}$$

where S is a given subset of Q. Indeed, suppose we are given a problem (GEOCP), and an optimal trajectory-control pair (ξ_*, η_*). Consider a new problem

$$(\mathrm{GEOCP}_{\,\mathrm{new}})\quad \boxed{\begin{array}{ll} \textit{Minimize} & \int_{a-1}^{b} L_{\mathrm{new}}(\xi_{\mathrm{new}}(t), \dot{\xi}_{\mathrm{new}}(t), t)dt \\ \textit{subject to} & \dot{\xi}_{\mathrm{new}}(t) = f_{\mathrm{new}}(\xi_{\mathrm{new}}(t), \eta_{\mathrm{new}}(t), t) \text{ for a.e. } t, \\ & (\xi_{\mathrm{new}}(t), \eta_{\mathrm{new}}(t)) \in Q_{\mathrm{new}} \times U_{\mathrm{new}} \text{ for all } t, \\ \text{and} & \xi_{\mathrm{new}}(a-1) = \bar{q}_{\mathrm{new}}, \quad \xi_{\mathrm{new}}(b) \in S_{\mathrm{new}}\,, \end{array}}$$

where $Q_{\mathrm{new}} = Q \times Q$, $U_{\mathrm{new}} = U \times \mathbb{R}^n$, $S_{\mathrm{new}} = S$, $\bar{q}_{\mathrm{new}} = (\xi_*(a), \xi_*(a))$, and, for $q^1 \in Q$, $q^2 \in Q$, $u \in U$, $v \in \mathbb{R}^n$, $a-1 \le t \le b$, we let

$$\begin{aligned} L_{\mathrm{new}}(q^1, q^2, u, v, t) &= \chi_{[a,b]}(t) L(q^1, u, t)\,, \\ f_{\mathrm{new}}(q^1, q^2, u, v, t) &= \chi_{[a,b]}(t).\Big(f(q^1, u, t), 0\Big) + (1-\chi_{[a,b]}(t)).(v, v)\,, \end{aligned}$$

where, if E is a set, then χ_E denotes the indicator function of E, i. e., $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.

It is then readily shown that the new reference trajectory-control pair, consisting of the curve $[a,b] \ni t \to \xi_{*,\mathrm{new}}(t) \stackrel{\mathrm{def}}{=} (\xi_*(t), \xi_*(a))$, and the corresponding control $[a,b] \ni t \to \eta_{*,\mathrm{new}}(t) \stackrel{\mathrm{def}}{=} (\eta_*(t), 0)$ (extended to the interval $[a-1, b]$ by letting $\xi_{*,\mathrm{new}}(t) = (\xi_*(a), \xi_*(a))$ and $\eta_{*,\mathrm{new}}(t) = (0,0)$ for $a-1 \le t < a$) is optimal for $(\mathrm{GEOCP}_{\,\mathrm{new}})$. Moreover, it is easy to see that $(\mathrm{GEOCP}_{\,\mathrm{new}})$ is of our more special form. Applying to our situation the conclusion of the Maximum Principle for problems of this special form, we get precisely the conclusion of the general result.

We are now ready to "geometrize" the problem by reducing it to a question about separation of sets.

Definition 5.5.1 We say that two subsets $\mathcal{R}$ and S of a topological space Q are *separated* at a point $\hat{q} \in Q$ if $\mathcal{R} \cap S = \{\hat{q}\}$, and that $\mathcal{R}$ and S are *locally separated* at $\hat{q}$ if there exists a neighborhood V of $\hat{q}$ such that $\mathcal{R} \cap S \cap V = \{\hat{q}\}$. $\diamond$

For a control system Σ with dynamical law

$$\dot{q} = f(q, u, t)\,, \quad q \in Q\,, \quad u \in U\,, \quad t \in I\,, \tag{22}$$

where I is a subinterval of $\mathbb{R}$ (and for the time being we take Q to be an open subset of $\mathbb{R}^n$), let us define $\mathcal{R}^{\Sigma}_{[a,b]}(\bar{q})$—the Σ-*reachable set from* $\bar{q}$ *over* $[a, b]$—to be the set of all points $\xi(b)$, for all possible trajectory-control pairs $\gamma = (\xi, \eta)$ of Σ such that ξ is defined on $[a, b]$ and $\xi(a) = \bar{q}$. Also, define H^{Σ}—the *Hamiltonian* of Σ—to be the function

$$Q \times U \times \mathbb{R}^n \times I \ni (q, u, p, t) \to H^{\Sigma}(q, u, p, t) = \langle p, f(q, u, t)\rangle\,. \tag{23}$$

Then it is not hard to show that the Maximum Principle follows from a necessary condition for a reachable set $\mathcal{R}^{\Sigma}_{[a,b]}(\bar{q})$ to be separated from some other set S.

Before we state this condition, we give a precise definition of the class of control systems to which it will apply.

Definition 5.5.2 A ***control system in*** $\mathbb{R}^n$ is a 5-tuple $\Sigma = (Q, U, I, \mathcal{U}, f)$ such that

(CS1) Q*—the "state space" of* Σ*—is an open subset of* $\mathbb{R}^n$*,*

(CS2) U*—the "space of control values" of* Σ*—is a set,*

(CS3) $I \subseteq \mathbb{R}$*—the "time interval" of* Σ*—is a subinterval of* $\mathbb{R}$*,*

(CS4) $\mathcal{U}$*—the "class of admissible controls" of* Σ*—is a set of maps* $\mathrm{Dom}(\eta) \ni t \to \eta(t) \in U$ *such that each* $\eta \in \mathcal{U}$ *is defined on a set* $\mathrm{Dom}(\eta) = [a_\eta, b_\eta]$ *which is a compact subinterval of* I*,*

and

(CS5) f*—the "dynamics" of* Σ*—is a map from* $Q \times U \times I$ *to* $\mathbb{R}^n$*.* $\diamond$

Remark 5.5.1 We emphasize that the control set U is *just an abstract set*, with no extra structure. Working on this level of generality has many advatages. (For a simple illustration, we refer the reader to the derivation of the equation of the Riemannian geodesics from the maximum principle, outlined in Example 5.10.2 below, where we take U to be the set of all smooth vector fields on a manifold.) $\diamond$

Then the following statement is the ***set separation version of the Maximum Principle***:

(SSMP)

Let $\Sigma = (Q, U, I, \mathcal{U}, f)$ be a control system in $\mathbb{R}^n$. Assume that $S \subseteq Q$, $a \leq b$, $a, b \in I$, $\xi_* : [a,b] \to Q$ is a trajectory of Σ corresponding to a control $\eta_* : [a,b] \to U$, and $\xi_*(b) \in S$. Let C be a tangent cone to S at $\xi_*(b)$ which is not a linear subspace of $\mathbb{R}^n$. Then a necessary condition for $\mathcal{R}^{\Sigma}_{[a,b]}(\xi_*(a))$ to be locally separated from S at $\xi_*(b)$ is that there exist a nonzero row-vector-valued map $[a,b] \ni t \to \psi(t) \in \mathbb{R}_n$ that satisfies:

1. *the* **Hamiltonian maximization condition**

(HMC)
$$H^{\Sigma}(\xi_*(t), \eta_*(t), \psi(t), t) = \max\Big\{ H^{\Sigma}(\xi_*(t), u, \psi(t), t) : u \in U \Big\},$$

2. *the* **adjoint equation**

(AE)
$$\dot{\psi}(t) = -\frac{\partial H^{\Sigma}}{\partial q}\Big(\xi_*(t), \eta_*(t), \psi(t), t\Big),$$

and

3. *the* **transversality condition**

(TC)
$$-\psi(b) \in C^{\dagger}.$$

Naturally, (SSMP) as stated is not yet a rigorous mathematical theorem, because some crucial technical hypotheses and definitions are still missing. What is needed to render (SSMP) precise is the following:

- (TH1) a specification of technical conditions on the map f and the class $\mathcal{U}$ of open-loop controls;
- (TH2) a precise definition of "trajectory-control pair of Σ;"
- (TH3) precise technical conditions on the reference trajectory-control pair (ξ_*, η_*);
- (TH4) a precise definition of "tangent cone;"
- (TH5) a precise specification of what it means, for an $\mathbb{R}_n$-valued map p on $[a,b]$,
  - (TH5.i) to satisfy the maximization condition (HMC), and
  - (TH5.ii) to be a "solution" of the adjoint equation (AE).

Remark 5.5.2 We include (TH3) as a separate item, because we wish to allow for the possibility that in the final rigorous version of (SSMP) the reference trajectory-control pair might be required to satisfy special conditions that are not necessarily true of general trajectory-control pairs.[10] Notice that, once (TH1,2,3,4,5) are taken care of, nothing else is needed

[10] This is a common situation in optimization theory. For example, we all learn in freshman calculus that the simplest necessary condition for a function f of one variable to have a local minimum at an interior point p of its domain is that $f'(p) = 0$, *provided that p is a point where f is differentiable. But f need not be differentiable anywhere else.*

to make (SSMP) precise, since the concepts of "polar cone," "local separation," and "reachable set" are unambiguously defined. $\diamond$

We claim that *(SSMP), once it is made precise by filling all the technical gaps, implies the usual Maximum Principle for optimal control.* In fact, each *version* V of (SSMP)—i. e., each true theorem obtained from (SSMP) under appropriate technical assumptions—implies a version V' of the usual Maximum Principle, with technical conditions corresponding in a natural way to those of V.

To see this, consider the standard optimal control problem (GEOCP), with the special endpoint condition (21). Suppose (ξ_*, η_*) is a solution. Form the "augmented system" $\Sigma^\# = (Q^\#, U, I, f^\#)$, where

(A.1) $Q^\# = \mathbb{R} \times Q$,

(A.2) $f^\# : Q^\# \times U \times I \to \mathbb{R}^{n+1}$ is the map

$$(r, q, t, u) \to (-L(q, u, t), f(q, u, t)),$$

(A.3) $S^\# \stackrel{\text{def}}{=} \Big\{(c, q) : q \in S\,,\, c \geq \hat{c} + \|q - \hat{q}\|^2\Big\}$,

(A.4) $c_*(t) = -\int_a^t L(\xi_*(s), \eta_*(s), s)\, ds$ for $a \leq t \leq b$,

(A.5) $\hat{c} = c_*(b)$ and $\hat{q} = \xi_*(b)$.

If we define $\xi_*^\#(t) = (c_*(t), \xi_*(t))$, then $(\xi_*^\#, \eta_*)$ is a trajectory-control pair for $\Sigma^\#$, such that $\xi_*^\#(a) = \bar{q}^\# \stackrel{\text{def}}{=} (0, \bar{q})$ and $\xi_*^\#(b) \in S^\#$. The fact that (ξ_*, η_*) is a solution of (GEOCP) implies that $\mathcal{R}_{[a,b]}^{\Sigma^\#}(\bar{q}^\#) \cap S^\# = \{\bar{q}^\#\}$. If we apply (SSMP), and make the reasonable assumption that

(P1) *if $S \subseteq \mathbb{R}^n$, $\hat{q} \in S$, $\hat{c} \in \mathbb{R}$, C is a tangent cone to S at $\hat{q}$, and $S^\#$ is defined by (A.3), then $C^\# \stackrel{\text{def}}{=} [0, +\infty[\times C$ is a tangent cone to $S^\#$ at $(\hat{c}, \hat{q})$,*

then we get the usual Maximum Principle. *Notice that for this application it is not necessary to require that the cone C not be a linear subspace, because $C^\#$ will never be a linear subspace, even if C is.*

Remark 5.5.3 The hypothesis that C is not a linear subspace of $\mathbb{R}^n$ is necessary, for otherwise (SSMP) can obviously fail for trivial reasons. To see this, take $Q = \mathbb{R}^n$, let S be a linear subspace of $\mathbb{R}^n$, let T be a complementary subspace of S, and consider the control system $\dot{q} = u$, $u \in T$, $q \in \mathbb{R}^n$. Then the reachable set from 0 over any interval $[a, b]$ with $a < b$ is T. So this set is separated from S at 0. On the other hand, the adjoint equation just says that $\dot{\psi} = 0$, and the Hamiltonian maximization condition implies that $\psi(t)$ must annihilate T. An obvious choice for C is S itself, and then the requirement that $-\psi(b) \in C^\dagger$ says that $\psi(b)$ must annihilate S. So $\psi(t) \equiv 0$, contradicting the nontriviality of ψ. $\diamond$

In addition to implying the usual necessary conditions for optimality, the separation theorem (SSMP) has the additional advantage that, properly interpreted, it also implies a *sufficient condition for local controllability along a reference trajectory*, i.e., equivalently, a *necessary condition for a system not to be locally controllable along a reference trajectory.*

Precisely, recall that Σ is *locally controllable* along a trajectory-control pair (ξ_*, η_*) defined on an interval $[a, b]$ if $\xi_*(b)$ is an interior point of $\mathcal{R}^{\Sigma}_{[a,b]}(\xi_*(a))$. So, if Σ is not locally controllable along (ξ_*, η_*), then there must exist a sequence $\{q_j\}$ of points of Q that converge to $\xi_*(b)$ and do not belong to $\mathcal{R}^{\Sigma}_{[a,b]}(\xi_*(a))$. By taking a subsequence, if necessary, we can assume that the limiting direction

$$v = \lim_{j\to\infty} \frac{q_j - \hat{q}}{\|q_j - \hat{q}\|} \tag{24}$$

exists. Let

$$S = \{\hat{q}\} \cup \{q_j : j = 1, 2, \ldots\} . \tag{25}$$

Then it is clear that the sets S and $\mathcal{R}^{\Sigma}_{[a,b]}(\xi_*(a))$ are separated at $\xi_*(b)$. Suppose we could take the tangent cone C to S at $\xi_*(b)$ to be the half-line $\{rv : r \geq 0\}$. Then we could apply (SSMP) to get the following *local controllability version of the Maximum Principle*:

(LCMP) *Let $\Sigma = (Q, U, I, \mathcal{U}, f)$ be a control system in $\mathbb{R}^n$. Assume that $a \leq b$, $a, b \in I$, and $\xi_* : [a, b] \to Q$ is a trajectory of Σ corresponding to a control $\eta_* : [a, b] \to U$. Then a necessary condition for $\mathcal{R}^{\Sigma}_{[a,b]}(\xi_*(a))$ not to be a neighborhood of $\xi_*(b)$ is that there exist a nonzero map $[a, b] \ni t \to \psi(t) \in \mathbb{R}_n$ such that (HMC) and (AE) hold.*

From now on we will take (SSMP) to be the "true" statement of the Maximum Principle, and will seek to understand its geometric meaning further. ***We will, however, insist on wanting (LCMP) to be a special case of (SSMP).*** This means that we need to use in (SSMP) a concept of "tangent cone" to a set S at a point $\hat{q} \in S$ having—in addition to (P1)—the property that:

(P2) *if S is given by (25), where $\{q_j\}$ is a sequence such that $q_j \to \hat{q}$, $q_j \neq \hat{q}$, and the limit (24) exists, then the half-line $\{rv : r \geq 0\}$ is a tangent cone to S at $\hat{q}$.*

Property (P2) excludes many of the concepts of "tangent cone to a set at a point" that appear in the literature, such as the Clarke tangent cone, or Boltyanskii's "approximating cones." The most natural concept of "tangent cone" for which (P2) holds is the Bouligand (or "contingent") tangent cone. It is well known, however, that this concept is too weak to be of use for the Maximum Principle. (For example, let us consider the control system $\dot{x} = u$, $\dot{y} = 0$ in $\mathbb{R}^2$, the initial point $(0, 0)$, and the terminal set $S = \{(x, y) : x \geq 0 \, , \, |y| \geq x^2\}$. Take $a = 0$, $b = 1$, and consider the trajectory $t \to (0,0)$, corresponding to the control $\eta(t) \equiv 0$. Then the Bouligand tangent cone to S at $(0,0)$ is the half-space $C = [0,+\infty[\times\mathbb{R}$. So the hypotheses of (SSMP) hold in this case, but the conclusion clearly does not.)

5.6 Weakly Approximating Cones and Transversality

It turns out that there is at least one concept of "tangent cone" that has all the desired properties. Suppose $S \subseteq \mathbb{R}^n$, $\hat{q} \in S$, and $C \subseteq \mathbb{R}^n$. We say that C is a *weakly approximating cone* to S at $\hat{q}$ if

(WAC.1) *C is a closed convex cone;*

(WAC.2) *there exist a closed subset C^* of C, and a continuous map $F : C^* \to S$, such that*

(WAC.2.a) $\lim_{\varepsilon\downarrow 0}\ \widetilde{\sup}\Big\{\|\xi(1)\| : \xi \in C^0([0,1], C\backslash C^*), \|\xi(0)\| \le \varepsilon\Big\} = 0\,,$

(WAC.2.b) $\qquad F(v) = \hat{q} + v + o(\|v\|) \qquad \text{as} \qquad v \to 0\,,\ v \in C^*\,.$

In this definition, $C^0([0,1], C\backslash C^*)$ is the set of continuous maps from $[0,1]$ to $C\backslash C^*$. The expression "$\widetilde{\sup}$" stands for "supremum in the set $[0,+\infty]$" so that $\widetilde{\sup}(\emptyset) = 0$.[11] It then follows that *(WAC.2.a) holds, in particular, if C^* is a full relative neighborhood of 0 in C.*

Notice that (WAC.2) implies in particular that $0 \in C^*$ because, if $0 \notin C^*$, then there would exist a convex neighborhood U of 0 such that $U \cap C^* = \emptyset$, since C^* is closed. But then there would be a curve $\xi \in C^0([0,1], C\backslash C^*)$ such that $\xi(0) = 0$ and $\xi(1) \neq 0$, contradicting (WAC.2.a).)

Condition (WAC.2.a) says, roughly, that "within C, it is not possible to join points that are close to 0 to points that are far from 0 without hitting the set C^*."

If the set C^* can be taken to be a full relative neighborhood of 0 in C, then we call C an *approximating cone* to S at $\hat{q}$. In other words, C is an *approximating cone* to S at $\hat{q}$ if

(AC.1) *C is a closed convex cone;*

(AC.2) *there exist a neighborhood U of 0 and a continuous map F from $C \cap U$ to S, such that $F(v) = \hat{q} + v + o(\|v\|)$ as $v \to 0$ via values in C.*

It turns out that (SSMP) is true if the words "tangent cone" are interpreted to mean "weakly approximating cone" as in our definition, provided that appropriate technical assumptions are made on the system Σ. To understand how this works, let us first quote the following ***separation theorem*** proved recently, in a more general setting, in Sussmann [47]:

[11] If we had used "sup" instead of "$\widetilde{\sup}$," then the supremum of the empty set would have been equal to $-\infty$.

Theorem 5.6.1 *Let C_1 be a closed convex cone in $\mathbb{R}^m$, let U be a neighborhood of 0 in $\mathbb{R}^m$, let $f : U \cap C_1 \to \mathbb{R}^n$ be a continuous map, and let $L : \mathbb{R}^m \to \mathbb{R}^n$ be linear. Suppose that f is differentiable at 0 with differential L, i.e. that*

$$f(x) = f(0) + L.x + o(||x||) \qquad \text{as} \qquad x \to_{C_1} 0\,. \tag{26}$$

Let S be a subset of $\mathbb{R}^n$ having a cone C_2 as weakly approximating cone at $f(0)$. Assume that C_2 is not a linear subspace of $\mathbb{R}^n$. Then a necessary condition for the sets $f(U \cap C_1)$ and S to be locally separated at $f(0)$ is that $(LC_1)^\dagger \cap (-C_2)^\dagger \neq \{0\}$. $\diamond$

In other words, if $f(U \cap C_1) \cap S \cap V = \{f(0)\}$ for some neighborhood V of $f(0)$, then there exists a nonzero row vector $\bar{z} \in \mathbb{R}_n$ such that $\bar{z}.L.v_1 \leq 0$ for all $v \in C_1$ and $\bar{z}.v_2 \geq 0$ for all $v \in C_2$.

5.7 A Streamlined Version of the Classical Maximum Principle

As a first step in our discussion of recent formulations of the Maximum Principle, we now present a rigorous version that resembles in spirit the "classical" version given in the 1962 book by Pontryagin *et al.* [35]. Actually, the theorem to be given here is much stronger than the classical result, especially because of the stronger form of the Transversality Condition, as will be explained below. It is, however, a distillation of what lies behind the method of proof of [35], whose main point is, in our view, that it relies on the ***construction of needle variations*** [12], and the ***differentiability in the classical sense of the reference flow and the variations.***

The proof strategy consists of constructing needle variations, studying the corresponding endpoint maps, computing their differentials, and then applying a separation result. The "classical" case arises when the differentials used are classical differentials. In that case, the appropriate separation result is our Theorem 5.6.1. In Section 5.8 we will show how, using essentially the same approach but relying on more general concepts of differential, it is possible to derive much stronger results that contain, extend, and strengthen various "nonsmooth" versions of the Maximum Principle.

First, let us define the *needle variation of a map $\eta : [a, b] \to U$ obtained by inserting a control value $u \in U$ at a time $s \in [a, b[$* to be the family $\{\nu_{s,u}(r)(\eta)\}_{0 \leq r \leq b-s}$ of maps $\nu_{s,u}(r)(\eta) : [a, b] \to U$ defined by

$$\nu_{s,u}(r)(\eta)(t) = \begin{cases} \eta(t) & \text{if} \quad t \in [a, b] \setminus]s, s + r]\,, \\ u & \text{if} \quad t \in]s, s + r]\,. \end{cases} \tag{27}$$

With this definition, each $\nu_{s,u}(r)$ —for $0 \leq r \leq b - s$—is a map from the set $MAP([a, b], U)$ of all maps $\eta : [a, b] \to U$ into itself, and $\nu_{s,u}(0)(\eta) = \eta$.

[12] The idea of using needle variations goes back to Weierstrass (1815-1897), who gave three series of lectures on the calculus of variations, in 1875, 1879, and 1882. Using needle variations, Weierstrass proved the "side condition," which amounts, in modern terms, to the Maximum Principle for the problem (SCV).

Clearly, we can consider composites of the maps $\nu_{s_j,u_j}(r_j)$. Precisely, if $a \le s_1 < s_2 < \cdots < s_m < b$, $\mathbf{u} = (u_1, \cdots, u_m) \in U^m$ is an m-tuple of control values, and $\mathbf{s} = (s_1, \ldots, s_m)$, we define maps

$$\nu_{\mathbf{s},\mathbf{u}}(r_1, \ldots, r_m) : MAP([a,b],U) \to MAP([a,b],U)\,,$$

if $0 \le r_i \le b - s_i$ for $i = 1, \cdots, m$, by letting

$$\nu_{\mathbf{s},\mathbf{u}}(r_1, \ldots, r_m) \stackrel{\text{def}}{=} \nu_{s_1,u_1}(r_1) \circ \nu_{s_2,u_2}(r_2) \circ \cdots \circ \nu_{s_m,u_m}(r_m)\,. \tag{28}$$

We then define a ***needle-variational neighborhood*** of η to be a set $\mathcal{N}$ such that, for every $m \in \mathbb{N}$, every choice of $\mathbf{s} = (s_1, \ldots, s_m)$ such that $a \le s_1 < s_2 < \cdots < s_m < b$, and every choice of $\mathbf{u} = (u_1, \ldots, u_m)$ in U^m, there exists an $\bar{r} > 0$ such that $\nu_{\mathbf{s},\mathbf{u}}(\mathbf{r})(\eta) \in \mathcal{N}$ for every m-tuple $\mathbf{r} = (r_1, \ldots, r_m) \in [0, \bar{r}]^m$, such that $\mathbf{s} + \mathbf{r} \in [a,b]^m$.

To every map $\eta : [a,b] \to U$, we associate a time-varying vector field $f_\eta : Q \times [a,b] \to \mathbb{R}^n$ by letting

$$f_\eta(q,t) \stackrel{\text{def}}{=} f(q, \eta(t), t)\,.$$

We say that a time-varying $\mathbb{R}^n$-valued vector field F, defined on some subset $\mathrm{Dom}(F)$ of $\mathbb{R}^n \times \mathbb{R}$, satisfies a ***$C^1$-Carathéodory condition*** near a curve $\xi : [a,b] \to \mathbb{R}^n$ if there exists a tube

$$\mathcal{T}(\xi, \varepsilon) \stackrel{\text{def}}{=} \{(q,t) : a \le t \le b, \|q - \xi(t)\| \le \varepsilon\}\,, \tag{29}$$

such that

(M-Car) *$F(q,t)$ is measurable with respect to t—on the compact set $\{t : (q,t) \in \mathcal{T}(\xi,\varepsilon)\}$—for each fixed q, and of class C^1 with respect to q—on the closed ball $\{q \in \mathbb{R} : \|q - \xi(t)\| \le \varepsilon\}$—for each fixed $t \in [a,b]$,*

(C^1-Car) *there is a function $\varphi \in L^1([a,b], \mathbb{R})$ such that*

$$\|F(q,t)\| + \left\|\frac{\partial F}{\partial q}(q,t)\right\| \le \varphi(t)$$

for all $(q,t) \in \mathcal{T}(\xi,\varepsilon)$.

We then assume that

(CTH.1) *The class $\mathcal{U}$ of admissible controls is a needle-variational neighborhood of η_*.*

(CTH.2) *For every control η which is either constant or equal to the reference control η_*, the corresponding time-varying vector field f_η satisfies a C^1-Carathéodory condition near ξ_*.*

(CTH.3) *A **trajectory** for a control $\eta \in \mathcal{U}$ is a locally absolutely continuous map $\xi : [a_\eta, b_\eta] \to Q$ such that $\dot{\xi}(t) = f(\xi(t), \eta(t), t)$ for almost all $t \in [a_\eta, b_\eta]$. A **trajectory-control pair** is a pair (ξ, η) such that $\eta \in \mathcal{U}$ and ξ is a trajectory for η.*

(CTH.4) *A* ***solution of the adjoint equation (AE) along the reference trajectory-control pair*** $\gamma_* = (\xi_*, \eta_*)$ *is an absolutely continuous map* $\psi : [a,b] \to \mathbb{R}_n$ *such that the equality*

$$-\dot{\psi}(t) = \psi(t).\frac{\partial f}{\partial q}(\xi_*(t), \eta_*(t), t)$$

holds for almost all $t \in [a_\eta, b_\eta]$.

(CTH.5) *The Hamiltonian minimization condition (HMC) is interpreted* ***weakly****, as the statement that*

(WHMC) *for every* $u \in U$, *the inequality*

$$\psi(t).f(\xi_*(t), \eta_*(t), t) \geq \psi(t).f(\xi_*(t), u, t)$$

holds for almost all $t \in [a,b]$.

(CTH.6) *"Tangent cone" means* ***weakly approximating cone****.*

(For simplicity, we shall refer to (CTH) as the "classical technical hypotheses," because they are roughly those of [35], although they differ from them in some important technical aspects that will be discussed later.)

Under (CTH), all the terms occurring in (SSMP) are now precisely defined, and the following rigorous result holds:

Theorem 5.7.1 *Under the classical technical hypotheses (CTH), the Maximum Principle (SSMP) is true.* ◇

We now outline the proof of Theorem 5.7.1, following the ideas of [35]. Let $\tilde{\mathcal{U}}$ be the union of $\{\eta_*\}$ and the set of all constant controls $\eta : [a,b] \to U$. Then the C^1-Carathéodory condition implies that every control $\eta \in \tilde{\mathcal{U}}$ gives rise to a *flow of class* C^1, i.e., to a family $\Phi^\eta = \{\Phi^\eta_{t,s}\}_{a \leq s \leq t \leq b}$ of maps of class C^1, each one of which is defined on an open subset $\mathrm{Dom}(\Phi^\eta_{t,s})$ of Q. The maps $\Phi^\eta_{t,s}$ are uniquely characterized by the condition that, if $a \leq s \leq t \leq b$, and $q \in Q$, then $\Phi^\eta_{t,s}(q)$ is defined if and only if the solution $\tau \to \xi_{q,\eta,s}(\tau)$ of the initial value problem $\dot{\xi}(\tau) = f(\xi(\tau), \eta(\tau), \tau)$, $\xi(s) = q$, is defined on $[s,t]$, and in that case $\Phi^\eta_{t,s}(q) = \xi_{q,\eta,s}(t)$. The flow maps $\Phi^\eta_{t,s}$ satisfy the obvious identities $\Phi^\eta_{t,t} = \text{identity}$ and $\Phi^\eta_{t_3,t_2} \circ \Phi^\eta_{t_2,t_1} = \Phi^\eta_{t_3,t_1}$ whenever $a \leq t_1 \leq t_2 \leq t_3 \leq b$.

The differentials $D\Phi^{\eta_*}_{t,s}$ of the maps $\Phi^{\eta_*}_{t,s}$ satisfy the *variational equation*:

$$D\Phi^{\eta_*}_{t,s}(q) = \text{identity} + \int_s^t \frac{\partial f}{\partial q}\Big(\Phi^{\eta_*}_{\tau,s}(q), \eta_*(\tau), \tau\Big).D\Phi^{\eta_*}_{\tau,s}(q)\, d\tau\,. \qquad (30)$$

Use γ_* to denote the reference trajectory-control pair (ξ_*, η_*). We then define the *linearized flow* $\mathbf{L}^{\gamma_*}$ to be the family $\mathbf{L}^{\gamma_*} = \{L^{\gamma_*}(t,s)\}_{a \leq s \leq t \leq b}$ of linear maps $L^{\gamma_*}(t,s) : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$L^{\gamma_*}(t,s) = D\Phi^{\eta_*}_{t,s}(\xi_*(s))\,.$$

Also, define

$$M^{\gamma_*}(t) = \frac{\partial f}{\partial q}\Big(\xi_*(t), \eta_*(t), t\Big)\,.$$

Then the linear maps $L^{\gamma_*}(t,s)$ satisfy the identities $L^{\gamma_*}(t,t) = $ identity and $L^{\gamma_*}(t_3,t_2)\circ L^{\gamma_*}(t_2,t_1) = L^{\gamma_*}(t_3,t_1)$ whenever $a \le t_1 \le t_2 \le t_3 \le b$. Also, (30) says that

$$L^{\gamma_*}(t,s) = \text{identity} + \int_s^t M^{\gamma_*}(\tau)L^{\gamma_*}(\tau,s)\,d\tau \;\text{ if }\; a \le s \le t \le b. \tag{31}$$

It then follows easily that

$$L^{\gamma_*}(t,s) = \text{identity} + \int_s^t L^{\gamma_*}(t,\sigma)M^{\gamma_*}(\sigma)\,d\sigma \;\text{ if }\; a \le s \le t \le b. \tag{32}$$

This implies, in particular, that *a map $[a,b] \ni t \to \psi(t) \in \mathbb{R}_n$ is a solution of the adjoint equation if and only if it satisfies the "integrated adjoint equation"*

$$\psi(t) = \psi(b).L^{\gamma_*}(b,t) \quad \text{for} \quad a \le t \le b. \tag{33}$$

Now, let $\hat{U}$ be a fixed finite subset of U. We then construct needle variations of the reference trajectory-control pair $\gamma_* = (\xi_*, \eta_*)$ using control values in $\hat{U}$. Precisely, we pick an m-tuple $\mathbf{s} = (s_1, \ldots, s_m)$ such that $a \le s_1 < s_2 < \cdots < s_m < b$, and an m-tuple $\mathbf{u} = (u_1, \ldots, u_m)$ of members of $\hat{U}$. Our hypotheses then imply that there exists an $\bar{r} > 0$ such that

1. the controls $\nu_{\mathbf{s},\mathbf{u}}(\mathbf{r})(\eta_*)$ are in $\mathcal{U}$ whenever $\mathbf{r} = (r_1, \ldots, r_m) \in [0,\bar{r}]^m$,
2. the ***endpoint map*** $\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*}$, given by

$$\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*}(q,\mathbf{r}) \stackrel{\text{def}}{=} \Phi_{b,a}^{\nu_{\mathbf{s},\mathbf{u}}(\mathbf{r})(\eta_*)}(q)\,, \tag{34}$$

   is defined and continuous on $B(\xi_*(a),\bar{r}) \times [0,\bar{r}]^m$, where

$$B(\xi_*(a),\bar{r}) \stackrel{\text{def}}{=} \{q \in \mathbb{R}^n : \|q - \xi_*(a)\| \le \bar{r}\}\,.$$

Moreover, one can prove that, if the times $s_1, \ldots, s_m$ are chosen in a special way, then the map $\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*}$ is differentiable at $(\xi_*(a),0)$, and the differential $D\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*}(\xi_*(a),0)$ is the linear map from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ given by

$$\begin{aligned} &D\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*}(\delta q, \delta r_1, \ldots, \delta r_m) \\ &= L^{\gamma_*}(b,a).\delta q + \sum_{i=1}^m \delta r_i.L^{\gamma_*}(b,s_i).v^{\gamma_*}(s_i,u_i)\,, \end{aligned} \tag{35}$$

where

$$v^{\gamma_*}(s,u) \stackrel{\text{def}}{=} f(\xi_*(s),u,s) - f(\xi_*(s),\eta_*(s),s)\,.$$

Precisely, what is required for (35) to hold is to choose the s_i belonging to the "good set" $G(\eta_*,\hat{U})$ of all times $s \in [a,b[$ such that

(G) there exist a compact subset K of $[a,b]$ and an integrable function $\varphi : [a,b] \to [0,\infty]$, such that

$$\lim_{r\downarrow 0} \frac{1}{r}\Big(\text{meas}(K\cap[s,s+r])\Big) = 1\,, \tag{36}$$

$$\lim_{r\downarrow 0} \frac{1}{r}\int_s^{s+r} |\varphi(t)-\varphi(s)|\,dt = 0\,, \tag{37}$$

the inequality

$$||f(q,u,t)|| \le \varphi(t) \tag{38}$$

holds whenever $t\in[s,s+r]$ and $u\in\hat{U}\cup\{\eta_*(t)\}$,

$$\lim_{q\to\xi_*(s),r\downarrow 0,s+r\in K} f(q,u,s+r) = f(\xi_*(s),u,s) \text{ for all } u\in\hat{U}\,, \tag{39}$$

and

$$\lim_{q\to\xi_*(s),r\downarrow 0,s+r\in K} f(q,\eta_*(s+r),s+r) = f(\xi_*(s),\eta_*(s),s)\,. \tag{40}$$

It follows from the Scorza-Dragoni theorem that, under our assumptions, the set $[a,b]\backslash G(\eta_*,\hat{U})$ has measure zero.

Now suppose that $\mathcal{R}^{\Sigma}_{[a,b]}(\xi_*(a))$ and S are locally separated at $\xi_*(b)$. Then *a fortiori* the sets $\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*}\Big(\{\xi_*(a)\}\times[0,\bar{r}]^m\Big)$ and S are locally separated at $\xi_*(b)$, if $\bar{r}>0$ is small enough. Applying Theorem 5.6.1, we find that there exists a covector $\bar{p}\in\mathbb{R}_n$ such that $||\bar{p}||=1$, $\bar{p}.v\ge 0$ for all $v\in C$, and

$$\bar{p}.L^{\gamma_*}(b,s_i).v^{\gamma_*}(s_i,u_i)\le 0 \quad \text{for} \quad i=1,\dots,m\,. \tag{41}$$

If we define

$$\psi(t) = \bar{p}.L^{\gamma_*}(b,t) \quad \text{for} \quad t\in[a,b]\,,$$

we have $\psi(b).v\ge 0$ for all $v\in C$, and

$$\psi(t).v^{\gamma_*}(s_i,u_i)\le 0 \quad \text{for} \quad i=1,\dots,m\,. \tag{42}$$

The conclusion of the Maximum Principle follows from this by an elementary compactness argument. Indeed, using Lusin's Theorem, we can write $G(\eta_*,\hat{U}) = \bigcup_{k=0}^{\infty} E_k$, where $\text{meas}(E_0)=0$ and, for $k>0$, the set E_k is compact and such that the restrictions to E_k of the maps $t\to v^{\gamma^*}(t,u)$ are continuous for every $u\in\hat{U}$. For $k>0$, let D_k be the set of points of density of E_k. Let $D=\bigcup_{k=1}^{\infty} D_k$, so $\text{meas}(D)=b-a$. If F is an arbitrary subset of $D\times\hat{U}$, let $\Pi(F,\hat{U})$ be the set of all $\bar{p}\in\mathbb{R}_n$ such that $||\bar{p}||=1$, $-\bar{p}\in C^{\dagger}$, and $\bar{p}.L^{\gamma_*}(b,t).v^{\gamma_*}(t,u)\le 0$ for all $(t,u)\in F$. Then $\Pi(F,\hat{U})$ is clearly compact, and

$$\Pi(F_1\cup F_2,\hat{U}) = \Pi(F_1,\hat{U})\cap\Pi(F_2,\hat{U}) \quad \text{for} \quad \text{all} \quad F_1,F_2\,.$$

We already know that $\Pi(F,\hat{U}) \neq \emptyset$ if $F = \{(s_1,u_1),\ldots,(s_m,u_m)\}$ with $s_1 < s_2 < \cdots < s_m$. If F is an arbitrary finite subset of $D \times \hat{U}$, we can write $F = \{(s_1,u_1),\ldots,(s_m,u_m)\}$ with $s_1 \leq s_2 \leq \cdots \leq s_m$. Each s_i belongs to $D_{k(i)}$ for some $k(i)$, so $s_i = \lim_{j\to\infty} s_{i,j}$, where $s_{i,j} \in D_{k(i)}$ and $s_{1,j} < s_{2,j} < \cdots < s_{m,j}$ for each j. If $F_j = \{(s_{1,j},u_1),\ldots,(s_{m,j},u_m)\}$, then $\Pi(F_j,\hat{U}) \neq \emptyset$. Pick $\bar{p}_j \in \Pi(F_j,\hat{U})$. Then $\|\bar{p}_j\| = 1$. Pick a subsequence $\{\bar{p}_{j(\ell)}\}$ of $\{\bar{p}_j\}$ that converges to a limit $\bar{p}$. Then

$$\bar{p}_{j(\ell)}.L^{\gamma_*}(b,s_{i,j(\ell)}).v^{\gamma_*}(s_{i,j(\ell)},u_i) \leq 0 \quad \text{for all} \quad i,\ell\,.$$

Since $s_{i,j} \to s_i$, $s_{i,j} \in E_{k(i)}$, $s_i \in E_{k(i)}$, and the function $s \to v^{\gamma_*}(s,u_i)$ is continuous on $E_{k(i)}$, we have

$$\bar{p}.L^{\gamma_*}(b,s_i).v^{\gamma_*}(s_i,u_i) \leq 0 \quad \text{for} \quad i = 1,\ldots,m\,.$$

Clearly, $\|\bar{p}\| = 1$ and $-\bar{p} \in C^\dagger$, so $\bar{p} \in \Pi(F,\hat{U})$. Therefore $\Pi(F,\hat{U}) \neq \emptyset$ if F is an arbitrary finite subset of $D \times \hat{U}$. So $\{\Pi(F,\hat{U})\}_{F\subseteq D\times\hat{U},\mathrm{card}(F)<\infty}$ is a family of compact subsets having the finite intersection property. This implies that the intersection of all the $\Pi(F,\hat{U})$, as F ranges over all finite subsets of $D \times \hat{U}$, is nonempty. Therefore

$$\Pi(D \times \hat{U}) \neq \emptyset\,.$$

Now, for an arbitrary subset $\hat{U}$ of U, let $\tilde{\Pi}(\hat{U})$ be the set of all $\bar{p} \in \mathbb{R}_n$ such that $\|\bar{p}\| = 1$, $-\bar{p} \in C^\dagger$, and, for every $u \in \hat{U}$, $\bar{p}.L^{\gamma_*}(b,t).v^{\gamma_*}(t,u) \leq 0$ for almost all $t \in [a,b]$. Once again, $\tilde{\Pi}(\hat{U})$ is compact for every $\hat{U}$, and $\tilde{\Pi}(\hat{U}_1 \cup \hat{U}_2)$ is equal to $\tilde{\Pi}(\hat{U}_1) \cap \tilde{\Pi}(\hat{U}_2)$ for all $\hat{U}_1, \hat{U}_2$. (The compactness of $\tilde{\Pi}(\hat{U})$ follows because, if $\bar{p}_j \in \tilde{\Pi}(\hat{U})$ and $\bar{p}_j \to \bar{p}$, then $\int_\alpha^\beta \bar{p}_j.L^{\gamma_*}(b,t).v^{\gamma_*}(t,u)\,dt \leq 0$ whenever $a \leq \alpha \leq \beta \leq b$ and $u \in \hat{U}$, so $\int_\alpha^\beta \bar{p}.L^{\gamma_*}(b,t).v^{\gamma_*}(t,u)\,dt \leq 0$ whenever $a \leq \alpha \leq \beta \leq b$ and $u \in \hat{U}$, and then, for each fixed u, the inequality $\bar{p}.L^{\gamma_*}(b,t).v^{\gamma_*}(t,u) \leq 0$ must hold for almost all $t \in [a,b]$.) So $\{\tilde{\Pi}(\hat{U})\}_{\hat{U}\subseteq U,\mathrm{card}(\hat{U})<\infty}$ is a family of compact subsets having the finite intersection property. This implies that the intersection of all the $\tilde{\Pi}(\hat{U})$, as $\hat{U}$ ranges over all finite subsets of U, is nonempty. Therefore $\tilde{\Pi}(U) \neq \emptyset$, which is precisely the desired conclusion. ◇

The proof of the Maximum Principle outlined above is essentially based on the ideas of [35]. There are, however, some major differences between our technical assumptions and those of [35], which make our version stronger.

First of all, in [35] f is assumed to depend on q and u only.[13]

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[13] As explained in Chapter 1 of [35], a result in which $f(q,u)$ is assumed to be of class C^1 with respect to the state q can be transformed immediately into a result in which f is allowed to be time-dependent—i. e., $f = f(q,u,t)$—*provided that f is of class C^1 with respect to q and t.* This is done by adding time as extra state variable q_0, obeying the equation $\dot{q}_0 = 1$. This procedure does not, however, allow the extra generality of considering systems where f is only measurable with respect to t.

Second, [35] assumes that U is a subset of some Euclidean space $\mathbb{R}^m$,[14] and both $f(q,u)$ and $\frac{\partial f}{\partial q}(q,u)$ are jointly continuous with respect to q and u on the product $Q \times U$. It is assumed that every admissible control is a bounded measurable U-valued map on $[a,b]$, where "bounded" means "with values in a compact subset of U." The class $\mathcal{U}$ of admissible controls is assumed to contain the reference control, and to be such that, if $\eta \in \mathcal{U}$ has domain $[a_\eta, b_\eta]$, t' and t'' are numbers such that $a_\eta \le t' \le t'' \le b_\eta$, and $u \in U$, then the control $[a_\eta, b_\eta] \ni t \to \eta(t) + \chi_{[t',t'']}(t)(u - \eta(t))$ also belongs to $\mathcal{U}$.

This is more than enough to guarantee that our technical hypotheses hold. Moreover, since U is a subset of $\mathbb{R}^m$, and $f(q,u)$ is continuous with respect to u, the "weak" Hamiltonian minimization condition of (CTH.5) implies the "strong" version of (HMC):

(SHMC) *The inequality*

$$\psi(t).f(\xi_*(t),\eta_*(t)) = \max\Big\{\psi(t).f(\xi_*(t),u) : u \in U\Big\}$$

holds for almost all $t \in [a,b]$.

(Indeed, U is a separable metric space. If ψ satisfies (WHMC), then we can pick a countable dense subset U_0 of U, and a "bad" subset $B \subseteq [a,b]$ of measure zero such that $\psi(t).f(\xi_*(t),\eta_*(t)) \ge \psi(t).f(\xi_*(t),u)$ whenever $t \in [a,b]\backslash B$ and $u \in U_0$. The continuity of f with respect to u then implies that (SHMC) holds.)

Third, *there is a fundamental difference between our version of the Transversality Condition (TC) and that of [35]*, arising from the fact that our version uses a much weaker concept of "tangent cone" than that of [35]. Indeed, the concept of tangent cone implicitly used in [35] in the formulation of (TC) is the one that we have called "approximating cone" in this discussion.[15]

The following example shows how this makes a difference.

Example 5.7.1 Consider the optimal control problem in $\mathbb{R}^2$—with coordinate point $q = (x,y)$—in which it is desired to minimize the integral

$$J = -\int_0^1 u(t)dt\,,$$

[14] The authors of [35] were obviously aware of the possibility of generalizing the Maximum Principle to the case when U is an arbitrary set. Indeed, they state in a footnote, on page 75, that their arguments "carry over without any change to the case where U is an arbitrary subset of some topological Hausdorff space with a denumerable basis." They then add that "an insignificant modification of the proof even makes it possible to remove the requirement that there exist a denumerable basis."

[15] The definition of "tangent cone" as "approximating cone" is not given explicitly in [35]. The crucial topological argument needed to establish the transversality conditions corresponds to our Theorem 5.6.1, with "approximating cone" instead of "weakly approximating cone." The argument is only sketched in [35], and the reader is referred for full details to [3]. Cf. Remark 5.7.1 below for a comparison of the two separation theorems.

subject to the dynamical law $\dot{x} = u$, $\dot{y} = v$ and the endpoint constraints $\xi(0) = (0,0)$, $\xi(1) \in S$, where

$$S = \left\{(x,y) : x > 0 \wedge y = x \sin \frac{1}{x}\right\} \cup \{(0,0)\}.$$

Then the trajectory $[0,1] \ni t \to (0,0)$, corresponding to the controls $u(t) \equiv v(t) \equiv 0$, obviously fails to be a minimizer, since the choice of controls $v(t) \equiv 0$, $u(t) \equiv \frac{\pi}{k}$, for any positive integer k, will yield a trajectory satisfying our endpoint constraints and having a lower cost.

On the other hand, *the version of the Maximum Principle given in [35] will not exclude this trajectory as a candidate for a minimum.* This can be seen by explicitly finding an adjoint vector that satisfies all the conditions, which is quite easy to do. There is, however, an even simpler argument. It suffices to notice that the only approximating cone for the set S at $(0,0)$ is $\{(0,0)\}$. Hence the conclusion of the Maximum Principle for our optimization problem is identical to that arising from any modified problem with the same dynamics and cost functional, and with the set S replaced by another set S' having $\{(0,0)\}$ as an approximating cone at $(0,0)$. In particular, we can take $S' = \{(0,0)\}$. Since the cost of any feasible trajectory $t \to \xi(t) = (x(t), y(t))$ is just $-x(1)$, we see that all feasible trajectories of our original problem that end at $(0,0)$ are actually minimizers for the modified problem, so they satisfy the conditions of the Maximum Principle of [35].

It turns out, however, that *our version of the transversality condition excludes the trajectory* $[0,1] \ni t \to (0,0)$. To see this, notice that the cone $C = \{(x,y) : |y| \leq x\}$ is a "tangent cone" to S at $(0,0)$ in our sense. To satisfy the necessary condition of the Maximum Principle we need a vector $(\psi_1, \psi_2) \in \mathbb{R}_2$ and a nonnegative ψ_0 such that, to begin with, $\psi_1 u + \psi_2 v + \psi_0 u$ is maximized by $u = v = 0$, for which we must have $\psi_2 = 0$ and $\psi_1 + \psi_0 = 0$. Moreover, since $(\psi_0, \psi_1, \psi_2) \neq (0,0,0)$, we need $\psi_0 > 0$ and $\psi_1 < 0$. But the transversality condition yields $-(\psi_1, \psi_2) \in C^\dagger$, so $\psi_1 \geq 0$, since $(1,0) \in C$. So we have reached a contradiction. ◇

For a second illustration of the extra power of our version of the "classical" Maximum Principle, we observe that, *if "tangent cone" is taken to mean "approximating cone," then Condition (P2) does not hold, so the local controllability result is **not** a special case of the general separation theorem, and must be handled separately.* On the other hand, *if "tangent cone" is taken to mean "weakly approximating cone," then (P2) holds, so **the local controllability result becomes a special case of the general maximum principle, and does not require a separate analysis.***

Remark 5.7.1 The most important technical difference between our version of the classical Maximum Principle and that of [35] is our use of weakly approximating cones. This, in our view, solves a puzzling question regarding the proof given in [35]. The crucial point of that proof is found on page 97 of [35], where the authors—assuming that the necessary condition of the Maximum Principle is violated—are trying to prove that the intersection of two sets contains more than one point. They do this by proving the existence of a *one-parameter family* of points q_ε, one for each sufficiently small positive value of ε. This argument strongly suggests that one ought to be able to prove the existence of a full curve of intersection points and get a result stronger than what is actually proved in [35]. It turns out that this is not quite true, but is nearly true. What actually follows when the argument is carried out with care, is the existence of a nontrivial ***connected*** set of intersection points. The set may, however, fail to be path-connected. The key result is the following theorem (cf. Leray-Schauder [29], Browder [9]), in which we use $\mathbb{B}^n$ to denote the closed unit ball in $\mathbb{R}^n$:

Theorem 5.7.2 *Let $F : [0,1] \times \mathbb{B}^n \to \mathbb{R}^n$ be a continuous map, such that*

1. $0 \notin F\Big([0,1] \times \partial\mathbb{B}^n\Big)$,
2. *the map $\partial\mathbb{B}^n \ni x \to F(0,x) \in \mathbb{R}^n \backslash \{0\}$ is homotopic (in the class of continuous maps from $\partial\mathbb{B}^n$ to $\mathbb{R}^n \backslash \{0\}$) to the inclusion map $x \to x$.*

Then there exists a compact connected subset Z of $[0,1] \times \mathbb{B}^n \to \mathbb{R}^n$ such that $Z \cap (\{0\} \times \mathbb{B}^n) \neq \emptyset$, $Z \cap (\{1\} \times \mathbb{B}^n) \neq \emptyset$, and $F(t,x) = 0$ whenever $(t,x) \in Z$.

In other words: if $\{F_t\}_{0 \leq t \leq 1}$ is a homotopy of maps from $\mathbb{B}^n$ to $\mathbb{R}^n$ that never take the value 0 on the boundary of $\mathbb{B}^n$, and F_0, restricted to the boundary of $\mathbb{B}^n$, is homotopic to the inclusion map, then not only is there a zero of F at each level of the homotopy—as implied by standard degree theory—but in addition there must exist a connected set of zeros of the map $(t,x) \to F_t(x)$ that contains a zero at each end of the homotopy.

Theorem 5.7.2 is the basic tool used in [47] to prove Theorem 5.6.1 and its generalizations. Our definition of "weakly approximating cone" is precisely what is needed to make this work. ◇

5.8 Clarke's Nonsmooth Version and the Łojasiewicz Improvement

Theorem 5.7.1 and its proof, as sketched in Section 5.7, admit an obvious generalization, obtained by observing that, in the proof, the C^1-Carathéodory condition for the controls other than the reference control is never used. In fact, the formula for the differential of the endpoint map is valid without any change if the "constant control" time-varying vector fields $(q,t) \to f(q,u,t)$ are only assumed to satisfy the following "Łojasiewicz technical hypothesis":

(LTH) *There exists an $\varepsilon_u > 0$ such that $f(q,u,t)$ is continuous in q for each fixed t and measurable in t for each fixed q on the tube $\mathcal{T}(\xi_*, \varepsilon_u)$, and the bound $\|f(q,u,t)\| \leq \varphi(t)$ holds for some integrable function $[a,b] \ni t \to \varphi(t)$,*

and to be such that

(U) *the ordinary differential equation $\dot{q} = f(q,u,t)$ has uniqueness of solutions.*

Condition (U) is only needed if one insists on having a single-valued continuous endpoint map that is differentiable in the classical sense, but can be dispensed with if one drops this requirement. Indeed, it turns out that there is a very nice class of *set-valued* maps for which a concept very similar to that of classical differentiability exists, and the analogues of Theorems 5.7.2 and 5.6.1 are still valid. Precisely, let us call a set-valued map F from a compact metric space X to a metric space Y *regular* if

(REG.1) *the graph* $GR(F) = \{(x, y) : y \in F(x)\}$ *is compact;*

(REG.2) *there exists a sequence* $\{F_j\}_{j=1}^{\infty}$ *of single-valued continuous maps from* X *to* Y *that converges to* F *in the graph sense, i.e. is such that*

$$\lim_{j\to\infty} \sup\left\{\operatorname{dist}\Big((x,y), GR(F)\Big) : (x,y) \in GR(F_j)\right\} = 0\,.$$

With this definition, it turns out that

Theorem 5.8.1 *Theorem 5.7.2 also holds if F is assumed to be set-valued and regular, rather than single-valued and continuous.* ◇

Remark 5.8.1 Actually, the "regular" version of Theorem 5.7.2 is more elegant and natural than the "continuous" version, because the resulting set Z can be taken to be of the form $\gamma([0,1])$, where γ is a regular set-valued map from $[0,1]$ to $[0,1] \times \mathbb{B}^n$. ◇

Moreover, ***Theorem 5.6.1 is true for regular maps.*** (The definition of "differential" is still the obvious analogue of that of the classical differential of a single-valued map: if F is a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $F(x) = \{y\}$, C is a closed convex cone in $\mathbb{R}^n$, and $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, we say that F is *differentiable at x in the direction of C, with differential L*, if

$$\sup\Big\{\|y' - y - L(x' - x)\| : y' \in F(x')\Big\} = o(\|x' - x\|)$$

as $x' - x \to_C 0$.)

The flow maps $(t, s, q) \to \Phi^u_{t,s}(q)$ turn out to be regular near (t, t, q) for every $(t, q) \in [a, b] \times Q$ and every constant control u, as long as the vector field $(q, t) \to f(q, u, t)$ satisfies (LTH). (The single-valued approximations required by (REG.2) are obtained by approximating the flow Φ^u by the regularized flows $\Phi^{u,\rho}$, where $\Phi^{u,\rho}$ is the flow of the regularized vector field $(q, t) \to f^\rho(q, u, t)$, with

$$f^\rho(q, u, t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(q + \rho h, u, t)\sigma(h)dh$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}$ is a smooth nonnegative function supported in the ball $\{x : \|x\| \le 1\}$ and such that $\int \sigma = 1$.) Moreover, even though the endpoint maps constructed in Section 5.7 no longer need to be single-valued, they turn out to be regular near $(\xi_*(a), 0)$ and, if the insertion times s_i of the needle variations are chosen as explained in Section 5.7, then the differentiation formula remains unchanged. So the proof given in Section 5.7 is valid in the more general setting, and we get the *classical Maximum Principle with the Lojasiewicz improvement*[16]:

Theorem 5.8.2 *Assume that the classical technical hypotheses (CTH) hold, except that only the reference control η_* is required to satisfy a C^1-Carathéodory condition near ξ_*, and for the constant controls $u \in U$ we only assume that the weaker condition (LTH) holds. Then the Maximum Principle (SSMP) is true.* ◇

A different extension of the classical version of the Maximum Principle was proposed by F.H. Clarke in [10] and [11], and has become known as the "nonsmooth Maximum Principle." In this version, Condition (CTH.2) is replaced by the weaker requirement that the vector fields $(q,t) \to f(q,\eta(t),t)$ be Lipschitz continuous, with an integral bound on the Lipschitz constant.

Precisely, let us say that a time-varying vector field F, defined on some subset $\mathrm{Dom}(F)$ of $\mathbb{R}^n \times \mathbb{R}$, satisfies a ***Lipschitz-Carathéodory condition*** near a curve $\xi : [a,b] \to \mathbb{R}^n$ if there exists a tube $\mathcal{T}(\xi,\varepsilon)$ such that (M-Car) holds, and

(Lip-Car) *there is a function $\varphi \in L^1([a,b],\mathbb{R})$ such that*

$$\|F(q,t)\| \le \varphi(t) \quad \text{and} \quad \|F(q',t) - F(q,t)\| \le \varphi(t).\|q'-q\|$$

whenever $(q,t) \in \mathcal{T}(\xi,\varepsilon)$ and $(q',t) \in \mathcal{T}(\xi,\varepsilon)$.

In the Clarke version of the Maximum Principle, Condition (CTH.2) is replaced by the followin:

(CTH.2.Cl) *For every control η that is either constant or equal to the reference control η_*, the corresponding time-varying vector field f_η satisfies a Lipschitz-Carathéodory condition near ξ_*.*

Naturally, when this weaker condition holds, it is no longer possible to interpret the adjoint equation in the classical sense, so (CTH.4) has to be modified. The modification proposed by Clarke was that the adjoint equation be interpreted as a *differential inclusion* with the classical Jacobian matrix replaced by a Clarke generalized Jacobian. In other words, Clarke suggested that, instead of (CTH.4), the following be used:

(CTH.4.Cl) *A* ***solution of the adjoint equation (AE) along the reference trajectory-control pair*** *$\gamma_* = (\xi_*,\eta_*)$ is an absolutely continuous map $\psi : [a,b] \to \mathbb{R}_n$ such that*

$$-\dot\psi(t) \in \psi(t).\partial_1 f(\xi_*(t),\eta_*(t),t) \ \text{for a.e. } t \in [a,b]. \qquad (43)$$

[16] Communicated to us by S. Łojasiewicz, Jr., in a personal conversation that took place in May 1992. Łojasiewicz's brilliant idea was explained to us for the much more general setting of the nonsmooth Maximum Principle, discussed below, and to our knowledge still remains unpublished.

Here $\partial_1 f(\bar{q}, u, t)$ is the *Clarke generalized Jacobian* (CGJ) at $\bar{q}$ of the map $q \to f(q, u, t)$.[17]

Clarke's own proof technique is not "classical," in the sense that it does not proceed as we did in Section 5.7, i. e., by constructing needle variations, differentiating the endpoint maps, and then using a separation theorem such as Theorem 5.6.1. It turns out, however, that a classical proof of Clarke's result can be given, and this classical proof has a number of advantages over the nonsmooth one, as we now show.

The link between Clarke's nonsmooth Maximum Principle and the classical approach is provided by J. Warga's theory of "derivate containers" (cf. Warga [52]-[55]). A *derivate container* of a Lipschitz-continuous map $\mu : Q \to \mathbb{R}^m$, where Q is open in $\mathbb{R}^n$, at a point $q \in Q$ is a kind of "generalized differential of μ at q." The precise definition is as follows:

Definition 5.8.1 Let Q be open in $\mathbb{R}^n$, let $\mu : Q \to \mathbb{R}^m$ be Lipschitz-continuous, and let $\bar{q} \in Q$. A compact set $\mathbf{L}$ of linear maps $\mathbb{R}^n \to \mathbb{R}^m$ is a *derivate container* of μ at $\bar{q} \in Q$ if for every $\varepsilon > 0$ there exist a neighborhood V_ε of $\bar{q}$ and a sequence μ_j of maps of class C^1 from V_ε to $\mathbb{R}^m$, such that $\mu_j \to \mu$ uniformly on V_ε as $j \to \infty$ and the differential $D\mu_j(q)$ satisfies $\mathrm{dist}(D\mu_j(q), \mathbf{L}) \leq \varepsilon$ for every j and every $q \in V_\varepsilon$. $\diamond$

What makes this concept useful is, to begin with, that it satisfies the *chain rule*: if Q_i are open in $\mathbb{R}^{n_i}$ for $i = 1, 2$, $\mu_i : Q_i \to \mathbb{R}^{n_{i+1}}$ are Lipschitz continuous, $\mu_1(Q_1) \subseteq Q_2$, $q_1 \in Q_1$, $q_2 = \mu_1(q_1)$, and $\mathbf{L}_i$ are, for $i = 1, 2$, derivate containers of μ_i at q_i, then

$$\mathbf{L}_2 \circ \mathbf{L}_1 \stackrel{\mathrm{def}}{=} \{L_2 \circ L_1 : L_2 \in \mathbf{L}_2\,,\ L_1 \in \mathbf{L}_1\}$$

is a derivate container of $\mu_2 \circ \mu_1$ at q_1.[18]

In addition, Warga's derivate containers provide a natural extension to flows of Lipschitz vector fields of the differentiation results of Section 5.7 for flows of vector fields of class C^1. Indeed, if (CTH.2.Cl) holds, then the reference flow $\Phi^{\eta_*} = \{\Phi^{\eta_*}_{t,s}\}_{a \leq s \leq t \leq b}$ consists of Lipschitz continuous maps, each one of which is defined on an open subset $\mathrm{Dom}(\Phi^{\eta_*}_{t,s})$ of Q. Moreover, derivate containers[19] $D\Phi^{\eta_*}_{t,s}(\xi_*(s))$ of the maps $\Phi^{\eta_*}_{t,s}$ at the points $\xi_*(s)$ can be computed by solving a *set of variational equations*. Precisely, define

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[17] For a Lipschitz continuous map $\mu : Q \to \mathbb{R}^m$, where Q is open in $\mathbb{R}^n$, the CGJ of μ at $q \in Q$ is the convex hull $\partial\mu(q)$ of the set of all linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $L = \lim_{j \to \infty} L_j$ for some sequence $\{L_j\}$ such that there are $q_j \in Q$ for which $q_j \to q$ and μ is differentiable at q_j with differential L_j. The existence of many such sequences $\{q_j\}$ follows because μ is differentiable almost everywhere by Rademacher's Theorem. The set $\partial\mu(q)$ is then nonempty, compact and convex.

[18] A somewhat similar rule for the composition of maps holds for Clarke generalized Jacobians: suppose that $Q_1, Q_2, q_1, q_2, \mu_1$, and μ_2 are as above; then the inclusion $\partial(\mu_2 \circ \mu_1)(q_1) \subseteq \mathrm{co}(\partial\mu_2(q_2) \circ \partial\mu_1(q_1))$ holds. This, however, is insufficient for the application to the Maximum Principle, since the operation of taking convex hull can produce sets that are too large. The advantage of the derivate containers is that the chain rule holds *without having to take convex hulls.*

[19] We are choosing our language carefully. Derivate containers are not unique, so one cannot talk about *the* derivate container of a map at a point. The expression $D\Phi^{\eta_*}_{t,s}(\xi_*(s))$ does not denote "the derivate container of $\Phi^{\eta_*}_{t,s}$ at $\xi_*(s)$." It is just the name we choose

$\mathbf{M}^{\gamma_*}(t) = \partial_1 f(\xi_*(t), \eta_*(t), t)$ for $a \le t \le b$. If $[a,b] \ni t \to M(t) \in \mathbf{M}^{\gamma_*}(t)$ is an arbitrary measurable selection of the set-valued map $\mathbf{M}^{\gamma_*}$, we define the *linearized flow* L_M to be the family $\{L_M(t,s)\}_{a \le s \le t \le b}$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$L_M(t,s) = \text{identity} + \int_s^t M(\tau) L_M(\tau, s)\, d\tau\,. \tag{44}$$

We then define $\mathbf{L}^{\gamma_*}(t,s)$ to be the set $\{L_M(t,s) : M \in \Gamma(\mathbf{M}^{\gamma_*})\}$, where $\Gamma(\mathbf{M}^{\gamma_*})$ is the set of all measurable selections of $\mathbf{M}^{\gamma_*}$. It is clear that each $\mathbf{L}^{\gamma_*}(t,s)$ is compact and nonempty. (The compactness follows from the facts that (i) $\Gamma(\mathbf{M}^{\gamma_*})$ is weakly compact in $L^1([a,b], \mathbb{R}^n)$, since $\mathbf{M}^{\gamma_*}$ is a measurable integrably bounded set-valued map with compact convex values, and (ii) the map $M \to L_M$ is continuous with respect to the weak topology for M and the topology of uniform convergence for L_M.)

Moreover, it is easy to see that the set-valued maps

$$(t,s) \to \mathbf{L}^{\gamma_*}(t,s) \subseteq \mathbb{R}^{n \times n}$$

satisfy the identities

$$\mathbf{L}^{\gamma_*}(t,t) = \{\text{identity}\} \quad \text{for} \quad t \in [a,b]\,, \tag{45}$$

and

$$\mathbf{L}^{\gamma_*}(t_3,t_2) \circ \mathbf{L}^{\gamma_*}(t_2,t_1) = \mathbf{L}^{\gamma_*}(t_3,t_1) \text{ whenever } a \le t_1 \le t_2 \le t_3 \le b\,. \tag{46}$$

It was proved by Warga that $\mathbf{L}^{\gamma_*}(t,s)$ *is a derivate container of* $\Phi^{\eta_*}_{t,s}$ *at* $\xi_*(s)$. So we choose to define

$$D\Phi^{\eta_*}_{t,s}(\xi_*(s)) \stackrel{\text{def}}{=} \mathbf{L}^{\gamma_*}(t,s)\,.$$

Moreover, it is easy to see that *a map* $[a,b] \ni t \to \psi(t) \in \mathbb{R}_n$ *is a solution of the adjoint equation in the sense of (CTH.4.Cl) if and only if it satisfies the "integrated adjoint equation"*

$$\psi(t) = \psi(b).L_M(b,t) \quad \text{for} \quad a \le t \le b \tag{47}$$

for some $M \in \Gamma(\mathbf{M}^{\gamma_*})$. (Indeed, if (47) holds, then an argument similar to that used in Section 5.7 shows that $\dot\psi(t) = -\psi(t).M(t)$ a.e., so (43) holds. Conversely, if (43) holds, then one can show that there is a measurable selection $M \in \Gamma(\mathbf{M}^{\gamma_*})$ such that $\dot\psi(t) = -\psi(t).M(t)$ for a.e. t, and then (47) holds.)

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to give to one particular derivate container of $\Phi^{\eta_*}_{t,s}$ at $\xi_*(s)$, to emphasize the analogy with the classical C^1 case. Notice that "$D\Phi^{\eta_*}_{t,s}(\xi_*(s))$" should not be read as "D of the map $\Phi^{\eta_*}_{t,s}$ at $\xi_*(s)$," since $D\Phi^{\eta_*}_{t,s}(\xi_*(s))$ is not uniquely determined by the map $\Phi^{\eta_*}_{t,s}$ and the point $\xi_*(s)$, because the same map could arise from two different flows, and in that case the derivate container defined here could depend on which flow is used.

Unfortunately, this is not enough yet to carry out the proof of §5.7 in the more general setting. The reason for this is that *the class of Lipschitz maps is not large enough to contain the endpoint maps defined by the needle variations.*

A class with the desired properties was introduced in Sussmann [42]-[45]. The maps in this class are called "semidifferentiable," and their generalized differentials are called "semidifferentials." The precise definition is quite simple:

Definition 5.8.2 Let F be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $\bar{q} \in \mathbb{R}^n$, and let C be a closed convex cone in $\mathbb{R}^n$. Let $\mathbf{L}$ be a compact set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. We say that F is *semidifferentiable at $\bar{q}$ in the direction of C, with semidifferential* $\mathbf{L}$—and write $\mathbf{L} \in SD_C(F;\bar{q})$—if there exist a compact neighborhood V of $\bar{q}$ in $\mathbb{R}^n$, and a Lipschitz map $H : V \to \mathbb{R}^m$, such that (a) the restriction of F to $V \cap (\bar{q}+C)$ is a regular set-valued map from $V \cap (\bar{q}+C)$ to $\mathbb{R}^m$, (b) H is a "first-order approximation to F in the direction of C near $\bar{q}$," in the sense that

$$\sup\Big\{\|y - H(q)\| : y \in F(q)\Big\} = o(\|q-\bar{q}\|) \quad \text{as} \quad q - \bar{q} \to_C 0\,,$$

and (c) $\mathbf{L}$ is a derivate container of H at $\bar{q}$. ◇

Semidifferentials also satisfy the *chain rule*: if

(a) F_i are set-valued maps from $\mathbb{R}^{n_i}$ to $\mathbb{R}^{n_{i+1}}$ for $i = 1, 2$,
(b) C_i are closed convex cones in $\mathbb{R}^{n_i}$,
(c) $q_i \in \mathbb{R}^{n_i}$,
(d) $F_1((q_1 + C_1) \cap U_1) \subseteq q_2 + C_2$ for some neighborhood U_1 of q_1,
(e) $\mathbf{L}_i$ belongs to $SD_{C_i}(F_i; q_i)$ for $i = 1, 2$,

then $\mathbf{L}_2 \circ \mathbf{L}_1 \in SD_{C_1}(F_2 \circ F_1; q_1)$.

Moreover, it is clear that a Lipschitz-continuous map is semidifferentiable in the direction of any closed convex cone, and any derivate container of the map is also a semidifferential. So Warga's result on derivate containers of flow maps arising from Lipschitz vector fields says in particular that these flow maps are semidifferentiable, and gives us an explicit construction—if (CTH.2.Cl) holds—of a family $\{\mathbf{L}^{\gamma_*}(t,s)\}_{a\le s\le t\le b}$ of compact sets of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$\mathbf{L}^{\gamma_*}(t,s) \in SD_{\mathbb{R}^n}(\Phi^{\eta_*}_{t,s};\xi_*(s)) \quad \text{whenever} \quad a \le s \le t \le b\,, \tag{48}$$

and in addition (45) and (46) hold.

It follows trivially from the definition of semidifferentiability that *if a continuous single-valued or regular set-valued map F is differentiable at a point $\bar{q}$ in the classical sense in the direction of a cone C with differential L, then* $\{L\} \in SD_C(F;\bar{q})$. Thus the theory of semidifferentials contains both that of derivate containers of Lipschitz continuous maps and that of regular classically differentiable maps, and brings them both into a unified framework where the chain rule holds.

In addition, it turns out that the semidifferentiable analogue of Theorem 5.6.1 holds:

Theorem 5.8.3 *Let C_1 be a closed convex cone in $\mathbb{R}^m$, let U be a neighborhood of 0 in $\mathbb{R}^m$, let $f : U \cap C_1 \to \mathbb{R}^n$ be a set-valued map, and let $\mathbf{L} \in SD_{C_1}(f;0)$. Let S be a subset of $\mathbb{R}^n$ having a cone C_2 as weakly approximating cone at $f(0)$. Assume that C_2 is not a linear subspace of $\mathbb{R}^n$. Then a necessary condition for the sets $f(U \cap C_1)$ and S to be locally separated at $f(0)$ is that $(LC_1)^\dagger \cap (-C_2)^\dagger \neq \{0\}$ for some $L \in \mathbf{L}$, i.e. that there exist an $L \in \mathbf{L}$ and a nonzero covector $\bar{p} \in \mathbb{R}_n$ such that $p.L.v_1 \leq 0$ for all $v_1 \in C_1$ and $p.v_2 \geq 0$ for all $v_2 \in C_2$.* ◇

(This is a special case of the more general theorem of Sussmann [47].)

Using these results, the proof of Section 5.7 carries over to the current setting. It suffices to observe that, if the endpoint maps $\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*}$ are defined as in Section 5.7, then $\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*}$ is semidifferentiable at $(\xi_*(a),0)$ in the direction of $\mathbb{R}^n \times \mathbb{R}^m_+$, and one member of $SD_{\mathbb{R}^n \times \mathbb{R}^m_+}(\mathcal{E}_{\mathbf{s},\mathbf{u},\eta_*};(\xi_*(a),0))$ is the set

$$\mathbf{D}_{\mathbf{s},\mathbf{u},\gamma_*} = \{E(L) : L \in \Lambda^{\gamma_*}\}\,,$$

where Λ^{γ_*} is the set of all "compatible selections of $\mathbf{L}^{\gamma_*}$" (i.e., the set of all families $L = \{L(t,s)\}_{a \leq s \leq t \leq b}$ such that $L(t,s) \in \mathbf{L}^{\gamma_*}(t,s)$ for each s,t, and $L(t_3,t_2) \circ L(t_2,t_1) = \bar{L}(t_3,t_1)$ whenever $a \leq t_1 \leq t_2 \leq t_3 \leq b$) and, for $L \in \Lambda$,

$$E(L)(\delta q, \delta r_1, \ldots, \delta r_m) \stackrel{\text{def}}{=} L(b,a).\delta q + \sum_{i=1}^{m} \delta r_i.L(b,s_i).v^{\gamma_*}(s_i,u_i)\,. \tag{49}$$

It can be proved that Λ^{γ_*} coincides with the set of all maps $L_M^{\gamma_*}$ defined earlier. Using this, the proof of the Maximum Principle given in Section 5.7 applies in this case, and we get the *nonsmooth Maximum Principle with the Łojasiewicz improvement*:

Theorem 5.8.4 *Assume that (i) the classical technical hypotheses (CTH.1), (CTH.3), (CTH.5), (CTH.6) hold; (ii) the concept of "solution of the adjoint equation" is interpreted in the sense of (CTH.4.Cl); (iii) the reference vector field f_{η_*} satisfies a Lipschitz-Carathéodory condition near ξ_*; and (iv) condition (LTH) holds for every $u \in U$. Then the Maximum Principle (SSMP) is true.* ◇

This result includes the nonsmooth Maximum Principle as a special case, but is more general because of the Łojasiewicz improvement. (The result of [10] requires the stronger assumption that $f(q,u,t)$ be Lipschitz with respect to q for all u, t.)

5.9 Multidifferentials, Flows, and a General Version of the Maximum Principle

Proving the nonsmooth version of the maximum principle by classical needle-variation techniques has several advantages, two of which should already be apparent from the previous discussion: (i) the proof is conceptually close in spirit to the elegant classical idea of geometrizing the argument by making everything revolve around the construction of a suitable "tangent cone" to the reachable set, and (ii) the proof yields a stronger result because of the Lojasiewicz improvement.

These, however, are only the first two of a long list of reasons that make it worthwhile to pursue the classical approach. It suffices to look at the basic structure of the proof to see that, with practically no change, one can go much farther and prove a more general theorem.

To begin with, the Lipschitz-Carathéodory conditions for the reference control are only needed to establish the semidifferentiability of the reference flow Φ^{η_*} and to show that $\mathbf{L}^{\gamma_*}$ is a semidifferential of the flow. Once this is understood, one can do away with the technical assumptions for the reference vector field and impose instead the much weaker requirement that the reference trajectory be part of a general *semidifferentiable flow* Φ. It is then no longer necessary to assume that the state space Q is fixed, and we can allow it to depend on t. The "reachable set" will just be a subset $\mathcal{R}$ of the terminal state space Q_b, and the Maximum Principle will give a necessary condition for $\mathcal{R}$ to be separated from some other set. This condition will involve the other basic ingredient of our general situation, namely, a set $\mathcal{V}$ of *variations*. The variations should then give rise to *endpoint maps*, and should be such that these endpoint maps, when applied starting at $\xi_*(a)$, always yield points of $\mathcal{R}$.

Finally, and most importantly, the proofs of Sections 5.7 and 5.8 have essentially the same structure, and differ mainly in the specific differentiation theory they use. This strongly suggests that everything might work equally well if, instead of the semidifferentials, we use an even more general differentiation theory, provided only that this theory has some basic properties, such as the chain rule and a transversality theorem such as Theorems 5.6.1 and 5.8.3.

This suggestion turns out to be true. At least one such theory, that of the *multidifferentials*, was introduced in Sussmann [47]. Using multidifferentials, one can prove an even more general version of the maximum principle, which applies to *systems of differential inclusions* as well as to control systems of the usual kind.

We now make all this precise, and begin by listing the basic definitions.

Multidifferentials. If X, Y are finite-dimensional real linear spaces, we write $L(X,Y)$ to denote the set of all linear maps from X to Y. If X is normed, we write $B_X \stackrel{\text{def}}{=} \{x \in X : \|x\| \leq 1\}$. If S is a convex subset of X, $\mathbf{D} \subseteq L(X,Y)$, and $k = 1, 2, \ldots$, we define $C^k_{\mathbf{D}}(S,Y)$ to be the set of all maps $H : S \to Y$ that admit an extension to a C^k map $\tilde{H} : \Omega \to Y$, defined on an open subset Ω of X such that $\mathrm{Clos}(S) \subseteq \Omega$, and having the property that

$$D\tilde{H}(x) \in \mathbf{D} \quad \text{for all} \quad x \in S\,. \tag{50}$$

(If $\mathbf{D}$ is closed in $L(X,Y)$, then of course (50) is equivalent to the requirement that $D\tilde{H}(x) \in \mathbf{D}$ for all $x \in \mathrm{Clos}(S)$.)

Definition 5.9.1 Let X, Y be finite-dimensional normed real linear spaces, and let X be normed. Let $F : X \to 2^Y$ be a set-valued map. Let $(\bar{x}, \bar{y})$ be a point in $X \times Y$, and let $\mathbf{D}$ be a compact subset of $L(X,Y)$. Let C be a closed convex cone in X. We say that $\mathbf{D}$ is a *multidifferential of F at $(\bar{x}, \bar{y})$*

in the direction of C, and write

$$\mathbf{D} \in MD_C(F;\bar{x},\bar{y})\,,$$

if the following is true:

(MD) for every neighborhood $\mathbf{D}'$ of $\mathbf{D}$ in $L(X,Y)$ there exists a pair (ρ,Θ) such that

(MMD.1) $\rho > 0$, and Θ is a function on the interval $]0,\rho]$ with values in $[0,\infty[$ and such that $\lim_{s\downarrow 0}\Theta(s) = 0$,

(MMD.2) for every $\varepsilon > 0$ there exist f, H such that

(MMD.2.1) f is a regular set-valued map from $\bar{x} + (C \cap \rho B_X)$ to Y;
(MMD.2.2) $\mathrm{Graph}(f) \subseteq \mathrm{Graph}(F)$;
(MMD.2.3) $H \in C^1_{\mathbf{D}'}(\bar{x} + (C\cap \rho B_X), Y)$;
(MMD.2.4) $H(\bar{x}) = \bar{y}$;
(MMD.2.5) the inequality

$$\sup\Big\{\|y-H(x)\| : y\in f(x)\Big\} \leq \varepsilon+\Theta(\|x-\bar{x}\|)\|x-\bar{x}\| \qquad (51)$$

holds whenever $x - \bar{x} \in C \cap \rho B_X$. ◇

Time interval. A *time interval* is a totally ordered set. In what follows, a time interval I will be fixed, and we will use "$\leq$" to denote the order relation. The reader may assume that I is a subinterval of $\mathbb{R}$, but this fact will never be used. We prefer the more general setting of a totally ordered set both because it is mathematically more natural and because in several applications this extra generality is actually useful.

State bundles. If I is a time interval, a *state space bundle*—or *state bundle*, for short—*over* I is a family $Q = \{Q_t\}_{t\in I}$ of nonempty sets, indexed by $t \in I$. If J is a subset of I, then J is a time-interval (i. e., a totally ordered set) as well, so the family $Q\lceil J \stackrel{\mathrm{def}}{=} \{Q_t\}_{t\in J}$ is a state bundle, called the *restriction* of Q to J.

Sections. Given a state bundle Q over a time interval I, a *section of* Q is a map $I \ni t \to \xi(t) \in Q_t$. *No requirement such as smoothness, or continuity, or even measurability, is made in the definition of section.* In fact, such a requirement would be meaningless in this general setting, since the Q_t are just sets, and could be all different. We use $\Xi(Q)$ to denote the set of all sections of Q. We should think of the members of $\Xi(Q)$ as "curves" on the time-varying state space Q_t, at least in some formal sense.

Flows. If I is a time interval, and Q is a state bundle over I, then a *flow* on Q is a family $\Phi = \{\Phi_{t,s} : s,t \in I, s \leq t\}$ of set-valued maps, such that

a. *for each* $s,t\in I$ *such that* $s \leq t$, $\Phi_{t,s}$ *is a map from* Q_s *to subsets of* Q_t,

b. *for each* $t \in I$, $\Phi_{t,t}$ *is the identity map of* Q_t, and

c. *the identity*

$$\Phi_{t_3,t_2} \circ \Phi_{t_2,t_1} = \Phi_{t_3,t_1} \tag{52}$$

holds whenever $t_1 \leq t_2 \leq t_3$ *and the* t_i *are in* I.

We use $FLOW(Q)$ to denote the set of all flows on Q.

If $\Phi \in FLOW(Q)$ and J is a subset of the time interval I of Q, then the family $\Phi\lceil J = \{\Phi_{t,s}\}_{s,t\in J, s\leq t}$ is a flow on $Q\lceil J$, called the *restriction* of Φ to J.

Trajectories. A *trajectory* of a flow Φ on Q is a section $\xi \in \Xi(Q)$ such that $\xi(t) \in \Phi_{t,s}(\xi(s))$ whenever $s,t \in I$, $s \leq t$. We write $TRAJ(\Phi)$ to denote the set of all trajectories of Φ.

Variations. If $k \in \mathbb{N}$, then a *k-parameter variation* in a set S is, simply, a map ν from the nonnegative orthant $\mathbb{R}^k_+$ of $\mathbb{R}^k$ into S. If $\nu(0) = s$, $s \in S$, then s is the *base point* of ν, and ν is said to be a *(k-parameter) variation of s in S*. We write $VAR^k(S)$ to denote the set of all k-parameter variations in S, and $VAR^k(S;s)$ to denote the set of all $\nu \in VAR^k(S)$ whose base point is s.

Given a flow Φ on a state bundle Q over a time interval I, $a, b \in I$, and $a \leq b$, a *k-parameter variation of* Φ *carried by the interval* $[a,b]$ is a k-parameter variation of $\Phi_{b,a}$ in the set $SVM(Q_a, Q_b)$ of all set-valued maps from Q_a to Q_b. We use $VAR^k(\Phi; a, b)$ to denote the set of all k-parameter variations of Φ carried by $[a,b]$.

The endpoint map of a variation. If $k \in \mathbb{N}$, Φ, Q, I, a, b are as above, and $\nu \in VAR^k(\Phi; a, b)$, the *endpoint map* $\mathcal{E}(\nu; \Phi, a, b)$ is the set-valued map from $Q_a \times \mathbb{R}^k_+$ to Q_b such that

$$\mathcal{E}(\nu; \Phi, a, b)(q, r_1, \ldots, r_k) = \nu(r_1, \ldots, r_k)(q) \tag{53}$$

for $q \in Q_a$, $r_1, \ldots, r_k \in [0, \infty[$.

Linear bundles and flows. If Q is a state bundle and the sets Q_t are linear spaces, then Q is called a *linear bundle.*[20] If all the linear spaces are finite-dimensional then Q is a *finite-dimensional linear state space bundle.*

Naturally, when Q is a real linear state space bundle we can talk about *linear flows*. We use $LFLOW(Q)$ to denote the set of all linear flows on Q.

Linear multiflows. A *linear multiflow* on a linear state bundle X over the time interval I is a family $\Lambda = \{\Lambda_{t,s} : s,t \in I, s \leq t\}$ such that each $\Lambda_{t,s}$ is a nonvoid set of linear maps from X_s to X_t, having the property that $\Lambda_{t,t} = \{\text{identity}_{X_t}\}$ for each t, and

$$\Lambda_{t_3,t_2} \circ \Lambda_{t_2,t_1} = \Lambda_{t_3,t_1} \tag{54}$$

whenever $t_1 \leq t_2 \leq t_3$ and the t_i are in I. If X is finite-dimensional, we call a linear multiflow Λ on X *compact* if all the sets $\Lambda_{t,s}$ are compact. We use $LMFLOW(X)$, $LMFLOW_c(X)$ to denote, respectively, the set of all linear multiflows on X, and that of all $\Lambda \in LMFLOW(X)$ that are compact.

[20] Throughout this discussion, the field of scalars is always $\mathbb{R}$. No other field is ever considered.

OFD bundles. An *open finite-dimensional (OFD) state space bundle* is a state bundle Q such that each Q_t is a nonempty open subset of a finite-dimensional linear space X_t. In that case, the bundle $X = \{X_t\}_{t\in I}$ is the *tangent bundle* of Q.

Differentiable flows. If Q is an OFD state bundle over a time interval I, with tangent bundle X, we define a flow Φ to be *differentiable* along a trajectory ξ of Φ if $\Phi_{t,s}$ is continuous near $\xi(s)$ and differentiable at $\xi(s)$ whenever $s,t \in I$, $s \leq t$. If Φ is differentiable along ξ, then we can define a flow $D\Phi(\xi) \in \mathit{LFLOW}(X)$—called the *differential*, or the *linearization*, of Φ along ξ—by letting $D\Phi(\xi)_{t,s}$ be the differential $D\Phi_{t,s}(\xi(s))$, so $D\Phi(\xi)_{t,s}$ is a linear map from X_s to X_t. The fact that $D\Phi(\xi)$ is a linear flow follows from the chain rule for classical differentials.

Multidifferentials of flows. Let Q be an OFD a state space bundle over a time interval I, and let X be its tangent bundle. We then define a *multidifferential of a flow* $\Phi \in \mathit{FLOW}(Q)$ *along a trajectory* $\xi \in \mathit{TRAJ}(\Phi)$ to be a linear multiflow $\Lambda = \{\Lambda_{t,s}\}_{s,t\in I, s\leq t} \in \mathit{LMFLOW}(X)$ such that $\Lambda_{t,s}$ belongs to $\mathit{MD}_{X_s}(\Phi_{t,s};\xi(s),\xi(t))$ whenever $s,t \in I$, $s \leq t$. (This implies, in particular, that $\Lambda \in \mathit{LMFLOW}_c(X)$.) We use $\mathit{MD}(\Phi;\xi)$ to denote the set of all multidifferentials of Φ along ξ, and call Φ *multidifferentiable along* ξ if $\mathit{MD}(\Phi;\xi) \neq \emptyset$.

Remark 5.9.1 It is possible to give counterexamples showing that if each map $\Phi_{t,s}$ is multidifferentiable at $\xi(s)$, then a family Λ of multidifferentials satisfying the compatibility condition (54) may fail to exist. For this reason, we impose the requirement that such a family exist as an extra condition in our definition of multidifferentiability of a flow along a trajectory. $\diamond$

Compatible selections. If X is a finite-dimensional linear state space bundle over I, and Λ is a linear multiflow over X, a *compatible selection* of Λ is a linear flow $L = \{L_{t,s} : s,t \in I, s \leq t\} \in \mathit{LFLOW}(X)$ such that $L_{t,s} \in \Lambda_{t,s}$ whenever $s \leq t$ and $s,t \in I$. We use $\mathit{CS}(\Lambda)$ to denote the set of all compatible selections L of Λ.

It is not hard to prove, using Zorn's Lemma together with the compactness of the sets $\Lambda_{t,s}$, that compatible selections always exist if Λ is compact. Moreover, *if* Λ *belongs to* $\mathit{LMFLOW}_c(X)$, *and* J *is a subset of* I, *then every compatible selection* $L \in \mathit{CS}(\Lambda\lceil J)$ *of the restriction* $\Lambda\lceil J = \{\Lambda_{t,s} : s \in J, t \in J, s \leq t\}$ *can be extended to a compatible selection* $L' \in \mathit{CS}(\Lambda)$.

Impulse infinitesimal variations. Given a finite-dimensional linear state bundle X over the time interval I, a *1-parameter impulse infinitesimal variation in* X is a pair $V = (t,v)$ such that $t \in I$ and $v \in X_t$. If $a \in I$, $b \in I$ are such that $a \leq b$ and $a \leq t \leq b$, then V is said to be *carried by the interval* $[a,b]$. If $k \in \mathbb{N}$, then a *k-parameter impulse infinitesimal variation in* X is a k-tuple $V = (V_1,\ldots,V_k)$ of 1-parameter infinitesimal variations in X. If $a \in I$, $b \in I$ are such that each V_j is carried by $[a,b]$, then we say that V is *carried by* $[a,b]$. We use $\mathit{IIV}^k(X)$ to denote the set of all k-parameter impulse infinitesimal variations in X, and $\mathit{IIV}^k(X;a,b)$ to denote the set of all $V \in \mathit{IIV}^k(X)$ that are carried by the interval $[a,b]$.

Given a flow Φ on an OFD state bundle Q over the time interval I with tangent bundle X, we will also refer to the members of $\mathit{IIV}^k(X)$ as k-

parameter impulse infinitesimal variations of Φ, and we will write $IIV^k(\Phi)$, $IIV^k(\Phi; a, b)$ instead of $IIV^k(X)$, $IIV^k(X; a, b)$.

The infinitesimal endpoint map determined by an impulse infinitesimal variation. Suppose we are given a compact linear multiflow X on a finite-dimensional linear state bundle X over the time interval I. Let $k \in \mathbb{N}$, and assume that $a, b \in I$, $a \leq b$. Let $V \in IIV^k(X; a, b)$, and suppose that $V = (V_1, \ldots, V_k)$, $V_j = (t_j, v_j)$ for $j = 1, \ldots, k$.

For each compatible selection $L \in CS(\Lambda)$, define a linear map $E(L; V, a, b)$ from $X_a \times \mathbb{R}^k$ to X_b by

$$E(L; V, a, b)(x, r_1, \ldots, r_k) = L_{b,a}x + \sum_{j=1}^{k} r_j L_{b,t_j} v_j \,. \tag{55}$$

The set $\mathbf{E}(\Lambda; V, a, b) = \{E(L; V, a, b) : L \in CS(\Lambda)\}$ is the *infinitesimal endpoint map associated to V and Λ*. It is clear that $\mathbf{E}(\Lambda; V, a, b)$ is a nonempty compact set of linear maps from $X_a \times \mathbb{R}^k$ to X_b.

Impulse infinitesimal variations generated by a variation. Suppose we are given I, X, a, b, Λ, k, $V = ((t_1, v_1), \ldots, (t_k, v_k))$, Q, Φ, ξ as above. Let $\nu \in VAR^k(\Phi_{b,a})$. We say that V is *generated by* $(\nu, \Phi, \xi, \Lambda, a, b)$ if

$$\mathbf{E}(\Lambda; V, a, b) \in MD_{X_a \times \mathbb{R}^k_+}(\mathcal{E}(\nu; \Phi, a, b)\,; (\xi(a), 0), \xi(b))\,. \tag{56}$$

Adjoint vectors. Assume that X is a finite-dimensional linear state bundle over the time interval I, and $\Lambda \in LMFLOW(X)$. We then define a Λ-*adjoint vector* to be a map $I \ni t \to \psi(t) \in X_t^*$—where X_t^* is the dual space of X_t—having the property that [21]

$$\psi(s) \in \psi(t) \circ \Lambda_{t,s} \quad \text{whenever} \quad s \in I, t \in I, s \leq t\,. \tag{57}$$

An equivalent characterization is the following: a Λ-*adjoint vector* is a map $I \ni t \to \psi(t) \in X_t^*$ such that there is a compatible selection $L \in CS(\Lambda)$ for which

$$\psi(s) = \psi(t) \circ L_{t,s} \quad \text{whenever} \quad s \in I, t \in I, s \leq t\,. \tag{58}$$

We use $ADJ(\Lambda)$ to denote the set of all Λ-adjoint vectors.

Strong admissibility. Let Q be an OFD state bundle with time interval I, and assume I has a minimum a and a maximum b. Let $\Phi \in FLOW(Q)$, $\xi \in TRAJ(\Phi)$, $\Lambda \in MD(\Phi; \xi)$. Let $\mathbf{V} \subseteq IIV^1(\Phi)$ be a finite set of one-parameter infinitesimal variations of Φ, and let $\mathcal{R}$ be a subset of Q_b. We say that $\mathbf{V}$ is *strongly admissible for* $(\mathcal{R}, \Phi, \xi, \Lambda)$ if:

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[21] Recall that $\Lambda_{t,s}$ is a set of linear maps from X_s to X_t, and $\psi(t)$ is a linear map from X_t to $\mathbb{R}$, so the composite $\psi(t) \circ \Lambda_{t,s} = \{\psi(t) \circ L : L \in \Lambda_{t,s}\}$ is well defined. The composite $\psi(t) \circ L$ is the usual pullback of the functional $\psi(t) : X_t \to \mathbb{R}$ to a functional from X_s to $\mathbb{R}$ via L. Often, one uses L^* to denote the pullback map, so what we call $\psi(t) \circ L$ would be called $L^*(\psi(t))$ instead, but we prefer to write $\psi(t) \circ L$. This notation has the advantage of giving the usual matrix formula $\psi(s) = \psi(t).L_{t,s}$ if the X_t are Euclidean spaces, thought of as spaces of column vectors, and then the X_t^* are regarded as spaces of row vectors.

(SADM) *There exist an integer* $k \in \mathbb{N}$, *a variation* $\nu \in VAR^k(\Phi)$, *a* $V = (V_1, \ldots, V_k) \in IIV^k(\Phi)$, *and an* $\bar{r} > 0$, *such that*

(1) $\mathcal{E}(\nu; \Phi, a, b)(\xi(a), r) \subseteq \mathcal{R}$ *whenever* $r = (r_1, \ldots, r_k) \in [0, \bar{r}]^k$,

(2) V *is generated by* $(\nu, \Phi, \xi, \Lambda, a, b)$,

and

(3) $\mathbf{V} \subseteq \{V_j : j = 1, \ldots, k\}$.

Convergence of impulse infinitesimal variations. Given a time interval I having a minimum a and a maximum b, a linear finite-dimensional state bundle X over I, and a compact linear multiflow Λ on I, we can define a topology $\mathcal{T}(X, \Lambda)$ on $IIV^1(X)$ as follows. We pick, in an arbitrary fashion, a compact neighborhood K of 0 in X_b^*. We then let $ADJ(\Lambda; K)$ denote the set of all adjoint vectors $\psi \in ADJ(\Lambda)$ such that $\psi(b) \in K$, equipped with the topology of pointwise convergence. Then $ADJ(\Lambda; K)$ is a topological subspace of the product space $\Pi_{t \in I} \Lambda_{b,t}^*(K)$, which is compact by Tikhonov's theorem. It is easy to prove that $ADJ(\Lambda; K)$ is itself compact. Each $V = (t, v) \in IIV^1(X)$ gives rise to a continuous map $\mu_V : ADJ(\Lambda; K) \to \mathbb{R}$, defined by letting $\mu_V(\psi) = \psi(t).v$. Then $\mathcal{T}(\Lambda, X)$ is the topology of uniform convergence of the functions μ_V on $ADJ(\Lambda; K)$. (This topology is easily seen to be independent of the choice of K. Notice, however, that $\mathcal{T}(\Lambda, X)$ need not be Hausdorff, since it may happen, for example, that $\Lambda_{b,t} = \{0\}$ for some $t < b$ but $\dim(X_t) > 0$, in which case, if $V = (t, v)$, with $v \in X_t$ and $v \neq 0$, and $V' = (t, 0)$, then the functions μ_V and $\mu_{V'}$ will coincide, even though $V \neq V'$.)

Admissibility. Let Q be an OFD state bundle with time interval I and tangent bundle X. Assume that I has a minimum a and a maximum b. Let $\Phi \in FLOW(Q)$, $\xi \in TRAJ(\Phi)$, $\Lambda \in MD(\Phi; \xi)$. Let $\mathbf{V}$ be a set of 1-parameter $\Lambda \in SD(\Phi; \xi)$. Let $\mathbf{V}$ be a set of 1-parameter impulse infinitesimal variations of Φ. Let $\mathcal{R}$ be a subset of Q_b. We call $\mathbf{V}$ *admissible for* $(\mathcal{R}, \Phi, \xi, \Lambda)$ if every finite subset of $\mathbf{V}$ can be approximated in the topology $\mathcal{T}(\Lambda, x)$ of $IIV^1(\Phi)$ by strongly admissible finite subsets of $IIV^1(\Phi)$. The precise meaning of this condition is as follows:

(ADM) *Let* K *be a fixed compact neighborhood of* 0 *in* X_b. *Then for every finite subset* $\mathbf{W}$ *of* $\mathbf{V}$ *and every* $\varepsilon > 0$ *there exists a finite subset* $\mathbf{W}_\varepsilon$ *of* $IIV^1(\Phi)$ *which is strongly admissible for* $(\mathcal{R}, \Phi, \xi, \Lambda)$ *and such that*

$$\max_{W \in \mathbf{W}} \min_{W' \in \mathbf{W}_\varepsilon} \max_{\psi \in ADJ(\Lambda; K)} |\mu_W(\psi) - \mu_{W'}(\psi)| \leq \varepsilon\,. \tag{59}$$

(It is easy to see that the validity of (ADM) does not depend on the choice of the compact set K.)

We are now ready to state the *flows, variations, and multidifferentials version of the Maximum Principle*:

Theorem 5.9.1 *Let* I *be a time interval having a minimum* a *and a maximum* b. *Let* Q *be an OFD state bundle over* I, *and let* X *be its tangent* *bundle. Let $\Phi \in FLOW(Q)$, $\xi \in TRAJ(\Phi)$, $\Lambda \in MD(\Phi;\xi)$. Let $\mathcal{R}$, S be subsets of Q_b. Let C be a closed convex cone in X_b which is a weakly approximating cone to S at $\xi(b)$ and is not a linear subspace of X_b. Let $\mathbf{V}$ be a subset of $I\!IV^1(\Phi)$ which is admissible for $(\mathcal{R}, \Phi, \xi, \Lambda)$. Assume that $\mathcal{R}$ and S are locally separated at $\xi(b)$. Then there exists a $\psi \in ADJ(\Lambda)$ such that $\psi(b) \neq 0$, $-\psi(b) \in C^\dagger$ and $\psi(t).v \leq 0$ for all $(t, v) \in \mathbf{V}$.* ◇

This theorem contains as special cases all the versions stated earlier, but is much more general in a number of ways. For example:

1. Non-Lipschitz and discontinuous reference vector fields. For control systems of the classical form $\dot{q} = f(q, u, t)$, Theorem 5.9.1 applies in many cases when the time-varying vector field f_{η_*} arising from the reference control η_* fails to be Lipschitz or even continuous with respect to q. Two simple illustrations of this are the *reflected brachystochrone problem* and the *light refraction problem*.

Example 5.9.1 ***(The reflected brachystochrone problem.)*** We consider the minimum-time optimal control problem in $\mathbb{R}^2$—with coordinates x, y—whose dynamical law is $\dot{x} = u\sqrt{|y|}$, $\dot{y} = v\sqrt{|y|}$, the controls u, v being subject to the constraint $u^2 + v^2 \leq 1$. (It is easy to reformulate the problem as one involving a fixed interval such as $I = [0, 1]$. It suffices to rewrite the dynamical equations as $\dot{x} = wu\sqrt{|y|}$, $\dot{y} = wv\sqrt{|y|}$ and seek to minimize $\int_0^1 w(t)dt$, where w is a new control with values in $[0, \infty[$.) In this case, the right-hand side of the dynamical equations is continuous but not Lipschitz continuous. Moreover, if one wants to join optimally a point A in the upper half-plane to a point B in the lower half-plane, then the optimal trajectory will obviously go from A to a point C in the x-axis and then from C to B. The parts from A to C and from C to B are solutions of the classical brachystochrone problem, so they are arcs of cycloids. This still leaves one free parameter, namely the choice of C. For a given C, the corresponding curve ξ_C and control η_C are such that the time-varying vector field f_{η_C} is not Lipschitz continuous with respect to the state. This means that we cannot even apply the nonsmooth maximum principle with Lojasiewicz modification to the pair (ξ_C, η_C), because the reference vector field is not Lipschitz. It turns out that Theorem 5.9.1 does apply, because the flow determined by η_C is multidifferentiable—and, in fact, differentiable in the classical sense—along ξ_C, provided that we restrict it to the totally ordered set $I - \{\tau\}$, where τ is the unique point such that $\xi_C(\tau)$ lies in the x axis.

An application of Theorem 5.9.1 gives the extra junction condition. Precisely, the cycloids satisfy a differential equation $(y')^2|y| = K - |y|$. The extra conditions is that the constants K_+ and K_- corresponding to the two arcs of cycloid that make up ξ_C be equal. ◇

Example 5.9.2 ***(The refraction problem.)*** This is similar to the reflected brachystochrone problem, except that the dynamical law is now $||\dot{q}|| = c_+$ if $y > 0$, $||\dot{q}|| = c_-$ if $y < 0$, where c_-, c_+ are positive constants such that $c_- \neq c_+$. Now the dynamical law has a *discontinuous* right-hand side. Once again, Theorem 5.9.1 applies. The answer is, of course, that of the well known Snell law of refraction, but this time the derivation of the result does not require an extra *ad hoc* argument and fits directly within the framework of the necessary conditions for optimality. ◇

Remark 5.9.2 Example 5.9.2 is also covered by the theory of "optimal multiprocesses" of Clarke and Vinter, cf. [12]. ◇

2. Stronger conclusions for nonsmooth problems. Even in situations where the nonsmooth version applies, it can happen that Theorem 5.9.1 gives strictly stronger results, as the following example shows.

Example 5.9.3 ***(A problem with Lipschitz-continuous data, for which the usual nonsmooth maximum principle does not give the best possible result.)*** We consider the one-dimensional optimal control problem in which the dynamical law is

$$\dot{x} = u + x^2 \sin \frac{1}{x}\,, \tag{60}$$

the control constraint is $u \in \{-1, 0, 1\}$, and the objective is to minimize the integral $\int_0^1 e^t u(t)dt$ subject to the endpoint constraints $x(0) = x(1) = 0$. Suppose we want to test the trajectory $[0,1] \ni t \to x(t) = 0$ and corresponding control $u(t) \equiv 0$ for optimality. The nonsmooth maximum principle will give the adjoint equation $\dot{\psi} \in [-|\psi|, |\psi|]$, one of whose solutions is $\tilde{\psi}(t) = e^t$. The Hamiltonian (assuming that $\psi_0 = 1$, which is easily justified for this problem) is $H = \psi.u + \psi.x^2 \sin \frac{1}{x} - e^t u$. So the Hamiltonian maximization condition is satisfied for $\tilde{\psi}$. Therefore the nonsmooth maximum principle does *not* exclude the zero trajectory as a candidate for optimality. On the other hand, Theorem 5.9.1 does, because the reference flow Φ is also differentiable in the classical sense, since $\Phi_{t,s}(x) = x + o(|x|)$ as $x \to 0$ for each t, s, as can easily be seen, for example, using Gronwall's inequality. So, using the classical differential in the role of Λ, we get the adjoint equation $\dot{\psi} = 0$, which is *not* satisfied by any Hamiltonian-maximizing choice of ψ. ◇

The preceding example shows the advantage of a theory with a nonunique "generalized differential": our version of the maximum principle gives a nontrivial Hamiltonian-maximizing adjoint vector for *every* multidifferential $\Lambda \in MD(\Phi, \xi_*)$; when the conditions of the nonsmooth maximum principle (NSMP) hold, the usual statement of the NSMP only gives an adjoint vector corresponding to one of these multidifferentials, whereas our version allows for the possibility of selecting other multidifferentials that may yield stronger conclusions.

Example 5.9.4 ***(A non-Lipschitz analogue of Example 5.9.3.)*** It is easy to modify Example 5.9.3 to get a problem where the conditions of the maximum principle are not satisfied but those of our Theorem 5.9.1 are. It suffices to substitute for x^2 in (60) a factor of the form $|x|^\alpha$, where $1 < \alpha < 2$. Then the right-hand side of the dynamical equation is not Lipschitz continuous, but Theorem 5.9.1 still applies. ◇

3. High-order variations. Theorem 5.9.1 is also stronger, for a different reason, in situations where the dynamics is very smooth, and the classical version applies. This is because the theorem uses a general, abstract definition of "variation," so that the usual needle variations are special cases, but other, "high-order" variations, expressed in terms of Lie brackets of the vector fields of the system under consideration, are also allowed. (For some simple examples of such variations, cf. Kawski and Sussmann [25]. For a more general discussion of high-order conditions, cf. Gabasov-Kirillova [17], Kelley, Kopp and Moyer [26], Knobloch [27], Krener [28].)

4. Jumps. Theorem 5.9.1 allows *jump* maps. For example, it could happen that at some time $\bar{t}$ the state jumps from q to $J(q)$, where J is a map. It could even happen that the state spaces $Q_{\bar{t}_-}$, $Q_{\bar{t}_+}$ before and after the jump are different. (To include such jumps in our framework it is convenient to change the time-interval I and treat $\bar{t}_-$ and $\bar{t}_+$ as different points, which, incidentally, is one of several reasons why it is preferable to work with general totally ordered sets rather than ordinary intervals.) If the jump time $\bar{t}$ and the map J are fixed in advance, and J is multidifferentiable at $(\xi_*(\bar{t}_-), \xi(\bar{t}_+))$ with multidifferential $\mathbf{D}$, then the corresponding condition on the adjoint covector ψ is $\psi(\bar{t}_-) \in \psi(\bar{t}_+) \circ \mathbf{D}$. If $\bar{t}$ and J are not fixed, then one gets extra inequality conditions involving the variations of the pair $(\bar{t}, J)$.

5. Differential inclusions. Our result also applies to *systems of differential inclusions*. For example, the following is a special case of Theorem 5.9.1.

Theorem 5.9.2 *Let $I = [a, b]$ be a compact subinterval of $\mathbb{R}$. Let $n \in \mathbb{N}$, let Q be an open subset of $\mathbb{R}^n$, and let $\mathbf{F} = \{F_\alpha : \alpha \in A\}$ be a family of set-valued maps $Q \times I \ni (q, t) \to F_\alpha(q, t) \subseteq \mathbb{R}^n$, such that each F_α is almost lower semicontinuous and locally integrably lower bounded. Let $\alpha_* \in A$, and let $\xi_* : [a, b] \to Q$ be a trajectory of F_{α_*} such that F_{α_*} is integrably pseudo-Lipschitz along ξ_*. Let $\mathcal{R} \subseteq Q$ be such that, whenever $k \in \mathbb{N}$, $a \leq \tau_1 < \ldots < \tau_k < b$, and $\alpha_1, \ldots, \alpha_k \in A$, there exists an $\bar{r} > 0$ such that, if $(r_1, \ldots, r_k) \in [0, \bar{r}]^k$, then every solution ξ of the inclusion*

$$\dot{\xi}(t) \in \sum_{j=1}^{k} \chi_{[\tau_j, \tau_j + r]}(t) F_{\alpha_j}(\xi(t), t) + \left(1 - \sum_{j=1}^{k} \chi_{[\tau_j, \tau_j + r]}(t)\right) F_{\alpha_*}(\xi(t), t)$$

for which $\xi(a) = \xi_*(a)$ satisfies $\xi(b) \in \mathcal{R}$. Let $S \subseteq Q$, and let C be a closed convex cone in $\mathbb{R}^n$ which is a weakly approximating cone to S at $\xi_*(b)$ and is not a linear subspace of $\mathbb{R}^n$. Assume that $\mathcal{R}$ and S are locally separated at $\xi_*(b)$. Then:

(i) If $\lim_{\delta \downarrow 0} \sup \left\{ \operatorname{dist}(\dot{\xi}_*(t), F_{\alpha_*}(x, t)) : a \leq t \leq b, \|x - \xi_*(t)\| \leq \delta \right\} = 0$, then there exists a time-varying vector field $(q, t) \to f(q, t)$, defined on some tube $\mathcal{T}(\xi_*, \varepsilon)$ (cf. Eq. (29)), that satisfies a Lipschitz-Carathéodory condition near ξ_* and is such that $\dot{\xi}(t) = f(\xi(t), t)$ for almost every t, and $f(q, t) \in \operatorname{co}(F(q, t))$ for all $(q, t) \in \mathcal{T}(\xi_*, \varepsilon)$,

(ii) For every f that satisfies the conditions of (i) there exists an absolutely continuous solution $\psi : [a, b] \to \mathbb{R}_n$ of the differential inclusion

$-\dot{\psi}(t) \in \psi(t) \cdot \partial f_t(\xi_*(t))$—*where* $f_t(q) \stackrel{\text{def}}{=} f(q,t)$—*such that* $\psi(b) \neq 0$, $-\psi(b) \in C^\dagger$, *and, for every* $\alpha \in A$,

$$\psi(t).\dot{\xi}_*(t) \geq \max\{\psi(t).v : v \in F_\alpha(\xi_*(t),t)\}$$

for almost all $t \in I$. ◇

(The definitions of "almost lower semicontinuous," "locally integrably lower bounded," and "integrably pseudo-Lipschitz" can be found, for example, in Sussmann [45].)

Remark 5.9.3 Theorem 5.9.2 gives a necessary condition for separation *in terms of the adjoint equation for a selection of the convexified inclusion.* So this condition is also the necessary condition for the relaxed problem. The observation that the condition arising from the relaxed problem is also necessary for the nonrelaxed one is known as the "Kaskosz maximum principle" (cf. Kaskosz [23], Warga [56]). ◇

Remark 5.9.4 A result similar to Theorem 5.9.2 but for a single Lipschitz nonconvex inclusion was proved by Q. J. Zhu in [58] and by H. D. Tuan in [51]. ◇

6. Hybrid problems. Theorem 5.9.1 applies to problems that are "hybrid" in the sense that the technical assumptions T_j of various existing necessary conditions for optimality (such as the classical smooth maximum principle with high-order conditions, the nonsmooth maximum principle, and the maximum principle for differential inclusions) hold near the reference trajectory on various subintervals I_j of its time interval I, but none of these technical conditions holds all the time. In this case, the other theorems will give for each j an adjoint vector ψ_j defined on I_j and satisfying the conclusions C_j of the theorem that applies on I_j, but it will not follow that these results can be combined into one that yields the existence of a global ψ satisfying C_j on I_j for each j.

Example 5.9.5 ***(High-order conditions for a problem whose dynamics is not everywhere smooth.)*** Consider a minimum-time problem with dynamical law

$$\dot{q} = f(q) + ug(q)$$

and control constraint $|u| \leq 1$. The high-order maximum principle gives an extra inequality—the Legendre-Clebsch condition—valid when f and g are sufficiently smooth, e.g. of class C^3. On the other hand, if f and g are only Lipschitz continuous, the nonsmooth maximum principle applies. It is natural to ask what happens if f and g are smooth in some open region Q_S of the state space, but just Lipschitz continuous in some other open region Q_L. Suppose $\xi : [a,b] \to Q_S \cup Q_L$ is a trajectory that is not entirely contained in either of the two regions and corresponds to a control $\eta : [a,b] \to [-1,1]$. A reasonable conjecture is that, if ξ is time-optimal, then there will exist a nontrivial field of covectors ψ that satisfies the conclusion of the nonsmooth principle while $\xi(t)$ is in Q_L, and those of the smooth high-order principle when $\xi(t) \in Q_S$. In other words, ψ *will satisfy the* *adjoint differential inclusion and the Hamiltonian maximization condition on the whole interval* $[a, b]$, *and in addition the Legendre-Clebsch condition*

$$\langle \psi(t), [g, [f, g]](\xi(t)) \rangle \geq 0$$

at every time t *such that* $|\eta(t)| < 1$, t *is a Lebesgue point of* η, *and* f *and* g *are of class* C^3 *near* $\xi(t)$. (For details of how to construct the variation that gives rise to the impulse infinitesimal variation $(t, -[g, [f, g]](\xi(t)))$, cf. [25].)

Neither the classical high-order maximum principle nor the nonsmooth maximum principle can be used to prove that the conjecture is true, but Theorem 5.9.1 applies. ◇

5.10 Three Ways to Make the Maximum Principle Intrinsic on Manifolds

So far, the state spaces have been open subsets of Euclidean spaces, and the formulations of the Maximum Principle have depended strongly on this fact. Naturally, it is desirable to have a formulation that works intrinsically on manifolds. This is essentially equivalent to having a formulation in Euclidean spaces that is invariant under arbitrary nonlinear changes of coordinates. Such a formulation would make it possible, even when we are working in $\mathbb{R}^n$, to follow Lagrange's strategy of expressing "the equations of every problem in the most simple form relative to each set of variables," and seeing "beforehand which variables one should use in order to facilitate the integration as much as possible."

It turns out that this can be done in several ways, all of which are useful and enhance the power of the Maximum Principle as a technical tool.

Let us first consider, for simplicity, the classical maximum principle. Since this case is discussed in great detail in Sussmann [46], we will only review it briefly.

Although the adjoint equation $-\dot{\psi}(t) = \frac{\partial H}{\partial q}(\xi_*(t), \eta_*(t), \psi(t), t)$ is not manifestly invariant under nonlinear coordinate changes, it is quite easy to prove its invariance, as we now show.

1. The Hamiltonian formulation. One way to prove invariance of the adjoint equation is to use the canonical symplectic structure of the cotangent bundle T^*Q of a manifold Q. This structure gives rise to a canonical fiber-preserving isomorphism $J : T^*(T^*Q) \to T(T^*Q)$, which enables us to asign to every covector $\omega \in T^*_z(T^*Q)$ a tangent vector $J_z(\omega) \in T_z(T^*Q)$. In particular, if $H : T^*Q \to \mathbb{R}$ is a function, and $z \in T^*Q$ is a point where H is differentiable, then there is a well defined tangent vector $\vec{H}(z) \in T_zT^*Q$, given by $\vec{H}(z) = J_z(dH(z))$.

Now suppose that we are given a control system $\dot{q} = f(q, u, t)$ on a manifold Q, and a reference trajectory-control pair $\gamma_* = (\xi_*, \eta_*)$ such that f_{η_*} satisfies a C^1-Carathéodory condition near ξ_*. For each u, t, define

$$\mathcal{H}_{u,t}(q, p) \stackrel{\text{def}}{=} H^{\Sigma}(q, u, p, t) \, (\stackrel{\text{def}}{=} p.f(q, u, t)) \, .$$

Then the function $\mathcal{H}_{\eta_*(t),t}$ is of class C^1 for almost every t. The system

$$\begin{aligned} \dot{\xi}(t) &= f(\xi(t), \eta_*(t), t)\,, && (61) \\ -\dot{\psi}(t) &= \psi(t).\frac{\partial f}{\partial q}(\xi(t), \eta_*(t), t) && (62) \end{aligned}$$

is then easily seen to be equivalent to

$$\dot{\zeta}(t) = \vec{\mathcal{H}}_{\eta_*(t),t}(\zeta(t))\,, \qquad (63)$$

if we write $\zeta(t) = (\xi(t), \psi(t))$. So the system (61,62) is clearly intrinsically defined in a coordinate-free way.

At this point, it is worth noticing that, if $\Sigma = (Q, U, I, \mathcal{U}, f)$ is a control system on a manifold Q, such that for every $\eta \in \mathcal{U}$ the corresponding vector field f_η satisfies a C^1-Carathéodory condition, then we can define

$$f^*(z, u, t) = \vec{\mathcal{H}}_{u,t}(z)\,, \quad \text{for} \quad (z, u, t) \in T^*Q \times U \times I\,,$$

and obtain a new control system $\Sigma^* = (T^*Q, U, I, \mathcal{U}, f^*)$ with state space T^*Q, called the *Hamiltonian lift* of Σ. Then the necessary condition given by the Maximum Principle can be viewed as stating that ***the reference trajectory-control pair*** $\gamma_* = (\xi_*, \eta_*)$ ***must be the projection of a reference trajectory-control pair*** $\Gamma_* = (\Xi_*, \eta_*)$ ***of the Hamiltonian lift such that*** Γ_* ***has some special properties***, namely, nontriviality, Hamiltonian maximization, and the transversality condition. This seemingly trivial remark has turned out to be useful because, once it is understood that the Maximum Principle relates certain trajectories of a control system Σ to special trajectories of Σ^*, then it is at least conceivable in principle that this construction may be iterated.

We now present an example—based on Sussmann [39, 40, 48]—showing how the idea of iterating Hamiltonian lifts can be used to prove a "weak regularity theorem" for trajectories:

Example 5.10.1 The following result was announced in [39] for the case when U is an interval and in [40] for the general case. A detailed proof appears in [48]:

Theorem 5.10.1 *Let Q be a real-analytic manifold, let U be a compact subanalytic subset of a real-analytic manifold $\hat{U}$, and let $f : Q \times U \to TQ$ be an analytic map such that $f(q, u) \in T_qQ$ for every $q \in Q$, $u \in U$. Let $\xi : [a, b] \to Q$ be an absolutely continuous curve that satisfies the equation $\dot{\xi}(t) = f(\xi(t), \eta(t))$ a.e. for some measurable control $\eta : [a, b] \to U$. Then there exists a trajectory-control pair $(\xi^\#, \eta^\#)$, defined on some interval $[a^\#, b^\#]$, such that $\xi^\#(a^\#) = \xi(a)$, $\xi^\#(b^\#) = \xi(b)$, and $\eta^\#$ is real-analytic on an open dense subset of $[a^\#, b^\#]$.*

Theorem 5.10.1 says, roughly, that for an analytic system, whenever a point $\bar{q}$ can be steered to some other point $\hat{q}$ by means of some measurable control it follows that this can also be done using a control that has an extra regularity property. This theorem can be applied to optimal control problems of the general form (FEOCP)—as long as f and L do not depend on t and are analytic with respect to q and u—by considering an augmented system in which the running cost is added as a new state variable. In that case, Theorem 5.10.1 implies that, *if $\bar{q}$ can be steered to $\hat{q}$ with cost c by means of some measurable control, then $\bar{q}$ can be steered to $\hat{q}$ with cost c by means of a control that is real-analytic on an open dense subset of its domain.* In particular, we can take c to be the optimal cost, and conclude—under the analyticity hypothesis—that if (FEOCP) has a solution then it has a solution that is real-analytic on an open dense subset of its domain. For problems having a unique solution, this result gives a *regularity property* of the solution.

The proof of Theorem 5.10.1 is quite long, and uses highly nontrivial facts about real-analytic maps, such as resolution of singularities. Without going into any details, we just single out one noteworthy fact: *the proof uses in a crucial way the idea of iterating the Hamiltonian lifting.* The way this idea enters the proof is roughly as follows. If $\xi(b)$ is an interior point of the reachable set $\mathcal{R}$ from $\xi(a)$, then it follows from general results about real-analytic systems that $\xi(b)$ is reachable from $\xi(a)$ by means of a piecewise constant control.[22] If $\xi(b)$ belongs to the boundary of $\mathcal{R}$, then the Maximum Principle says that ξ is the projection of a "special" trajectory Ξ of the lifted Hamiltonian system. What makes Ξ special is the Hamiltonian maximization condition, according to which $\eta(t)$ cannot be an arbitrary point of U, but must belong to $U_{\max}(\Xi(t))$, where

$$U_{\max}(q,p) = \Big\{u \in U : p.f(q,u) = \max\{p.f(q,v) : v \in U\}\Big\}.$$

So Ξ is a trajectory of a system $\Sigma^{*,\max}$ which is of the same kind as Σ, but has a "smaller" control space. The trouble with this is that the new control space $U_{\max}(q,p)$ depends on the point (q,p) in the state space T^*Q of the new system, so $\Sigma^{*,\max}$ is not the kind of control system to which the Maximum Principle can be applied. At this point, the machinery of real-analytic stratifications and desingularization turns out to be helpful, by enabling us to construct a partition $\mathcal{P}$ of T^*Q into connected embedded real-analytic submanifolds such that (a) for every t in an open dense subset J of $[a,b]$, the restriction of the curve Ξ to a neighborhood $V(t)$ of t is entirely contained in one of the members of $\mathcal{P}$, and (b) for every $P \in \mathcal{P}$, the set $U_{\max,P}(z) = \{u \in U_{\max}(z) : \vec{\mathcal{H}}_u(z) \in T_zP\}$ is "the same for all $z \in P$" and is "smaller than U." The expressions "the same" and "smaller" are, of course, vague, but they can be given a precise definition, using desingularization theory. It then turns out that locally, on an open dense subset of $[a,b]$, the pair (Ξ,η) is "exactly like" the pair (ξ,η), but with a smaller control set. One can then iterate the construction, and keep making the control set smaller and smaller. Moreover, this process can be proved to terminate in a finite number of steps, and Theorem 5.10.1 follows. $\diamond$

2. Connections along curves. A second way to reformulate the adjoint equation invariantly is to use *connections along curves*, as explained in

[22] It is not necessarily true that $\xi(b)$ is reachable from $\xi(a)$ by means of a piecewise constant control *in the same time* $b-a$.

Sussmann [46]. Precisely, recall that, if Q is a smooth manifold, then a *vector field along a curve* $\xi : [a,b] \to Q$ is a map $[a,b] \ni t \to \theta(t) \in T_{\xi(t)}Q$ that assigns to each $t \in [a,b]$ a tangent vector $\theta(t)$ to Q at $\xi(t)$. (The definition of a *field of covectors* is similar, except that in this case the value at each t must belong to the cotangent space $T^*_{\xi(t)}Q$.)

Definition 5.10.1 Let Q be a smooth manifold. A *connection along an absolutely continuous curve* $\xi : [a,b] \to Q$ is a map ∇ that assigns to every absolutely continuous vector field θ along ξ an integrable vector field $\nabla\theta$ along ξ, in such a way that $\nabla(\theta_1 + \theta_2) = \nabla(\theta_1) + \nabla(\theta_2)$ for all θ_1, θ_2, and $\nabla(\varphi\theta) = \varphi\nabla\theta + \dot{\varphi}\theta$ whenever θ is an absolutely continuous vector field along ξ and $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function. $\diamond$

It is easy to see that, if ξ is an integral curve of a time-varying vector field $(q,t) \to f(q,t)$ that satisfies a C^1-Carathéodory condition near ξ, then the pair (ξ, f) gives rise in a canonical way to a connection $\nabla_{\xi,f}$ along ξ, characterized by the fact that, if X is a smooth vector field on Q, and $X \circ \xi$ is the vector field along ξ defined by $(X \circ \xi)(t) = X(\xi(t))$, then

$$\nabla_{\xi,f}(X \circ \xi)(t) = [f_t, X](\xi(t))$$

for $t \in [a,b]$, where f_t is the vector field $q \to f_t(q) \stackrel{\text{def}}{=} f(q,t)$.

A connection along a curve induces an operator of "parallel translation" along the curve. Precisely, if $\xi : [a,b] \to Q$ is a curve and ∇ is a connection along ξ, then a vector field θ along ξ is ∇-*parallel* if $\nabla\theta \equiv 0$. It is easy to see that if $s \in [a,b]$ and $v \in T_{\xi(s)}Q$, then there is a unique vector field θ along ξ which is ∇-parallel and satisfies $\theta(s) = v$. If we define $\tau_{\xi,\nabla,t,s}(v) = \theta(t)$, then the map $\tau_{\xi,\nabla,t,s}$ is a linear isomorphism from $T_{\xi(s)}Q$ to $T_{\xi(t)}Q$ for each $s,t \in [a,b]$. The maps $\tau_{\xi,\nabla,t,s}$ satisfy the obvious flow identities (i.e., $\tau_{\xi,\nabla,t,t} = \text{identity}$, and $\tau_{\xi,\nabla,t_1,t_2} \circ \tau_{\xi,\nabla,t_2,t_3} = \tau_{\xi,\nabla,t_1,t_3}$).

Although a connection ∇ along a curve ξ is defined as a differential operator on vector fields along ξ, such an object induces in a standard way differential operators on fields of covectors and on fields of higher-order tensors. If θ is a field of vectors and ω is a field of covectors along ξ, then the "product rule"

$$\frac{d}{dt}\Big(\omega(t).\theta(t)\Big) = (\nabla\omega)(t).\theta(t) + \omega(t).(\nabla\theta)(t)$$

holds. In particular, if θ and ω are parallel (i.e. if $\nabla\omega \equiv 0$ and $\nabla\theta \equiv 0$) then the function $t \to \omega(t).\theta(t)$ is a constant.

If ξ is an integral curve of a time-varying vector field f, and $\nabla = \nabla_{\xi,f}$, then the equations $\nabla\theta = 0$ and $\nabla\omega = 0$ are, respectively, the *variational equation* and the *adjoint equation* along (ξ, f). In the setting of the maximum principle, when f is the vector field f_{η_*} corresponding to the reference control η_*, the adjoint equation that appears in the maximum principle is precisely the equation

$$\nabla_{\xi_*,f_{\eta_*}}\psi \equiv 0\,.$$

We now give a few examples—borrowed from Sussmann [46], where the reader can find a much more detailed discussion—to illustrate the advantages of the connection point of view.

Example 5.10.2 ***(The geodesic equation for Riemannian metrics.)*** The maximum principle, formulated in terms of connections along curves, can be used to give a very elegant, completely coordinate-free derivation of the geodesic equation for Riemannian metrics that elucidates the special role of the Levi-Civita connection without ever having to write formulas for the Christoffel symbols.

Suppose Q is a Riemannian manifold. Let U be the set of all smooth vector fields u on M such that $||u(q)|| \leq 1$ for all q. Let Σ be the control system $\dot{q} = f(q,u)$, $u \in U$, where $f(q,u) \stackrel{\text{def}}{=} u(q)$, and the class of admissible controls is the set $\mathcal{U}$ of all maps $\eta : [a_\eta, b_\eta] \to U$ such that the map $(q,t) \to f(q,\eta(t))$ satisfies a C^1-Carathéodory condition on every compact subset of Q. Then the minimum-time trajectories of Σ are exactly the length-minimizers parametrized by arc-length. Let ξ be a minimum-time trajectory of Σ, and let ξ satisfy $\dot{\xi}(t) = f(\xi(t),\eta(t))$ for some control η. Let ψ be a field of covectors given by the maximum principle. Then the Hamiltonian maximization condition implies that $\psi = \rho.G(\dot{\xi})$, where G is the metric, regarded as a vector bundle homomorphism from TQ to T^*Q, and ρ is a positive function. The constancy of the minimized Hamiltonian implies that ρ is constant, so we may assume that $\rho \equiv 1$. The adjoint equation says that

$$\nabla_{\xi,f_\eta}\psi \equiv 0\,.$$

Let ∇^{LC} be the Levi-Civita connection. Then, if X is any smooth vector field on Q, we have

$$\begin{aligned}
&\Big\langle(\nabla_{\xi,f_\eta}\psi)(t), X(\xi(t))\Big\rangle\\
&\quad= \frac{d}{dt}\Big\langle\psi(t), X(\xi(t))\Big\rangle - \Big\langle\psi(t), \Big(\nabla_{\xi,f_\eta}(X\circ\xi)\Big)(t)\Big\rangle\\
&\quad= \frac{d}{dt}\Big\langle\psi(t), X(\xi(t))\Big\rangle - \Big\langle\psi(t), [f_{\eta(t)}, X](\xi(t))\Big\rangle\\
&\quad= \frac{d}{dt}\Big\langle\psi(t), X(\xi(t))\Big\rangle - \Big\langle\psi(t), (\nabla^{LC}_{f_{\eta(t)}}X - \nabla^{LC}_X f_{\eta(t)})(\xi(t))\Big\rangle\,,
\end{aligned}$$

using the fact that ∇^{LC} is torsion-free.

For each t, the function $q \to ||f_{\eta(t)}(q)||^2$ has a local maximum at the point $q = \xi(t)$, so its derivative in any direction vanishes at $\xi(t)$. But

$$X.||f_{\eta(t)}||^2 = 2\Big\langle G(f_{\eta(t)}), \nabla^{LC}_X f_{\eta(t)}\Big\rangle\,,$$

since $\nabla^{LC}G = 0$. So $\Big\langle\psi(t), (\nabla^{LC}_X f_{\eta(t)})(\xi(t))\Big\rangle = 0$. Then

$$\begin{aligned}
\Big\langle\nabla_{\xi,f_\eta}\psi(t), X(\xi(t))\Big\rangle &= \frac{d}{dt}\Big\langle\psi(t), X(\xi(t))\Big\rangle - \Big\langle\psi(t), (\nabla^{LC}_{f_{\eta(t)}}X)(\xi(t))\Big\rangle\\
&= \Big\langle\nabla^{LC}_{f_\eta}\psi(t), X(\xi(t))\Big\rangle\,.
\end{aligned}$$

Since X is arbitrary, we conclude that $\nabla_{\xi,f_\eta}\psi \equiv \nabla^{LC}_{f_\eta}\psi$. Therefore the adjoint equation $\nabla_{\xi,f_\eta}\psi = 0$ implies $(\nabla^{LC}_{f_{\eta(t)}}\psi)(t) = 0$ for all t.

Since $\psi = G(\dot\xi)$ and $\nabla^{LC}G = 0$, we get $\nabla^{LC}_{f_{\eta(t)}}\dot\xi \equiv 0$, i.e., $\nabla^{LC}_{\dot\xi(t)}\dot\xi \equiv 0$, which is the usual geodesic equation. ◇

Example 5.10.3 ***(How the adjoint equation treats all vector fields equally.)*** The adjoint equation $\nabla_{\xi_*, f_{\eta_*}}\psi = 0$ implies, for any smooth vector field X on Q, the equation

$$\frac{d}{dt}\Big(\psi(t).X(\xi_*(t))\Big) = \psi(t).[f_{\eta_*(t),t}, X](\xi_*(t))\,, \tag{64}$$

where we write $f_{u,t}(q) \stackrel{\text{def}}{=} f(q,u,t)$. In fact, it is easy to prove that *the adjoint equation holds if and only if (64) holds for all smooth vector fields X.* The usual way of writing the adjoint equation in coordinates corresponds to singling out some special vector fields X, namely, the coordinate vector fields. Indeed, if $q \to (x^1(q), \ldots, x^n(q))$ is a chart, and we write $\psi(t) = \sum_j \psi_j(t)dx^j$, $f(q,u,t) = \sum_j f^j(q,u,t)\partial_j$, where ∂_j is the vector field usually called $\frac{\partial}{\partial x^j}$, then $\psi(t).\partial_j = \psi_j(t)$, and

$$[f_{\eta_*(t),t}, \partial_j](\xi_*(t)) = -\sum_i \partial_j f^i(\xi_*(t), \eta_*(t), t)\partial_i\,,$$

so (64), for $X = \partial_j$, says that

$$\dot\psi_j = -\sum_i \psi_i.\partial_j f^i(\xi_*(t), \eta_*(t), t)\,. \tag{65}$$

Naturally, the usual form of the adjoint equation says that (65) holds for $j = 1, \ldots, n$. Comparing with (64), we see that ***(64) says that something happens for all vector fields, whereas (65) says that the same thing happens for the special vector fields arising from a coordinate chart.*** In other words, (64) treats all vector fields equally, whereas (65) singles out some vector fields attached to a chart.

Paraphrasing the statement by Lagrange quoted earlier, one may expect that the more general formulation (64) might make it possible to

> express the equations of every optimal control problem in the most simple form relative to each set of variables and enable us to see beforehand which vector fields one should use in order to facilitate the analysis as much as possible.

It may happen that, to analyze a particular problem, the functions that play the most important role are the "momentum functions" $\psi.X_i$ arising from certain vector fields X_i that are not the ∂_j corresponding to a given chart. (Even more strongly, if the X_i do not commute then they cannot arise as coordinate vector fields of *any* chart.) ◇

Example 5.10.4 ***(A simple example of trajectory analysis using Lie brackets and the intrinsic Maximum Principle.)*** In Sussmann [46], we discuss many examples of how the qualitative analysis of the optimal trajectories becomes much simpler by using (64). The simplest of them is given by the proof of the following theorem:

Theorem 5.10.2 *Consider a control system* $\Sigma: \ \dot{q} = f(q) + ug(q)$ *in an open subset* Q *of* $\mathbb{R}^2$*, with control constraint* $|u| \leq 1$*. Suppose that*

(I) $g(q)$ *and* $[f,g](q)$ *are linearly independent at every* $q \in Q$*.*

Then every time-optimal trajectory of Σ *is bang-bang with finitely many switchings. If, in addition,*

(II) $f(q)$ *and* $g(q)$ *are linearly independent at every* $q \in Q$*,*

then every time-optimal trajectory of Σ *is bang-bang with at most one switching.*

Proof. Let $\xi : [a,b] \to Q$ be time-optimal, and let $\psi : [a,b] \to \mathbb{R}_2$ be an adjoint vector given by the Maximum Principle. Let

$$\varphi_1(t) = \psi(t).g(\xi(t)), \quad \varphi_2(t) = \psi(t).[f,g](\xi(t)), \quad \varphi_3(t) = \psi(t).f(\xi(t)).$$

Then (64) tells us that $\dot{\varphi}_1 = \varphi_2$, and (I) implies that φ_1 and φ_2 never vanish simultaneously. So all zeros of φ_1 are isolated, and then ξ is bang-bang, as stated. Now suppose that (II) holds as well. Then we can write $[f,g] = \alpha.f + \beta.g$, where α and β are smooth functions on Q. Condition (II) implies that α never vanishes along ξ, so the function $t \to \alpha(\xi(t))$ has constant sign. Clearly, $\varphi_2 = \alpha.\varphi_3 + \beta.\varphi_1$ along ξ. Suppose φ_1 vanishes more than once. Let t_1, t_2 be two consecutive zeros. Then $\dot{\varphi}_1(t_1).\dot{\varphi}_1(t_2) < 0$. So $\varphi_2(t_1).\varphi_2(t_2) < 0$. But

$$\varphi_2(t_1).\varphi_2(t_2) = \alpha(\xi(t_1)).\alpha(\xi(t_2)).\varphi_3(t_1).\varphi_3(t_2).$$

Since α has constant sign along ξ, we find $\varphi_3(t_1).\varphi_3(t_2) < 0$. On the other hand, the Hamiltonian H is equal to $\varphi_3 + u\varphi_1$, and this function must be constant along (ξ, ψ). In particular, this implies that $\varphi_3(t_1) = \varphi_3(t_2)$, so $\varphi_3(t_1).\varphi_3(t_2) \geq 0$. This contradiction proves our statement. $\diamond$

3. The intrinsic form of the general "flows, variations, and multidifferentials" version of the Maximum Principle. So far, we have concentrated on seeking intrinsic formulations of the classical smooth Maximum Principle. The next natural step would be to write intrinsic formulations of Clarke's nonsmooth principle, and other versions that generalize the classical one.

This is not hard to do. For example, Clarke's nonsmooth principle can be formulated in Hamiltonian form, by observing that the Clarke generalized gradient $\partial h(z)$ of a Lipschitz-continuous function $h : T^*Q \to \mathbb{R}$ at a point $z \in T^*Q$ is a well-defined nonempty compact convex subset of $T_z T^*Q$, and then rewriting the combined system (61,62), which now becomes a differential inclusion, as $\dot{\zeta}(t) \in J\Big(\partial \mathcal{H}_{\eta_*(t),t}(\zeta(t))\Big)$. Alternatively, we can use the connections approach, and observe that the set $\Gamma(\mathbf{M}^{\gamma_*})$ now becomes a set of connections along ξ_*, rather than a single connection as in the classical case.

It is better, however, to jump directly to the most general version of the Maximum Principle, namely, Theorem 5.9.1. It turns out that in this case very little work needs to be done, because *Theorem 5.9.1 is already almost intrinsic as it stands.* To render it completely intrinsic, it suffices to observe that

1. The concept of a *multidifferential* at a point $(q,q') \in Q \times Q'$ of a set-valued map $F : Q \to 2^{Q'}$ between manifolds of class C^1 is well defined, and the multidifferentials $\mathbf{D} \in MD(F;q,q')$ are nonempty compact subsets of $L(T_qQ, T_{q'}Q')$, the space of linear maps from T_qQ to $T_{q'}Q'$. (To see this, choose charts $x : U \to \mathbb{R}^n$, $x' : U' \to \mathbb{R}^{n'}$ of Q, Q', defined on neighborhoods U, U' of q, q'. Then, if we let $G = x' \circ F \circ x^{-1}$, it follows that G is a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^{n'}$, and $MD(G;x(q),x'(q'))$ is a well-defined set of nonempty compact subsets of $L(\mathbb{R}^n,\mathbb{R}^{n'})$. Then
$$MD(F;q,q') \stackrel{\text{def}}{=} \Big\{ Dx'(q') \circ \mathbf{D} \circ (Dx(q))^{-1} : \mathbf{D} \in MD(G;x(q),x'(q')) \Big\}$$
is clearly a well defined set of nonempty compact subsets of the space $L(T_qQ, T_{q'}Q')$. Moreover, the Chain Rule for multidifferentials implies that $MD(F;q,q')$ is independent of the choice of the charts x, x'.) More generally, if C is a closed convex cone in $\mathbb{R}^m$, then the concept of a multidifferential at $((q,0),q')$ of a set-valued map from $Q \times C$ to subsets of Q' is well defined.

2. There is an obvious way to define what is meant by a "C^1-manifold bundle" $Q = \{Q_t\}_{t\in I}$ over a time interval I, by just requiring the Q_t to be manifolds of class C^1.

3. If $Q = \{Q_t\}_{t\in I}$ is a C^1-manifold bundle, Φ is a flow on Q, and $\xi \in TRAJ(\Phi)$, then the *tangent bundle of* Q *along* ξ is the linear space bundle
$$T_\xi Q \stackrel{\text{def}}{=} \{T_{\xi(t)}Q_t\}_{t\in I} \,. \tag{66}$$
Then the concept of a *multidifferential of* Φ *along* ξ is well defined, and we can write $MD(\Phi;\xi)$ to denote the set of all such multidifferentials. The members of $MD(\Phi;\xi)$ are compact linear multiflows on the linear bundle $T_\xi Q$.

4. The concepts of variation, impulse infinitesimal variation, infinitesimal endpoint map, strong admissibility, admissibility, and adjoint vector, carry over with no change to this more general setting.

5. The concept of a "weakly approximating cone" $C \subseteq T_qQ$ to a subset S of a C^1 manifold Q at point $q \in Q$ makes sense intrinsically.

We then get the following ***intrinsic verson of the general Maximum Principle***:

Theorem 5.10.3 *Let I be a time interval having a minimum a and a maximum b. Assume that Q is a C^1-manifold state space bundle over I and Φ is a flow on Q. Let $\xi \in TRAJ(\Phi)$, and let X be the tangent bundle $T_\xi Q$, so $X_t = T_{\xi(t)}Q_t$ for $t \in I$. Let $\Lambda \in MD(\Phi;\xi)$ be a multidifferential of Φ along ξ. Let $\mathcal{R}$, S be subsets of Q_b. Let C be a closed convex cone in X_b which is a weakly approximating cone to S at $\xi(b)$ and is not a linear subspace of X_b. Let $\mathbf{V}$ be a subset of $\mathbb{IIV}^1(\Phi)$ which is admissible for*

$(\mathcal{R}, \Phi, \xi, \Lambda)$. *Assume that* $\mathcal{R}$ *and* S *are locally separated at* $\xi(b)$. *Then there exists a* $\psi \in ADJ(\Lambda)$ *such that* $\psi(b) \neq 0$, $-\psi(b) \in C^{\dagger}$ *and* $\psi(t).v \leq 0$ *for all* $(t, v) \in \mathbf{V}$. $\diamond$

5.11 Conclusion

Optimal control, in its earlier form known as the calculus of variations, has been the driving force behind some of the most important developments that have taken place in geometry. The systematic study of curves that minimize "length" turned out to be the key that unlocked the door opening up to modern differential geometry. This study was made possible by the availability of techniques invented by the practitioners of the calculus of variations for the analysis of even more general curve optimization problems. Besides, it was in the context of the calculus of variations that the first important discoveries were made of results that are "invariant under arbitrary nonlinear coordinate changes," leading to the general notions of covariance and invariance introduced in the nineteenth century.

After the calculus of variations expanded its scope in the 1950s and took on the new name of optimal control theory, it benefitted in many ways from the incorporation of geometric insights. The necessary conditions for optimality have been geometrized in two ways, namely, (a) by reformulating the maximum principle as a separation result for a reachable set, and (b) by making it intrinsically invariant on manifolds. The combined effect of these two geometrizations, coupled with

1. the use of results from homotopy theory (Leray-Schauder [29], Browder [9]) to get a stronger form of the transversality condition,

2. the adaptation of the needle-variations method of [35] to a broader setting, involving a theory of generalized differentials obtained by extending Warga's theory of derivate containers (Warga [52]-[55]),

3. the use of selection theorems for almost lower semicontinuous inclusions, based on extending the ideas of Bressan [5]-[7],

and, finally,

4. the use of uniform approximation theorems for relaxed trajectories of pseudo-Lipschitz differential inclusions, based on extending the ideas of Fryszkowski and coworkers [13, 16],

has led to the formulation of a nonsmooth, intrinsic maximum principle of great generality and power.

Much work remains to be done. For example, the ideas of the previous sections should be extended to problems with state space constraints. Also, a more systematic analysis is needed of the high-order variations that can occur. Finally, the precise relation between our version of the maximum principle and other recent nonsmooth versions (e.g. Ioffe [18], Ioffe-Rockafellar [19], Loewen-Rockafellar [30, 31], Rockafellar [38]) is not yet completely clear, and it is not known whether these other versions can be combined with ours in a truly unified framework.

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