WHY REAL ANALYTICITY IS IMPORTANT IN CONTROL THEORY

H. J. Sussmann

1. Introduction .

In the recent development of nonlinear control theory, properties of real analytic functions and maps, and their associated classes of sets —i.e. analytic, semianalytic and subanalytic sets— have come to play an increasingly important role. The purpose of this survey is to explain why this is so, by outlining some results where analyticity matters, and by giving general reasons why it ought to matter.

The paper is organized as follows. After introducing some basic notations and definitions, we describe some facts about integral manifolds, reachable sets, and nonlinear realization theory, which make use of analyticity in an "elementary way," i.e. through the analytic continuation principle. We then turn to deeper properties, such as the stratification theory of semianalytic and subanalytic sets, and present results in control theory where methods based on these results play a role. Finally, we turn to the even deeper theory of resolution of singularities, and some of its applications in control. In all cases, we try to explain why analyticity is crucial, but we do not give details of proofs.

2. Basic notations .

Throughout this paper, "smooth" means "of class C^∞," and "analytic" means "real analytic." *Manifolds* are supposed to be Hausdorff and countable at infinity, but *not* of pure dimension, so that a manifold M consists of a finite or countable number of connected components, each of which is paracompact, and these components are allowed to be of different dimensions. It is *not* required that there be an upper bound for these dimensions.

If M is a smooth manifold and $x \in M$, then T_xM denotes the tangent space to M at x. If X is a smooth vector field on M, and $x \in M$, then xX will mean the same as $X(x)$, i.e. the value of X at x. Similarly, if $\Phi : M \to N$ is a (possibly partially defined) smooth map, then $x\Phi$ will be an alternative notation for $\Phi(x)$.

We use exponential notation for the *flow* of a vector field, so that $t \to x\mathrm{e}^{tX}$ is the integral curve of X that goes through x at time $t=0$. If $v \in T_xM$, then $v\Phi$ denotes the value at Φ of the differential of Φ at x, i.e. the vector usually denoted by $\Phi_*(v)$ or $d\Phi(x)v$.

Since X is a first-order differential operator acting on smooth functions on M, the result of this action on a particular function f —sometimes denoted L_Xf and called the "Lie derivative" of f with respect to X— will simply be denoted Xf.

The *Lie bracket* $[X,Y]$ of two smooth vector fields X and Y is the differential operator $[X,Y] \stackrel{\mathrm{def}}{=} XY - YX$. The function $t \to x\mathrm{e}^{tX}Y\mathrm{e}^{-tX}$ takes values in the linear space T_xM. It turns out that this function is smooth, and its derivative is given by

$$\frac{d}{dt}\left(x\mathrm{e}^{tX}Y\mathrm{e}^{-tX}\right) = x\mathrm{e}^{tX}[X,Y]\mathrm{e}^{-tX} \ . \tag{2.1}$$

A *polysystem* is a differential equation

$$\dot{x} = f(x,u) \ , \ x \in M \ , \ u \in U \ , \tag{2.2}$$

where M is a smooth manifold, U is a set, and $f(x,u) \in T_xM$ for each $(x,u) \in M \times U$. To such a system we associate the collection $V_f = \{f_u : u \in U\}$ of vector fields f_u, where $f_u(x) \stackrel{\mathrm{def}}{=} f(x,u)$. The polysystem (2.2) is *smooth* (resp. *analytic*) if all the vector fields f_u are smooth (resp. analytic). If (2.2) is smooth, then the *controllability Lie algebra of* f —denoted by Λ_f— is the Lie algebra of vector fields generated by V_f.

A *control* for a polysystem (2.2) is a U-valued function η defined on some compact interval $[a,b] \subseteq \mathbb{R}$. Two controls that coincide almost everywhere are identified. A *trajectory* for a control $\eta : [a,b] \to U$ is an absolutely continuous curve $\gamma : [a,b] \to M$ such that $\dot{\gamma}(t) = f(\gamma(t), \eta(t))$ for almost all $t \in [a,b]$. A *control system* consists of the specification of a polysystem (2.2) together with a collection $\mathcal{U}$ of controls. We call the members of $\mathcal{U}$ *admissible controls*, and require that the class $\mathcal{U}$ be invariant under time translations (i.e. if $\eta : [a,b] \to U$ is in $\mathcal{U}$, $\tau \in \mathbb{R}$, and $\eta_\tau : [a+\tau, b+\tau] \to U$ is defined by $\eta_\tau(t) = \eta(t-\tau)$, then $\eta_\tau \in \mathcal{U}$) and concatenations (i.e. if $\eta : [a,b] \to U$ and $c \in]a,b[$ are such that the restrictions of η to $[a,c[$ and $]c,b]$ are in $\mathcal{U}$, then $\eta \in \mathcal{U}$), and we also demand that $\mathcal{U}$ contain all constant controls. It then follows that $\mathcal{U}$ contains all piecewise constant controls as well. A trajectory that corresponds to an admissible control is an *admissible trajectory.* If φ is any function with domain $[a,b]$ (e.g. a control or a trajectory), then we write $T_-(\varphi)$ for a, $T_+(\varphi)$ for b. When φ is a trajectory, we write $x_-(\varphi)$, $x_+(\varphi)$ for $\varphi(a)$, $\varphi(b)$, respectively.

If Σ is a control system, (2.2) the associated polysystem, and $\mathcal{U}$ the class of admissible controls, we write f_Σ, V_Σ, Λ_Σ, M_Σ, U_Σ, $\mathcal{U}_\Sigma$, for f, V_f, Λ_f, M, U, $\mathcal{U}$, respectively.

We use $\mathcal{A}_\Sigma$ to denote the set of all *admissible pairs* of Σ, i.e. of all pairs $\Gamma = (\gamma, \eta)$ such that $\eta \in \mathcal{U}$ and γ is a trajectory for η. (We then use $T_\pm(\Gamma)$, $x_\pm(\Gamma)$, to denote $T_\pm(\gamma)$ and $x_\pm(\gamma)$, respectively.) The system Σ has the *piecewise constant approximation property* (PCAP) if, whenever γ is an admissible trajectory, and $c \in [a,b]$, then there exists a sequence $\{\gamma_j\}$ of trajectories corresponding to piecewise constant controls, such that $T_-(\gamma_j) = a$, $T_+(\gamma_j) = b$, $\gamma_j(c) = \gamma(c)$, and $\gamma_j \to \gamma$ uniformly on $[a,b]$ as $j \to \infty$.

If Σ is a control system, $x \in M_\Sigma$ and $T \geq 0$, we use $\mathcal{R}_\Sigma(x, \leq T)$ to denote the set of all points of the form $\gamma(t)$, for all admissible trajectories γ and all t such that $x(\gamma) = x$, $T_-(\gamma) = 0$, $T_+(\gamma) = t \leq T$.

3. Analytic continuation, integral manifolds and reachable sets .

Undoubtedly, the simplest property of analytic maps is the *analytic continuation principle* (ACP) for functions of one variable, according to which, if f is a real- or vector-valued analytic function on an open interval $I \subseteq \mathbb{R}$, and the set of zeros of f has an accumulation point in I, then $f \equiv 0$. This result already has an important control-theoretic implication that makes analytic control systems fundamentally different from C^∞ ones, as we now show.

Recall that, if Λ is any set of smooth vector fields on a smooth manifold M, an *integral manifold* of Λ is a smooth connected embedded submanifold S of M such that T_xS is equal to the linear span of $\Lambda(x)$ for each $x \in S$. (Naturally, $\Lambda(x) \stackrel{\text{def}}{=} \{X(x) : X \in \Lambda\}$.) An integral manifold S is *maximal* if, whenever S' is another integral manifold such that $S' \cap S \neq \emptyset$, it follows that S' is an open subanifold of S. Then the ACP implies:

Theorem 3.1. *Let Λ be a Lie algebra of analytic vector fields. Then through every point $x \in M$ there passes a unique maximal integral manifold of Λ.*

We will use $I(\Lambda, x)$ to denote the unique maximal integral manifold whose existence is assured by Theorem 3.1. The above result is due to Hermann [12] and Nagano [20]. We now sketch one method of proof based on the results of [26], in order to make it clear that the only fact about analytic functions being used is the ACP.

It is shown in [26] that, if Λ is any set of smooth vector fields, then the *orbits* of Λ are smooth submanifolds. By definition, the orbit $\mathcal{O}(\Lambda, x)$ of Λ through a point $x \in M$ is the set of all $y \in M$ of the form

$$y = x\mathrm{e}^{t_1 X_1}\mathrm{e}^{t_2 X_2}\ldots\mathrm{e}^{t_m X_m} \ , \tag{3.1}$$

for all choices of m, of the X_i in Λ and the $t_i \in \mathbb{R}$. Moreover, if we define $\Phi^x_{\mathbf{X}}(\mathbf{t})$ to be the right-hand side of (3.1) (with $\mathbf{X} = (X_1, X_2, \ldots, X_m)$, $\mathbf{t} = (t_1, t_2, \ldots, t_m)$), then the tangent space $T_y\mathcal{O}(\Lambda, x)$ of $\mathcal{O}(\Lambda, x)$ at y is spanned by the images of the differentials $d\Phi^x_{\mathbf{Z}}(\boldsymbol{\tau})$, ranging over all choices of $\mathbf{Z}$ and $\boldsymbol{\tau}$ such that $\Phi^x_{\mathbf{Z}}(\boldsymbol{\tau}) = y$. In addition, it is obvious that $\Lambda(y) \subseteq T_y\mathcal{O}(\Lambda, x)$. It is easy to show that every integral manifold of Λ is contained in an orbit. Using all this, it is clear that Theorem 3.1 will follow if we show that, if M is analytic and Λ is a Lie algebra of analytic vector fields, then $\operatorname{Im} d\Phi^x_{\mathbf{X}}(\mathbf{t}) \subseteq \Lambda(\Phi^x_{\mathbf{X}}(\mathbf{t}))$ for each $x \in M$ and each choice of $\mathbf{X}$, $\mathbf{t}$. But $\operatorname{Im} d\Phi^x_{\mathbf{X}}(\mathbf{t})$ is spanned by the vectors $v_k = y_k X_k \mathrm{e}^{t_{k+1}X_{k+1}}\ldots\mathrm{e}^{t_m X_m}$ for $k = 1, \ldots, m$, where $y_k = x\mathrm{e}^{t_1 X_1}\ldots\mathrm{e}^{t_k X_k}$. So all we need is to show that, if $X \in \Lambda$, then $xX\mathrm{e}^{t_1 X_1}\ldots\mathrm{e}^{t_m X_m} \in \Lambda(x\mathrm{e}^{t_1 X_1}\ldots\mathrm{e}^{t_m X_m})$ for all choices of x, m, the X_i and the t_i. This is clearly equivalent to the statement that $v = z\mathrm{e}^{-t_m X_m}\ldots\mathrm{e}^{-t_1 X_1}X\mathrm{e}^{t_1 X_1}\ldots\mathrm{e}^{t_m X_m} \in \Lambda(z)$ for all z, m, X_i, t_i. Clearly, $v \in T_z M$ and, if we fix z, m and the X_i, then v is an analytic function of the t_i whose domain is a connected open subset of $\mathbb{R}^m$. Moreover, it follows from (2.1) that

$$\left.\frac{\partial^{\nu_1+\ldots+\nu_m} v}{\partial t_1^{\nu_1}\ldots t_m^{\nu_m}}\right|_{t_1=\ldots=t_m=0} = \mathrm{ad}^{\nu_1}_{X_1}\ldots\mathrm{ad}^{\nu_m}_{X_m}(X) \tag{3.2}$$

for all $\nu_1, \ldots, \nu_m$, where ad_Y is the operator $Z \to [Y, Z]$. So all the partial derivatives of v with respect to the t_i, of all orders, are in $\Lambda(z)$. Because of the ACP, it follows that $v \in \Lambda(z)$, and the proof is complete. ∎

Theorem 3.1 provides some interesting information about properties of reachable sets. To see this, we first quote two results that are valid for smooth systems as well. The first one is the "positive form of Chow's Theorem" (cf. [14] for a proof). Recall that the system Σ is said to satisfy the *Lie algebra full rank condition* (LAFRC) at a point x if $\Lambda_\Sigma(x) = T_x M_\Sigma$. Then we have

Theorem 3.2. *Assume that Σ satisfies the LAFRC at x. Then for every $T > 0$ the interior of the reachable set $\mathcal{R}_\Sigma(x, \leq T)$ is nonvoid.* ∎

The second result is elementary, so we omit the proof.

Theorem 3.3. *If a smooth control system Σ satisfies the PCAP, then every trajectory of Σ is entirely contained in an orbit of V_Σ.* ∎

Remark 3.1. The PACP plays a crucial role in Theorem 3.3. To see this, consider the system $\dot{x} = ux$, $x \in \mathbb{R}$, $u \in \mathbb{R}$, with $\mathcal{U}$ the class of all measurable $\mathbb{R}$-valued functions. Then $x = 1$ is reachable from $x = 0$ using the control $\eta(t) = \frac{1}{2t}$ and corresponding trajectory $\gamma(t) = t^2$, but no trajectory that corresponds to a piecewise constant —or Lebesgue integrable— control and goes through $x = 0$ can ever leave $x = 0$. Therefore Σ does not have the *PCAP*. It is clear that $\{0\}$ is an orbit, but γ is not entirely contained in it. ∎

Combining these results, and using $I(\Sigma, x)$ to denote the integral manifold $I(\Lambda_\Sigma, x)$, and $I(\Sigma, x)$-closure (resp. $I(\Sigma, x)$-interior) to denote relative closure (resp. relative interior) with respect to $I(\Sigma, x)$, one gets immediately the following theorem (cf. [25]):

Theorem 3.4. *For an analytic system Σ, if $x \in M_\Sigma$ and $T > 0$, then the reachable set $\mathcal{R}_\Sigma(x, \leq T)$ is a subset of $I(\Sigma, x)$ and is actually contained in the $I(\Sigma, x)$-closure of its $I(\Sigma, x)$-interior.*

The proof of Theorem 3.4 uses analyticity only once, to conclude that $I(\Sigma, x)$ exists. So Theorem 3.4 is "elementary," in the sense that it only depends on the ACP. However, the theorem already gives interesting control-theoretic information. For instance, it implies in particular that the reachable sets $\mathcal{R}_\Sigma(x, \leq T)$ have integer dimension (using any reasonable dimension theory, e.g. Hausdorff). Also, it shows that the reachable sets do not have "smaller dimensional tails," (i.e. nonvoid relatively open subsets of smaller dimension). The latter conclusion fails for smooth systems, as can be seen by considering the system $\dot{x} = 1$, $\dot{y} = \varphi(x)u$, where $u \in \mathbb{R}$ and φ is a smooth function which vanishes for $x \leq 0$ and is > 0 for $x > 0$. (The reachable set from $(-1,0)$ in time ≤ 2 is the union of the segment $\{(x,0) : -1 \leq x \leq 0\}$ and the strip $\{(x,y) : 0 < x \leq 1\}$.) It is not known whether the former conclusion —on the integrality of the dimension— holds for smooth systems as well.

4. Controllability and minimal realizations .

The theorem on the existence of integral manifolds makes it possible to develop a theory of minimal realizations for nonlinear analytic systems which retains some of the features of linear realization theory. Recall that in the linear theory the key concepts are *controllability* and *observability.* A linear system $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, with output $y = Cx$, is defined to be *minimal* if it is both controllable and observable. One of the fundamental results of the theory is that every linear input-output map arising from a linear system as above can also be obtained from a minimal one, and this *minimal realization* is unique up to an obvious diffeomorphism. To extend the theory to the nonlinear case one has to look for the appropriate extensions to the nonlinear framework of the concepts of controllability and observability, and then seek to prove results on minimal realizations.

For linear systems, controllability can be characterized in several equivalent ways. For instance, one could use the algebraic criterion provided by the Kalman condition (i.e. $\operatorname{rank}(B, AB, A^2B, \ldots, A^{n-1}B) = n$), or the property of *controllability from the origin* (i.e. that the reachable set from the origin is the whole space), or *complete controllability* (i.e. the condition that from every point of the state space it is possible to reach every point), or the *accessibility property* (i.e. the condition that the reachable set from the every point has a nonvoid interior). These four properties are not equivalent in the nonlinear case. When we seek a nonlinear generalization, it is clear that the Kalman condition, as formulated in terms of matrices, is too tied to linearity to even make sense in the nonlinear setting (although, as will be pointed out below, it does have a good nonlinear generalization if properly reformulated in terms of Lie brackets). The second and third properties have an obvious drawback: suppose we start with a very simple nonlinear system such as $\dot{x} = y^2$, $\dot{y} = u$, with an output (say $z = x$). The system is not "controllable" in the sense of one of these conditions. By analogy with linear theory we would like to make it "controllable" by making the state space smaller. This means that we would have to find a subset S of $\mathbb{R}^2$ such that the restriction of our system to S is "controllable," and then replace the original system by its restriction to this subset. We can certainly make the system controllable from zero by taking S to be the reachable set from zero, but this set is no longer a manifold. To pursue the theory along these lines would force us to consider systems on very general state spaces, including at least all reachable sets. This is quite unpleasant, and in addition would make the next step — quotienting out the unobservable part to achieve observability— impossible. If we seek to make our system completely controllable rather than just controllable from zero, then the situation is even worse: there is *no subset* of M to which our system can be restricted and become completely controllable. It turns out, however, that the system can easily be made "controllable" in the sense of the fourth definition, i.e. the accessibility property. In fact, our system has this property, as can be verified by computing Lie brackets and showing that the controllability Lie algebra has dimension two at every point, or directly by studying the reachable sets.

It turns out that, in general, if one wishes to develop a good theory of nonlinear minimal realizations, then *the "right" class of nonlinear systems is that of complete analytic systems, and the accessibility property is the "right" concept of controllability for such systems.* (A system Σ is *complete* if all the vector fields in V_Σ are complete. The first one to single out the accessibility property as a crucial element in the theory was D. L. Elliott; cf., e.g. [9].) We will discuss the first assertion below but first let us present the evidence for the second one. To begin with, it is always true that, if we are given an *initialized analytic system with outputs* (i.e. an analytic system Σ together with a point $\bar{x} \in M_\Sigma$ and an analytic "output" function $\varphi : M_\Sigma \to N$, where N is some other analytic manifold), then one can restrict Σ to a submanifold S of M_Σ so that the new system will have the accessibility property and give rise to the same input-output map as the original one. (Indeed, this can be achieved by just taking S to be the maximal integral manifold of Λ_Σ through $\bar{x}$.) Moreover, once this is done, then it is possible to "quotient out the unobservable part" and achieve observability as well. (Remarkably, the difficulties we encountered when trying to define controllability in the nonlinear setting do not arise for observability. The concept of observability has a very natural nonlinear generalization: we call (Σ, φ) *observable* if, whenever x_1, x_2 are two different initial states, it follows that they are distinguishable. Here we call x_1, x_2 *distinguishable* if there are admissible pairs $\Gamma_i = (\gamma_i, \eta_i)$, $i = 1, 2$, such that $x_-(\gamma_i) = x_i$, $\eta_1 \equiv \eta_2$, but $\varphi(\gamma_1(t)) \neq \varphi(\gamma_2(t))$ for some t.) In addition, the accessibility property, as opposed to the other candidates for a nonlinear analogue of controllability, lends itself to a simple algebraic characterization, because it holds if and only if the LAFRC is satisfied at every point. (Not surprisingly, for linear systems the LAFRC turns out to be exactly the Kalman controllability criterion.)

We now turn to the assertion that analyticity, rather than smoothness, is the natural property of nonlinear systems that is required in order to have a good theory. Having explained that a good theory exists in the analytic case, we must now show why such a theory cannot exist for C^∞ systems. In our outline of the real analytic theory, the first point where analyticity comes in is in the existence of integral manifolds of Λ_Σ. This might appear not to be a serious obstacle, because in the C^∞ case something quite similar to the integral manifolds still exists, namely, the orbits. Notice, however, that it is not true that, if we restrict a smooth system to one of its orbits, then the resulting restriction has the accessibility property. This objection might be countered by trying a new definition of controllability. Define a system Σ to be *orbit-minimal* if the state space M_Σ is itself an orbit of Σ. (This is equivalent to requiring complete controllability with piecewise constant controls of the *symmetrized system* $\Sigma^{\rm sym}$, i.e. the system arising from the polysystem $\{f_u : u \in U_\Sigma\} \cup \{-f_u : u \in U_\Sigma\}$.) We could then remark that the properties of accesibility and orbit-minimality are equivalent for analytic systems, and seek to extend the theory to smooth systems by using the latter property to be our nonlinear analogue of controllability. If we do this, then we have a way of restrciting an arbitrary smooth intialized system $(\Sigma, \bar{x})$ to a submanifold of M_Σ to make it "controllable," but *it becomes impossible to carry out the other two steps.*

To see why this is so, consider first the step of "quotienting out the unobservable part." The precise meaning of this is that we define an equivalence relation $\sim$ by declaring x_1 and x_2 to be equivalent if they are not distinguishable, i.e. if for every input both states give rise to the same output. One then needs to know that $\sim$ is "regular," in the sense that the quotient $M_\Sigma/\sim$ is still a manifold. As stated before, this is true in the analytic case. However, in the C^∞ case $\sim$ *may fail to be regular.* For instance, if $\varphi : \mathbb{R} \to \mathbb{R}$ is smooth, $\varphi(x) = 0$ for $x \geq 0$, and $\varphi'(x) < 0$ for $x < 0$, and we consider the system $\dot{x} = 1$ with output φ, then $x_1 \sim x_2$ iff $x_1 = x_2$ or x_1 and x_2 are both ≥ 0.

In the analytic case, $\sim$ is regular because of the main theorem of [27], according to which an equivalence relation $\sim$ on a manifold M is regular if (a) $\sim$ is closed as a subset of $M \times M$, and (b) $\sim$ admits a sufficiently large class V of "symmetry vector fields." We recall that a *symmetry vector field* for $\sim$ is a vector field X such that, if $x \sim \tilde{x}$, then it follows that $xe^{tX} \sim \tilde{x}e^{tX}$ for all t for which both sides are defined. "Sufficiently large" means that $V(x) = T_xM$ for all x. Notice that X is a symmetry vector field if and only if, whenever $p = (x, \tilde{x}) \in \sim$ —where $\sim$ is regarded as a subset of $M \times M$— then it follows that the integral trajectory of $X \oplus X$ through p is entirely contained in $\sim$. This shows that $\sim$ is a union of orbits of the set $\mathbf{S}$ of all $X \oplus X$ such that X is a symmetry vector field. It then follows that, if L_{sym} is the Lie algebra of vector fields generated by the symmetry vector fields, then every member of L_{sym} is a symmetry vector field as well. If Σ is analytic and has the accessibility property, the existence of a sufficiently large class of symmetry vector fields will follow if we prove that every f_u is a symmetry vector field. Now, it is easy to see from the definition of indistinguishablity that, if $x \sim \tilde{x}$, then $xe^{tf_u} \sim \tilde{x}e^{tf_u}$ for all $t \geq 0$. Using analyticity one can then conclude that the same is true for $t < 0$, and therefore f_u is indeed a symmetry vector field.

Finally, we claim that the third step, i.e. the equivalence of minimal realizations, also depends on analyticity. It is easy to show by means of examples that equivalence can fail in the smooth case. To understand why it works in the analytic case, we simply observe that the input-output map only depends on the control system and the output function *on the reachable set from the initial state*. Hence it will be impossible to show that two systems are equivalent using only the equality of the input-output maps, unless one can somehow infer, when the systems coincide on the reachable sets from the initial states, that they coincide everywhere. This can be done under the hypothesis of analyticity, because in that case the reachable sets have novoid interior —due to accessibility— and then the vector fields f_u and the output map φ are determined everywhere —because of analytic continuation— once they are known on the reachable sets.

For a detailed account of the existence and uniqueness results, the reader is referred to [28].

5. The Weierstrass Preparation Theorem and its consequences .

Real analytic functions of one variable "behave locally like polynomials." This can be made precise in several ways, e.g by observing that, if φ is analytic, then for y near a point x, $\varphi(y)$ is equal to $p(y)u(y)$, where $p(y)$ is a polynomial and u is an analytic function such that $u(x) \neq 0$. The higher dimensional generalization of this fact is the *Weierstrass Preparation Theorem* (WPT): if φ is analytic real-valued on an open set $\Omega \subseteq \mathbb{R}^n$, and

$\bar{x} \in \Omega$, then, after making a linear change of coordinates, one can write $\varphi(x) = p(x)u(x)$ on some neigborhood of $\bar{x}$, where u is a analytic function such that $u(\bar{x}) \neq 0$ and p is a *distinguished polynomial*, i.e. $p(x) = (x_1 - \bar{x}_1)^m + \sum_{i=1}^m p_i(x^*)(x_1 - \bar{x}_1)^{m-i}$, where $x^* \stackrel{\text{def}}{=} (x_2, \ldots, x_m)$, and the p_j are analytic functions near $\bar{x}^* = (\bar{x}_2, \ldots, \bar{x}_m)$ that vanish at $\bar{x}^*$.

An important consequence of the WPT is the *Noetherian property* of the ring of germs of analytic functions. Precisely, let $\mathcal{O}_x$ denote, for $x \in \mathbb{R}^n$, the set of germs at x of analytic functions. (By definition, a member of $\mathcal{O}_x$ is an equivalence class of real-valued analytic functions φ defined on some neighborhood N_φ of x, the equivalence relation being the one that identifies two such functions φ, ψ, if there some neighborhood N of x such that $N \subseteq N_\varphi \cap N_\psi$ and $\varphi \equiv \psi$ on N.) Then it can be proved, using the WPT, that *the ring $\mathcal{O}_x$ is Noetherian* (i.e. every ideal is finitely generated).

The Noetherian property of the rings of germs implies theorems about the number of zeros of certain functions, which in turn have control theoretic consequences. We illustrate this by briefly describing a bang-bang theorem proved in [31]. (Several other theorems giving bounds on the number of switchings have been proved, also making heavy use of analyticity, cf. e.g. [34], [36], [37], [42], [43], [44].)

Consider a system $\dot{x} = f(x) + ug(x)$, $-1 \leq u \leq 1$, with f and g analytic. A natural condition one can impose in order to guarantee that all time-optimal trajectories will be bang-bang is the following (cf. [31] for an explanation of why the condition is natural):

(C) for every x, k there is a neighborhood N of x and analytic functions α_{ik} for $i = 0, \ldots, k+1$, such that $[g, \mathrm{ad}_f^k(g)] = \sum_{i=0}^{k+1} \alpha_{ik}\, \mathrm{ad}_f^i(g)$ on N, and $|\alpha_{k,k+1}(x)| < 1$.

If (C) holds, then it was proved in [31] that, whenever there is a control η that steers a point x time-optimally to a point y, then there is an η' that steers x to y time-optimally and is bang-bang with at most ν switchings, where ν can be chosen to be a fixed number as long as $T = T_+(\eta) - T_-(\eta)$ stays bounded and the trajectory γ that corresponds to η and starts at x stays in a fixed compact set. The proof of this result is rather technical, and we shall limit ourselves here to describing the role of analyticity. If (γ, η) is time-optimal, let $t \to \lambda(t)$ be an adjoint vector, as specified by the Pontryagin Maximum Principle(cf. [21]). The switching function $\varphi(t)$ is then given by $\varphi(t) =$

$\langle\lambda(t), g(\gamma(t))\rangle$. Differentiation yields $\dot{\varphi}(t) = \langle\lambda(t), [f,g](\gamma(t))\rangle$. Differentiating again we get $\ddot{\varphi}(t) = \langle\lambda(t), [f,[f,g]](\gamma(t))\rangle + \eta(t)\langle\lambda(t), [g,[f,g]](\gamma(t))\rangle$. If we are in a neighborhood N of some point z where the conclusion of Condition (C) holds, then we can write $[g,[f,g]] = \alpha_{01}g + \alpha_{11}[f,g] + \alpha_{21}[f,[f,g]]$, and express $\ddot{\varphi}(t)$ as a linear combination of $\langle\lambda(t), g(\gamma(t))\rangle$, $\langle\lambda(t), [f,g](\gamma(t))\rangle$ and $\langle\lambda(t), [g,[f,g]](\gamma(t))\rangle$, the coefficients being $\eta(t)\alpha_{01}(t)$, $\eta(t)\alpha_{11}(t)$ and $1+\eta(t)\alpha_{21}(t)$. If N is chosen small enough so that $|\alpha_{21}| \leq c_2 < 1$ throughout N, then the last coefficient is positive and bounded away from zero. If we let $\varphi_k(t) = \langle\lambda(t), \mathrm{ad}_f^{k-1}(g)(\gamma(t))\rangle$ (so that $\varphi_1 \equiv \varphi$), then we have shown that $\dot{\varphi}_1 = \varphi_2$ and $\dot{\varphi}_2 = \theta_{21}\varphi_1 + \theta_{22}\varphi_2 + \theta_{23}\varphi_3$, where the functions θ_{2i} are measurable and bounded by a fixed constant which does not depend on η, and θ_{23} is also positive and bounded below by a fixed constant. One can differentiate further and write $\dot{\varphi}_k = \sum_{i=1}^{k+1}\theta_{ki}\varphi_i$, with the θ_{ki} measurable and bounded, and $\theta_{k,k+1}$ bounded below. Notice that, in principle, each equation involves a new φ_j, and it appears that the system of differential equations might never close. However, due to the Noetherian property, the system does indeed close: the space of germs at z of analytic vector fields is a finitely generated module over the Noetherian ring $\mathcal{O}_z$. Hence every submodule is finitely generated. If we apply this to the submodule generated by the germs of the vector fields $\mathrm{ad}_f^k(g)$, we see that eventually one of these germs will be a linear combination of the preceding ones. This implies that one of the φ_k's will be a linear combination of the φ_j for $j < k$. So the system of differential equations described above does close after all. Therefore, for some k, we can write $\dot{\Phi}_k(t) = M(t)\Phi_k(t)$, where Φ_k is the vector $(\varphi_1, \ldots, \varphi_k)$, and M is a matrix whose entries m_{ij} vanish whenever $j > i+1$. Moreover, all the m_{ij} are measurable and bounded by a fixed constant, and the $m_{i,i+1}$ are also positive and bounded below by a fixed constant. Notice that one particular case of this situation would be $m_{i,i+1} \equiv 1$ and $m_{ij} \equiv 0$ if $i < k$ and $j \neq i+1$. In that case, if we let $\beta_i = \theta_{ki}$, we see that φ_1 satisfies the k-th order differential equation $\varphi_1^{(k)} = \beta_1\varphi_1 + \beta_2\dot{\varphi}_1 + \ldots + \beta_k\varphi_1^{(k-1)}$, and the functions β_i are measurable and bounded by a fixed constant. In this particular case we can therefore conclude that there is a fixed bound on the number of zeros of φ_1, using well known results on zeros of solutions of linear differential equations. It turns out that the general case is not fundamentally different: we can allow the entries above the main diagonal —i.e. the $m_{i,i+1}$— to be measurable functions not necessarily equal to 1, as long as they are positive, bounded, and bounded below, and we can allow the entries in the main diagonal and below —i.e. the m_{ij} with $j \leq i$— to be nonzero, as long as they stay bounded. (The crucial fact is that the m_{ij} for $j > i+1$ have to vanish.) Under these conditions, we proved in [31] that the number of zeros of φ_1 has to stay bounded, and this of course gives the bound on the number of switchings.

6. Analytic sets and stratifications .

Many finer properties of analytic maps are *finiteness results*. The simplest such result for functions φ of one variable is the fact that, if φ is analytic on an open interval I, then the set $Z(\varphi)$ of zeros of φ is discrete, unless $Z(\varphi) = I$. This follows, for instance, from the observation that φ can be written locally as a product of a polynomial and an invertible function. In higher dimensions, the analogue of this observation is the WPT, and this also has deep consequences about sets of zeros. Roughly speaking, the set of zeros $Z(\varphi)$ of an analytic function must be very special. More precisely, it can be shown that $Z(\varphi)$ has, locally, a finite number of connected components. One can actually prove results that are much stronger, in that they apply to larger classes of sets, and that they give stronger conclusions. To make this precise, define a *semianalytic subset* of an analytic manifold M to be a set $S \subseteq M$ with the property that, if x is any point of M, then there exists a neighborhood N of x and a finite collection $\mathcal{F}$ of real-valued analytic functions on N, such that $S \cap U$ belongs to the Boolean algebra of subsets of N generated by the sets $\{y : f(y) = 0\}$, $\{y : f(y) > 0\}$, for all $f \in \mathcal{F}$. It follows easily from the definition that the class $\mathrm{Sem}\, M$ of semianalytic subsets of M is a Boolean algebra (i.e. is closed under the operations of complementation, finite union, and finite intersection).

It turns out that semianalytic sets have a "very special structure," in the sense that they admit *stratifications*, as we now explain. Define a *stratum* in a smooth manifold M to be a smooth connected embedded submanifold of M. The *frontier* $\mathrm{Fron}\, S$ of a subset S of M is the set $\mathrm{Clos}\, S \cap (M - S)$. A *stratification* in M is a set $\mathcal{S}$ of pairwise disjoint strata, such that (i) $\mathcal{S}$ is locally finite (i.e. for every compact subset K of M, the set $\mathcal{S}_K = \{S : S \in \mathcal{S}\,,\ S \cap K \neq \emptyset\}$ is finite), and (ii) for every $S \in \mathcal{S}$, the frontier of S is a union of members of $\mathcal{S}$ of dimension strictly less than $\dim S$. The *support* $|\mathcal{S}|$ of $\mathcal{S}$ is the union of all members of $\mathcal{S}$. If $|\mathcal{S}| = A$ then we call $\mathcal{S}$ a *stratification of* A. We call $\mathcal{S}$ *analytic* if every stratum is a analytic submanifold of M, and *semianalytic* if every stratum is a semianalytic subset of M.

A stratification $\mathcal{S}$ is *compatible* with a subset A of M if A is a union of members of $\mathcal{S}$, i.e. if $A = \bigcup\{S : S \in \mathcal{S}_A\}$. With this terminology, one of the main theorems on semianalytic sets can be stated simply as follows: if $A \subseteq M$ is semianalytic, then there exists an analytic, semianalytic stratification $\mathcal{S}$ of M which is compatible with A. Actually, a much stronger result is still true: first of all, if one has a finite, or even locally finite collection $\mathcal{A}$ of semianalytic sets, then one can use the same stratification $\mathcal{S}$ for all of them. Second, $\mathcal{S}$ can be chosen to have a number of extra properties, such as the Whitney conditions, and the fact that all the strata are actually "nice." (For instance, one can take $\mathcal{S}$ such that all the strata are diffeomorphic to balls, and one can even construct an $\mathcal{S}$ which is a *triangulation.*) We shall not pursue these details, which can be found, e.g., in the work of Lojasiewicz (cf. [17]), and we limit ourselves to stating the following relatively weaker result:

Theorem 6.1. *If M is an analytic manifold and $\mathcal{A}$ is a locally finite set of semianalytic subsets of M, then there exists an analytic, semianalytic stratification $\mathcal{S}$ of M which is compatible with all the members of $\mathcal{A}$.* ∎

7. Real analyticity and trajectory regularity .

It was stated in §6 that the set $Z(\varphi)$ of zeros of an analytic function is very special. Theorem 6.1 enables us to make this precise in a number of ways. Indeed, the set $Z(\varphi)$ is clearly semianalytic, and so Theorem 6.1 tell us, for instance, that its intersection with every compact semianalytic set has finitely many connected components.

This observation has at least one important consequence, on the conceptual level, pertaining to one of the most fundamental issues in Control Theory, namely, the problem of the *regularity properties of optimal trajectories.* Consider an optimal control problem obtained by the specification of a control system Σ together with a "Lagrangian" L, i.e. a function $L : M_\Sigma \times U \to \mathbb{R}$. Asssume that, for every admissible pair $\Gamma = (\gamma, \eta) \in \mathcal{A}_\Sigma$, the Lebesgue integral $J_{\Sigma,L}(\Gamma) = \int_{T_-(\gamma)}^{T_+(\gamma)} L(\gamma(t), \eta(t))\, dt$ is defined (i.e., if $h(t) = L(\gamma(t), \eta(t))$, then h is measurable and at least one of the functions $h_+ = \max(h, 0)$, $h_- = \max(-h, 0)$ is integrable). Then we call a particular Γ *optimal* if $J_{\Sigma,L}(\Gamma) \leq J_{\Sigma,L}(\Gamma')$ for every admissible Γ' such that $x_-(\Gamma') = x_-(\Gamma)$ and $x_+(\Gamma') = x_+(\Gamma)$. It is known that, under suitable technical conditions, optimal trajectories must satisfy certain special properties, known as *necessary conditions for optimality*, of which the Pontryagin Maximum Principle (cf. [21] or [16]) is the best known example. One may then ask whether these conditions actually impose some nontrivial *a priori* restriction on optimal controls. For instance, one might begin by asking whether, for problems in which U is a reasonably smooth or piecewise smooooth set (e.g. a closed cube in $\mathbb{R}^m$), and f and L are smooth in x and u, it follows that, if $\Gamma = (\gamma, \eta)$ is optimal, then η is necessarily piecewise smooth. The answer to this question is "no," as shown, e.g., the well known example of "Fuller's problem" (cf. [19], [15]).

But one may then ask whether the answer might be "yes" if less regularity is required. For instance, does optimality imply that the set of points of discontinuity of η is necessarily countable? Or perhaps it may be uncountable but it has to be of measure zero?

It is clear that the answer to the above questions, as formulated, is "no" for a very trivial reason, namely, that there are "degenerate" systems for which *every* admissible pair is optimal. (This happens, e.g., if $L \equiv 0$, or if we are dealing with a minimum time problem —i.e. $L \equiv 1$— but one of the coordinates —say x_1— satisfies an equation $\dot{x}_1 = 1$.) In order to rule out such situations, we reformulate the question in terms of the possibility of *choosing* a special η. Given a class $\mathcal{P}$ of optimal control problems (Σ, L), all of them having the same control space U and the same class $\mathcal{U}$ of admissible controls, we call a subset $\mathcal{U}'$ of $\mathcal{U}$ *sufficient* for $\mathcal{P}$ if, whenever $(\Sigma, L) \in \mathcal{P}$ and $\Gamma = (\gamma, \eta)$ is optimal for (Σ, L), then there exists an admissible $\Gamma' = (\gamma', \eta')$ such that $x_-(\Gamma') = x_-(\Gamma)$, $x_+(\Gamma') = x_+(\Gamma)$, $J_{\Sigma,L}(\Gamma') = J_{\Sigma,L}(\Gamma)$, and $\eta' \in \mathcal{U}'$.

We can, for instance, take U to be the unit cube in $\mathbb{R}^m$, and $\mathcal{U}$ to be the class of all measurable U-valued functions defined on some compact subinterval of $\mathbb{R}$, and define $\mathcal{P}^\infty$ (resp. $\mathcal{P}^\omega$) to be the class of all (Σ, L) such that M_Σ is a smooth (resp. analytic) manifold and f, L are smooth (resp. analytic) as functions of x and u. We then ask whether there exists *some* class $\mathcal{U}'$ which is a *proper subset* of $\mathcal{U}$ and is sufficient for $\mathcal{P}^\infty$ or for $\mathcal{P}^\omega$.

It turns out that *the answer is "no" for $\mathcal{P}^\infty$ but "yes" for $\mathcal{P}^\omega$.* We will prove the first assertion, because the proof is quite simple and shows very clearly why the realm of C^∞ systems allows pathology that is definitely excluded in the analytic case. To establish the first assertion, it suffices to show that, if an arbitrary measurable function $\eta : [a, b] \to U$ is given, then we can construct a $(\Sigma, L) \in \mathcal{P}^\infty$, and states $x_\pm \in M_\Sigma$ such that η optimally steers x_- to x_+ but no other control does. We will take Σ to be of the special form $\dot{x}_0 = 1$, $\dot{x}_1 = u_1$, ..., $\dot{x}_m = u_m$, $\dot{x}_{m+1} = \varphi(x_0, x_1, \ldots, x_m)$, where φ is a smooth function to be chosen in a special way. We will select φ so that there are states $x_\pm$ such that η steers x_- to x_+ and no other control does. One can then choose L arbitrarily (e.g. $L \equiv 0$, or $L \equiv 1$), and the desired conclusion will follow.

In order to choose φ, we first define $\zeta : [a,b] \to \mathbb{R}^m$ by $\zeta(t) = \int_a^t \eta(s)\,ds$, and we let $K = \{(x_0, \zeta(x_0)) : a \le x_0 \le b\}$. Then ζ is a continuous function on $[a,b]$, and K is its graph, so K is a compact subset of $\mathbb{R}^{m+1}$. We then use the fact that, given any closed subset C of a smooth manifold M, then there exists a smooth function on M whose set of zeros is exactly C. Clearly, this function can be taken to be > 0 outside C. We apply this with $C = K$, $M = \mathbb{R}^{m+1}$, thereby obtaining our desired φ. We then take $x_- = (a, 0, \ldots, 0)$, $x_+ = (b, \zeta(b), 0)$. It is then easily verified that a control η' steers x_- to x_+ iff $\eta' = \eta$ a.e.

The preceding construction works *because every closed subset of a Euclidean space is the zero set of some smooth function.* As explained above, a similar conclusion is not true for analytic functions, since the zero sets of such functions, besides being closed, are very special. Naturally, this does not yet prove the existence of nontrivial special properties of optimal controls for analytic problems (i.e. the existence of nontrivial classes $\mathcal{U}'$ that are sufficient for $\mathcal{P}^\omega$) but it strongly supports the conjecture that such properties might exist. This suggestion has actually turned out to be a true theorem, as discussed below.

8. Subanalytic sets, reachable sets and synthesis .

The theory of semianalytic sets and their associated stratifications turns out to lack one important ingredient. This is most easily seen by considering the analogous theory of *semialgebraic sets.* Recall that a semialgebraic subset of $\mathbb{R}^n$ is a set which belongs to the Boolean algebra generated by the sets $\{x : p(x) = 0\}$ and $\{x : p(x) > 0\}$, where $p : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary polynomial function. It is then more or less clear that the theories of semialgebraic and semianalytic sets are somewhat similar, the latter being a kind of “local” analogue of the former, in which polynomials are replaced by analytic germs. Since analytic germs are in some sense like polynomials —e.g. thanks to the WPT— it is natural to expect that the two theories will be similar, except possibly for the fact that theorems that hold globally for semialgebraic sets will hold locally for semianalytic sets. This is indeed true to some extent. For instance, there is a *stratification theorem*, according to which, if $\mathcal{A}$ is a finite family of semialgebraic subsets of $\mathbb{R}^n$, then there exists a *finite* stratification $\mathcal{S}$ of $\mathbb{R}^n$ by semialgebraic subsets which are analytic submanifolds, such that $\mathcal{S}$ is compatible with every member of $\mathcal{A}$. It is clear that Theorem 6.1 is, in a very precise sense, a local analogue of this result.

There is, however, one very important property of semialgebraic sets whose semianalytic analogue is false. The *Tarski-Seidenberg Theorem* says that, if $P : \mathbb{R}^n \to \mathbb{R}^m$ is a polynomial map, and $S \subseteq \mathbb{R}^n$ is semialgebraic, then $P(S)$ is semialgebraic in $\mathbb{R}^m$. The "local semianalytic" analogue of this would be the statement that, if $F : M \to N$ is a proper analytic map between analytic manifolds, then the image under F of a semianalytic subset S of M is semianalytic in N. (The restriction to proper maps is a natural one, because one is seeking a local theory. Moreover, if we do not impose such a restriction the results fails for trivial reasons, as seen, e.g., by taking $F : \mathbb{R}^2 \to \mathbb{R}$ to be the projection onto the first coordinate, and $S = \{(1/n, n) : n = 1, 2, \ldots\}$.) Unfortunately, this statement is false. However, it can be made true by enlarging the class of semianalytic sets and defining a new type of sets, called *subanalytic* (cf. [10], [11], [13], [49]). By definition, a subanalytic subset of a analytic manifold M is a subset S such that there exist a analytic manifold N, a proper analytic map $F : N \to M$, and a semianalytic subset S' of N, such that $F(S') = S$. It is then obvious from this definition that the image of a subanalytic set under a proper analytic map is subanalytic, and that every semianalytic set is subanalytic. It is slightly less obvious, but true and easy to prove, that the class of subanalytic sets is closed under finite unions, finite intersections, and inverse images by analytic maps. It is a lot less obvious, but still true, that the complement of a subanalytic set is subanalytic. Finally, there is a stratification theorem for subanalytic sets which is identical to Theorem 6.1, except only that the word "semianalytic" has to be replaced throughout by "subanalytic."

This suggests that the class of subanalytic sets perhaps qualifies better than that of semianalytic sets as the analytic analogue of the class of polynomials. Moreover, we claim that *the property of being closed under images is precisely what is needed in many applications in control theory*, and therefore the class of subanalytic sets is bound to play a fundamental role in control.

To see why this should be so, let us first observe that *subanalytic sets are, by definition, images, whereas reachable sets are also images.* (Assuming global existence and uniqueness of trajectories, the reachable set in time let $\leq T$ from a point x is the image of the set $\mathcal{U}^T$ of all admissible controls whose domain is $[0, t]$ for some $t \in [0, T]$ under the "terminal state map" X_+^x which assigns to each control η the terminal point $x_+(\gamma^{\eta,x})$, where $\gamma^{\eta,x}$ is the trajectory for η such that $\gamma(0) = x$.) This suggests that reachable sets might be subanalytic, in which case we could apply the stratification theorems to derive nontrivial conclusions about their structure. Perhaps more importantly, one ought to be able to get conclusions about the regularity of the value function for optimal control problems. Indeed, given an optimal control problem (Σ, L), we can consider the *augmented control system* Σ^L obtained from Σ by adding a new scalar variable y and the equation $\dot{y} = -L(x, u)$. It is clear that (x, y) can be driven to $(x', 0)$ by means of a trajectory of Σ^L iff x can be driven to x' by a trajectory of Σ with cost y. In particular, if we use $\mathcal{C}_\Sigma(x)$ to denote, for a control system Σ and a point $x \in M_\Sigma$, the set of all points that can be driven to x, then the graph G of the value function for (Σ, L) with target point x is the "lower boundary" $\partial_- \mathcal{C}_{\Sigma_L}(x, 0) = \{(z, y) : y = \inf\{y' : (z, y') \in \mathcal{C}_{\Sigma_L}(x, 0)\}\}$. Since $\mathcal{C}_{\Sigma_L}(x, 0)$ is a reachable set (the reachable set from $(x, 0)$ for the "backward" system whose trajectories are those of Σ^L run backwards), we see that subanalyticity theorems for reachable sets are likely to imply results on the structure of value functions.

Unfortunately, the observation that both subanalytic sets and reachable sets are images is no more than a very vague analogy, from which we have no right to conclude anything. While it may be true that both kinds of sets are images, they are images of very different things. In one case they are images of semianalytic sets, i.e. in particular of finite-dimensional sets, whereas in the other case they are images of infinite-dimensional function spaces. So there appears to be no serious reason to expect that the theory of subanalytic sets will apply to the study of reachable sets.

This is a very valid objection, but it can be countered with the following argument, which is still far from rigorous but represents a definite improvement over our initial observation. If we wish to understand the structure of reachable sets, then what matters is their boundary. And the trajectories that go from a point x to the boundary of the reachable set from x have to satisfy a necessary condition, as specified by the Prontryagin Maximum Principle. This condition says that there exists an adjoint vector with certain properties. The important thing for us is that the system of equations for the state vector x and the adjoint vector λ is a system of ordinary differential equations of the usual kind, with existence and uniqueness of solutions, except for the fact that the right-hand side involves the control u, which is to be chosen as a function of x and λ so as to minimize the Hamiltonian. The minimizing value of u need not be unique and, even when it is unique, the vector field in (x, λ) space that results from plugging it in need not be locally Lipschitz, or even continuous, so existence and uniqueness of solutions is not guaranteed. If we ignore this last problem for the time being, we see that there are serious reasons to suspect that the boundary of the reachable set might be, after all, an image of a finite-dimensional set, namely, the set of all pairs $(\lambda(0), t)$.

If we want to turn the above remarks into a rigorous theory, it is now clear that what needs to be done is to study the question of *finite-dimensional analytic reduction* (FDAR) of a control system, i.e. the question whether we can find a finite dimensional analytic manifold A_0, a subanalytic subset A of A_0, and a family $\{\gamma_\alpha : \alpha \in A\}$ of trajectories, such that (a) whenever it is possible to go from a point x to a point x' by *some* trajectory γ of Σ, then this can actually be done by means of one of the γ_α, and (b) the maps $X_- : \alpha \to x_-(\gamma_\alpha)$ and $X_+ : \alpha \to x_+(\gamma_\alpha)$ are analytic, i.e. —to be completely precise— have analytic extensions to some open set containing the closure of A_0. (If in addition the γ_α of (a) can be chosen so that $T_+(\gamma_\alpha) - T_-(\gamma_\alpha) = T_+(\gamma) - T_-(\gamma)$, then we will say that we have an *equal-time* FDAR.)

If such a family can be found, then it is clear that the reachable set from any point x is subanalytic, provided only that an extra properness condition holds. Indeed, the reachable set from x is just the image under X_+ of the set $\{\alpha \in A_0 : X_-(\alpha) = x\}$, which is clearly subanalytic in A. Therefore the reachable set will be subanalytic if X_+ is proper, or even under the weaker condition that the map $\alpha \to (X_-(\alpha), X_+(\alpha))$ is proper. Similarly, if we just consider the reachable set up to time T, then this set will be subanalytic if our FDAR is equal-time and the time function $\tau : \alpha \to T_+(\gamma_\alpha) - T_-(\gamma_\alpha)$ is analytic, provided —as before— that a properness condition holds. Indeed, the set $\mathcal{R}_\Sigma(x, \leq T)$ is just the image under X_+ of $\{\alpha : X_-(\alpha) = x\, , \tau(\alpha) \leq T\}$, so $\mathcal{R}_\Sigma(x, \leq T)$ is subanalytic provided that the map $\alpha \to (X_-(\alpha), X_+(\alpha), \tau(\alpha))$ is proper.

To see how this can actually be applied, we now outline, using our terminology, some of the results of Brunovsky [7], who was the first to notice that the theory of subanalytic sets was bound to play an important role in control theory. We consider the simple example of a linear control system $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, with a control constraint $u \in K$, where K is a *polyhedron*, i.e. the convex hull of a finite set $\mathcal{K}$ of points. It is then easy to prove a *bang-bang theorem with a bound on the number of switchings*, according to which there exist $c_1 > 0$, $c_2 > 0$ with the property that, whenever it is possible to go from a point x to another point y in time T, then it is also possible to go from x to y in time T by means of a bang-bang trajectory with at most $c_1 + c_2 T$ switchings. Using this, we take our manifold A to have connected components A^σ, where σ ranges over the set of all *bang-bang* *control strategies*, i.e. all finite sequences $\sigma = (u_1, \ldots, u_r)$ (of arbitrary length $r \stackrel{\text{def}}{=} |\sigma|$) of points of $\mathcal{K}$. For each σ, A^σ is just a copy of $\mathbb{R}^n \times \mathbb{R}^{|\sigma|}$. (The resulting manifold is not of pure dimension, and has connected components of arbitrarily high dimension but, as explained in §2, this is permitted.) The set A_0 is the union of sets $A_0^\sigma \subseteq A^\sigma$, where $A_0^\sigma = \{\alpha = (x, t_1, \ldots, t_{|\sigma|}) : x \in \mathbb{R}^n\,,\, t_1 \geq 0\,,\, \ldots,\,,\, t_{|\sigma|} \geq 0\,,\, 1 + c_1 + c_2 T \geq |\sigma|\}$. For each $\sigma = (u_1, \ldots, u_r)$, $\alpha = (x, t_1, \ldots, t_r) \in A_0^\sigma$ we let γ_α be the bang-bang trajectory that starts at x at time 0, then is a trajectory corresponding to the constant control u_1 during time t_1, then a trajectory for the control u_2 during time t_2, and so on. This is indeed an equal-time FDAR of our control system, due to the bound on the number of switchings. The properness condition follows because, if x and T stay bounded —say $x \in J$, with J compact, and $0 \leq T \leq \bar{T}$— then $\{\alpha \in A_0^\sigma : X_-(\alpha) \in J\,,\, \tau(\alpha) \leq \bar{T}\}$ is obviously compact for each σ *and in addition is void for all but finitely many* σ (precisely, for all σ such that $|\sigma| > 1 + c_1 + c_2\bar{T}$). Notice that, if we had not included the requirement that $1 + c_1 + c_2 T \geq |\sigma|$ in the definition of A_0^σ, we would still have obtained an equal time FDAR, but the properness would have failed. This shows that the role of the bound on the number of switchings is to give the necessary properness.

The above construction shows that, for a linear system with a polyhedral control constraint as above, the reachable sets $\mathcal{R}_\Sigma(x, \leq T)$ are compact and subanalytic. The stratification theorem then implies that these sets are finite unions of pairwise disjoint connected embedded analytic submanifolds.

The time-optimal control problem for linear systems with a polyhedral control constraint can be studied by a similar method. This results in the proof of the existence, for such systems, of a "regular synthesis." (The precise definition of "regular synthesis" needed here is a modification of the one given by Boltyanski in [1], and can be found, e.g. in Brunovsky's papers [7], [8], or in [38] or [39]. For other results on synthesis, also making heavy use of real analyticity, cf. [30], [33], [36], [37], [42], [43], [44].)

The previous result for linear systems is of course very special, but it is clear that it only depends on some features of the linear problem, notably the bang-bang theorem with bounds on the number of switchings. Moreover, the fact that the controls used are bang-bang is not really very important: the same result will clearly follow if singular arcs are also allowed, provided that these arcs are analytically parametrized by a finite-dimensional parameter. This has led to renewed interest in a question which, in any case, ought to have been regarded from the very beginning as one of the most basic problems in optimal control, namely, that of the structure of optimal trajectories, and in particular under what conditions such trajectories can be proved to be finite concatenations of bang-bang and singular arcs, with bounds on the number of switchings. (For recent results on the structure of trajectories, cf. [2], [3], [4], [5], [6], [22], [23], [45], [47], [51], [52].)

9. Some results on optimal trajectories .

Motivated by the relationship with the problem of the structure of reachable sets and the properties of the value functions, we now return to the question of the structure of optimal trajectories, which was briefly touched upon in §5. We showed in §5 that, for general smooth systems, one cannot derive nontrivial regularity properties for optimal controls. The situation is quite different for analytic systems, as shown by the following theorem:

Theorem 9.1. *Consider a system* $\dot{x} = f(x,u)$, $x \in M$, $u \in U$, *where* M *is an analytic manifold,* U *is a compact subanalytic subset of an analytic manifold* N, *and* f *is analytic as function of* x *and* u. *Assume that* x, y *are points of* M *such that there is a trajectory* γ *corresponding to some measurable control* $\eta : [a,b] \to U$, *that goes from* x *to* y. *Then there is a trajectory from* x *to* y *that corresponds to a control* η' *that is analytic on an open dense subset of the interval* $[T_-(\eta'), T_+(\eta')]$.

This result is proved in [46] for systems $\dot{x} = f(x) + ug(x)$, $-1 \leq u \leq 1$, and in [48] for the general case. (Cf. also [50] for a stronger result for bang-bang trajectories.) As is to be expected —since the result is *not* true for smooth systems— the proof makes heavy use of properties of analytic maps and associated sets. The proof for the case considered in [46] uses stratifications, and the proof of the general case requires even deeper tools, namely *desingularization* (cf. below).

It is a very interesting open problem to decide to what extent the very weak regularity result of Theorem 9.1 can be improved. For instance, is it possible to improve it by having "open dense subset of full measure" instead of just "open dense subset"? Can one do even better and obtain an open subset whose complement is *countable*? Ideally, one would want to obtain controls with finitely many switchings, but this is known to be impossible, as shown by Fuller's example.

10. Desingularization and its applications .

H. Hironaka's *resolution of singularities theorem* (cf. [13], and [49] for an elementary account) says that, if M is a connected analytic manifold and $\varphi : M \to \mathbb{R}$ is an analytic function which does not vanish identically, then φ admits a *desingularization*, i.e. there exists an analytic manifold N and a proper, surjective, analytic map $\Phi : N \to M$ which is an analytic diffeomorphism on an open dense subset Ω of N, and is such that the composite function $\varphi\Phi$ is *locally monomial*, i.e. for every point $z \in N$ there is an analytic coordinate chart $(x_1, \ldots, x_n)$ on a neighborhood Z of z such that, on Z, $\varphi\Phi$ is given by $\varphi\Phi(x) = H(x)x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$, where the a_i are nonnegative integers and $H(x)$ nowhere vanishes.

One of the consequences of the resolution of singularities theorem is the *desingularization of subanalytic sets.* A subanalytic subset of M is a locally finite union of sets $\psi_j(C_j)$, where the C_j are open cubes in Euclidean spaces $\mathbb{R}^{n_j}$, and each ψ_j is an analytic map on a neigborhood Z_j of the closure of C_j. In particular, it follows easily that a compact subanalytic set is the image of a compact manifold under an analytic map.

The desingularization theorems have had at least three applications to significant control theory problems. First, desingularization plays a crucial role in the proof of Theorem 9.1. Second, the theory has been used to prove a regularity theorem for linear time-optimal control problems with a non-polyhedral control constraint. The results of [18] showed that, even for a very simple control set, such as the set $\{(u_1, u_2) : u_2^2 - 1 \leq u_1 \leq 1 - u_2^2\}$, it is possible for the reachable sets $\mathcal{R}_\Sigma(x, \leq T)$ to fail to be subanalytic, and for the value function for the minimum time problem not to be subanalytic either. However, in [40] it was shown that, if the control set U is compact, convex, and subanalytic, then the value function is analytic on the complement of a finite or countable union of connected analytic submanifolds of positive codimension.

Finally, the third application is to the theory of synthesis. It is well known that, if a family of Pontryagin extremals $\{(\gamma_x, \eta_x) : x \in \Omega\}$, with $x_(\gamma_x) = x$, is "sufficiently smooth" in some sense, and satisfies a normality condition, then the gradient at a point $\bar{x}$ of the cost function $V : x \to J(\gamma_x, \eta_x)$ is equal to the value at $T_-(\gamma_x)$ of the adjoint variable $t \to \lambda_x(t)$ whose existence is asserted by the Pontryagin Maximum Principle. This fact plays a crucial role in the proof of sufficiency theorems for optimality, as shown, e.g., in [1]. It is an important question to find good conditions under which the existence of a family of optimal trajectories with the appropriate smoothness can be guaranteed. It turns out that, using desingularization, one can prove the existence of a family that has the desired properties on a reasonably large set, provided the optimal control problem admits a FDAR which satisfies some extra technical conditions. (For a more detailed account, cf. [53]).

11. Other applications of analyticity .

We conclude by briefly listing some other developments where analyticity has played an important role:

1. the existence of universal inputs was proved in [32] using stratification theory,
2. the convergencew of the Chen series for analytic systems has applications in local controllability theory (cf., e.g., [35], [41]),
3. the existence of piecewise analytic stabilizing feedback controls for completely controllable systems was established in [29] using subanalytic sets,
4. a complete analysis of the structure of time-optimal trajectories for analytic systems $\dot{x} = f(x) + ug(x)$, $|u| \leq 1$ in the plane was carried out in [42], [43], [44].

REFERENCES

[1] Boltyansky, V.G., "Sufficient conditions for optimality and the justification of the Dynamic Programming Principle," *S.I.A.M. J. Control* **4** (1966), pp. 326-361.

[2] Bressan, A., "A high-order test for optimality of bang-bang controls," *S.I.A.M. J. Control Opt.* **23** (1985), pp. 38-48.

[3] Bressan, A., "The generic local time-optimal stabilizing controls in dimension 3," *S.I.A.M. J. Control Opt.* **24** (1986), pp. 177-190.

[4] Bressan, A., "Local asymptotic approximation of nonlinear control systems," *Int. J. Control*, 41: 1331-1336 (1985).

[5] Bressan, A., "Directional convexity and finite optimality conditions," *J. Math. Anal. Appl.*, 80: 102-129 (1981).

[6] Bressan, A., "Dual variational methods in optimal control theory," to appear in *Finite Dimensional Controllability and Optimal Control*, H.J. Sussmann ed., to be published by M. Dekker, Inc.

[7] Brunovsky, P., "Every normal linear system has a regular synthesis," *Mathematica Slovaca*, **28** (1978), pp. 81-100.

[8] Brunovsky, P., "Existence of regular synthesis for general problems," *J. Diff. Equations*, **38** (1980), pp. 317-343.

[9] Elliott, D. L., "A consequence of controllability," J. Differential Equations **10** (1971), pp. 364-370.

[10] Gabrielov, A., "Projections of semianalytic sets," *Funct. Anal. Appl.* **2**, No. 4 (1968), pp. 282-291.

[11] Hardt, R., "Stratifications of real analytic maps and images," *Inventiones Math.* **28** (1975), pp. 193-208.

[12] Hermann, R., "On the accessibility problem in control theory," in *International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics*, J.P. LaSalle and S. Lefschetz Eds., Academic Press, New York (1963).

[13] Hironaka, H., *Subanalytic Sets*, Lect. Notes Istituto Matematico "Leonida Tonelli," Pisa, Italy (1973).

[14] Krener, A. J., "A generalization of Chow's Theorem and the bang-bang principle," *S.I.A.M. J. Control* **12** (1974), pp. 43-52.

[15] Kupka, I., "The ubiquity of Fuller's phenomenon," to appear in *Finite Dimensional Controllability and Optimal Control*, H.J. Sussmann ed., M. Dekker, Inc.

[16] Lee, E. B., and L. Markus, *Foundations of Optimal Control Theory*, J. Wiley, New York, 1967.

[17] Lojasiewicz, S., *Ensembles semianalytiques*, Notes Inst. Hautes Études, Bures-sur-Yvette (1965).

[18] Lojasiewicz Jr., S. and H. J. Sussmann, "Some examples of reachable sets and optimal cost functions that fail to be subanalytic," *S.I.A.M. J. Control and Optimization* **23**, No. 4 (1985), pp. 584-598.

[19] Marchal, C., "Chattering arcs and chattering controls," *J. Optim Theory Appl.* **11** (1973), pp. 441-468.

[20] Nagano, T., "Linear differential systems with singularities and an application to transitive Lie algebras," *J. Math. Soc. Japan* 18: 398-404 (1966).

[21] Pontryagin, L.S., V.G. Boltyansky, R.V. Gamkrelidze and E.F. Mischenko, *The Mathematical Theory of Optimal Processes*, J. Wiley (1962).

[22] Schättler, H., "On the local structure of time-optimal bang-bang trajectories in $\mathbb{R}^3$," *S.I.A.M. J. Control Opt.* **26** (1988), pp. 186-204.

[23] Schättler, H., "The local structure of time-optimal trajectories in $\mathbb{R}^3$ under generic conditions," *S.I.A.M. J. Control Opt.* **26** (1988), pp. 899-918.

[24] Schättler, H., and H.J. Sussmann, "On the regularity of optimal controls," *J. Appl. Math. Physics (ZAMP)* **38** (1987), pp. 292-301.

[25] Sussmann, H. J., and V. Jurdjevic, "Controllability of nonlinear systems," *J. Diff. Equations* **12** (1972), pp. 95-116.

[26] Sussmann, H. J., "Orbits of families of vector fields and integrability of distributions," *Trans. Amer. Math Soc.* **180** (1973), pp. 171-188.

[27] Sussmann, H. J., "A generalization of the closed subgroup theorem to quotients of arbitrary manifolds," *J. Diff. Geometry* **10** (1975), pp. 151-166.

[28] Sussmann, H. J., "Existence and Uniqueness of minimal realizations of nonlinear systems," *Math. Systems Theory* **10** (1977), pp. 263-284.

[29] Sussmann, H. J., "Subanalytic sets and feedback control," *J. Diff. Equations* **31**, No.1 (1979), pp. 31-52.

[30] Sussmann, H. J., "Analytic stratifications and control theory," in *Proc. 1978. Int. Congress of Mathematicians*, Helsinki (1980), pp. 865-871.

[31] Sussmann, H. J., "A bang-bang theorem with bounds on the number of switchings," *S.I.A.M. J. Control and Optimization* **17**, No. 5 (1979), pp. 629-651.

[32] Sussmann, H. J., "Single-input observability of continuous-time systems," *Math. Systems Theory* **12** (1979), pp. 371-393.

[33] Sussmann, H. J., "Les sémigroupes sousanalytiques et la régularité des commandes en boucle fermée," *Astérisque (Soc. Math. de France)* **75-76** (1980), pp. 219-226.

[34] Sussmann, H. J., "Bounds on the number of switchings for trajectories of piecewise analytic vector fields," *J. Diff. Equations* **43** (1982), pp. 399-418.

[35] Sussmann, H. J., "Lie brackets and local controllability: a sufficient condition for scalar input systems," *S.I.A.M. J. Control and Optimization* **21**, No. 5 (1983), pp. 686-713.

[36] Sussmann, H. J., "Time-optimal control in the plane," in *Feedback Control of Linear and Nonlinear Systems*, D. Hinrichsen and A. Isidori Eds., Springer-Verlag Lect. Notes in Control and Information Sciences, No. 39, New York (1982), pp. 244-260.

[37] Sussmann, H. J., "Subanalytic sets and optimal control in the plane," in *Proc. 21st I.E.E.E. Conference on Decision and Control*, Orlando, Fla. (Dec 1982), pp. 295-299.

[38] Sussmann, H. J., "Lie brackets, real analyticity and geometric control theory," in *Differential Geometric Control Theory*, R.W. Brockett, R.S. Millman and H.J. Sussmann Eds., Birkhäuser Boston Inc. (1983), pp. 1-115.

[39] Sussmann, H. J., "Lie Brackets and real analyticity in control theory," in *Mathematical Control Theory*, C. Olech Ed., Banach Center Publications, Volume 14, PWN-Polish Scientific Publishers,Warsaw, Poland, 1985, pp. 515-542.

[40] Sussmann, H. J., "Resolution of singularities and linear time-optimal control," in *Proceedings of the 23rd I.E.E.E. Conference on Decision and Control*, Las Vegas, Nevada (Dec. 1984), pp. 1043-1046.

[41] Sussmann, H. J., "A general theorem on local controllability," *S.I.A.M. J. Control and Optimization* **25**, No. 1 (1987), pp. 158-194.

[42] Sussmann, H. J., "The structure of time-optimal trajectories for single-input systems in the plane : the C^∞ nonsingular case," *S.I.A.M. J. Control and Optimization* **25**, No. 2 (1987), pp. 433-465.

[43] Sussmann, H. J., "The structure of time-optimal trajectories for single-input systems in the plane: the general real-analytic case," *S.I.A.M. J. Control and Optimization* **25**, No. 4, 1987,pp. 868-904.

[44] Sussmann, H. J., "Regular synthesis for time-optimal control of single-input real-analytic systems in the plane," *S.I.A.M. J. Control and Optimization* **25**, No. 5, 1987, pp. 1145-1162.

[45] Sussmann, H. J., "Envelopes, conjugate points and optimal bang-bang extremals," in *Algebraic and Geometric Methods in Nonlinear Control Theory*, M. Fliess and M. Hazewinkel Eds., D. Reidel Publishing Co., Dordrecht, The Netherlands, 1986, pp.325-346.

[46] Sussmann, H. J., "A weak regularity theorem for real analytic optimal control problems," *Revista Matemática Iberoamericana* **2**, No. 3, 1986, pp. 307-317.

[47] Sussmann, H. J., "Recent developments in the regularity theory of optimal trajectories," in *Linear and Nonlinear Mathematical Control Theory*, Rendiconti del Seminario Matematico, Università e Politecnico di Torino, Fascicolo Speciale 1987, pp. 149-182.

[48] Sussmann, H. J., "Trajectory regularity and real analyticity: some recent results," in *Proceedings of the 25th I.E.E.E. Conference on Decision and Control*, Athens, Greece, Dec. 1986, pp. 592-595.

[49] Sussmann, H. J., "Real analytic desingularization and subanalytic sets: an elementary approach," to appear in *Transactions of the A.M.S.*

[50] Sussmann, H. J., "Hamiltonian lifts and optimal trajectories," Rutgers Center for Systems and Control Technical Report 88-05, *Proceedings of the 27th I.E.E.E. Conference on Decision and Control*, Austin, Texas, Dec. 1988, pp. 1182-1186.

[51] Sussmann, H. J., "A counterexample on optimal control regularity for real analytic systems," Rutgers Center for Systems and Control Technical Report 88-07, September 1988.

[52] Sussmann, H. J., "Envelopes, higher-order optimality conditions, and Lie brackets," Rutgers Center for Systems and Control Technical Report Sycon 89-18, September 1989. To appear in the Proceedings of the 1989 IEEE CDC.

[53] Sussmann, H. J., "Synthesis, presynthesis, sufficient conditions for optimality and subanalytic sets," Rutgers Center for Systems and Control Technical Report 89-20, September 1989. To appear in *Finite Dimensional Controllability and Optimal Control*, H.J. Sussmann ed., M. Dekker, Inc.

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, U.S.A.

E-mail: sussmann@math.rutgers.edu .