Notes on second-order open mapping theorems

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1 Quadratic maps and the Hessian

NOTE: This section is "pure algebra." So we work with linear spaces over an arbitrary field $\mathbb{K}$, assuming only that $\mathbb{K}$ is not of characteristic 2.

1.1 Quadratic and linear+quadratic maps.

If X and Y are linear spaces over $\mathbb{K}$, and $X \times X \ni (x, x') \mapsto B(x, x') \in Y$ is a symmetric bilinear map, we will write Q_B to denote the quadratic map associated with B, i.e., the map

$$X \ni x \mapsto \frac{1}{2} B(x, x) \stackrel{\text{def}}{=} Q_B(x) \in Y\,.$$

It is well known that B is completely determined by Q_B, since

$$B(x, y) = Q_B(x + y) - Q_B(x) - Q_B(y)\,. \tag{1}$$

Definition 1.1 *Let X, Y be linear spaces over $\mathbb{K}$. A* ***quadratic map*** *from X to Y is a map Q such that $Q = Q_B$ for some bilinear symmetric map $B : X \times X \mapsto Y$ (which is then unique, as explained above). If Q is a quadratic map, then we will use B^Q to denote the corresponding symmetric bilinear map.*

A ***linear+quadratic map*** *from X to Y is a map $M : X \mapsto Y$ which is the sum of a linear map and a quadratic map.* $\diamondsuit$

If $M : X \mapsto Y$ is a linear+quadratic map, then the linear map $L : X \mapsto Y$ and the quadratic map $Q : X \mapsto Y$ such that $M = L + Q$ are uniquely determined by M. (Indeed, fix $x \in X$; for $\varepsilon \in \mathbb{K}$ write

$$p(\varepsilon) = M(\varepsilon x) = \varepsilon L(x) + \varepsilon^2 Q(x)\,;$$

then

$$p(1) = L(x) + Q(x) \qquad \text{and} \quad p(2) = 2L(x) + 4Q(x)\,,$$

so

$$L(x) = \frac{4p(1) - p(2)}{2} \qquad \text{and} \quad Q(x) = p(1) - L(x)\,;$$

therefore $L(x)$ and $Q(x)$ are determined by M. Since this is true for all $x \in X$, we see that L and Q are determined by M. Notice that this argument depends quite strongly on the fact that $\mathbb{K}$ is not of characteristic 2.) The linear map L and the quadratic map Q are, respectively, the ***linear part*** and the ***quadratic part*** of M.

1.2 The differential of a linear+quadratic map.

Suppose that X and Y are linear spaces over the field $\mathbb{K}$, and $M : X \mapsto Y$ is a linear+quadratic map with linear part L and quadratic part Q. Let x_0 be a point of X. The ***differential*** of M at x_0 is the linear map

$$X \ni x \mapsto DM(x_0)(x) \stackrel{\mathrm{def}}{=} L(x) + B_Q(x_0, x) \in Y\,.$$

This definition is justified by the fact that, if we write $p(\varepsilon) = M(x_0 + \varepsilon x)$, then $DM(x_0)(x)$ is the coefficient of ε in $p(\varepsilon)$, since

$$\begin{aligned} M(x_0+\varepsilon x) &= \Big(L(x_0)+\varepsilon L(x)\Big)+\Big(\frac{1}{2}B_Q(x_0,x_0)+\varepsilon B_Q(x_0,x)+\frac{\varepsilon^2}{2}B_Q(x_0,x_0)\Big) \\ &= L(x_0)+\frac{1}{2}B_Q(x_0,x_0)+\varepsilon\Big(L(x)+B_Q(x_0,x)\Big)+\frac{\varepsilon^2}{2}B_Q(x_0,x_0)\,. \end{aligned}$$

1.3 Zeros and regular zeros of a linear+quadratic map.

Definition 1.2 *If S is a set and Y is a linear space, a* ***zero*** *of a map $\mu : S \mapsto Y$ is a point $s \in S$ such that $\mu(s) = 0$.*

$\diamondsuit$

Definition 1.3 *Suppose that X, Y are linear spaces, and $M : X \mapsto Y$ is a linear+quadratic map. A* ***regular point*** *of M is a point $x_0 \in X$ such that the linear map $DM(x_0) : X \mapsto Y$ is surjective. A* ***regular zero*** *of M is a zero of M which is is regular point.* $\diamondsuit$

Equivalently, if L, Q are the linear and quadratic parts of M, a regular zero of Q is a point $x_0 \in X$ such that

a. $M(x_0) = 0$

and

b. the linear map $X \ni x \mapsto L(x) + B_Q(x_0, x) \in Y$ is surjective.

1.4 The Hessian.

For a better understanding of this subection and the next one, the reader should think of the special case when L and Q as the first derivative $Df(x_0)$ and one-half of the second derivative $D^2f(x_0)$ at a point x_0 of a map of class C^2 from an open subset U containing x_0 of a Banach space X to a Banach space Y.

Assume that

(H1) X and Y are linear spaces over $\mathbb{K}$;

(H2) $L : X \mapsto Y$ is a linear map;

(H3) $Q : X \mapsto Y$ is a quadratic map.

Definition 1.4 *If (H1,2,3) hold, then the* ***Hessian*** *of the pair (L, Q) is the quadratic map $H_{L,Q} : K_L \mapsto C_L$ (where $K_L = \ker L$, $C_L = \operatorname{coker} L = Y/\operatorname{im} L$) given by*

$$H_{L,Q}(k) = \pi(Q(k)) \qquad \text{for} \qquad k \in K\,, \tag{2}$$

where π is the canonical projection from Y to the quotient space C_L. $\diamondsuit$

Remark 1.5 As we pointed out above, an important special case of our general definition of the Hessian occurs when $\mathbb{K} = \mathbb{R}$, X and Y are real Banach spaces, $L = Df(x_0)$ and $Q = \frac{1}{2}D^2f(x_0)$, where f is a map of class C^2 from an open subset U of X to Y, and U contains x_0. In that case, the Hessian $H_{L,Q}$ is called ***the Hessian of*** f ***at*** x_0, and we use the alternative notation $H_f(x_0)$ for $H_{L,Q}$ (i.e., for $H_{Df(x_0),\frac{1}{2}D^2f(x_0)}$). $\diamondsuit$

Remark 1.6 Under the conditions of the previous remark, the Hessian H_f is a quadratic map from $\ker Df(x_0)$ to $\operatorname{coker} Df(x_0)$. Clearly, $\ker Df(x_0)$ is a closed subspace of X, but in principle there is no reason for $\operatorname{im} Df(x_0)$ to be closed in Y, and if $\operatorname{im} Df(x_0)$ is not closed in Y then the quotient $\operatorname{coker} Df(x_0) = Y/\operatorname{im} Df(x_0)$ is not a Banach space.

This is probably why in several papers on this subject (e.g. [1, 2]) the authors impose the extra requirement that $\operatorname{im} Df(x_0)$ be closed. It turns out, however, that *this assumption is never needed.* As we have shown, the concept of a "regular zero of H_f" can be defined in a purely algebraic way. Furthermore, in the next subsection we show that the existence of a regular zero of H_f is exactly equivalent to the existence of a regular zero of another linear+quadratic map $\Phi_{L,Q}$ (that is, $\Phi_{Df(x_0),\frac{1}{2}D^2f(x_0)}$), and this second map is automatically a map between Banach spaces, because it goes from the product $X \times \ker Df(x_0)$ to Y. $\diamondsuit$

Remark 1.7 The Hessian $H_f(x_0)$ of a map f of class C^2 at a point x_0 is defined in terms of the best linear approximation $Df(x_0)(h)$ of $f(x_0+h)-f(x_0)$ and the best quadratic approximation $\frac{1}{2}D^2f(x_0)(h)$ of $f(x_0+h)-f(x_0)-Df(x_0)(h)$ near $h=0$. However, the formula for $H_f(x_0)$ only makes use of the values $D^2f(x_0)(h)$ for $h \in \ker Df(x_0)$. This suggests that the existence of the second derivative in directions not belonging to $\ker Df(x_0)$ should not be needed. It turns out, in fact, that the proper setting for the theory is much more general. All that is needed (assuming, for simplicity, that $x_0 = 0$ and $f(x_0) = 0$) is the existence of a continuous linear map $L : X \mapsto Y$ and a continuous quadratic map $Q : K_L \mapsto Y$ (where $K_L = \ker L$) such that, for $(x,k) \in X \times K_L$,

$$f(x+k) = Lx + Q(k) + o(|||(x,k)|||^2) \qquad \text{as} \quad (x,k) \to (0,0) \in X \times K_L\,, \tag{3}$$

where $|||(x,k)||| \stackrel{\text{def}}{=} \|x\|^{1/2} + \|k\|$.

This condition is of course satisfied when f is of class C^2 near 0. (*Proof.* Write

$$f(x+k) = L(x+k) + Q(x+k) + o(\|x\|^2 + \|k\|^2)\,,$$

where $L = Df(x_0)$ and $Q = \frac{1}{2}D^2f(x_0)$. Then $L(x+k) = L(x)$ because $k \in \ker L$. Furthermore,

$$Q(x+k) = \frac{1}{2}B_Q(x+k, x+k) = \frac{1}{2}B_Q(x,x) + B_Q(x,k) + \frac{1}{2}B_Q(k,k)\,,$$

and

$$\frac{1}{2}B_Q(x,x) + B_Q(x,k) = o(\|x\|)\,,$$

so

$$f(x+k) = L(x) + Q(k) + o(\|x\| + \|k\|^2)\,.$$

And, finally, a quantity which is $o(\|x\| + \|k\|^2)$ is *a fortiori* $o(|||(x,k)|||^2)$, since $\|x\| + \|k\|^2 \leq (\|x\|^{1/2} + \|k\|)^2$.)

The map f must be required to be continuous, but it is not necessary that it be of class C^2 or even of class C^1. (For example, let $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ be a nowhere differentiable function. Define $f : \mathbb{R}^2 \mapsto \mathbb{R}$ by letting

$$f(x,y) = x + y^2 + (x^4 + y^4)\varphi(x,y)\,.$$

Then f clearly is continuous and has a quadratic approximation of the desired type, while on the other hand f is not differentiable at any point other than $(0,0)$.) $\diamondsuit$

Remark 1.8 The expression $|||(x,k)|||$ is in fact a "homogeneous norm" relative to a group of dilations on $X \times K_L$. (Precisely, define $\delta_t : X \times K_L \mapsto X \times K_L$, for $t > 0$, by letting $\delta_t(x,k) = (t^2 x, tk)$. Let $|||(x,k)||| = \|x\|^{1/2} + \|k\|$. Then $|||\delta_t(x,k)||| = t|||(x,k)|||$. Furthermore, $|||\cdot|||$ satifies the triangle inequality $|||(x_1+x_2, k_1+k_2)||| \leq |||(x_1,k_1)||| + |||(x_2,k_2)|||$, as well as the condition that $|||(x,k)||| = 0$ if and only if $(x,k) = (0,0)$.)

So (3) is precisely the statement that $f(x+k)$ has a linear+quadratic approximation with an error which is a "small o of the norm squared." In addition, relative to the dilations δ_t, the linear+quadratic approximation is just "quadratic homogeneous," in the sense that, if we define $M(x,k) = L(x) + Q(k)$, then $M(\delta_t(x,k)) = t^2 M(x,k)$.

This approach is pursued in Sussmann [3], where it is proved that, when an approximation of this kind exists, and the approximating map has a regular zero, then the map itself is open at 0, provided that the target space Y is finite-dimensional. (In the infinite-dimensional case, stronger conditions are needed, analogous to those of Graves' theorem on strictly differentiable maps. This is also discussed in [3].) $\diamondsuit$

1.5 Regular zeros of the Hessian.

The regular zeros of a Hessian $H_{L,Q}$ can also be characterized as the regular zeros of a linear+quadratic map $\Phi_{L,Q} : X \times K_L$ to Y as follows. We define a map $\Phi_{L,Q} : X \times K_L$ to Y by

$$\Phi_{L,Q}(x,k) = Lx + Q(k)\,.$$

Then $\Phi_{L,Q}$ is clearly linear+quadratic.

Theorem 1.9 *Assume that X, Y, L, Q are such that (H1,2,3) hold, and K_L, C_L, $H_{L,Q}$ are as in Definition 1.4. Then a point $k_0 \in K_L$ is a regular zero of $H_{L,Q}$ if and only if there exists an $x_0 \in X$ such that (x_0, k_0) is a regular zero of $\Phi_{L,Q}$. In particular, $H_{L,Q}$ has a regular zero if and only if $\Phi_{L,Q}$ has a regular zero.*

Proof. Let $\pi : Y \mapsto C_L$ be the canonical projection. Suppose k_0 is a regular zero of $H_{L,Q}$. Then $k_0 \in K_L$, $H_{L,Q}(k_0) = 0$ as a member of C_L, and the linear map $K_L \ni k \mapsto \pi(B_Q(k_0,k)) \in C_L$ is surjective. The fact that $H_{L,Q}(k_0) = 0$ as a member of C_L says that $Q(k_0) = 0$ modulo $\operatorname{im} L$, that is, that there exists $x_0 \in X$ such that $L(x_0) + Q(k_0) = 0$ in Y. Hence (x_0, k_0) is a zero of $\Phi_{L,Q}$. To show that (x_0, k_0) is a regular zero of $\Phi_{L,Q}$ we must prove the surjectivity of the differential $D\Phi_{L,Q}(x_0, k_0)$, which is a linear map from $X \times K_L$ to Y. So let us pick an arbitrary $y \in Y$. The surjectivity of $DH_{L,Q}(k_0)$ implies that we can find $k \in K_L$ such that $DH_{L,Q}(k_0)(k) = \pi(y)$. But $DH_{L,Q}(k_0)(k) = \pi(B_Q(k_0,k))$, so $B_Q(k_0,k) = y$ modulo $\operatorname{im} L$. Then there exists $x \in X$ such that $L(x) + B_Q(k_0,k) = y$, so $y = D\Phi_{L,Q}(x_0,k_0)(x,k)$, completing the proof that $D\Phi_{L,Q}(x_0,k_0)$ is surjective.

To prove the converse, suppose (x_0, k_0) is a regular zero of $\Phi_{L,Q}$. Then $0 = \Phi_{L,Q}(x_0,k_0) = L(x_0) + Q(k_0)$, so $Q(k_0) = 0$ modulo $\operatorname{im} L$, and then $H_{L,Q}(k_0) = 0$, so k_0 is a zero of $H_{L,Q}$. To prove regularity, we have to show that the linear map $K_L \ni k \mapsto \pi(B_Q(k_0,k)) \in C_L$ is surjective. So we pick $\eta \in C_L$ and try to express η as $\pi(B_Q(k_0,k))$ for some $k \in K_L$. Pick $y \in Y$ such that $\pi(y) = \eta$. Since $D\Phi_{L,Q}(x_0,k_0)$ is surjective, we can find $x \in X$, $k \in K_L$, such that $L(x) + B_Q(k_0,k) = y$. But then $B_Q(k_0,k) = y$ modulo $\operatorname{im} L$, so $\pi(B_Q(k_0,k)) = \pi(y) = \eta$. $\diamondsuit$

2 A second-order open mapping theorem

In this section we do analysis, so now the field of scalars is $\mathbb{R}$.

If A, B are topological spaces, f is a map from A to B, and $a \in A$, we say that f is ***open at*** a if, whenever W is an open subset of A such that $a \in W$, it follows that $f(a)$ is an interior point of $f(W)$.

Theorem 2.1 *Let X, Y be Banach spaces such that $\dim Y < \infty$. Let f be a Y-valued map of class C^2 from an open subset U of X such that $0 \in U$. Assume that $f(0) = 0$. Suppose that the Hessian $H_f(0)$ has a regular zero. Then f has regular zeros arbitrarily close to 0. In particular f is open at 0.*

Remark 2.2 The openness part of the statement follows from the assertion about the regular zeros, because if W is open and $0 \in W$ then we can find a regular zero x_0 of f belonging to W and then $f(x_0)$ will be an interior point of $f(W)$, i.e., $\in \operatorname{Int} W$.

It turns out, however, that it is more convenient to prove the openness first and then use it to deduce the result about the regular zeros as a corollary.

For the optimal control application, what we really need is the openness result. So the reader may ask why we do not simply omit the result on the regular zeros, since in any case it is proved *after* rather than before the openness assertion. The reason is that the existence of regular zeros will be needed in our inductive proof of the main theorem on the index. $\diamondsuit$

PROOF OF THEOREM 2.1. Write

$$\begin{aligned} L &= Df(0)\,, \\ Q &= \frac{1}{2}D^2 f(0)\,, \\ K &= \ker L\,, \\ C &= \operatorname{coker} L\,. \end{aligned}$$

We know from Theorem 1.9 that the existence of a regular zero of H_f is equivalent to the existence of points $x_0 \in X$, $k_0 \in K$ such that

(E1) $\Phi(x_0, k_0) = 0$,

and

(E2) the linear map $X \times K \ni (x, k) \mapsto L(x) + B_Q(k_0, k) \in Y$ is surjective,

where Φ is the map $X \times K \ni (x, k) \mapsto L(x) + Q(k) \in Y$.

Let Ψ be the linear map of (E2), so $\Psi = D\Phi(x_0, k_0)$. Using the fact that $f \in C^2$, write

$$f(x+k) = L(x) + Q(k) + o(|||(x,k)|||^2) \qquad \text{as} \quad (x,k) \to (0,0)\,,$$

where $|||(x,k))||| = \sqrt{\|x\|} + \|k\|$ (cf. Remark 1.7). Also, let

$$\delta_t(x,k) = (t^2 x, tk) \qquad \text{for} \quad (x,k) \in X \times K\,,\ t \in \mathbb{R}\,,\ t \geq 0\,.$$

Then $|||\delta_t(x,t)||| = t|||(x,k)|||$ and $\Phi(\delta_t(x,k)) = t^2\Phi(x,k)$ for all x, k, t.

Fix an open subset W of X such that $0 \in W$ and $W \subseteq U$. Find an open subset $\mathcal{W}$ of $X \times K$ such that $x + k \in W$ whenever $(x,k) \in \mathcal{W}$. Define $F(x,k) = f(x+k)$, so F is defined on $\mathcal{W}$.

Let $s \mapsto \sigma(s)$ be a function, defined for s in some interval $]0, \bar{s}]$ with $\bar{s} > 0$, such that $\lim_{s\downarrow 0} \sigma(s) = 0$, and having the property that the conditions

$$(x,k) \in \mathcal{W} \quad \text{and} \qquad \|F(x,k) - L(x) - Q(k)\| \leq \sigma(\|x\| + \|k\|)|||(x,k)|||^2$$

hold whenever $\|x\| + \|k\| \leq \bar{s}$.

For $t > 0$, let $\xi(t) = \delta_t(x_0, k_0)$, i.e., $\xi(t) = (t^2 x_0, t k_0)$. Then all the points $\xi(t)$ are regular zeros of Φ. We are going to prove that there exist regular zeros $\zeta(t)$ of F "close" to the $\xi(t)$ in a sense to be made precise below.

Let $B_Y(r)$ be the closed unit ball of Y with center 0 and radius r. Since $\Phi(x_0, k_0) = 0$ and the linear map $D\Phi(x_0, k_0) = \Psi : X \times K \mapsto Y$ is surjective, the usual implicit function theorem (which is applicable here since Y is finite-dimensional, so the closed linear subspace $\ker \Psi$ of X has a complement), we can find a positive number $\bar{r}$ and a smooth map $\Theta : B_Y(\bar{r}) \mapsto X \times K$ such that $\Phi \circ \Theta = \mathrm{id}_{B_Y(\bar{r})}$ and $\Theta(0) = (x_0, k_0)$. Since Y is finite-dimensional, $B_Y(\bar{r})$ is compact, so $\Theta(B_Y(\bar{r}))$ is compact. Hence the exists a positive constant κ such that $\|\Theta(y)\| \leq \kappa$ whenever $y \in B_Y(\bar{r})$. (NOTE: the norm $\|\cdot\|$ on $X \times K$ is the map $(x, k) \mapsto \|x\| + \|k\|$.)

We now define maps $\mu_t : B_Y(\bar{r}) \mapsto Y$, for $t > 0$, t small, as follows.

$$\mu_t(y) = \frac{1}{t^2} F\Big(\delta_t(\Theta(y))\Big) \ \text{ if } \ y \in B_Y(\bar{r})\,,\, 0 < t \leq \tau\,.$$

Here the constant τ is chosen so that

$$\delta_t(\Theta(B_Y(\bar{r}))) \subseteq \{(x, k) : \|x\| + \|k\| \leq \bar{s}\} \quad \text{whenever} \quad 0 < t \leq \tau\,.$$

(This is poosible because the set $\Theta(B_Y(\bar{r}))$ is bounded and $\delta_t \to 0$ as $t \downarrow 0$ uniformly on bounded sets.)

Let us study the map ν_t obtained by substituting Φ for F in the definition of μ_t, and compare ν_t with μ_t. We have

$$\nu_t(y) = \frac{1}{t^2} \Phi\Big(\delta_t(\Theta(y)\Big) \ \text{ if } \ y \in B_Y(\bar{r})\,,\, 0 < t \leq \tau\,.$$

Since

$$\Phi\Big(\delta_t(\Theta(y))\Big) = t^2 \Phi(\Theta(y)) = t^2 y\,,$$

we see that

$$\nu_t(y) = y \ \text{ if } \ y \in B\,,\, 0 < t \leq \tau$$

Furthermore, if $y \in B_Y(\bar{r})$, $0 < t \le \tau$, we have

$$\begin{aligned}
\|\mu_t(y) - \nu_t(y)\| &= \left\| \frac{1}{t^2} F\Big(\delta_t(\Theta(y)\Big) - \frac{1}{t^2}\Phi\Big(\delta_t(\Theta(y)\Big)\right\| \\
&= \frac{1}{t^2}\left\| F\Big(\delta_t(\Theta(y)\Big) - \Phi\Big(\delta_t(\Theta(y)\Big)\right\| \\
&\le \frac{1}{t^2}\sigma\Big(\|\delta_t(\Theta(y)\|\Big)\,|||\delta_t(\Theta(y)|||^2 \\
&= \frac{1}{t^2}\sigma\Big(\|\delta_t(\Theta(y)\|\Big)\,t^2\,|||\Theta(y)|||^2 \\
&= \sigma\Big(\|\delta_t(\Theta(y)\|\Big)\,|||\Theta(y)|||^2 .
\end{aligned}$$

Since $\|\delta_t(\Theta(y))\| \to 0$ as $t \downarrow 0$, uniformly with respect to $y \in B_Y(\bar{r})$, and $\Theta(B_Y(\bar{r}))$ is bounded, it follows that $\|\mu_t(y) - \nu_t(y)\| \to 0$ as $t \downarrow 0$, uniformly with respect to $y \in B_Y(\bar{r})$, and $\Theta(B_Y(\bar{r}))$. In other words,

$$\lim_{t\downarrow 0} \beta(t) = 0\,, \tag{4}$$

where

$$\beta(t) = \sup\Big\{\|\mu_t(y) - y\| : y \in B_Y(\bar{r})\Big\} = 0\,. \tag{5}$$

Let $\bar{\tau}$ be such that $0 < \bar{\tau} \le \tau$ and $2\beta(t) < \bar{r}$ for $0 < t \le \bar{\tau}$. Then, if $0 < t \le \bar{\tau}$, and $z \in B_Y(\beta(t))$, the map

$$B_Y(2\beta(t)) \ni y \mapsto y - \mu_t(y) + z \stackrel{\text{def}}{=} \omega_{z,t}(y)$$

satisfies $\|\omega_{z,t}(y)\| \le 2\beta(t)$ for all $y \in B_Y(2\beta(t))$. Hence $\omega_{z,t}$ is a continuous map from $B_Y(2\beta(t))$ to $B_Y(2\beta(t))$. Therefore the Brouwer fixed point theorem implies that there exists a $y_{z,t} \in B_Y(2\beta(t))$ such that $\omega_{z,t}(y_{z,t}) = y_{z,t}$. i.e., that $\mu_t(y_{z,t}) = z$.

Let $\zeta_z^*(t) = \delta_t(\Theta(y_{z,t}))$. Then $F(\zeta_z^*(t)) = t^2 z$. It follows that, if $0 < t \le \bar{\tau}$, $w \in B_Y(t^2\beta(t))$, and we define $\zeta_w(t) = \zeta^*_{w/t^2}(t)$, then $F(\zeta_w(t)) = w$.

In particular, we have shown that the ball $B_Y(t^2\beta(t))$ is contained in $F(\mathcal{W})$. Since $F(\mathcal{W}) \subseteq f(W)$, we have shown that $B_Y(t^2\beta(t)) \subseteq f(W)$ if t is small enough, and this proves the openness of f at 0.

Furthermore, if we let $\zeta(t) = \zeta_0(t)$, the points $\zeta(t)$ satisfy $F(\zeta(t)) = 0$, so the $\zeta(t)$ are zeros of F. It follows that, if we write $\zeta(t) = (x(t), k(t))$, then $x(t) + k(t)$ is a zero of f.

We now show that if t is small enough then $x(t) + k(t)$ is a regular zero of f. For this purpose, we want to show that the points $\zeta(t)$ are "very close" to the $\xi(t)$.

By definition, $\zeta(t) = \zeta_0^*(t) = \delta_t(\Theta(y_{0,t}))$. So $\delta_t^{-1}(\zeta(t)) = \Theta(y_{0,t})$. As $t \downarrow 0$, $y_{0,t}$ goes to 0 (because $y_{0,t} \in B_Y(t^2\beta(t))$). Hence $\Theta(y_{0,t})$ goes to $\Theta(0)$, i.e., to (x_0, k_0). So we have shown that

$$\lim_{t\downarrow 0} \delta_t^{-1}(\zeta(t)) = (x_0, k_0)\,. \tag{6}$$

Since $\xi(t) = \delta_t(x_0, k_0)$, we have $(x_0, k_0) = \delta_t^{-1}(\xi(t)))$. So we can rewrite (6) as

$$\lim_{t\downarrow 0} \delta_t^{-1}\Big(\zeta(t) - \xi(t)\Big) = 0\,. \tag{7}$$

Formulas (6) and (7) tell us that $\zeta(t)$ is very close to $\xi(t)$ is a very precise sense. If we write $\zeta(t) = (x(t), k(t))$ as before, (6) says that

$$\lim_{t\downarrow 0} \frac{x(t)}{t^2} = x_0 \qquad \text{and} \qquad \lim_{t\downarrow 0} \frac{k(t)}{t} = k_0\,. \tag{8}$$

In other words, the points $(x(t), k(t))$ that we have found satisfy

$$x(t) = t^2 x_0 + o(t^2) \qquad \text{and} \qquad k(t) = tk_0 + o(t)\,. \tag{9}$$

We now prove that

(#) *For sufficiently small t, $x(t) + k(t)$ is a regular point of f.*

To prove (#), we first observe that it suffices to show that

(#') *For sufficiently small t, $\zeta(t)$ is a regular point of F.*

(Indeed, it follows from the chain rule that the differential $DF(\zeta(t)) : X \times K \mapsto Y$ is the composite map $Df(x(t)+k(t)) \circ \Sigma$, where $\Sigma : X \times K \mapsto X$ is the map $(x, k) \mapsto x+k$. Hence $Df(x(t)+k(t))$ is necessarily surjective if $DF(x(t), k(t))$ is surjective.)

We now prove (#'). Using the fact that f is of class C^2, write

$$F(x, k) = f(x + k) = Df(0)(x + k) + \frac{1}{2}D^2 f(0)(x + k) + R(x, k)\,,$$

where the remainder R is a map of class C^2 such that $R(0, 0)$, $DR(0, 0)$, and $D^2R(0, 0)$ vanish.

Then $DF(x, k) = Df(0) \circ \Sigma + \Lambda_{x,k} + DR(x, k)$, where Σ is, as above, the map $X \times K \ni (x, k) \mapsto x + k \in X$, and $\Lambda_{x,k}$ is the linear map $X \times K \ni (\Delta x, \Delta k) \mapsto B_Q(x + k, \Delta x + \Delta k) \in X$.

For $t > 0$, let Ξ_t be the linear map

$$X \times K \ni (x, k) \mapsto \Big(x, \frac{k}{t}\Big) \in X \times K\,.$$

Then

$$DF(x(t), k(t)) \circ \Xi_t = M(t) + DR(x(t), k(t)) \circ \Xi_t\,, \tag{10}$$

where $M(t)$ is the linear map

$$X \times K \ni (\Delta x, \Delta k) \mapsto DF(0)\Big(\Delta x + \frac{\Delta k}{t}\Big) + B_Q\Big(x(t) + k(t), \Delta x + \frac{\Delta k}{t}\Big) \in Y\,.$$

Then, if $(\Delta x, \Delta k) \in X \times K$, we have

$$\begin{aligned} M(t)(\Delta x, \Delta k) &= DF(0)\Big(\Delta x + \frac{\Delta k}{t}\Big) + B_Q\Big(x(t) + k(t), \Delta x + \frac{\Delta k}{t}\Big) \\ &= DF(0)(\Delta x) + B_Q\Big(x(t) + k(t), \frac{t\Delta x + \Delta k}{t}\Big) \\ &= DF(0)(\Delta x) + B_Q\Big(\frac{x(t) + k(t)}{t}, t\Delta x + \Delta k\Big)\,, \end{aligned}$$

where we have used the bilinearity of B_Q and the fact that $DF(0)(\Delta k) = 0$ since $\Delta k \in \ker Df(0)$.

In view of (9),

$$\lim_{t\downarrow 0} \Big(\frac{x(t) + k(t)}{t}\Big) = k_0\,.$$

Therefore

$$\lim_{t\downarrow 0} M(t) = M(0)\,, \tag{11}$$

where $M(0)$ is the linear map

$$X \times K \ni (\Delta x, \Delta k) \mapsto DF(0)\Delta x + B_Q\Big(k_0, \Delta k) \in Y\,,$$

which is exactly the linear map of (E2). It follows that $M(0)$ is surjective.

In addition, the fact that $R(0, 0)$, $DR(0, 0)$, and $D^2R(0, 0)$ vanish implies that $DR(x, k) = o(\|x\| + \|k\|)$ as $(x, k) \to (0, 0)$. Then $\|DR(x(t), k(t))\| = o(t)$ as $t \downarrow 0$, since $\|x(t)\| + \|k(t))\| = O(t)$. Since $\|\Xi_t\| = \frac{1}{t}$, we can conclude that

$$\lim_{t\downarrow 0} DR(x(t), k(t)) \circ \Xi_t = 0\,. \tag{12}$$

Then (10), (11) and (12) imply

$$\lim_{t\downarrow 0} DF(x(t), k(t)) \circ \Xi_t = M(0)\,. \tag{13}$$

Since $M(0)$ is surjective, the map $DF(x(t), k(t)) \circ \Xi_t$ must be surjective for t small enough. This in turn implies that $DF(x(t), k(t))$ is surjective for t small enough. Hence we have proved (#'), and the proof of our theorem is complete.

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