A General Result on the Stabilization of Linear Systems Using Bounded Controls[1]

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ABSTRACT

We present two constructions of controllers that globally stabilize linear systems subject to control saturation. We allow essentially arbitrary saturation functions. The only conditions imposed on the system are the obvious necessary ones, namely that no eigenvalues of the uncontrolled system have positive real part and that the standard stabilizability rank condition hold. One of the constructions is in terms of a "neural-network type" one-hidden layer architecture, while the other one is in terms of cascades of linear maps and saturations.

Key words: linear systems, saturated actuators, bounded controls, neural nets

October, 1992. Revised February 1994.

[1] This research was supported in part by US Air Force Grant 91-0343 and by NSF Grant DMS-8902994.
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1 Introduction

We consider linear time-invariant continuous-time systems

$$\Sigma : \quad \dot{x} = Ax + Bu\,, \tag{1.1}$$

where (i) $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, for some integers n (the dimension of the system) and m (the number of inputs), and (ii) the control values u are restricted to satisfy a bound $|u| \leq C$, where C is a given positive constant.

The study of such systems is motivated by the possibility of actuator saturation or constraints on actuators, reflected sometimes also in bounds on available power supply or rate limits. These systems cannot be naturally dealt with within the context of standard (algebraic) linear control theory, but are ubiquitous in control applications. To quote the recent textbook [8] (page 171): "saturation is probably the most commonly encountered nonlinearity in control engineering." Mathematically, control questions become nontrivial, as only control values bounded by C are allowed into the underlying linear system.

We will present results on *global stabilization,* concentrating on several explicit architectures for controllers. Of course, there are general limits as to what can be achieved, no matter what type of control law is allowed. An obvious necessary condition for stabilizability is that Σ be *asymptotically null-controllable with bounded controls (ANCBC).* (We call Σ *ANCBC with bound* C if for every $x \in \mathbb{R}^n$ there exists an open-loop control $u : [0,\infty) \to \mathbb{R}^m$ that steers x to the origin in the limit as $t \to +\infty$ and satisfies $|u(t)| \leq C$ for all t. It turns out —cf. Remark 1.1 below— that if Σ has this property for some $C \in (0,\infty)$ then it has it for every $C \in (0,\infty)$, so we can simply talk about Σ being ANCBC, without mentioning C.) The ANCBC property is equivalent to the following algebraic condition:

(ANCBC*) (i) A has no eigenvalues with positive real part, and (ii) the pair (A,B) is stabilizable in the ordinary sense (i.e. all the uncontrollable modes of Σ have strictly negative real parts).

(The theory of controllability of linear systems with bounded controls is a well-studied topic; see e.g. the fundamental paper [6], as well as the different, more algebraic approach discussed in [9].) Notice that under Condition (ANCBC*) there may very well be nontrivial Jordan blocks corresponding to critical eigenvalues, so the system $\dot{x} = Ax$ need not be asymptotically stable or even Lyapunov-stable. This is what makes the problem interesting, and allows inclusion of examples of practical importance such as systems involving integrators.

In very special cases, including all one- and two-dimensional systems, stabilization is possible by simply using a saturated linear feedback law of the type:

$$u \;=\; \overline{\sigma}(Fx)\,, \tag{1.2}$$

where F is an $m \times n$ matrix and $\overline{\sigma}$ is a function that computes a saturation in each coordinate of the vector Fx, for instance, $u_i = \mathrm{sat}\,((Fx)_i)$ —where $\mathrm{sat}\,(s) \;=\; \mathrm{sign}\,(s)\min\{|s|,1\}$— or $u_i = \tanh((Fx)_i)$. A similar solution is possible for systems that are neutrally stable (i.e. such that the Jordan form of A has no off-diagonal ones corresponding to imaginary eigenvalues), using the "Jurdjevic-Quinn" approach (see [2] and [7]). Thus it is natural to ask if simple

control laws such as (1.2) can also be used for more general systems. This was negatively answered in a paper by A.T. Fuller as far back as the late 1960s. He showed in [1] that already for triple integrators such saturated linear feedback is not sufficient, at least under certain assumptions on the saturation σ. (A stronger negative result, which applies to basically arbitrary σ's, was more recently given, independently, in [14].)

The fact that linear feedback laws when saturated can lead to instability has motivated a large amount of research. (See for instance [3] and [4], and references therein, for estimates of the size of the regions of attraction that result when using linear saturated controllers.) Here we take a different approach. Rather than working with linear saturated control laws $u = \sigma(Fx)$, and trying to show that they are globally stabilizing, or to estimate their domains of attraction, we allow more general bounded (and hence necessarily nonlinear) laws. This is not a new idea since, for example, optimal control techniques can be and have been applied. Optimal control laws, however, may be highly discontinuous. *But by ignoring optimality questions, one may hope to find more regular and/or simpler controllers.* Indeed, taking this point of view, we were able to obtain, in the previous work [12], a general result on bounded stabilization by means of infinitely differentiable feedback laws. The result of [12] holds under the weakest possible conditions, namely, for ANCBC systems.

Unfortunately, the construction in [12] relied on a complicated and far from explicit inductive procedure. On the other hand, since saturated linear feedbacks suffice for up to two dimensions, it is natural to look, in higher dimensions, for control designs based on combining saturation nonlinearities in simple ways, for example by taking *linear combinations* or *compositions*. (In the language of neural networks, one wants control laws that are implementable by feedforward nets with "hidden layers", rather than by the "perceptrons" represented by (1.2)). Recently, and motivated in part by [12] and [14], Andrew Teel showed in [15] how, in the particular case of single-input multiple integrators, such combinations of saturations are indeed sufficient to obtain stabilizing feedback controllers. Here we obtain a general solution of the same type, for the full case treated in [12]. The approach is explicit and constructive. Our solution is inspired by the techniques introduced in [15] for the particular case treated there, but the details are far more complicated, due to the possibilities of having both multiple inputs and (perhaps multiple) purely imaginary eigenvalues, and the need to deal with arbitrary saturations.

We present two types of control designs, labeled "Type $\mathcal{F}$" and "Type $\mathcal{G}$," involving, respectively, compositions and linear combinations of saturated linear functions. (In neural network terms, a Type $\mathcal{G}$ design involves a "single hidden layer net.") We also study the output stabilization problem, in which only partial measurements $y = Cx$ are available for control. Under suitable detectability conditions, the standard Luenberger observer construction is shown to carry over to this case, and a separation principle is proved which allows the use of the saturated control design given earlier in the paper.

Our result was first announced in [17], where we considered a very special type of feedbacks for which the saturations are exactly linear near 0 and, when a system has a pure imaginary eigenvalue, a saturation with three different slopes may be needed. For the results proved here, the saturations σ are essentially arbitrary, since they are only required to be locally Lipschitz, bounded away from 0 as $s \to \pm\infty$, differentiable at 0, and such that $\sigma'(0) > 0$ and $s\sigma(s) > 0$ whenever $s \neq 0$. So, mathematically, our results show, for example, that one can

use real analytic functions to implement feedback laws, a fact that would not follow from the conclusions of [17] or [15]. From an engineering point of view, they insure that rather general components can be employed in the feedback design, subject only to mild conditions which are robustly satisfied. In the terminology of current "artificial neural networks" technology, our results allow the implementation of feedback controllers using very general types of activation (neuron characteristic) functions. For a detailed application of the results given in this paper to a model aircraft control example, see [13].

The paper is organized as follows. Our main result on state feedback is stated in Section 2 and proved in Section 4, using two technical lemmas proved in Section 3. In Section 5 we provide an algorithm to find a stabilizing feedback when saturations are employed, and describe the structure of our two kinds of feedbacks by means of block diagrams. The algorithm is then applied in Section 6 to the case of multiple integrators. Section 7 contains the statement and proof of the result on output feedback stabilization. Finally, in Section 8 we present applications to the stabilization of cascaded systems.

Remark 1.1 If $C > 0$, let Σ_C denote the system Σ with control values u restricted to satisfy $|u| \leq C$. Let $C_1 > 0$, $C_2 > 0$, and write $r = \frac{C_2}{C_1}$. Then it is easy to see that if $t \to x(t)$ is a trajectory of Σ_{C_1}, then $t \to rx(t)$ is a trajectory of Σ_{C_2}. In particular, a state x_0 can be asymptotically steered to 0 by means of a control bounded by C_1 if and only if rx_0 can be asymptotically steered to 0 using a control bounded by C_2. Therefore, if *all* initial states x_0 can be steered to 0 using open-loop or feedback controls bounded by C_1, then the same is true using controls bounded by C_2. So *the property that Σ is ANCBC with bound C holds for one C if and only if it holds for every C, and the same is true for the property that Σ is stabilizable by means of a smooth feedback bounded by C.* □

Acknowledgement

This paper benefited from many constructive suggestions made by several referees, the cognizant Associate Editor, and especially Andrew Teel. (In particular, the fact that no Lipschitz continuity assumptions on k are required in Theorem 7.1 was observed by Teel.) We wish to express our gratitude to all of them.

2 Statement of the Main Results

We first define $\mathcal{S}$ to be the class of all locally Lipschitz functions $\sigma : \mathbb{R} \to \mathbb{R}$ such that $s\sigma(s) > 0$ whenever $s \neq 0$, σ is differentiable at 0, $\sigma'(0) > 0$, and $\liminf_{|s|\to\infty} |\sigma(s)| > 0$.

For any finite sequence $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_k)$ of functions in $\mathcal{S}$, we define a set $\mathcal{F}_n(\boldsymbol{\sigma})$ of functions f from $\mathbb{R}^n$ to $\mathbb{R}$ inductively as follows:

- if $k = 0$ (i.e. if $\boldsymbol{\sigma}$ is the empty sequence), then $\mathcal{F}_n(\boldsymbol{\sigma})$ consists of one element, namely, the zero function from $\mathbb{R}^n$ to $\mathbb{R}$,
- $\mathcal{F}_n(\sigma_1)$ consists of all the functions $h : \mathbb{R}^n \to \mathbb{R}$ of the form $h(x) = \sigma_1(g(x))$, where $g : \mathbb{R}^n \to \mathbb{R}$ is linear,

- for every $k > 1$, $\mathcal{F}_n(\sigma_1, \cdots, \sigma_k)$ is the set of all functions $h : \mathbb{R}^n \to \mathbb{R}$ that are of the form $h(x) = \sigma_k(f(x) + cg(x))$, with f linear, $g \in \mathcal{F}_n(\sigma_1, \cdots, \sigma_{k-1})$, and $c \geq 0$.

We also define $\mathcal{G}_n(\boldsymbol{\sigma})$ to be the class of functions $h : \mathbb{R}^n \to \mathbb{R}$ of the form

$$h(x) = a_1\sigma_1(f_1(x)) + a_2\sigma_2(f_2(x)) + \cdots + a_k\sigma_k(f_k(x))\,,$$

where $f_1, \cdots, f_k$ are linear functions and $a_1, \cdots, a_k$ are nonnegative constants such that $a_1 + \cdots + a_k \leq 1$.

Next, for an m-tuple $\boldsymbol{l} = (l_1, \cdots, l_m)$ of nonnegative integers, define $|\boldsymbol{l}| = l_1 + \cdots + l_m$. For a finite sequence $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_{|\boldsymbol{l}|}) = (\sigma_1^1, \cdots, \sigma_{l_1}^1, \cdots, \sigma_1^m, \cdots, \sigma_{l_m}^m)$ of functions in $\mathcal{S}$, we let $\mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$, $\mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$ denote, respectively, the set of all functions $h : \mathbb{R}^n \to \mathbb{R}^m$ such that $h_i \in \mathcal{F}_n(\sigma_1^i, \cdots, \sigma_{l_i}^i)$ for $i = 1, 2, \cdots, m$, and the set of those $h : \mathbb{R}^n \to \mathbb{R}^m$ such that $h_i \in \mathcal{G}_n(\sigma_1^i, \cdots, \sigma_{l_i}^i))$ for $i = 1, 2, \cdots, m$, where $h_1, \ldots, h_m$ are the components of h. (It is clear that $\mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma}) = \mathcal{F}_n(\boldsymbol{\sigma})$ and $\mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma}) = \mathcal{G}_n(\boldsymbol{\sigma})$ if $m = 1$.) For a system (1.1), a feedback control law $u = -k(x)$ will be said to be *of type* $\mathcal{F}$ (or *of type* $\mathcal{G}$) if $k \in \mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$ (or $k \in \mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$) for some $\boldsymbol{l}$ and some finite sequence $\boldsymbol{\sigma}$ of bounded functions belonging to $\mathcal{S}$.

Let $\delta > 0$. Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}^n$ be a vector-valued function on I. We say that f is *eventually bounded by* δ (and write $|f| \leq_{\text{ev}} \delta$) if there exists a $T \in \mathbb{R}$ such that $[T, +\infty) \subseteq I$ and $|f(t)| \leq \delta$ for all $t \geq T$. Given a control system $\Sigma : \dot{x} = f(x, u)$ in $\mathbb{R}^n$, with inputs in $\mathbb{R}^m$, we say that Σ is $SISS$ ("small-input small-state") if for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $e : [0, +\infty) \to \mathbb{R}^m$ is bounded, measurable, and eventually bounded by δ, then every maximally defined solution $t \to x(t)$ of $\dot{x} = f(x, e(t))$ is eventually bounded by ε. For $\Delta > 0, N > 0$, we say that Σ is $SISS_L(\Delta, N)$ if, whenever $0 < \delta \leq \Delta$, it follows that, if $e : [0, +\infty) \to \mathbb{R}^m$ is bounded, measurable, and eventually bounded by δ, then every maximally defined solution of $\dot{x} = f(x, e(t))$ is eventually bounded by $N\delta$. A system is $SISS_L$ ("SISS with linear gain") if it is $SISS_L(\Delta, N)$ for some $\Delta > 0, N > 0$. A differential equation $\dot{x} = f(x)$ will be called $SISS$, $SISS_L(\Delta, N)$, or $SISS_L$, if the control system $\dot{x} = f(x) + u$ is, respectively, $SISS$, $SISS_L(\Delta, N)$ or $SISS_L$.

Remark 2.1 We will frequently use the fact that *if a system* $\dot{x} = f(x)$ *is* $SISS_L(\Delta, N)$, *and* $\lambda > 0$, *then the system* $\dot{x} = \lambda f(\frac{x}{\lambda})$ *is* $SISS_L(\lambda\Delta, N)$. To prove this, assume that x takes values in $\mathbb{R}^n$, and let $e : [0, \infty) \to \mathbb{R}^n$ be bounded, measurable, and eventually bounded by a δ such that $\delta \leq \lambda\Delta$. Let $I \ni t \to x(t) \in \mathbb{R}^n$ be a maximally defined solution of $\dot{x}(t) = \lambda f(\frac{x(t)}{\lambda}) + e(t)$. Let $y(t) = \lambda^{-1}x(t)$ for $t \in I$. Then $\dot{y}(t) = f(y(t)) + \lambda^{-1}e(t)$, and it is clear that y is a maximally defined solution of this equation, since any extension of y to a larger interval yields in an obvious way an extension of x. Since $|\lambda^{-1}e(t)| \leq_{\text{ev}} \lambda^{-1}\delta \leq \Delta$, we see that there is a T such that $[T, \infty) \subseteq I$ and $|y(t)| \leq N\lambda^{-1}\delta$ for $t \geq T$. But then $|x(t)| \leq N\delta$ for $t \geq T$, and our conclusion follows. □

Remark 2.2 The terminology "SISS" should not be confused with the different –but closely related– notion of "input to state stability" (ISS) given in [10] and other recent papers. It should also be possible to restate and prove the results given in this paper in terms of the ISS property, but the property called here SISS was exactly the one needed for the induction step in the proof of Theorem 2.3. □

For a system $\dot{x} = f(x,u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, we say that a feedback $u = k(x)$ is *stabilizing* if 0 is a globally asymptotically stable equilibrium of the closed-loop system $\dot{x} = f(x, k(x))$. If, in addition, this closed-loop system is $SISS_L$, then we will say that k is *$SISS_L$-stabilizing.*

For a square matrix A, let $\mu(A) = s(A) + z(A)$, where $s(A)$ is the number of conjugate pairs of nonzero purely imaginary eigenvalues of A (counting multiplicity) and $z(A)$ is the multiplicity of zero as an eigenvalue of A. We recall that an eigenvalue λ of a matrix A is *stable* if $\mathrm{Re}\,\lambda < 0$, *unstable* if $\mathrm{Re}\,\lambda > 0$, and *critical* if $\mathrm{Re}\,\lambda = 0$. Our main result is:

Theorem 2.3 *Let Σ be a linear system $\dot{x} = Ax + Bu$ with state space $\mathbb{R}^n$ and input space $\mathbb{R}^m$. Assume that Σ is ANCBC, i.e. that Σ is stabilizable and A has no unstable eigenvalues. Let $\mu = \mu(A)$. Let $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_\mu)$ be an arbitrary sequence of bounded functions belonging to $\mathcal{S}$. Then there exists an m-tuple $\boldsymbol{l} = (l_1, \cdots, l_m)$ of nonnegative integers such that $|\boldsymbol{l}| = \mu$, for which there are $SISS_L$-stabilizing feedbacks $u = -k_{\mathcal{F}}(x)$ and $u = -k_{\mathcal{G}}(x)$ such that $k_{\mathcal{F}} \in \mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$, $k_{\mathcal{G}} \in \mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$. Moreover, $k_{\mathcal{F}}$ and $k_{\mathcal{G}}$ can be chosen so that the linearizations at 0 of the corresponding closed-loop systems are asymptotically stable.*

Precise procedures to compute these feedbacks, and diagrams describing their structure, will be discussed in later sections.

Remark 2.4 The definitions of $\mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$ and $\mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$ clearly imply that every member of $\mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma}) \cup \mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$ is differentiable at the origin, so the linearizations referred to in the statement of Theorem 2.3 are well defined. □

Remark 2.5 We chose to formulate our main result in terms of the $SISS_L$ property for two main reasons, namely, (a) that, even if we were only interested in stabilization, our inductive proof of Theorem 2.3 requires that we prove the $SISS_L$ property at each step in order to carry out the induction, and (b) because it will be needed later in our proof of Theorem 7.1 on output feedback. □

Remark 2.6 *A stabilizing feedback need not have the SISS property, even if it is linear near the origin.* To illustrate this, consider the double integrator $\Sigma : \dot{x} = y, \dot{y} = u$. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be an odd continuous function such that $s\sigma(s) > 0$ for $s \neq 0$, $\sigma(s) = s$ for $|s| < \frac{1}{2}$ and $\sigma(s) = \frac{1}{s}$ for $s > 1$. Then the feedback $u = -\sigma(x+y)$ stabilizes Σ. (This can be proved by verifying that $V(x,y) = \int_0^{x+y} \sigma(s)\,ds + \frac{1}{2}y^2$ is a Lyapunov function for the closed-loop system with $u = -\sigma(x+y)$, and applying the LaSalle Invariance Principle.) Let $e(t) = \sigma\Big(\log(t+1) + \frac{1}{t+1}\Big) - \frac{1}{(t+1)^2}$. Then clearly $e(t) \to 0$ as $t \to \infty$. But not every solution of $\dot{x} = y$, $\dot{y} = -\sigma(x+y) + e(t)$ converges to zero. For example, $x(t) = \log(t+1)$, $y(t) = \frac{1}{t+1}$ is a solution, but $x(t) \to \infty$ as $t \to \infty$. □

3 Two Technical Lemmas

The proof of Theorem 2.3 will be based on two lemmas.

Lemma 3.1 *Let $\Sigma : \dot{x} = Ax + bu$ be an n-dimensional linear single-input system. Suppose that (A, b) is a controllable pair and all the eigenvalues of A are critical. Fix a $\nu > 0$.*

(i) If 0 is an eigenvalue of A, then there is a linear change of coordinates $Tx = (y_1, \cdots, y_n)' = (\bar{y}', y_n)'$ of $\mathbb{R}^n$ that puts Σ in the form

$$\begin{aligned} \dot{\bar{y}} &= A_1\bar{y} + b_1(y_n + \nu u) \ , \\ \dot{y}_n &= u \ , \end{aligned} \tag{3.1}$$

where the pair (A_1, b_1) is controllable and y_n is a scalar variable.

(ii) If A has an eigenvalue of the form $i\omega$, with $\omega > 0$, then there is a linear change of coordinates $Tx = (y_1, \cdots, y_n)' = (\bar{y}', y_{n-1}, y_n)'$ of $\mathbb{R}^n$ that puts Σ in the form:

$$\begin{aligned} \dot{\bar{y}} &= A_1\bar{y} + b_1(y_n + \nu u) \ , \\ \dot{y}_{n-1} &= \omega y_n \ , \\ \dot{y}_n &= -\omega y_{n-1} + u \ , \end{aligned} \tag{3.2}$$

where the pair (A_1, b_1) is controllable and y_{n-1}, y_n are scalar variables.

PROOF. We first prove (i). If 0 is an eigenvalue of A, then there exists a nonzero n-dimensional row vector v such that $vA = 0$. Let $\xi : \mathbb{R}^n \to \mathbb{R}$ be the linear function $x \to vx$. Then $\dot{\xi} = (vb)u$ along trajectories of Σ. Controllability of (A, b) implies that $vb \neq 0$. So we may assume that $vb = 1$. Make a linear change of coordinates $Tx = (\bar{z}', z_n)'$ so that $z_n \equiv \xi$. Then the system equations are of the form $\dot{\bar{z}} = A_1\bar{z} + z_n\tilde{b}_1 + u\tilde{b}_2$, $\dot{z}_n = u$. It is clear that every eigenvalue of A_1 also has zero real part. Now change coordinates again by letting $\bar{y} = \bar{z} + z_n\tilde{b}_3$, $y_n = z_n$, where the vector $\tilde{b}_3$ will be chosen below. Then the system equations become $\dot{\bar{y}} = A_1\bar{y} + y_n(\tilde{b}_1 - A_1\tilde{b}_3) + u(\tilde{b}_2 + \tilde{b}_3)$, $\dot{y}_n = u$. Choose $\tilde{b}_3$ to be a solution of $\tilde{b}_2 + \tilde{b}_3 = \nu(\tilde{b}_1 - A_1\tilde{b}_3)$, i.e. $\nu A_1\tilde{b}_3 + \tilde{b}_3 = \nu\tilde{b}_1 - \tilde{b}_2$. (This is possible because $\nu A_1 + I$ is nonsingular.) Let $b_1 = \tilde{b}_1 - A_1\tilde{b}_3$. The equations now become $\dot{\bar{y}} = A_1\bar{y} + (y_n + \nu u)b_1$, $\dot{y}_n = u$, as desired.

We now prove (ii). Let $\omega > 0$ be such that $i\omega$ is an eigenvalue of A. Then $-\omega^2$ is an eigenvalue of A^2. So there is a nonzero n-dimensional row vector $\tilde{v}$ such that $\tilde{v}A^2 = -\omega^2\tilde{v}$. Let $\tilde{w} = \omega^{-1}\tilde{v}A$. Then $\tilde{w}A = -\omega\tilde{v}$ and $\tilde{w}A^2 = -\omega^2\tilde{w}$. Moreover, $\tilde{w}$ cannot be a multiple of $\tilde{v}$ because, if $\tilde{w} = \lambda\tilde{v}$, then $\tilde{w}A = -\omega\tilde{v}$ would imply that $\tilde{v}A = -\lambda^{-1}\omega\tilde{v}$, so $-\lambda^{-1}\omega$ would be a nonzero real eigenvalue of A. So the linear span S of $\tilde{v}$ and $\tilde{w}$ is a two-dimensional subspace, all whose members v satisfy $vA^2 = -\omega^2 v$. In particular, we can choose $v \in S$ such that $vb = 0$ but $v \neq 0$. If we then define w by $w = \omega^{-1}vA$, we get $wA = -\omega v$. Moreover, wb cannot vanish for, if it did, the subspace $\{x : vx = wx = 0\}$ would contain b and be invariant under A, contradicting controllability. So, after multiplying both v and w by a constant, if necessary, we may assume that $wb = 1$. Let ξ, η be the linear functionals $x \to vx$, $x \to wx$. Then, along trajectories of Σ, $\dot{\xi} = \omega\eta$ and $\dot{\eta} = -\omega\xi + u$. Make a linear change of coordinates $Tx = (\bar{z}', z_{n-1}, z_n)'$ so that $z_{n-1} \equiv \xi$, $z_n \equiv \eta$. Then the system equations are of the form $\dot{\bar{z}} = A_1\bar{z} + z_{n-1}\tilde{b}_1 + z_n\tilde{b}_2 + u\tilde{b}_3$, $\dot{z}_{n-1} = \omega z_n$, $\dot{z}_n = -\omega z_{n-1} + u$, and every eigenvalue of A_1 has zero real part. Now change coordinates again by letting $\bar{y} = \bar{z} + z_{n-1}\tilde{b}_4 + z_n\tilde{b}_5$, $y_{n-1} = z_{n-1}$, $y_n = z_n$, where the vectors $\tilde{b}_4$, $\tilde{b}_5$ will be chosen below. Then the system equations become

$$\begin{aligned} \dot{\bar{y}} &= A_1\bar{y} + y_{n-1}(\tilde{b}_1 - A_1\tilde{b}_4 - \omega\tilde{b}_5) + y_n(\tilde{b}_2 - A_1\tilde{b}_5 + \omega\tilde{b}_4) + u(\tilde{b}_3 + \tilde{b}_5), \\ \dot{y}_{n-1} &= \omega y_n \ , \\ \dot{y}_n &= -\omega y_{n-1} + u. \end{aligned} \tag{3.3}$$

If we could choose $\tilde{b}_4$, $\tilde{b}_5$ such that

$$\tilde{b}_1 - A_1\tilde{b}_4 - \omega\tilde{b}_5 = 0 \quad \text{and} \quad \tilde{b}_3 + \tilde{b}_5 = \nu(\tilde{b}_2 - A_1\tilde{b}_5 + \omega\tilde{b}_4), \tag{3.4}$$

then we could let $b_1 = \tilde{b}_2 - A_1\tilde{b}_5 + \omega\tilde{b}_4$, and (3.3) would become $\dot{\bar{y}} = A_1\bar{y} + (y_n + \nu u)b_1$, $\dot{y}_{n-1} = \omega y_n$, $\dot{y}_n = -\omega y_{n-1} + u$, as desired. To prove the existence of $\tilde{b}_4$ and $\tilde{b}_5$, we rewrite the second equation of (3.4) as $(\nu A_1 + I)\tilde{b}_5 = \nu\tilde{b}_2 - \tilde{b}_3 + \nu\omega\tilde{b}_4$, multiply both sides by ω, and plug in the value of $\omega\tilde{b}_5$ given by the first equation of (3.4), namely, $\omega\tilde{b}_5 = \tilde{b}_1 - A_1\tilde{b}_4$. We end up with the equation $(\nu A_1^2 + A_1 + \nu\omega^2 I)\tilde{b}_4 = \nu A_1\tilde{b}_1 + \tilde{b}_1 - \nu\omega\tilde{b}_2 + \omega\tilde{b}_3$. Since all the eigenvalues of A_1 are critical, the matrix $\nu A_1^2 + A_1 + \nu\omega^2 I$ is nonsingular, so $\tilde{b}_4$ exists. □

Lemma 3.2 *Let $\omega > 0$. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $s\sigma(s) > 0$ whenever $s \neq 0$, $\liminf_{|s|\to\infty} |\sigma(s)| > 0$, $\liminf_{s\to 0}(\frac{\sigma(s)}{s}) > 0$, and $\limsup_{s\to 0}(\frac{\sigma(s)}{s}) < \infty$. Then the 3-input control system*

$$\dot{x} = \omega y + v_3, \quad \dot{y} = -\omega x - \sigma(y - v_1) + v_2 \tag{3.5}$$

has the $SISS_L$ property.

Proof. First, we remark that, if e_1, e_2, e_3 are bounded measurable real-valued functions on $[0,\infty)$, then every maximally defined solution $\gamma = (x(\cdot), y(\cdot)) : I \to \mathbb{R}^2$ of the system

$$\dot{x} = \omega y + e_3(t)\,, \quad \dot{y} = -\omega x - \sigma(y - e_1(t)) + e_2(t), \tag{3.6}$$

defined on a subinterval I of $[0,\infty)$ such that $0 \in I$, is in fact defined on $[0,\infty)$. (To see this, assume that $I = [0,a)$ with $a < \infty$. Let $V(x,y) = x^2 + y^2$. Then $\lim_{t\to a-} V(\gamma(t)) = +\infty$. On the other hand, the derivative $\dot{V}$ of V along a solution is easily seen to be equal to $2x(t)e_3(t) + 2y(t)e_2(t) - 2y(t)\sigma(y(t) - e_1(t))$. Let $M > 0$ be such that $|e_i(t)| \leq M$ for all $i = 1,2,3$, $t \in [0,\infty)$, and let $N > 0$ be such that $|\sigma(s)| \leq N$ whenever $|s| \leq 2M$. Then $y\sigma(y - e_1(t)) \geq 0$ if $|y| \geq M$, and $|y\sigma(y - e_1(t))| \leq MN$ if $|y| \leq M$, so $\dot{V} \leq V + 2M^2 + 2NM$. But then $t \to V(\gamma(t))$ is bounded on I, and we have reached a contradiction.)

In view of the above observation, it clearly suffices to find positive constants $\Delta, \nu_1, \nu_2, \nu_3$ with the property that

(P) whenever $0 < \delta < \Delta$, and e_1, e_2, e_3 are bounded measurable real-valued functions on $[0,\infty)$ such that

$$\limsup_{t\to+\infty} |e_i(t)| \leq \frac{\delta}{\nu_i} \quad \text{for } i = 1,2,3\,, \tag{3.7}$$

then, if $\gamma = (x(\cdot), y(\cdot)) : [0,\infty) \to \mathbb{R}^2$ is any solution of (3.6), it follows that $\limsup_{t\to+\infty} ||\gamma(t)|| \leq \delta$.

Actually, it suffices to find $\Delta, \nu_1, \nu_2, \nu_3$ such that

(I) if $0 < \delta < \Delta$, and $\gamma : [0,\infty) \to \mathbb{R}^2$ is a solution of (3.6) for some triple of functions $e_1, e_2, e_3 : [0,\infty) \to \mathbb{R}$ such that $||e_i||_{L^\infty} \leq \frac{\delta}{\nu_i}$ for $i = 1,2,3$, then $\limsup_{t\to+\infty} ||\gamma(t)|| \leq \delta$.

(Indeed, if (I) holds, e_1, e_2, e_3 are bounded measurable on $[0,\infty)$, (3.7) holds for some $\delta \in (0,\Delta)$, and $\gamma : [0,\infty) \to \mathbb{R}^2$ is a solution of (3.6), then for every $\tilde{\delta} \in (\delta,\Delta)$ there is a $T \geq 0$ such that the restrictions $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ of e_1, e_2, e_3 to $[T,\infty)$ satisfy $||\tilde{e}_i||_{L^\infty} \leq \frac{\tilde{\delta}}{\nu_i}$. Applying (I) to the curve $t \to \gamma(t+T)$ we find that $\limsup_{t\to+\infty} ||\gamma(t)|| \leq \tilde{\delta}$.)

For $a > 0$, $\omega > 0$, we let $A_{a,\omega}$ be the matrix $\begin{pmatrix} 0 & \omega \\ -\omega & -a \end{pmatrix}$, and define $P_{a,\omega}$ to be the symmetric matrix P that satisfies the Lyapunov equation $A_{a,\omega}^\dagger P + P A_{a,\omega} = -2I$, so

$$P_{a,\omega} = \begin{pmatrix} \frac{a}{\omega^2} + \frac{2}{a} & \frac{1}{\omega} \\ \\ \frac{1}{\omega} & \frac{2}{a} \end{pmatrix} . \tag{3.8}$$

Then $P > 0$. We write $B_1 = (1,0)$, $B_2 = (0,1)$, $\beta_{i,a,\omega} = ||B_i P_{a,\omega}||$, so

$$\beta_{1,a,\omega} = \sqrt{\frac{5}{\omega^2} + \frac{a^2}{\omega^4} + \frac{4}{a^2}} \quad \text{and} \quad \beta_{2,a,\omega} = \frac{1}{a\omega}\sqrt{a^2 + 4\omega^2} .$$

We let $p^-_{a,\omega}$ and $p^+_{a,\omega}$ denote, respectively, the smallest and largest eigenvalues of $P_{a,\omega}$. Then $||P_{a,\omega}|| = p^+_{a,\omega}$, and

$$p^\pm_{a,\omega} = \frac{a}{2}\beta^2_{2,a,\omega} \pm \frac{a}{2\omega}\beta_{2,a,\omega} . \tag{3.9}$$

For each $\Lambda > 0$, we define three numbers $a(\Lambda)$, $b(\Lambda)$, $c(\Lambda)$ by

$$a(\Lambda) = \inf\left\{\frac{\sigma(s)}{s} : |s| \leq \Lambda\right\}, \quad b(\Lambda) = \sup\left\{\frac{\sigma(s)}{s} : |s| \leq \Lambda\right\}, \quad c(\Lambda) = \inf\{|\sigma(s)| : |s| \geq \Lambda\} . \tag{3.10}$$

Our hypotheses on σ imply that $a(\Lambda)$, $b(\Lambda)$ and $c(\Lambda)$ are finite and nonzero and, moreover, $a(\Lambda) \leq b(\Lambda)$, with equality holding iff σ is linear on $[-\Lambda,\Lambda]$.

From now on we fix a $\Lambda_0 > 0$ and an $\omega > 0$, and let $a = a(\Lambda_0)$, $b = b(\Lambda_0)$, $c = c(\Lambda_0)$, $A = A_{a,\omega}$, $P = P_{a,\omega}$, $p^\pm = p^\pm_{a,\omega}$, $\beta_i = \beta_{i,a,\omega}$. We let

$$\bar{\Lambda} = \min\left(\Lambda_0, \frac{c}{a}\right) . \tag{3.11}$$

Then

$$\Lambda \leq \bar{\Lambda} \Rightarrow \Big(|\sigma(s)| \geq a\Lambda \ \text{ for all } \ s \ \text{ such that } \ |s| \geq \Lambda\Big) . \tag{3.12}$$

This implies that

$$\text{if } \ 0 < \Lambda \leq \bar{\Lambda} , \ \text{ then } \ |as - \sigma(s)| \leq \frac{s\,\sigma(s)}{\Lambda} + (b-a)|s|\chi_\Lambda(s) \ \text{ for all } \ s \in \mathbb{R} , \tag{3.13}$$

where χ_Λ is the indicator function of the interval $[-\Lambda,\Lambda]$. We will also need the trivial estimate

$$|\sigma(s)| \leq \frac{s\,\sigma(s)}{\Lambda} + b|s|\chi_\Lambda(s) \ \text{ whenever } \ s \in \mathbb{R} , \ 0 < \Lambda \leq \bar{\Lambda} . \tag{3.14}$$

For $K > 0$, $0 < \Lambda \leq \bar{\Lambda}$, we define a function $V_{K,\Lambda} : \mathbb{R}^2 \to \mathbb{R}$ by

$$V_{K,\Lambda}(X) = K||X||^3 + \Lambda\, X^\dagger P X \quad \text{for} \quad X = (x, y)^\dagger \in \mathbb{R}^2\,. \tag{3.15}$$

Then the derivative $\dot{V}_{K,\Lambda}$ of $V_{K,\Lambda}$ along a solution of (3.6) is given by

$$\dot{V}_{K,\Lambda} = 3K||X||\Big(EX - \sigma(z)y\Big) - 2\Lambda||X||^2 + 2\Lambda(E + (ay - \sigma(z))B_2)PX\,, \tag{3.16}$$

where $z = y - e_1$, $E = (e_3, e_2)$. Writing $y = z + e_1$, and using (3.13) and $|B_i P X| \leq \beta_i||X||$, we get

$$\begin{aligned} \dot{V}_{K,L} \quad \leq \quad & 3K||X||^2||E|| + 3K||X||\,|e_1|\,|\sigma(z)| - 3K||X||\,z\sigma(z) \\ & - 2\Lambda||X||^2 + 2\Lambda(\beta_1|e_3| + \beta_2|e_2|)||X|| + 2\Lambda a\beta_2\,|e_1|\,||X|| \\ & + 2\beta_2||X||\,z\,\sigma(z) + 2\Lambda\beta_2(b-a)||X||\,|z|\chi_\Lambda(z)\,. \end{aligned} \tag{3.17}$$

It is clear that $|z|\chi_\Lambda(z) \leq \Lambda$. In addition, (3.14) implies that the term $|e_1|\,|\sigma(z)|$ can be replaced by $|e_1|\,\frac{z\sigma(z)}{\Lambda} + |e_1|\,b|z|\chi_\Lambda(z)$, which is bounded by $|e_1|\,\frac{z\sigma(z)}{\Lambda} + |e_1|\,b\Lambda$. Assume $|e_i| \leq C_i\Lambda$ for $i = 1, 2, 3$, where the C_i are constants to be chosen below, and write $||X|| = \xi\Lambda$. Let $C = \sqrt{C_2^2 + C_3^2}$. Then (3.17) implies

$$\dot{V}_{K,L} \leq \mathcal{K}_1\Lambda\xi z\sigma(z) + \Lambda^3\xi(\mathcal{K}_2 - \mathcal{K}_3\xi)\,, \tag{3.18}$$

where $\mathcal{K}_1 = 2\beta_2 - 3K(1 - C_1)$, and

$$\mathcal{K}_2 = 3KC_1b + 2C_3\beta_1 + 2C_2\beta_2 + 2C_1a\beta_2 + 2\beta_2(b-a)\,, \qquad \mathcal{K}_3 = 2 - 3KC\,. \tag{3.19}$$

Now choose the C_i such that

$$0 \leq C_1 < 1\,, \quad C_2 \geq 0\,, \quad C_3 \geq 0\,, \quad \text{and} \quad C_1 + \beta_2\sqrt{C_2^2 + C_3^2} < 1\,, \tag{3.20}$$

and observe that (3.20) guarantees the existence of a K such that

$$3K(1 - C_1) \geq 2\beta_2 \quad \text{and} \quad 3CK < 2\,. \tag{3.21}$$

Choose K so that (3.21) holds. Then $\mathcal{K}_1 \leq 0$, $\mathcal{K}_2 > 0$, and $\mathcal{K}_3 > 0$. So, if we define

$$\Xi = \frac{\mathcal{K}_2}{\mathcal{K}_3}\,, \tag{3.22}$$

we see that if $\xi > \Xi$ then

$$\dot{V}_{K,\Lambda}(X) \leq -\Lambda^3\mathcal{K}_3\xi(\xi - \Xi) < 0\,. \tag{3.23}$$

Now let $\mathcal{T}_\Lambda$ be the set of all trajectories $\gamma : [0, \infty) \to \mathbb{R}^2$ of (3.6) corresponding to inputs $e_i(t)$ that satisfy and $||e_i||_{L^\infty} \leq C_i\Lambda$ for $i = 1, 2, 3$. For $r > 0$, write $\varphi(r) = Kr^3 + \Lambda p^+ r^2$, and let $S_r = \{X : V_{K,\Lambda}(X) \leq \varphi(r)\}$. Choose $r > \Xi\Lambda$. Then the equality $V_{K,\Lambda}(X) = \varphi(r)$ implies $||X|| \geq r$, so (3.23) implies that S_r is forward-invariant under all trajectories in $\mathcal{T}_\Lambda$. If $\gamma \in \mathcal{T}_\Lambda$, then (3.23) implies that $\gamma(t) \in S_r$ for all sufficiently large t. On the other hand, every $X \in S_r$ satisfies the inequalities $K||X||^3 + p^-\Lambda||X||^2 \leq V_{K,\Lambda}(X) \leq Kr^3 + p^+\Lambda r^2$. Therefore

$$\limsup_{t\to+\infty}\,(K||X(t)||^3 + p^-\Lambda||X(t)||^2) \leq Kr^3 + p^+\Lambda r^2\,.$$

Since this is true for all $r > \Xi\Lambda$, we can conclude that

$$\lim_{t\to+\infty} \sup (K||X(t)||^3 + p^-\Lambda||X(t)||^2) \le \Lambda^3(K\,\Xi^3 + p^+\,\Xi^2)\,.$$

Now let $\tilde{\Xi}$ be the number characterized by

$$\tilde{\Xi} > 0 \quad \text{and} \quad K\tilde{\Xi}^3 + p^-\tilde{\Xi}^2 = K\Xi^3 + p^+\,\Xi^2\,. \tag{3.24}$$

Then

$$\lim_{t\to+\infty} \sup (K||X(t)||^3 + p^-\Lambda||X(t)||^2) \le \Lambda^3(K\,\tilde{\Xi}^3 + p^-\tilde{\Xi}^2)\,.$$

Writing $||X(t)|| = \Lambda\xi(t)$, we find that $\limsup_{t\to+\infty}(K\xi(t)^3+p^-\xi(t)^2) \le K\tilde{\Xi}^3+p^-\tilde{\Xi}^2$. Therefore $\limsup_{t\to+\infty}\xi(t) \le \tilde{\Xi}$, and then $\limsup_{t\to+\infty}||X(t)|| \le \tilde{\Xi}\Lambda$. So we have shown that (I) holds with $\Delta = \tilde{\Xi}\bar{\Lambda}$, and $\nu_i = \frac{\tilde{\Xi}}{C_i}$ for $i = 1, 2, 3$. □

Remark 3.3 The proof of Lemma 3.2 actually yields explicit formulas for possible choices of ν_1, ν_2, ν_3 and Δ. The constants C_i can be chosen in an arbitrary fashion, subject only to the condition that (3.20) hold, and then one can take K to be any number that satisfies (3.21). Formulas (3.19), (3.22) and (3.24) then determine $\mathcal{K}_2$, $\mathcal{K}_3$, Ξ and $\tilde{\Xi}$. We can then let $\Delta = \tilde{\Xi}\bar{L}$, $\nu_i = \frac{\tilde{\Xi}}{C_i}$.

A particularly simple choice of Δ and the ν_i results from taking

$$C_1 = \frac{1}{3}\,,\;\; C = \frac{1}{3\beta_2}\,,\;\; C_2 = \alpha_2 C\,,\;\; C_3 = \alpha_3 C\,,\;\; K = \beta_2\,, \tag{3.25}$$

where $\alpha_2 \ge 0$, $\alpha_3 \ge 0$, and $\alpha_2^2 + \alpha_3^2 = 1$. Then (3.20) and (3.21) clearly hold, and $\mathcal{K}_3 = 1$. So

$$\Xi = \mathcal{K}_2 = \frac{2\alpha_2}{3} + \frac{2\alpha_3\beta_1}{3\beta_2} + \Big(3b - \frac{4a}{3}\Big)\beta_2\,. \tag{3.26}$$

The value of $\tilde{\Xi}$ can then be found by solving (3.24). This equation implies the inequalities $\Xi < \tilde{\Xi} < \sqrt{\frac{p^+}{p^-}}\Xi$, and it is easy to see that

$$\sqrt{\frac{p^+}{p^-}} = \frac{\sqrt{a^2 + 4\omega^2} + a}{2\omega} \le 1 + \frac{a}{\omega}\,,$$

from which we get the bounds

$$\Xi < \tilde{\Xi} < \Big(1 + \frac{a}{\omega}\Big)\Xi\,.$$

We can then take $\nu_1 = 3\tilde{\Xi}$, $\alpha_2\nu_2 = \alpha_3\nu_3 = 3\beta_2\tilde{\Xi}$, and $\Delta = \tilde{\Xi}\bar{\Lambda}$. Moreover, it is clear that if we replace Δ by a smaller number, and the ν_i by larger ones, then the resulting values still have the desired properties. So we can also take $\Delta = \Xi\bar{\Lambda}$, $\nu_1 = 3(1 + \frac{a}{\omega})\Xi$, $\alpha_2\nu_2 = \alpha_3\nu_3 = 3\beta_2(1 + \frac{a}{\omega})\Xi$. Using the value of Ξ computed earlier, we get

$$\Delta = \frac{\bar{\Lambda}}{3}\left(2\alpha_2 + \frac{2\alpha_3\beta_1}{\beta_2} + (9b - 4a)\beta_2\right), \quad \nu_1 = \left(2\alpha_2 + \frac{2\alpha_3\beta_1}{\beta_2} + (9b - 4a)\beta_2\right)\left(1 + \frac{a}{\omega}\right), \tag{3.27}$$

$$\alpha_2\nu_2 = \alpha_3\nu_3 = \beta_2\nu_1\,. \tag{3.28}$$

A particularly simple formula for a possible choice of Δ is obtained by replacing the value of Δ given in (3.27) by the lower bound obtained using the inequalities $b \geq a$, $a\beta_2 \geq 2$, $\beta_1 \geq \beta_2$, and $\alpha_2 + \alpha_3 \geq 1$. This leads to the choice

$$\Delta = 4\bar{\Lambda}\,. \tag{3.29}$$

Remark 3.4 In Remark 3.3 we allow the possibility that $\alpha_i = 0$ for $i = 2$ or 3. In that case, the corresponding constant ν_i is infinite, which means that the corresponding input e_i has to vanish. For example, if we choose $\alpha_2 = 1$, $\alpha_3 = 0$, we obtain constants Δ, ν_1, ν_2 with the property that, whenever $0 < \delta < \Delta$, and e_1, e_2 are bounded measurable real-valued functions on $[0,\infty)$ such that $\limsup_{t\to+\infty} |e_i(t)| \leq \frac{\delta}{\nu_i}$ for $i = 1, 2$, then, if $\gamma = (x(\cdot), y(\cdot)) : [0,\infty) \to \mathbb{R}^2$ is any solution of $\dot{x} = \omega y$, $\dot{y} = -\omega x - \sigma(y - e_1(t)) + e_2(t)$, it follows that $\limsup_{t\to+\infty} ||\gamma(t)|| \leq \delta$.

Corollary 3.5 *Let $\sigma : \mathbb{R} \to \mathbb{R}$ satisfy the assumptions of Lemma 3.2. For $n = 1$ or 2, let J be an $n \times n$ skew-symmetric matrix. Let $b = 1$ if $n = 1$, and $b = (0,1)'$ if $n = 2$. If $n = 2$, assume that $J \neq 0$. Then the control system*

$$\dot{x} = Jx - \sigma(x_n - v_1)b + v_2\,, \quad x \in \mathbb{R}^n\,,\ v_1 \in \mathbb{R}\,,\quad v_2 \in \mathbb{R}^n\,, \tag{3.30}$$

is $SISS_L$.

PROOF. When $n = 2$, the conclusion follows from Lemma 3.2. Assume $n = 1$. Pick $\Lambda_0, \bar{\Lambda}, a, b, c$ as in the proof of Lemma 3.2. Let $\Delta = 2\bar{L}$, $\nu_1 = 2$, $\nu_2 = \frac{3}{a}$. Let $0 < \delta < \Delta$. Let $V(x) = x^2$. Let $x(\cdot) : [0,\infty) \to \mathbb{R}$ be a solution of $\dot{x} = -\sigma(x - e_1(t)) + e_2(t)$ for some pair of measurable real-valued functions on $[0,\infty)$ such that $\limsup_{t\to\infty} |e_i(t)| \leq \frac{\delta}{\nu_i}$ for $i = 1, 2$. Pick $\tilde{\delta}$ such that $\delta < \tilde{\delta} < \Delta$, and then $T \geq 0$ such that $|e_i(t)| \leq \frac{\tilde{\delta}}{\nu_i}$ for $i = 1, 2$ and $t \geq T$. Then, if $|x(t)| \geq \tilde{\delta}$, it follows that

$$\dot{V}(x(t)) = -2x(t)(\sigma(x(t) - e_1(t)) - e_2(t)) < -\frac{a\tilde{\delta}^2}{3}\,.$$

Therefore, $|x(t)| \leq \tilde{\delta}$ for sufficiently large t. Since $\tilde{\delta}$ can be taken arbitrarily close to δ, we conclude that $\limsup |x(t)| \leq \delta$. □

Corollary 3.6 *Let J, b, n, σ be as in the statement of Corollary 3.5. Let $e_1 : [0,\infty) \to \mathbb{R}$ and $e_2 : [0,\infty) \to \mathbb{R}^n$ be bounded measurable functions such that $\lim_{t\to\infty} e_i(t) = 0$. Let $\gamma : [0,\infty) \to \mathbb{R}^n$ be a solution of (3.30) with $v_1 = e_1(t)$, $v_2 = e_2(t)$. Then $\lim_{t\to\infty} \gamma(t) = 0$.* □

Remark 3.7 The above proof of Lemma 3.2 is an adaptation of ideas of [5]. Using a similar technique, it can be shown that Corollary 3.5 also holds in higher dimensions, for controllable pairs (J, b) such that J is skew-symmetric, although in the general case it becomes more difficult to get explicit formulas for the constants. □

4 The Proof of Theorem 2.3

First, we notice that under the conditions of the theorem there exists a linear change of coordinates of the state space that transforms Σ into the block form

$$\Sigma: \quad \begin{cases} \dot{x}_1 &= A_1 x_1 + B_1 u, \ x_1 \in \mathbb{R}^{n_1}, \\ \dot{x}_2 &= A_2 x_2 + B_2 u, \ x_2 \in \mathbb{R}^{n_2}, \end{cases}$$

where (i) $n_1 + n_2 = n$, (ii) all the eigenvalues of A_1 are critical, (iii) all the eigenvalues of A_2 are stable, and (iv) (A_1, B_1) is a controllable pair. Suppose that we find a $SISS_L$-stabilizing feedback $u = -k(x_1)$ of Type $\mathcal{F}$ or Type $\mathcal{G}$ for the system $\dot{x}_1 = A_1 x_1 + B_1 u$ such that the linearization at 0 of the resulting closed-loop system is asymptotically stable. Then it is clear that this same feedback law will work for Σ as well. Thus, in order to stabilize Σ, it is enough to stabilize the "critical subsystem" $\dot{x}_1 = A_1 x_1 + B_1 u$. Without loss of generality, in our proof of the theorem we will suppose that Σ is already in this form.

We start with the single-input case, and prove the theorem by induction on the dimension of the system. As discussed earlier, we may assume that all the eigenvalues of A have zero real part and the pair (A, B) is controllable.

For dimension zero, there is nothing to prove. Now assume that we are given a single-input n-dimensional system, $n \geq 1$, and suppose that Theorem 2.3 has been established for all single-input systems of dimension $\leq n-1$. We consider separately the following two possibilities:

(i) zero is an eigenvalue of A,

(ii) zero is not an eigenvalue of A.

Recall that $\mu = \mu(A)$. We want to prove that for any finite sequence $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_\mu)$ of bounded functions in $\mathcal{S}$ there are $SISS_L$-stabilizing feedbacks $u = -k_{\mathcal{F}}(x)$ and $u = -k_{\mathcal{G}}(x)$ such that $k_{\mathcal{F}} \in \mathcal{F}_n(\boldsymbol{\sigma})$, $k_{\mathcal{G}} \in \mathcal{G}_n(\boldsymbol{\sigma})$, and the linearizations at 0 of the resulting closed-loop systems are asymptotically stable.

Assume that $\sigma'_\mu(0) = \kappa > 0$. In Case (i) we apply Part (i) of Lemma 3.1 and rewrite our system in the form

$$\begin{aligned} \dot{\bar{y}} &= A_1 \bar{y} + (y_n + \tfrac{1}{\kappa} u) b_1, \\ \dot{y}_n &= u, \end{aligned} \tag{4.1}$$

where $\bar{y} = (y_1, \cdots, y_{n-1})'$. (Notice that if $n = 1$ then only the second equation occurs.) In Case (ii), since $n > 0$, A has an eigenvalue of the form $i\omega$, with $\omega > 0$. So we apply Part (ii) of Lemma 3.1 and make a linear transformation that puts Σ in the form

$$\begin{aligned} \dot{\bar{y}} &= A_1 \bar{y} + (y_n + \tfrac{1}{\kappa} u) b_1, \\ \dot{y}_{n-1} &= \omega y_n, \\ \dot{y}_n &= -\omega y_{n-1} + u, \end{aligned} \tag{4.2}$$

where $\bar{y} = (y_1, y_2, \cdots, y_{n-2})'$. (Naturally, in the special case when $n = 2$, the first equation will be missing.) So in both cases Lemma 3.1 enables us to rewrite our system in the form

$$\begin{aligned} \dot{\bar{y}} &= A_1 \bar{y} + (y_n + \tfrac{1}{\kappa} u) b_1, \\ \dot{\tilde{y}} &= J \tilde{y} + u b_0, \end{aligned} \tag{4.3}$$

where J is a skew-symmetric matrix, (J, b_0) is a controllable pair, and we have $\tilde{y} = y_n, b_0 = 1$ in Case (i) and $\tilde{y} = (y_{n-1}, y_n)', b_0 = (0,1)'$ in Case (ii).

Let $u = -\sigma_\mu(y_n - \xi v) - \eta v$, where ξ and η are constants such that $\xi\eta = 0, \xi + \eta = 1$, and v is to be chosen later. Define $g(s) = \sigma_\mu(s) - \kappa s$. Then

$$y_n + \frac{1}{\kappa}u = (\xi - \frac{1}{\kappa}\eta)v - \frac{1}{\kappa}g(y_n - \xi v)\,.$$

Therefore the first equation of (4.3) becomes

$$\dot{\bar{y}} = A_1\bar{y} + vb_2 - g(y_n - \xi v)b_3\,, \tag{4.4}$$

where $b_2 = (\xi - \frac{1}{\kappa}\eta)b_1$, $b_3 = \frac{1}{\kappa}b_1$. Notice that $\xi - \frac{1}{\kappa}\eta \neq 0$, and therefore (A_1, b_2) is still a controllable pair. From the inductive hypothesis we know that, if $\mathcal{H}$ is either $\mathcal{F}$ or $\mathcal{G}$, then there exist $\bar{k} \in \mathcal{H}_n(\sigma_1, \cdots, \sigma_{\mu-1})$ such that the system $\dot{\bar{y}} = A_1\bar{y} - \bar{k}(\bar{y})b_2$ is $SISS_L(\bar{\Delta}, \bar{N})$ for some $\bar{\Delta}, \bar{N} > 0$, and the linearization at 0 of the closed-loop system is asymptotically stable. (For instance, in Case (ii) there exists a $SISS_L$-stabilizing feedback $u = -\bar{k}_{\mathcal{F}}$ for $\dot{\bar{y}} = A_1\bar{y} + b_2 v$ such that $\bar{k}_{\mathcal{F}} \in \mathcal{F}_{n-2}(\sigma_1, \cdots, \sigma_{\mu-1})$. Then $\bar{k}_{\mathcal{F}}$ can also be viewed in an obvious way as a function from $\mathbb{R}^n$ to $\mathbb{R}$. Thus $\bar{k}_{\mathcal{F}} \in \mathcal{F}_n(\sigma_1, \cdots, \sigma_{\mu-1})$.) Let $\lambda > 0$. Then $\dot{\bar{y}} = A_1\bar{y} - \lambda\bar{k}(\frac{\bar{y}}{\lambda})b_2$ is $SISS_L(\lambda\bar{\Delta}, \bar{N})$.

Also, since $\bar{k}$ is bounded and locally Lipschitz, and $\bar{k}(0) = 0$, it follows that there exist $L, M > 0$ such that the inequalities $|\bar{k}(\bar{y})| \leq L|\bar{y}|$ and $|\bar{k}(\bar{y})| \leq M$ hold for all $\bar{y}$.

On the other hand, the second equation of (4.3) becomes

$$\dot{\tilde{y}} = J\tilde{y} - \sigma_\mu(y_n - \xi v)b_0 - \eta v b_0\,. \tag{4.5}$$

Corollary 3.5 implies that there exist $\Delta_1, \nu_1, \nu_2 > 0$ such that

(A) *whenever $\delta \leq \Delta_1$, and $e_1, e_2 : [0,\infty) \to \mathbb{R}$ are bounded measurable functions such that $|e_i| \leq_{\text{ev}} \frac{\delta}{\nu_i}$, $i = 1, 2$, it follows that all the solutions of $\dot{\tilde{y}} = J\tilde{y} - \sigma_n(y_n - e_1)b_0 + e_2 b_0$ are eventually bounded by δ.*

Recall that σ_μ is differentiable at 0 and $\sigma'_\mu(0) = \kappa$. Then for every $\varepsilon > 0$ there exists $\Delta_2 > 0$ such that

$$|g(s)| \leq \varepsilon|s| \quad \text{for } |s| \leq \Delta_2\,. \tag{4.6}$$

Now, fix a positive number $\alpha < 1$. Choose $\varepsilon > 0$ such that

$$\varepsilon\Big(M(\xi\nu_1 + \eta\nu_2 + \xi) + \Delta_1\Big)||b_3|| < \alpha\bar{\Delta}\,, \tag{4.7}$$

and

$$\varepsilon\bar{N}L(\xi\nu_1 + \eta\nu_2 + \xi) < \frac{1}{4}\,. \tag{4.8}$$

Then there exists a $\Delta_2 > 0$ such that (4.6) is satisfied. (Notice that in the special case when σ_μ is linear near 0 the existence of a $\Delta_2 > 0$ such that (4.6) holds is also guaranteed if $\varepsilon = 0$, and then of course (4.7) and (4.8) hold as well. So in this case we can actually choose $\varepsilon = 0$.)

Now, choose $\delta_0 > 0$ so that

$$\delta_0 \leq \min\{\Delta_1, \alpha\Delta_2\}\,, \tag{4.9}$$

and let λ be such that $0 < \lambda < 1$ and

$$\lambda\xi M \leq (1-\alpha)\Delta_2\,, \tag{4.10}$$

$$\lambda(\xi\nu_1 + \eta\nu_2)M \leq \alpha\delta_0\,. \tag{4.11}$$

Then define

$$\Delta = \min\{\delta_0, \frac{(1-\alpha)\lambda\delta_0}{\nu_2}, (1-\alpha)\lambda\bar{\Delta}\}\,. \tag{4.12}$$

Now choose $v = -\lambda\bar{k}(\frac{\bar{y}}{\lambda})$. Then the closed-loop system arising from (4.4) and (4.5) is

$$\begin{aligned}\dot{\bar{y}} &= A_1\bar{y} - \lambda\bar{k}(\tfrac{\bar{y}}{\lambda})b_2 - g(y_n + \lambda\xi\bar{k}(\tfrac{\bar{y}}{\lambda}))b_3\,,\\ \dot{\tilde{y}} &= J\tilde{y} - \sigma_\mu(y_n + \lambda\xi\bar{k}(\tfrac{\bar{y}}{\lambda}))b_0 + \lambda\eta\bar{k}(\tfrac{\bar{y}}{\lambda})b_0\,.\end{aligned} \tag{4.13}$$

We will prove that (4.13) is $SISS_L(\Delta, N)$ for some $N > 0$. This will imply the conclusion of Theorem 2.3 for the single-input case, as we now show.

Notice that the linearization Σ_L of (4.13) at 0 is obviously asymptotically stable. (Indeed, if we linearize the first equation we just obtain the linearization at 0 of the system $\dot{\bar{y}} = A_1\bar{y} - \bar{k}(\bar{y})b_2$, and this linearization is asymptotically stable by the inductive hypothesis. If we linearize the second equation and discard the terms involving $\bar{y}$, we get the equation $\dot{\tilde{y}} = J\tilde{y} - \kappa y_n$, which is also asymptotically stable. The asymptotic stability of Σ_L follows easily.) As a consequence, the local asymptotic stability of (4.13) is guaranteed. To get a Type $\mathcal{F}$ feedback we take $\xi = 1, \eta = 0$. Then the feedback $u = -\sigma_\mu(y_n + \lambda\bar{k}(\frac{\bar{y}}{\lambda}))$ is as desired. To get a Type $\mathcal{G}$ feedback we take $\xi = 0, \eta = 1$. Then, for any $r > 0$, the closed-loop system corresponding to (4.3) and the feedback

$$u = -r\sigma_\mu(\frac{y_n}{r}) - r\lambda\bar{k}(\frac{\bar{y}}{r\lambda})\,, \tag{4.14}$$

is $SISS_L(r\Delta, N)$. Choosing r sufficiently small, we can insure that the sum of the coefficients of all saturations in (4.14) is bounded by 1. Therefore the feedback defined by (4.14) is as desired. This completes the proof that the conclusion of Theorem 2.3 for the single-input case follows if (4.13) is $SISS_L(\Delta, N)$ for some $N > 0$.

We now show that (4.13) is $SISS_L(\Delta, N)$ for some $N > 0$. Given $\delta \leq \Delta$, let $\bar{e}, \tilde{e}$ be bounded measurable vector-valued functions on $[0, \infty)$, eventually bounded by δ, and of the same dimensions as $\bar{y}, \tilde{y}$, respectively. We consider the system:

$$\begin{aligned}\dot{\bar{y}} &= A_1\bar{y} - \lambda\bar{k}(\tfrac{\bar{y}}{\lambda})b_2 - g(y_n + \lambda\xi\bar{k}(\tfrac{\bar{y}}{\lambda}))b_3 + \bar{e}\,,\\ \dot{\tilde{y}} &= J\tilde{y} - \sigma_\mu(y_n + \lambda\xi\bar{k}(\tfrac{\bar{y}}{\lambda}))b_0 + \lambda\eta\bar{k}(\tfrac{\bar{y}}{\lambda})b_0 + \tilde{e}\,.\end{aligned} \tag{4.15}$$

From (4.11) we get

$$\lambda\xi|\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\text{ev}} \lambda\xi M \leq \frac{\delta_0}{\nu_1}\,, \tag{4.16}$$

and (because $\delta \leq \frac{(1-\alpha)\lambda\delta_0}{\nu_2}$)

$$|\lambda\eta\bar{k}(\frac{\bar{y}}{\lambda})b_0 + \tilde{e}| \leq_{\text{ev}} \lambda\eta M + \delta \leq \frac{\delta_0}{\nu_2}\,. \tag{4.17}$$

So applying (A) to the second equation of (4.15), we conclude that

$$|\tilde{y}| \leq_{\rm ev} \max\{\lambda\xi M\nu_1, \lambda\eta M\nu_2 + \delta\nu_2\} \leq \lambda M(\xi\nu_1 + \eta\nu_2) + \delta\nu_2\,, \tag{4.18}$$

and therefore

$$|y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\rm ev} \lambda M(\xi\nu_1 + \eta\nu_2) + \delta\nu_2 + \lambda\xi M\,. \tag{4.19}$$

Inequality (4.11) and $\delta\nu_2 \leq (1-\alpha)\delta_0$ then imply that the right-hand side of (4.18) is bounded by δ_0. Therefore, it follows from (4.9) that

$$|\tilde{y}| \leq_{\rm ev} \min\{\Delta_1, \alpha\Delta_2\}\,. \tag{4.20}$$

Then (4.20) and (4.10) imply that $|y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\rm ev} \Delta_2$. From (4.6) we then get

$$|g(y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))| \leq_{\rm ev} \varepsilon|y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})|\,. \tag{4.21}$$

Since $\delta\nu_2 \leq (1-\alpha)\lambda\delta_0$ and $\delta_0 \leq \Delta_1$, using (4.19) we obtain

$$|y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\rm ev} \lambda\Big(M(\xi\nu_1 + \eta\nu_2 + \xi) + (1-\alpha)\Delta_1\Big)\,,$$

and then

$$|g(y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))b_3 + \bar{e}| \leq_{\rm ev} \lambda\varepsilon\Big(M(\xi\nu_1 + \eta\nu_2 + \xi) + (1-\alpha)\Delta_1\Big)||b_3|| + \delta\,. \tag{4.22}$$

From (4.7) and (4.12) we see that the right-hand side of (4.22) is bounded by $\lambda\bar{\Delta}$. Since the system $\dot{\bar{y}} = A_1\bar{y} - \lambda\bar{k}(\frac{\bar{y}}{\lambda})b_2$ is $SISS_L(\lambda\bar{\Delta}, \bar{N})$, it follows that

$$|\bar{y}| \leq_{\rm ev} \lambda\bar{N}\bar{\Delta}\,. \tag{4.23}$$

Now, suppose $\limsup_{t\to\infty} |\bar{y}(t)| = \rho > 0$. Then $|\bar{y}| \leq_{\rm ev} 2\rho$. So $|\bar{k}(y)| \leq L|y|$ implies that

$$|\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\rm ev} 2\xi L\rho \tag{4.24}$$

and

$$|\lambda\eta\bar{k}(\frac{\bar{y}}{\lambda})b_0 + \tilde{e}| \leq_{\rm ev} 2\eta L\rho + \delta\,. \tag{4.25}$$

We now apply (A) again to the second equation in (4.15), but this time we use (4.24) and (4.25). We then obtain

$$|\tilde{y}| \leq_{\rm ev} \max\{2\xi\nu_1 L\rho, 2\eta\nu_2 L\rho + \nu_2\delta\} \leq 2(\xi\nu_1 + \eta\nu_2)L\rho + \nu_2\delta\,. \tag{4.26}$$

(Notice that if the right-hand side of (4.26) is greater than Δ_1, then the inequality is trivial because of (4.20).) From (4.21), (4.24), and (4.26) it follows that

$$|g(y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))| \leq_{\rm ev} \varepsilon\Big(2(\xi\nu_1 + \eta\nu_2 + \xi)L\rho + \nu_2\delta\Big)\,.$$

Now, if

$$\varepsilon\Big(2(\xi\nu_1 + \eta\nu_2 + \xi)L\rho + \nu_2\delta\Big) + \delta \leq \lambda\bar{\Delta}\,, \tag{4.27}$$

then, since the system $\dot{\bar{y}} = A_1\bar{y} - \lambda\bar{k}(\frac{\bar{y}}{\lambda})b_2$ is $SISS_L(\lambda\bar{\Delta}, \bar{N})$, it follows that

$$|\bar{y}| \leq_{\text{ev}} \varepsilon\bar{N}\Big(2(\xi\nu_1 + \eta\nu_2 + \xi)L\rho + \nu_2\delta\Big) + \bar{N}\delta\ . \tag{4.28}$$

If (4.27) is not satisfied, then $\varepsilon\bar{N}\Big(2(\xi\nu_1 + \eta\nu_2 + \xi)L\rho + \nu_2\delta\Big) + \bar{N}\delta > \lambda\bar{N}\bar{\Delta}$. However, from (4.23) we see that (4.28) still holds. So we have established (4.28) in all cases. From (4.8) we then get

$$|\bar{y}| \leq_{\text{ev}} \frac{1}{2}\rho + \bar{N}(\varepsilon\nu_2 + 1)\delta\ . \tag{4.29}$$

Taking the $\limsup_{t\to\infty}$ of the left-hand side of (4.29), we get the inequality $\rho \leq \frac{1}{2}\rho + \bar{N}(\varepsilon\nu_2+1)\delta$, i.e. $\rho \leq 2\bar{N}(\varepsilon\nu_2 + 1)\delta$. Substituting this into (4.26) and (4.29), we find that $|\tilde{y}| \leq_{\text{ev}} \Big(4(\xi\nu_1 + \eta\nu_2)L\bar{N}(\varepsilon\nu_2 + 1) + \nu_2\Big)\delta$ and $|\bar{y}| \leq_{\text{ev}} 2\bar{N}(\varepsilon\nu_2 + 1)\delta$. So, if we take $N = 2\bar{N}(\varepsilon\nu_2 + 1)(1 + 2L(\xi\nu_1 + \eta\nu_2)) + \nu_2$, then (4.13) is $SISS_L(\Delta, N)$. As explained before, this establishes the conclusion of Theorem 2.3 for the single-input case.

Next, we deal with the general m-input case, and prove Theorem 2.3 by induction on m. We already know that the theorem is true if $m = 1$. Assume that Theorem 2.3 has been established for all k-input systems, for all $k \leq m - 1$, and consider an m-input system $\Sigma : \dot{x} = Ax + Bu$.

Assume without loss of generality that the first column b_1 of B is nonzero and consider the Kalman controllability decomposition of the system $\Sigma_1 : \dot{x} = Ax + b_1u$ (see [11], Lemma 3.3.3). We conclude that, after a change of coordinates $y = T^{-1}x$, Σ_1 has the form

$$\begin{aligned} \dot{y}_1 &= A_1y_1 + A_2y_2 + \bar{b}_1u\ , \\ \dot{y}_2 &= A_3y_2\ , \end{aligned}$$

where $(A_1, \bar{b}_1)$ is a controllable pair. In these coordinates, Σ has the form

$$\begin{aligned} \dot{y}_1 = {} & A_1y_1 + A_2y_2 \quad + \bar{b}_1u_1 + \bar{B}_1\bar{u}\ , \\ \dot{y}_2 = {} & \qquad A_3y_2 \qquad\quad + \bar{B}_2\bar{u}\ , \end{aligned} \tag{4.30}$$

where $\bar{u} = (u_2, \cdots, u_m)'$ and $\bar{B}_1, \bar{B}_2$ are matrices of the appropriate dimensions. So it suffices to prove the conclusion for (4.30). Let n_1, n_2 denote the dimensions of y_1, y_2, respectively. Recall that $\mu = \mu(A)$. Let $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_\mu)$ be any finite sequence of bounded functions in $\mathcal{S}$. Let $\mathcal{H}$ stand for either $\mathcal{F}$ or $\mathcal{G}$. Then, for the single-input controllable system

$$\dot{y}_1 = A_1y_1 + \bar{b}_1u_1$$

there is a feedback

$$u_1 = -k_1(y_1) \tag{4.31}$$

such that (i) $k_1 \in \mathcal{H}_{n_1}(\sigma_1, \cdots, \sigma_{\mu_1})$, where $\mu_1 = \mu(A_1)$; (ii) the resulting closed-loop system is $SISS_L(\Delta_1, N_1)$ for some $\Delta_1, N_1 > 0$; (iii) the linearization at 0 of the closed-loop system is asymptotically stable. Since (4.30) is controllable, we conclude that the $(m-1)$-input subsystem $\dot{y}_2 = A_3y_2 + \bar{B}_2\bar{u}$ is controllable as well. By the inductive hypothesis, this subsystem can be stabilized by a feedback

$$\bar{u} = -\bar{k}(y_2) = -(k_2(y_2), \cdots, k_m(y_2)) \tag{4.32}$$

such that (i) $\bar{k} \in \mathcal{H}^{\bar{\boldsymbol{l}}}_{n_2}(\sigma_{\mu_1+1}, \cdots, \sigma_\mu)$, where $\bar{\boldsymbol{l}} = (\mu_2, \cdots, \mu_m)$ is an $(m-1)$-tuple of nonnegative integers and $|\bar{\boldsymbol{l}}| = \mu - \mu_1$; (ii) the resulting closed-loop system Σ_2 is $SISS_L(\Delta_2, N_2)$ for some $\Delta_2, N_2 > 0$; (iii) the linearization at 0 of the closed-loop system is asymptotically stable. Let C be such that $|A_2 y_2 - \bar{B}_1 \bar{k}(y_2)| \leq_{\text{ev}} C|y_2|$ whenever $|y_2| \leq N_2\Delta_2$. (Such a C exists because $\bar{k}$ is locally Lipschitz.) Let Δ_3 be such that $0 < \Delta_3 \leq \Delta_2$ and $(CN_2+1)\Delta_3 < \Delta_1$. Then Σ_2 is also $SISS_L(\Delta_3, N_2)$. If e_1, e_2 are bounded measurable vector functions of the appropriate dimensions that are eventually bounded by $\delta \leq \Delta_3$, and $t \to (y_1(t), y_2(t))$ is a solution of

$$\begin{array}{rcrr} \dot{y}_1 = & A_1 y_1 + A_2 y_2 & -\bar{b}_1 k_1(y_1) - \bar{B}_1 \bar{k}(y_2) + e_1 \ , \\ \dot{y}_2 = & A_3 y_2 & - \bar{B}_2 \bar{k}(y_2) + e_2 \ , \end{array} \tag{4.33}$$

then the fact that $\dot{y}_2 = A_3 y_2 - \bar{B}_2 \bar{k}(y_2)$ is $SISS_L(\Delta_3, N_2)$ implies that $|y_2| \leq_{\text{ev}} N_2\delta$. Then $|A_2 y_2 - \bar{B}_1 \bar{k}(y_2) + e_1| \leq_{\text{ev}} (CN_2+1)\delta \leq (CN_2+1)\Delta_3 \leq \Delta_1$. Since $\dot{y}_1 = A_1 y_1 - \bar{b}_1 k_1(y_1)$ is $SISS_L(\Delta_1, N_1)$, we conclude that $|y_1| \leq_{\text{ev}} N_1(CN_2+1)\delta$. So $|y| \leq_{\text{ev}} N\delta$, where $N = N_1 + N_2 + CN_1N_2$. So the feedback given by (4.31) and (4.32) globally stabilizes (4.30), and the resulting closed-loop system is $SISS_L(\Delta_3, N)$. Therefore, if we let $\boldsymbol{l} = (\mu_1, \mu_2, \cdots, \mu_m)$ and $k = (k_1(y_1), k_2(y_2), \cdots, k_m(y_2))$, then $k \in \mathcal{H}^{\boldsymbol{l}}_n(\boldsymbol{\sigma})$ satisfies all the required properties. □

5 An Algorithm

We now present a two-step procedure for computing stabilizing feedbacks of the kind described in the statement of Theorem 2.3. The first step is to transform Σ into a special form by means of a linear change of coordinates. The second step is the construction of a stabilizing feedback for the transformed system. This essentially amounts to keeping track of the constants that occur in the proof of Lemma 3.2. All the relevant information about this proof has already been summarized in Remark 3.3 and Corollary 3.5, so our discussion will be based on these results —and on the proof of Theorem 2.3— but not on the proof of Lemma 3.2.

We will need the following corollary of the Kalman controllability decomposition:

Lemma 5.1 *Let Σ be a linear system of the form (1.1) with state space $\mathbb{R}^n$ and input space $\mathbb{R}^m$. Suppose that all the eigenvalues of A have nonpositive real parts and all the eigenvalues of the uncontrollable part of Σ have strictly negative real parts. Then there exists a linear change of coordinates which transforms (1.1) into the following form*

$$\begin{array}{rcl} \dot{x}_0 &=& A_{00}x_0 + A_{01}x_1 + A_{02}x_2 + \cdots + A_{0m}x_m + b_{01}u_1 + b_{02}u_2 + \cdots + b_{0m}u_m \ , \\ \dot{x}_1 &=& A_{11}x_1 + A_{12}x_2 + \cdots + A_{1m}x_m + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m \ , \\ \dot{x}_2 &=& A_{22}x_2 + \cdots + A_{2m}x_m + b_{22}u_2 + \cdots + b_{2m}u_m \ , \\ &\vdots& \\ \dot{x}_m &=& A_{mm}x_m + b_{mm}u_m \ , \end{array} \tag{5.1}$$

where all the eigenvalues of A_{00} have negative real part, all the eigenvalues of A_{ii} with $i \geq 1$ have zero real part, and all the pairs (A_{ii}, b_{ii}), $i \geq 1$, are controllable. (The coordinate x_i may have zero dimension. In that case, there is no equation for x_i in (5.1).) □

From the proof of Theorem 2.3 we see that, if we find $SISS_L$-stabilizing feedbacks $u_i = k_i(x_i)$ of Type $\mathcal{F}$ or Type $\mathcal{G}$ for the systems $\Sigma_i : \dot{x}_i = A_{ii}x_i + b_{ii}u_i$ and let $k = (k_1, \cdots, k_m)$ then k is $SISS_L$-stabilizing for Σ. So what we need is to stabilize each Σ_i separately, making sure that the $SISS_L$ property holds. To simplify the notation, we consider again an n-dimensional single-input controllable system $\Sigma : \dot{x} = Ax + bu$ such that all the eigenvalues of A have zero real part. Our goal is to find $SISS_L$-stabilizing feedbacks of Type $\mathcal{F}$ and Type $\mathcal{G}$ for Σ. For this purpose, we will use the following corollary of Lemma 3.1.

Lemma 5.2 *Let $\Sigma : \dot{x} = Ax + bu$, $x \in \mathbb{R}^n$, be a controllable single-input linear system. Suppose that all eigenvalues of A are critical. Let $\pm\omega_1 i, \pm\omega_2 i, \cdots, \pm\omega_q i$ be the nonzero eigenvalues of A, and let p be the multiplicity of 0 as an eigenvalue, so $p + 2q = n$. Let $s = p + q$, and let $(\nu_1, \cdots, \nu_{s-1})$ be a finite sequence of positive numbers. Let $\theta_{j,k} = 1$ for $k = j + 1$, and $\theta_{j,k} = \prod_{h=j}^{k-2} \nu_h$ for $j + 2 \leq k \leq s + 1$. Then there is a linear change of coordinates that puts Σ in the form:*

$$\begin{aligned}
\dot{z}_j &= \omega_j y_j \ , && j = 1, 2, \cdots, q \, , \\
\dot{y}_j &= -\omega_j z_j + \sum_{k=j+1}^{s} \theta_{j,k}\, y_k + \theta_{j,s+1}\, u \ , && j = 1, 2, \cdots, q \, , \\
\dot{y}_j &= \sum_{k=j+1}^{s} \theta_{j,k}\, y_k + \theta_{j,s+1}\, u \ , && j = q+1, q+2, \cdots, s-1 \, , \\
\dot{y}_s &= u \ .
\end{aligned} \tag{5.2}$$

To derive this from Lemma 3.1, we notice that, if $p > 0$, then from Part (i) of Lemma 3.1 it follows that there is a linear change of coordinates $x \to (\bar{y}', y_n)$ that puts Σ in the form

$$\dot{\bar{y}} = A_1 \bar{y} + b_1(y_n + \nu_{s-1} u) \, , \quad \dot{y}_n = u \, .$$

If $p > 1$, then relabeling $\bar{y}$ as x and applying Part (i) of Lemma 3.1 to the system $\Sigma_1 : \dot{x} = A_1 x + b_1 v$, with ν_{s-2} in the role of ν, we find a linear change of coordinates $x \to (\bar{y}', y_{n-1})$ that puts Σ_1 in the form $\dot{\bar{y}} = A_2 \bar{y} + b_2(y_{n-1} + \nu_{s-2} v)$, $\dot{y}_{n-1} = v$, where (A_2, b_2) is controllable. Substituting $v = y_n + \nu_{s-1} u$, we see that Σ has been transformed into

$$\begin{aligned}
\dot{\bar{y}} &= A_2 \bar{y} + b_2(y_{n-1} + \nu_{s-2} y_n + \nu_{s-2}\nu_{s-1} u) \ , \\
\dot{y}_{n-1} &= y_n + \nu_{s-1} u \ , \\
\dot{y}_n &= u \ .
\end{aligned}$$

Continuing in this way, we apply Part (i) of Lemma 3.1 to each pair (A_ℓ, b_ℓ), with $\nu_{s-\ell-1}$ in the role of ν, until zero is no longer an eigenvalue of A_ℓ, thereby generating a system whose last p equations are the last p equations of (5.2). We then apply Part (ii) q times, until we obtain the representation (5.2). □

Now let $(\sigma_1, \cdots, \sigma_s)$ be a finite sequence of bounded functions in $\mathcal{S}$. Suppose $\sigma_i'(0) = \kappa_i$. Choose $\nu_i = \frac{1}{\kappa_{i+1}}$, and then use Lemma 5.2 to put Σ in the form (5.2). (To find a linear transformation that puts Σ in this form simply let $(\tilde{A}, \tilde{b})$ denote the controllable pair corresponding to (5.2). Then $y = T^{-1}x$, with $T = \mathbf{R}(A, b)\mathbf{R}(\tilde{A}, \tilde{b})^{-1}$, is the desired transformation, where $\mathbf{R}(A, b)$ denotes the controllability matrix of (A, b). See Section 3.3 of [11].)

To get a stabilizing feedback of Type $\mathcal{F}$, we will choose positive constants $\lambda_1, \lambda_2, \cdots, \lambda_{s-1}$ and let

$$u = -\sigma_s\left(y_s + \lambda_{s-1}\sigma_{s-1}\left(\frac{y_{s-1}}{\lambda_{s-1}} + \lambda_{s-2}\sigma_{s-2}\Big(\frac{y_{s-2}}{\lambda_{s-2}\lambda_{s-1}} + \cdots + \lambda_1\sigma_1(\frac{y_1}{\lambda_1\lambda_2\cdots\lambda_{s-1}})\cdots\Big)\right)\right). \quad (5.3)$$

To get a Type $\mathcal{G}$ stabilizing feedback we choose positive numbers $\lambda_1, \lambda_2, \cdots, \lambda_s$ and let

$$u = -\lambda_s\sigma_s\left(\frac{y_s}{\lambda_s}\right) - \lambda_{s-1}\lambda_s\sigma_{s-1}\left(\frac{y_{s-1}}{\lambda_{s-1}\lambda_s}\right) - \cdots - (\lambda_1\lambda_2\cdots\lambda_s)\,\sigma_1\left(\frac{y_1}{\lambda_1\lambda_2\cdots\lambda_s}\right) \quad (5.4)$$

(See Figures 1 and 2.)

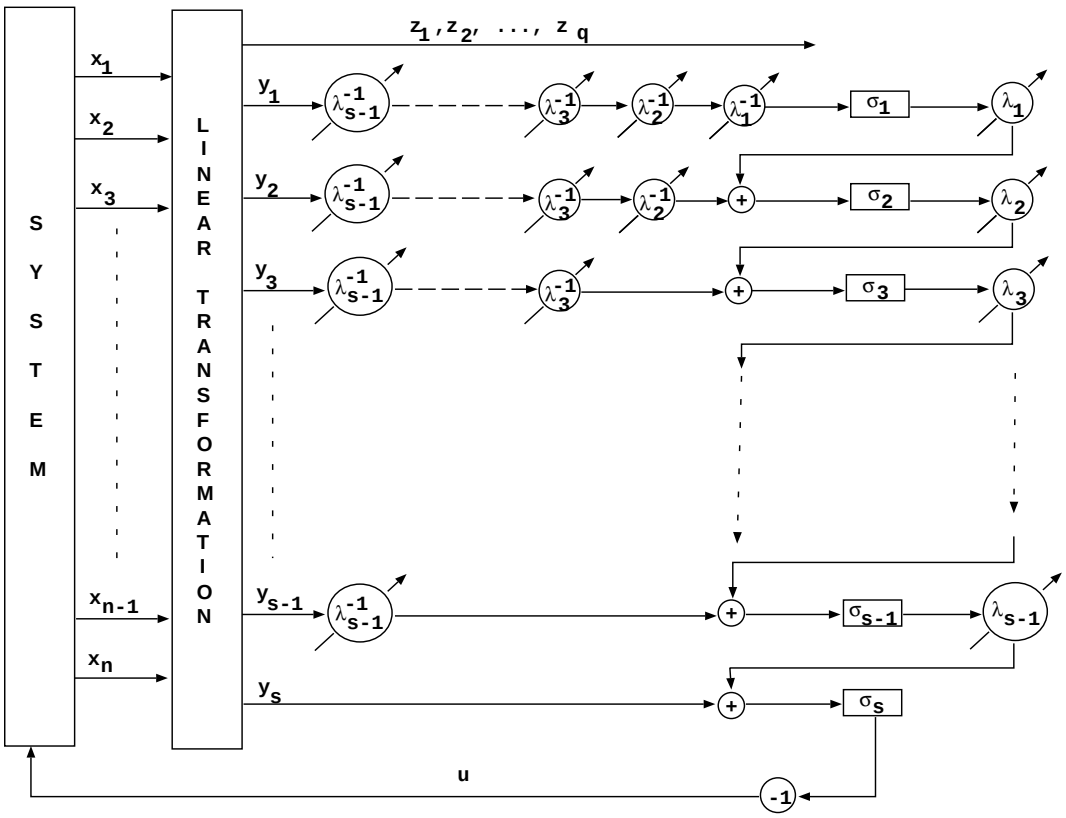

Figure 1: *Block diagram for (5.3)*

In both cases, the crucial question is how to choose the constants λ_j. We now describe in detail how to find values of $\lambda_1, \lambda_2, \cdots, \lambda_{s-1}$ for (5.3) that achieve the desired goal. (The procedure for (5.4) is similar.)

The λ_j will be chosen recursively. As in the proof of Theorem 2.3, we first find (Δ_1, N_1) so that the (z_1, y_1)-subsystem of (5.2) with $u = -\sigma_1(y_1)$ is $SISS_L(\Delta_1, N_1)$, then find λ_1 so that the (z_1, y_1, z_2, y_2)-subsystem of (5.2) with $u = -\sigma_2(y_2 + \lambda_1\sigma_1(\frac{y_1}{\lambda_1}))$ or $u = -\sigma_2(y_2) - \lambda_1\sigma_1(\frac{y_1}{\lambda_1})$ is $SISS_L(\Delta_2, N_2)$ for some $\Delta_2, N_2 > 0$, and so on.

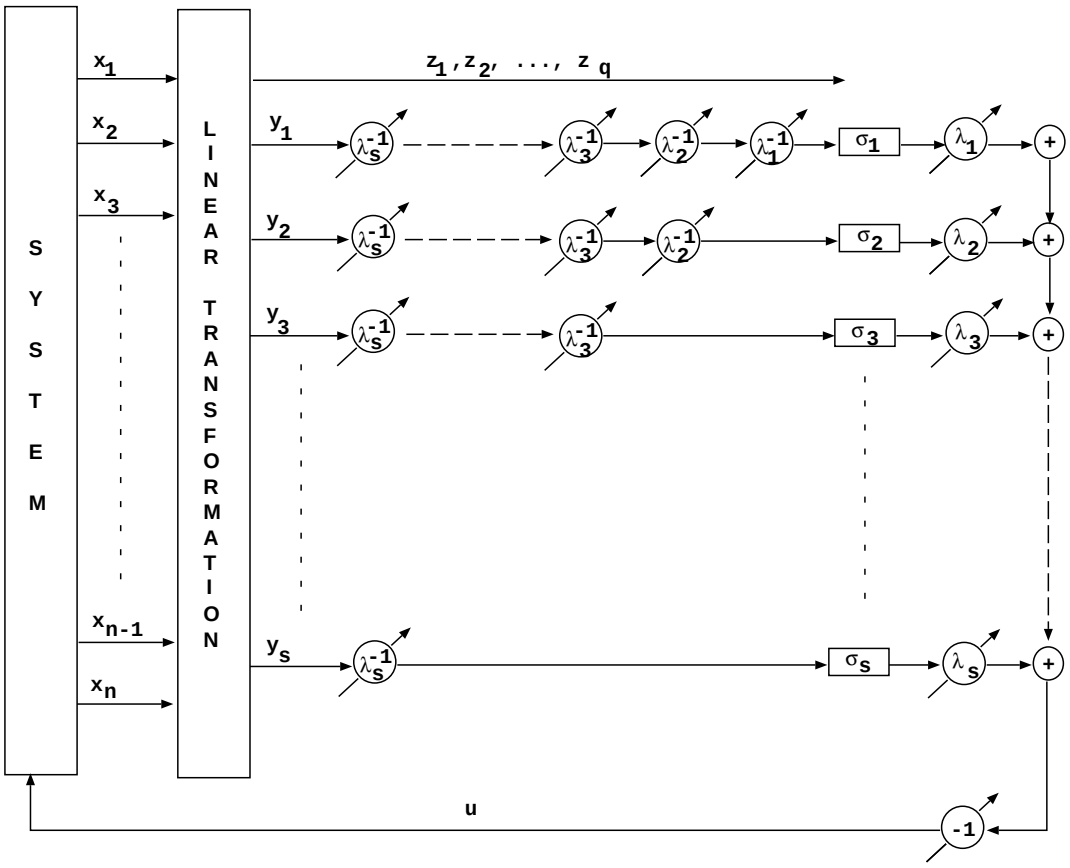

Figure 2: *Block diagram for (5.4)*

Step 1: For $i = 1, 2, \cdots, s$, find $\tilde{L}_i \geq 1, M_i > 0$ such that $|\sigma_i(t)| \leq \tilde{L}_i|t|$ and $|\sigma_i(t)| \leq M_i$ for all t. Notice that $\tilde{L}_i$ exists because σ_i is locally Lipschitz and bounded. Then define $L_i = \tilde{L}_1\tilde{L}_2 \cdots \tilde{L}_i$. The numbers L_i, M_i will play the roles of L, M in the proof of Theorem 2.3.

Step 2: For $i = 1, 2, \cdots, s$, we apply Remark 3.3 (with $\alpha_2 = 1$, $\alpha_3 = 0$, cf. Remark 3.4) and Corollary 3.5 to find $\Delta_1^i, \nu_1^i, \nu_2^i > 0$ such that, whenever e_1, e_2 are two bounded real-valued measurable functions on $[0, \infty)$ such that $|e_1| \leq_{\text{ev}} \frac{\delta}{\nu_1^i}$ and $|e_2| \leq_{\text{ev}} \frac{\delta}{\nu_2^i}$ for some $\delta \leq \Delta_1^i$, it follows that the state ϕ of the system $\dot{\phi} = J_i\phi - \sigma_i(y_i + e_1)b_i + e_2 b_i$ is eventually bounded by δ, where (i) $J_i = \begin{pmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{pmatrix}$ and $b_i = (0,1)^\dagger$ for $i = 1, \ldots, q$ and (ii) $J_i = 0$ and $b_i = 1$ for $i = q+1, \ldots, s$.

Step 3: Let $\Delta_0 = \infty, N_0 = 0$ and fix $\alpha \in (0,1)$. We now describe how to find $\varepsilon_i, \Delta_2^i, \delta_0^i, \lambda_i, \Delta_i, N_i$ recursively for $i = 1, 2, \cdots, s-1$.

Let $\varepsilon_i > 0$ satisfy

$$\varepsilon_i(M_i(\nu_1^i + 1) + \Delta_1^i)\frac{1}{\kappa_i} < \alpha\Delta_{i-1} \quad \text{and} \quad \varepsilon_i N_{i-1} L_i(\nu_1^i + 1) < \frac{1}{4}\,, \tag{5.5}$$

and find $\Delta_2^i > 0$ such that

$$|\sigma_i(t) - \kappa_i t| \leq \varepsilon_i |t| \quad \text{for } |t| \leq \Delta_2^i \,. \tag{5.6}$$

(If σ_i is linear near 0 then it is also possible to take $\varepsilon_i = 0$.)

Then let $\delta_0^i, \lambda_i, \Delta_i, N_i > 0$ satisfy the conditions of the proof of Theorem 2.2, namely,

$$\begin{aligned}
\delta_0^i &\leq \min\{\Delta_1^i, \alpha\Delta_2^i\}\,, \\
\lambda_i &\leq \min\{1, \tfrac{\alpha\delta_0^i}{\nu_1^i M_i}, \tfrac{(1-\alpha)\Delta_2^i}{M_i}\}\,, \\
\Delta_i &\leq \min\{\delta_0^i, \tfrac{(1-\alpha)\lambda_i\delta_0^i}{\nu_2^i}, (1-\alpha)\lambda_i\Delta_{i-1}\}\,, \\
N_i &\geq 2N_{i-1}(\varepsilon_i\nu_2^i + 1)(1 + 2L_i\nu_1^i) + \nu_2^i\,.
\end{aligned}$$

With the above choices, the constants λ_i, $i = 1, 2, \cdots, s-1$ are such that (5.3) globally stabilizes (5.2), and the closed-loop system is $SISS_L$. □

Remark 5.3 When all σ_i, $i = 1, 2, \cdots, s-1$, are linear near the origin, the algorithm described above is very simple. As an example, we consider the case of $\sigma_i(s) = \text{sat}\,(s)$, where $\text{sat}\,(s) = \text{sign}\,(s)\min\{|s|, 1\}$. Then $L_i = M_i = \kappa_i = 1$. When $i \leq q$, we use (3.27) and (3.29), and observe that we can take $\bar{\Lambda} = 1$, $a = b = 1$, and then $\beta_2 = \sqrt{4 + \frac{1}{\omega_i^2}}$. So we can choose $\Delta_1^i = 4$, and

$$\nu_1^i = \left(2 + 5\sqrt{4 + \frac{1}{\omega_i^2}}\right)\left(1 + \frac{1}{\omega_i}\right)\,. \tag{5.7}$$

When $i > q$, we take $\Delta_1^i = 2$ and $\nu_1^i = 2$. Next, we choose $\varepsilon_i = 0$, and $\Delta_2^i = 1$ in Step 3 of the above algorithm. (Notice that the choice $\varepsilon_i = 0$ automatically guarantees that (5.5) and (5.6) hold, irrespective of the values of Δ_{i-1}, N_{i-1}. So there is no need to compute the Δ_i and N_i.) For every i, the facts that $\Delta_1^i > 1$ and $\Delta_2^i = 1$ imply that, after fixing $\alpha \in (0, 1)$, we can take $\delta_0^i = \alpha$, and then it suffices to choose the λ_i such that $\lambda_i \leq \min\{\frac{\alpha^2}{\nu_1^i}, 1 - \alpha\}$. Then the feedback law (5.3) stabilizes (5.2). □

Example 5.4 Consider an oscillator with multiplicity two:

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = -x_1 + x_3\,, \quad \dot{x}_3 = x_4\,, \quad \dot{x}_4 = -x_3 + u\,, \tag{5.8}$$

where u is required to satisfy the constraint $|u| \leq \varepsilon$.

To get a feedback of the form

$$u = -\text{sat}\left(f_1(x) + \lambda\,\text{sat}\,(\frac{f_2(x)}{\lambda})\right)\,,$$

we need to find a linear transformation that puts (5.8) in the form (5.2). In our case, (5.2) reduces to

$$\dot{y}_1 = y_2\,, \quad \dot{y}_2 = -y_1 + y_4 + u\,, \quad \dot{y}_3 = y_4\,, \quad \dot{y}_4 = -y_3 + u\,. \tag{5.9}$$

We now write (5.8) as $\dot{x} = Ax + bu$ and (5.9) as $\dot{y} = \tilde{A}y + \tilde{b}u$. A simple computation then shows that

$$\mathbf{R}(A,b) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \text{ and } \quad \mathbf{R}(\tilde{A},\tilde{b}) = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} .$$

If we let $T = \mathbf{R}(A,b)\mathbf{R}(\tilde{A},\tilde{b})^{-1}$, then the coordinate change $y = T^{-1}x$ transforms (5.8) into (5.9). The matrix T^{-1} is easily computed, and the transformation $y = T^{-1}x$ turns out to be

$$y_1 = x_2 + x_3\,, \;\; y_2 = -x_1 + x_3 + x_4\,, \;\; y_3 = x_3\,, \;\; y_4 = x_4\,.$$

We now need to find $\lambda > 0$ so that

$$u = -\text{sat}\left(y_4 + \lambda\,\text{sat}\,(\frac{y_2}{\lambda})\right) \tag{5.10}$$

stabilizes (5.9). To compute λ, we follow the steps of Remark 5.3. Notice that $\omega_i = 1$. So we can take $\Delta_1^1 = 4$, $\nu_1^1 = 4 + 10\sqrt{5}$, and we can also replace the value of ν_1^1 by any larger number, so we can actually take $\nu_1^1 = 27$. Let $\alpha \in (0,1)$. Then $\lambda_1 = \min\{\frac{\alpha^2}{27}, 1-\alpha\}$. Choosing $\alpha = \frac{28}{29}$ we obtain $\lambda_1 = \frac{1}{29}$, so we can choose $\lambda = \frac{1}{29}$. Then the feedback defined by (5.10) globally stabilizes (5.9) and the resulting closed-loop system is $SISS$. Transforming back to the original x-coordinates, we have shown that the feedback

$$u = -\varepsilon\,\text{sat}\left(\frac{x_4}{\varepsilon} + \frac{1}{29}\text{sat}\left(\frac{29}{\varepsilon}(-x_1 + x_3 + x_4)\right)\right)$$

is $SISS_L$-stabilizing for the system (5.8). □

6 Multiple Integrators

For special systems, the general method of Section 5 can be often be pushed further, yielding simpler formulas with better constants. We will now illustrate this by showing a simple procedure to stabilize multiple integrators.

Theorem 6.1 *Let $0 < \varepsilon \leq \frac{1}{4}$. Then for the n-th order integrator*

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = x_3\,, \quad \ldots\,, \quad \dot{x}_n = u\,, \tag{6.1}$$

there exists a feedback law of the form

$$u = -\sum_{i=1}^{n} \varepsilon^i\,\text{sat}\,(h_i(x_1,\cdots,x_n))\,, \tag{6.2}$$

where each $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\cdots,n$, is a linear function, such that the origin is a globally asymptotically stable state for the resulting closed-loop system.

PROOF. We first apply Lemma 5.2 to (6.1) and conclude that for every $\varepsilon > 0$ there exists a linear change of coordinates $(x_1, \cdots, x_n) \to (y_1, \cdots, y_n)$ which transforms (6.1) into the form

$$\begin{aligned} \dot{y}_1 &= \varepsilon^{n-1} y_2 + \varepsilon^{n-2} y_3 + \cdots + \varepsilon y_n + u \ , \\ \dot{y}_2 &= \varepsilon^{n-2} y_3 + \cdots + \varepsilon y_n + u \ , \\ &\vdots \\ \dot{y}_{n-1} &= \varepsilon y_n + u \ , \\ \dot{y}_n &= u \ . \end{aligned} \tag{6.3}$$

(We only need to set $\nu_i = \frac{1}{\varepsilon}$ in (5.2) and then make a dilation $(x_1, x_2, \cdots, x_n) \to (\varepsilon^{1+2+\cdots+(n-1)} x_1, \varepsilon^{1+2+\cdots+(n-2)} x_2, \cdots, x_n)$.) We will show that, when $\varepsilon \leq \frac{1}{4}$, the feedback

$$u = k(y) = -\varepsilon \mathrm{sat}\,(y_n) - \varepsilon^2 \mathrm{sat}\,(y_{n-1}) - \cdots - \varepsilon^n \mathrm{sat}\,(y_1) \tag{6.4}$$

stabilizes (6.3).

To prove this, we observe that, for any trajectory $t \to y(t)$ of the resulting closed-loop system of (6.3) with the feedback given in (6.4), the n-th coordinate $y_n(t)$ will enter and stay in the interval $(-\frac{1}{2}, \frac{1}{2})$ after a finite time. This is obvious from the facts that the sign of $k(y)$ is opposite to that of y_n if $|y_n| \geq \frac{1}{2}$, since $|\varepsilon^2 \mathrm{sat}\,(y_{n-1}) + \cdots + \varepsilon^n \mathrm{sat}\,(y_1)| \leq \frac{4}{3}\varepsilon^2 \leq \frac{1}{3}\varepsilon < \varepsilon |y_n|$, and that $\frac{d}{dt} y_n^2(t) \leq -\frac{1}{6}\varepsilon$ if $|y_n(t)| \geq \frac{1}{2}$. So, after a finite time, $\mathrm{sat}\,(y_n)$ will be equal to y_n, and the expression for $k(y)$ gives

$$k(y) = -\varepsilon y_n - \varepsilon^2 \mathrm{sat}\,(y_{n-1}) - \cdots - \varepsilon^n \mathrm{sat}\,(y_1) \, . \tag{6.5}$$

Next, we consider the equation for $\dot{y}_{n-1}$. From (6.5), it follows that, after a finite time, this equation has the form

$$\dot{y}_{n-1} = -\varepsilon^2 \mathrm{sat}\,(y_{n-1}) - \cdots - \varepsilon^n \mathrm{sat}\,(y_1) \, .$$

With an analysis similar to that done for y_n, we conclude that the coordinate $y_{n-1}(t)$ will stay in the interval $(-\frac{1}{2}, \frac{1}{2})$ after a finite time, and then $k(y)$ will be given by the expression

$$k(y) = -\varepsilon y_n - \varepsilon^2 y_{n-1} - \varepsilon^3 \mathrm{sat}\,(y_{n-2}) - \cdots - \varepsilon^n \mathrm{sat}\,(y_1) \, .$$

Continuing in this way, we see that after a finite time $k(y)$ becomes linear in all the coordinates of y, and is given by

$$k(y) = -\varepsilon y_n - \varepsilon^2 y_{n-1} - \cdots - \varepsilon^n y_1 \, . \tag{6.6}$$

It is clear that the closed-loop system of (6.3) with the feedback $u = k(y)$ given in (6.6) is asymptotically stable. So the proof is complete. □

In Section 5 we described how to find the linear transformation that puts (6.1) in the form (6.3). We now develop another method to get the transformation for multiple integrators, which will also allow us to illustrate the stabilization of multiple integrators by means of a concrete systems diagram.

Consider the n-th order integrator (6.1). Let $\varepsilon > 0$. Define linear functions $f_1, f_2, \cdots, f_n$ as follows:

$$\begin{aligned} f_1(s_1) &= s_1 \ , \\ f_2(s_1, s_2) &= \varepsilon f_1(s_1) + f_1(s_2) \ , \\ &\vdots \\ f_n(s_1, s_2, \cdots, s_n) &= \varepsilon^{n-1} f_{n-1}(s_1, s_2, \cdots, s_{n-1}) + f_{n-1}(s_2, s_3, \cdots, s_n) \ . \end{aligned}$$

Then the change of coordinates $(x_1, \cdots, x_n) \to (y_1, \cdots, y_n)$ in $\mathbb{R}^n$ given by

$$\begin{aligned} y_n &= f_1(x_n) \ , \\ y_{n-1} &= f_2(x_{n-1}, x_n) \ , \\ &\vdots \\ y_1 &= f_n(x_1, x_2, \cdots, x_n) \ , \end{aligned} \tag{6.7}$$

puts (6.1) in the form (6.3).

Indeed, the definition of y_k implies that $y_k - y_{k+1} = \varepsilon^{n-k} f_{n-k}(x_k, \cdots, x_{n-1})$ for $k = 1, 2, \cdots, n-1$. Since $f_{n-k}(s_1, \cdots, s_{n-k})$ is a linear function, it follows that $\frac{\partial f_{n-k}}{\partial s_i}$ is the coefficient of s_i in f_{n-k}. Notice that $\dot{x}_i = x_{i+1}$ for $i = 1, 2, \cdots, n-1$, so we end up with the equality $\dot{f}_{n-k}(x_k, \cdots, x_{n-1}) = f_{n-k}(x_{k+1}, \cdots, x_n)$. Therefore

$$\dot{y}_k - \dot{y}_{k+1} = \varepsilon^{n-k} y_{k+1} \ , \qquad k = 1, 2, \cdots, n-1 \ . \tag{6.8}$$

Adding (6.8) for $k = i, i+1, \cdots, n-1$, we conclude that

$$\dot{y}_i - u = \varepsilon^{n-i} y_{i+1} + \varepsilon^{n-i-1} y_{i+2} + \cdots + \varepsilon y_n \ ,$$

which is the i^{th} equation in (6.3).

The change of coordinates given in (6.7) leads to the design of a bounded stabilizing feedback for multiple integrators shown in Figure 3.

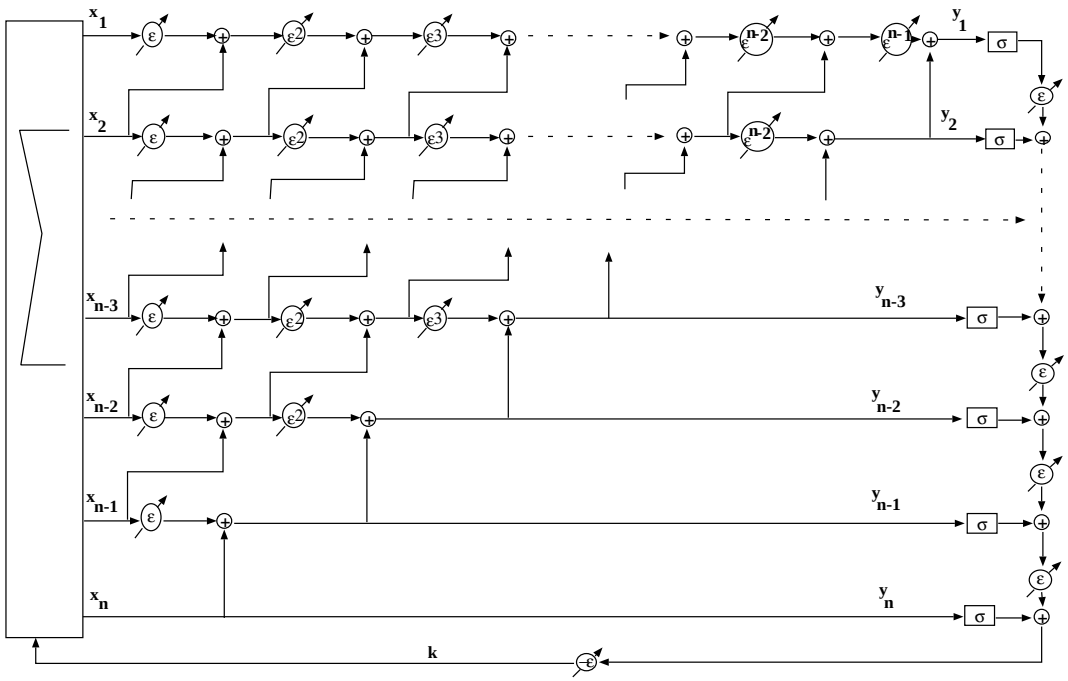

Figure 3: *The design of stabilizing feedback for multiple integrators*

7 Output Feedback

We now turn to the problem of dynamic output feedback stabilization.

Suppose we have a system $\dot{x} = Ax + Bu$ with an $\mathbb{R}^p$-valued output $y = Cx$. We assume that (A, B) is stabilizable, (A, C) is detectable, and A has no unstable eigenvalues.

We proceed as in the classical case of stabilization by a linear feedback. That is, we consider the composite system

$$\begin{array}{rcl} \dot{x} &=& Ax + Bu \ , \\ y &=& Cx \ , \\ \dot{z} &=& Az + Bu + L(y - Cz) \ , \\ u &=& k(z) \ , \end{array} \tag{7.1}$$

where k is any stabilizing feedback for the system $\dot{x} = Ax + Bu$ such that the closed-loop system is $SISS$, and L is any matrix such that all the eigenvalues of $A - LC$ have negative real parts. (Such a matrix exists because (A, C) is detectable.) As usual, it is better to analyze this system by using as state variables the coordinates z and $e = z - x$ (which is interpreted as observation error). In these variables, the equations are

$$\begin{array}{rcl} \dot{e} &=& (A - LC)e \ , \\ \dot{z} &=& Az + Bk(z) - LCe \ . \end{array} \tag{7.2}$$

We claim that the origin of this system is a globally asymptotically stable equilibrium. Local asymptotic stability is immediate, because (7.2) has the form of a cascade of two asymptotically stable subsystems. So we only need to show that all trajectories approach the origin. For this, it is enough to show that every solution of $\dot{z} = Az + Bk(z) - LCe(t)$ for which $e(t) \to 0$ as $t \to \infty$ must converge to zero. The $SISS$ property insures the desired result. Summarizing, we have established:

Theorem 7.1 *Consider a linear system of the form $\dot{x} = Ax + Bu$, $y = Cx$, such that A has no unstable eigenvalues, (A, B) is stabilizable and (A, C) is detectable. Then, for any observer system $\dot{z} = Az + Bu + L(y - Cz)$, and for any stabilizing feedback $u \to k(z)$ given by Theorem 2.3, it follows that the composite system $\dot{x} = Ax + Bk(z)$, $\dot{z} = Az + Bk(z) + L(Cx - Cz)$ has the origin as a globally asymptotically stable equilibrium .* □

We remark that, since the stabilizing feedback given in this paper causes the resulting closed-loop system to have the $SISS$ property, Theorem 7.1 holds for *any* linear observer. In [12], the same problem was also considered, but the result in [12] required that all observer poles have real part less than a certain margin. So Theorem 7.1 answers a question posed in [12], namely, whether an arbitrary linear observer can be used for stabilization.

8 An Application to Cascaded Systems

We now show how to extend our result to other classes of systems, such as those obtained by having a saturation nonlinearity in a forward path of integrators, as in Figure 4.

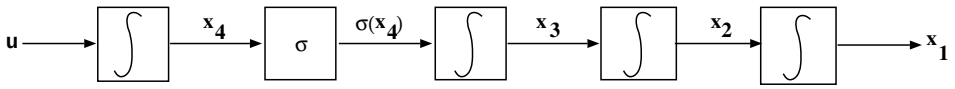

Figure 4: *Cascaded system*

Consider an $(n+m)$-dimensional system in the following partitioned form

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i \sigma_i(z_i)\,, \quad \dot{z} = u\,, \tag{8.1}$$

where $x \in \mathbb{R}^n$, $z = (z_1, \cdots, z_m)' \in \mathbb{R}^m$, $u = (u_1, \cdots, u_m)' \in \mathbb{R}^m$, and all $\sigma_i : \mathbb{R} \to \mathbb{R}$ are globally Lipschitz functions that are strictly monotonic near zero and satisfy $\sigma_i(0) = 0$. Let B denote $(b_1, \cdots, b_m)$. Assume that (A, B) is stabilizable and A has no unstable eigenvalues. We claim that *(8.1) is stabilizable and, moreover, if all the σ_i's and their inverses are smooth functions near* 0*, then there exists a smooth feedback that $SISS_L$-stabilizes (8.1).*

The stabilizability assertion follows trivially. In fact, the assumptions on the σ_i's, imply that there is an $\varepsilon > 0$ such that all σ_i's are strictly monotonic on the interval $(-\varepsilon, \varepsilon)$. From Theorem 2.3 we know that there exists an analytic feedback $v = \bar{k}(x) = (\bar{k}_1(x), \cdots, \bar{k}_m(x))'$, where $|\bar{k}_i(x)| < \varepsilon$ for $x \in \mathbb{R}^n$, which $SISS_L$- stabilizes the system $\dot{x} = Ax + Bv$. If we regard z as a control in the first equation of (8.1), then the feedback $z = k(x) = \left(\sigma_1^{-1}(\bar{k}_1(x)), \cdots, \sigma_m^{-1}(\bar{k}_m(x))\right)'$ gives rise to a $SISS_L$-stable closed-loop system

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i \sigma_i(k_i(x))\,, \tag{8.2}$$

where $k_i = \sigma_i^{-1}(\bar{k}_i(x))$. Using the standard "backstepping" (or "adding an integrator") construction (see, e.g., Lemma 4.8.3 in [11]), we conclude that (8.1) is stabilizable.

However, in order to build a stabilizer for (8.1) using backstepping, we would first have to find a Lyapunov function for (8.2). Since this is usually hard to do, it is desirable to be able to build a stabilizer without using Lyapunov functions. We now show how this can be done.

Since (8.2) is $SISS_L$ and the σ_i's are globally Lipschitz, the control system

$$\dot{x} = f(x, v) = Ax + \sum_{i=1}^{m} b_i \sigma_i(k_i(x) + v_i) \tag{8.3}$$

is $SISS_L$ as well. Let $w = z - k(x)$. Then, in the coordinates $(x', w')'$ in the state space, the equations of System (8.1) are

$$\begin{array}{rl} \dot{x} & = Ax + \sum_{i=1}^{m} b_i \sigma_i(k_i(x) + w_i), \\ \dot{w} & = u - \bigtriangledown k(x) \cdot \left(Ax + \sum_{i=1}^{m} b_i \sigma_i(k_i(x) + w_i)\right). \end{array} \tag{8.4}$$

Now let

$$u = -w + \bigtriangledown k(x) \cdot \left(Ax + \sum_{i=1}^{m} b_i \sigma_i(k_i(x) + w_i)\right) = k(x) - z + \bigtriangledown k(x) \cdot \left(Ax + \sum_{i=1}^{m} b_i \sigma_i(z_i)\right). \tag{8.5}$$

Then the resulting closed-loop system for (8.1) becomes

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i \sigma_i(k_i(x) + w_i)\,, \quad \dot{w} = -w\,,$$

which is stable because $w(t) \to 0$ as $t \to \infty$ and the (8.3) is $SISS_L$-stable. So (8.5) is a stabilizer for (8.1).

As an example to illustrate the above procedure, we consider the system

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = x_3\,, \quad \dot{x}_3 = \tanh(x_4)\,, \quad \dot{x}_4 = u\,, \tag{8.6}$$

which is of the form shown in Figure 4. (See also [16], where the author considers the case when "tanh" in (8.6) is replaced by any function σ satisfying $\sigma'(0) > 0$.) First, we need to find a smooth feedback that stabilizes the triple integrator

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = x_3\,, \quad \dot{x}_3 = u\,, \tag{8.7}$$

and has absolute value less than one. To do this, we use the transformation:

$$y_1 = \frac{1}{8}x_1 + \frac{3}{4}x_2 + x_3\,, \quad y_2 = \frac{1}{2}x_2 + x_3\,, \quad y_3 = x_3\,.$$

Then (8.7) takes the form

$$\dot{y}_1 = \frac{1}{4}y_2 + \frac{1}{2}y_3 + u\,, \quad \dot{y}_2 = \frac{1}{2}y_3 + u\,, \quad \dot{y}_3 = u\,. \tag{8.8}$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function with the property that

$$\varphi(s) = \begin{cases} s & \text{if } |s| < \frac{3}{4} \\ \operatorname{sign}(s) & \text{if } |s| > 1\,. \end{cases}$$

For every $\varepsilon > 0$, we define $\varphi_\varepsilon(s) = \varepsilon\varphi(\frac{s}{\varepsilon})$. Then it is not difficult to see that the feedback

$$u = -\frac{1}{2}\Big(\varphi_1(y_3) + \varphi_{\frac{1}{2}}(y_2) + \varphi_{\frac{1}{4}}(y_1)\Big)$$

stabilizes (8.8). Therefore

$$u = \bar{k}(x_1, x_2, x_3) = -\frac{1}{2}\Big(\varphi_1(x_3) + \varphi_{\frac{1}{2}}(\frac{1}{2}x_2 + x_3) + \varphi_{\frac{1}{4}}(\frac{1}{8}x_1 + \frac{3}{4}x_2 + x_3)\Big)$$

stabilizes (8.7). Clearly, the magnitude of this feedback is less than one. Let $k(x_1, x_2, x_3) = \tanh^{-1}(\bar{k}(x_1, x_2, x_3))$. Then from (8.5), we conclude that the feedback

$$u = k(x_1, x_2, x_3) - x_4 + x_2\frac{\partial k}{\partial x_1}(x_1, x_2, x_3) + x_3\frac{\partial k}{\partial x_2}(x_1, x_2, x_3) + \tanh(x_4)\frac{\partial k}{\partial x_3}(x_1, x_2, x_3)$$

stabilizes (8.6). □

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