Lie- and chronologico-algebraic tools for studying stability of time-varying systems

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Abstract

We will study stability and asymptotic stability for time-varying systems described by ODE's of the form $\dot{x} = f(\varepsilon^{-1}t, x)$, where $f(t, x)$ is 1-periodic with respect to t and $\varepsilon > 0$ is a small parameter. Since the discovery of stabilizing effect of vibration in the reverse pendulum example, there have been a lot of study regarding stability of such systems and design of fast-oscillating stabilizing feedback laws.

In this paper we suggest an approach which is kind of high-order averaging procedure based on Lie algebraic formalism and the formalism of chronological calculus. This latter is a method of asymptotic analysis for flows generated by time-variant ODE. We apply the approach to study stability issues for linear and nonlinear systems. In particular we derive conditions of stability for the second and third order linear differential equations with periodic fast-oscillating coefficients, we study output-feedback stabilization of bilinear systems and consider high-order averaging procedure for nonlinear systems under homogeneity assumptions. At the end we study the problem of stabilization of nonholonomic (control-linear) systems by means of time-varying feedbacks.

1 Introduction and general problem setting

1.1 Introduction

We will study stability and asymptotic stability properties of periodically time-varying systems $\dot{x} = f(t, x)$. In particular we deal with *fast oscillating* systems of the form $\dot{x} = f(\varepsilon^{-1}t, x, \varepsilon)$, where $\varepsilon > 0$ is a small parameter (an inverse to the rate of oscillation) and $f(t, x)$ is 1-periodic in t.

The systems of this type draw much attention for a number of reasons. Differential equations with periodic terms and the corresponding averaging tech-

niques were studied since the 18th century. Stability issues for periodic differential equations have been addressed since long ago. Substantial contributions to the theory were made by A.M.Lyapunov; further results have been obtained by O.Perron, N.Chetaev, I.Malkin, N.Erugin, E.Barbashin, R.Bellman, N.Krasovsky, J.LaSalle, J.Massera, L.Cesari among others.

In control theory and mechanics the discovery of stabilizing effect of vibration in the reverse pendulum example inspired the study of time varying feedback laws. This area became much more active after the discovery of obstructions (e.g. Brockett's criterion) to time-invariant stabilizability and especially after J.-M.Coron obtained ([5]) a general result on time-variant stabilizability for controllable nonlinear control systems (partial results of this kind appeared earlier for example in [18]). Since then many important contributions to the problem of design of time-varying feedbacks for the control systems have been done but still this problem remains (in our opinion) far from being completely solved. Some more comment will be provided at the end of this introduction.

In this paper we suggest kind of high-order averaging procedure for time-varying systems with the objective of studying their stability properties. Obviously of special (and only) interest are those cases where the first-order (standard) averaging leads to a system with critical equilibria. For example this turns out always to be the case in the problem of time-varying stabilization of a nonholonomic system.

The technique we utilize is based on so called *chronological calculus* - asymptotic expansions for the flows generated by time-varying nonlinear vector fields; this calculus has been developed by A.A.Agrachev and R.V.Gamkrelidze in 70's ([1]). This approach is Lie algebraic in its nature: the high-order averaging - the *logarithm of the time-1 map* - is computed via Lie brackets of the vector fields appearing in the right-hand side of the differential equation. The computation of the logarithm of a flow generated by a time-varying vector field is technically rather involved. In Section 4 we collect the main constructions of *chronological algebras* (see [1–3] for details) utilized for this computation and provide formulae for the first three terms of the expansion of the logarithm.

An obvious alternative to this approach is direct Lyapunov method applied to time-variant systems. One can find many interesting results of this kind in numerous publications by D.Aeyels, R.M'Closkey, L.Moreau, R.Murray, J.Peuteman, J.-B. Pomet, C.Samson (see for example [17,14,18] and references therein). On the other side an extensive survey of various averaging techniques can be found in the monograph of Sanders and Verhulst [21].

To illustrate the technique of high-order averaging we start with obtaining new (up to our knowledge) stability results for time-varying linear systems $\dot{x} = A(\varepsilon^{-1}t)x$ with fast oscillating coefficients. In particular we study sta-

bility issues for the second- and third-order linear differential equations with periodic fast-oscillating coefficients and for bilinear systems in an example of R.Brockett (see [4]).

Then we pass to the nonlinear case which is usually much more complex for treatment by most part of the techniques. In the scope of the high-order averaging approach we manage to handle the nonlinear case similarly to the linear one (see for example the formulation of nonlinear Floquet theorem in the Section 3) with one principal difference: one has to deal with infinite-dimensional Lie group of diffeomorphisms instead of a matrix Lie group. We illustrate the method by establishing a local asymptotic stability condition for an equilibrium of time-periodic nonlinear vector field. Our result generalizes a result by R.M'Closkey and P.Morin from their recent publication [12].

Finally we address the problem of time-varying feedback stabilization. Here we do not provide any survey; readers may consult [6] for a broad collection of results, open problems and references regarding this and similar topics.

Up to our knowledge there are essentially two approaches to designing time-varying stabilizing feedbacks. One makes use of time-variant Lyapunov functions (see for example [18,19]). Another approach developed recently by P. Morin, J.-B. Pomet and C. Samson ([15]) is 'converting the open-loop control techniques' of nonholonomic motion planning (the most developed technique is probably the one suggested by W.Liu and H.J.Sussmann; see [11] for the presentation) for constructing time-varying feedbacks. The feedbacks coming out of this approach are very complicated; it is not surprising because the open-loop controls were designed for accomplishing much more delicate task of tracking a given curve than just achieving asymptotic stability.

The material of the last section of this paper illustrates how the high-order averaging can be used to design of stabilizing time-varying feedbacks for drift-less nonholonomic systems. More applications of this approach to the problem of time-varying stabilization will appear in forthcoming publications.

1.2 General problem setting: properties $\mathbf{S}_{as}, \mathbf{S}_0, \mathbf{S}_{us}$

Consider the system of the form

$$\dot{x} = f(\varepsilon^{-1}t, x),$$

where $\varepsilon > 0$ is a small parameter, x belongs to $\mathbb{R}^n$ or to an n-dimensional manifold M. Since in the nonlinear case our treatment will be always local, we identify M with $\mathbb{R}^n$. The right-hand side is assumed to be 1-periodic: $f(t+1, x) = f(t, x),\ f(t, 0) = 0,\ \forall t$. For our objectives it suffices that $f(\tau, x)$

is 'integrable' with respect to t and C^k-smooth (with k sufficiently large) with respect to x. To simplify the presentation we will consider $f(\cdot, x)$ to be continuous and $f(\tau, \cdot)$ to be *real-analytic*.

We shall study the problem of asymptotic stability and/or of stability for the zero solution of such a system for small values of $\varepsilon > 0$. We shall distinguish the following three variants of the **property S**:
$\mathbf{S}_{as}$: *there exists* $\varepsilon_0 > 0$ *such that for all positive* $\varepsilon < \varepsilon_0$ *the zero solution is locally asymptotically stable;*
$\mathbf{S}_0$: *there exists* $\varepsilon_0 > 0$ *such that for all positive* $\varepsilon < \varepsilon_0$ *the zero solution is stable;*
$\mathbf{S}_{us}$: *there exists* $\varepsilon_0 > 0$ *such that for all positive* $\varepsilon < \varepsilon_0$ *the zero solution is unstable.*

2 Linear case: monodromy matrix, stability, Floquet theorem.

The problem has been set for nonlinear systems, but for linear systems with periodic coefficients similar questions arise. [1]

Consider the system

$$(dx/dt =)\, \dot{x}(t) = A(t/\varepsilon)x(t), \tag{1}$$

where $x \in \mathbb{R}^n$ or $\mathbb{C}^n$, $A(\tau)$ is $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$-valued function continuous and 1-periodic with respect to τ, $\varepsilon > 0$ is a small parameter.

What one can say about asymptotic stability of such a system?

The standard averaging result says that, if for the corresponding averaged matrix $\int_0^1 A(\tau)d\tau$ all the eigenvalues are located in the open left complex half-plane, then one can assert asymptotic stability of the system (1) for all sufficiently small $\varepsilon > 0$. *We want to find out what happens in the critical cases where some of the eigenvalues of this averaged matrix are located on the imaginary axis (for example vanish)?*

This question can be reduced to a more explicit one. Proceeding with the time substitution $\tau = t/\varepsilon$ and denoting the derivative $dx/d\tau$ again by $\dot{x}$ we arrive to the equation

$$\dot{x}(\tau) = \varepsilon A(\tau)x(\tau) \tag{2}$$

[1] Obviously the notions of global and local asymptotic stability coincide for linear systems

with 1-periodic coefficients. Consider the corresponding matrix differential equation

$$\dot{X}(\tau) = \varepsilon A(\tau)X(\tau),\ X \in \mathbb{R}^{n\times n} \text{ or, } X \in \mathbb{C}^{n\times n} \tag{3}$$

and introduce the *monodromy matrix* $M_\varepsilon = X(1)$. This monodromy matrix determines stability or instability of the system (2). *If the eigenvalues of M_ε (multiplicators) all belong to the interior of the unit circle, then the system is asymptotically stable. If at least one of these eigenvalues lies outside the unit circle, then the system is unstable.*

Obviously the principal difficulty in utilizing this result is impossibility to compute the spectra of the matrix M_ε. The only feasible idea is to deal with an expansion of M_ε with respect to ε; as it is known for small $\varepsilon > 0$ the matrix M_ε can be represented as a convergent power series in ε.

Instead of dealing with M_ε we will deal with the *logarithm* [2] Λ_ε of the matrix M_ε. The corresponding stability criterion becomes as follows:

the system (1) (with fixed $\varepsilon > 0$) is asymptotically stable if all the eigenvalues of the logarithm Λ_ε are located in the open left half-plane and unstable if any of them belongs to the right open half-plane.

A finer result holds.

Floquet theorem. *The fundamental solution $X_\varepsilon(t)$ of the system (3) can be represented as*

$$X_\varepsilon(t) = e^{t\Lambda_\varepsilon}\Phi_\varepsilon(t),$$

where $\Phi_\varepsilon(t)$ is a 1-periodic matrix-valued function.

The matrix Λ_ε admits a power series representation:

$$\Lambda_\varepsilon = \varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} + \cdots.$$

(It is easy to see that $\Lambda^{(1)} = A^1 = \int_0^1 A(\tau)d\tau$.) One can come to a conclusion about the properties $\mathbf{S}_{as}$, $\mathbf{S}_0$, $\mathbf{S}_{us}$ by analyzing the truncations of the latter series. Standard averaging results says that if the system $\dot{x} = \Lambda^{(1)}x$ is asymptotically stable (resp. unstable) then the system (1) possesses property S_{as} (resp. property S_{us}).

[2] The logarithm is more suitable object for us. As it will be seen the notion can be generalized onto the nonlinear case, where the monodromy map is a nonlinear diffeomorphism. In this latter case the 'logarithm' of this map, (if exists!) is a vector field - an element of a Lie algebra with natural linear structure suitable for expansions into series. It is invariant with respect to coordinate changes.

Obviously the logarithm Λ_ε of the monodromy matrix M_ε provides a complete information on a turnpike or averaged behaviour of the solution of the time-variant system (1). According to Floquet theorem this solution 'oscillates around' the trajectory $e^{t\Lambda_\varepsilon}$ of the time-invariant system $\dot{X} = \Lambda_\varepsilon X,\ X(0) = I$. Therefore Λ_ε can be seen as kind of 'complete averaging' of the system (2), while the truncations of its expansion can be seen as partial averagings of the system.

One can not always come to a conclusion regarding stability by looking at a given higher-order truncation of the series.

Example. Consider the equation

$$\ddot{x} + \sin(2\pi t/\varepsilon)x = 0. \tag{4}$$

After time substitution $\tau = t/\varepsilon$ it is equivalent to the system

$$dx/d\tau = \varepsilon y,\ dy/d\tau = -\varepsilon \sin(2\pi\tau)x.$$

We will study stability properties for this system (asymptotic stability is excluded by Liouville theorem). We will invoke the terms of the corresponding logarithm Λ_ε (see subsection 6.1 for their computation):

$$\varepsilon\Lambda^{(1)} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix},\ \varepsilon^2\Lambda^{(2)} = \begin{pmatrix} -\varepsilon^2/2\pi & 0 \\ 0 & \varepsilon^2/2\pi \end{pmatrix},\ \varepsilon^3\Lambda^{(3)} = \begin{pmatrix} 0 & 0 \\ -3\varepsilon^3/8\pi^2 & 0 \end{pmatrix}.$$

The matrix $\Lambda^{(1)}$ is 2×2 Jordan block with zero eigenvalue; therefore the origin is unstable equilibrium for the first order averaging. It is unstable as well for the second-order averaging $\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)}$, whose eigenvalues are $\mp\frac{\varepsilon^2}{2}$. Still one can not conclude the property $\mathbf{S}_{us}$, because the eigenvalues of the matrix can be affected by a term of order ε^3. It happens indeed! The third-order averaging $\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} + \varepsilon^3\Lambda^{(3)}$ has zero trace and positive determinant equal $\varepsilon^4/8\pi^2$. The contributions of the terms $\Lambda^{(i)},\ i \geq 4$, to the determinant are $o(\varepsilon^4)$ and hence we may assert the property $\mathbf{S}_0$ for the system (4).

A major problem is the computation of the expansion for Λ_ε. In the Section 4 we will proceed with this computation in nonlinear setting.

3 Nonlinear time-periodic systems. Time-1-Map. Nonlinear Floquet theorem

Let us consider the nonlinear system

$$\dot{x} = f(\varepsilon^{-1}t, x),$$

where $f(\tau, x)$ is 1-periodic with respect to τ. Introducing the fast time variable $\tau = t/\varepsilon$ we come to the equation

$$dx/d\tau = \varepsilon f(\tau, x). \tag{5}$$

Here $f(\tau, x)$ is time-variant vector field. Let P_ε^τ be the flow generated by it, which means that P_ε^τ is the family of diffeomorphisms satisfying the *linear* operator differential equation

$$dP_\varepsilon^\tau/d\tau = P_\varepsilon^\tau \circ \varepsilon f(\tau, \cdot),\ P_\varepsilon^0 = I.$$

Here we identify a diffeomorphism $P \in Diff(\mathbb{R}^n)$ with the automorphism $\varphi(\cdot) \mapsto \varphi(P(\cdot))$ of the algebra of smooth functions $\varphi(\cdot) \in C^\infty(\mathbb{R}^n)$. In its turn a vector field f (time variant-vector field $f(\tau, \cdot)$) is identified with a derivation (resp. with a locally integrable family of derivations) of the same algebra (see [1,3] for more details).

Sometimes we will denote this flow by $P_\varepsilon^\tau = \overrightarrow{\exp} \int_0^\tau \varepsilon f(\theta, \cdot) d\theta$ and call it, following [1], *right chronological exponential*. The diffeomorphism $M_\varepsilon = P_\varepsilon^1$ is a *time*-1-*map* or a *monodromy map*. Obviously $P_\varepsilon^{\tau+1} = M_\varepsilon \circ P_\varepsilon^\tau$ and the origin is a fixed point of M_ε.

Proposition 3.1 *The origin O is asymptotically stable for the system if it is an asymptotically stable fixed point for the map M_ε, i.e. $\forall \Delta > 0\ \exists \delta > 0$ such that $\rho(x, O) < \delta \Rightarrow \rho(M_\varepsilon^n(x), 0) < \Delta$ and $\rho(M_\varepsilon^n(x), 0) \to 0$ as $n \to +\infty$.*

The question is reduced to the one of stability of the origin for the diffeomorphism M_ε. Again it will be more convenient for us to deal with a linear object - the *logarithm of the diffeomorphism M_ε*. Time-invariant vector field Λ_ε being a logarithm of the diffeomorphism M_ε means that M_ε is included in the flow $e^{\Lambda_\varepsilon t}$ generated by the vector field Λ_ε: $M_\varepsilon = e^{\Lambda_\varepsilon t}\Big|_{t=1} = e^{\Lambda_\varepsilon}$. Again we may introduce the expansion of Λ_ε into the power series

$$\Lambda_\varepsilon = \sum_{i=1}^{\infty} \varepsilon^i \Lambda^{(i)} \tag{6}$$

and try to proceed in a similar way as in the linear case.

There are additional difficulties on this way. In contrast to the linear case *it is not always that the logarithm* Λ_ε *exists*. Indeed as it is known (see [10, p.456]) the exponential map of the infinite-dimensional Lie algebra $\mathrm{Vect}\mathbb{R}^n$ of real analytic vector fields into the infinite-dimensional Lie group of real analytic diffeomorphisms $\mathrm{Diff}\mathbb{R}^n$ is *not locally onto*.

Still if one manages to construct a formal power expansion for Λ_ε, then the exponentials of its truncations provide nice asymptotics for the diffeomorphism M_ε (see [1]) and we often are able to conclude asymptotic stability or instability.

This existence issue (or a possibility to include a diffeomorphism in a flow generated by a *time-invariant* vector field) is subtle (see for example [16]). A remark made in [22, Subsection 2.1] turns out to be not valid in the generality in which it is stated. This does not invalidate any of (stability) results obtained in [22].

In the rest of this section we assume that the vector field Λ_ε exists, although in the subsequent sections we assume (unless anything else is explicitly stated) existence of the logarithm in formal category. Obviously the origin is an equilibrium point for the vector field Λ_ε and the following result holds.

Theorem 3.1 *If the origin is an asymptotically stable equilibrium point for* Λ_ε *then it is an asymptotically stable equilibrium point for the system* $\dot{x} = f(\varepsilon^{-1}t, x)$.

As in the linear case one can not conclude automatically $\mathbf{S}_{as}$ property for a nonlinear system basing on the stability properties of a truncated series $\sum_{i=1}^{N} \varepsilon^i \Lambda_\varepsilon^{(i)}$, unless this truncation includes a Hurwitzian linear part. We illustrate the phenomenon with the following elementary example.

Example. Consider a system

$$\dot{x}_1 = -x_1 + x_2 \sin 2\pi t/\varepsilon, \;\; \dot{x}_2 = x_1 \cos 2\pi t/\varepsilon + x_2^3.$$

The first-order averaging of this system - the first term of the expansion of the corresponding logarithm - is the vector field $\Lambda^{(1)} = -x_1\partial/\partial x_1 + x_2^3\partial/\partial x_2$. The origin is an *unstable* equilibrium for $\Lambda^{(1)}$. Still in contrast to this fact *the system (13) is asymptotically stable*, for sufficiently small $\varepsilon > 0$.

To prove the asymptotic stability we may consider the system as a perturbation of its linear part by the nonlinearity $x_2^3\partial/\partial x_2$. We will establish in the next section (see Example 2) that for a sufficiently small $\varepsilon > 0$ the linear part is asymptotically stable, i.e. spectrum of the corresponding averaging Λ_ε is in the open left half-plane. Since the linear system is time-periodic, it is reducible and hence *Lyapunov regular*, i.e. the sum of the characteristic numbers equals

to the average of the trace of its coefficient matrix. Then by a standard result of Lyapunov the nonlinear system has an asymptotically stable equilibrium at the origin. □

We finish this section with a complete analogue of Floquet theorem for nonlinear time-periodic system.

Theorem 3.2 (nonlinear Floquet theorem) *Assume that P^τ is the flow generated by a time-periodic differential equation $dx/d\tau = f(\tau, x)$, where $f(\tau + 1, x) = f(\tau, x)$. If the vector field Λ is the logarithm of the diffeomorphism P^1, then the flow P^τ can be represented as a composition $P^\tau = e^{\Lambda\tau} \circ \Phi^\tau$ of the flow $e^{\Lambda\tau}$ and of the 1-periodic flow Φ^τ (1-periodicity of a flow Φ^τ means $\Phi^\tau = \Phi^{\tau+1}$).*

The proof goes essentially along the same line as the proof of standard (linear) Floquet theorem. Instead of matrix multiplication one has to consider compositions of flows. Certainly we have to assume that these compositions and flows exist.

4 Computation of the logarithm of a diffeomorphism

For a diffeomorphism included into the flow generated by time-varying ODE one can compute the logarithm via 'formal chronological series'. It has been accomplished in [1]); in [3] another method of its computation has been provided. This latter is based on so called *chronological product* of time-variant vector fields (see [2]).

Definition 5 *For two time-varying vector fields $X_t(x), Y_t(x)$, which are absolutely continuous with respect to t, the chronological product $(X_\cdot * Y_\cdot)_\cdot$ equals:*

$$(X_\cdot * Y_\cdot)_t = \int_0^t \left[X_\tau, \frac{\partial}{\partial \tau} Y_\tau \right] d\tau, \tag{7}$$

where $[\cdot, \cdot]$ stays for the Lie bracket of two vector fields.

This is a nonassociative product, satisfying the *chronological identity*

$$X * (Y * Z) - Y * (X * Z) = (X * Y - Y * X) * Z.$$

In their recent paper [9] M.Kawski and H.J.Sussmann defined (right) chronological algebra by means of the identity $u \cdot (v \cdot w) = (u \cdot v) \cdot w + (v \cdot u) \cdot w$. A typical example of the chronological algebra in this latter sense is a space of integrable functions on R with a product $(u \cdot v)(t) = v(t) \int_0^t u(\tau) d\tau$. This

operation is important for design of time-variant feedbacks but in the rest of this paper we shall use another operation in the space of integrable functions:

$$(u \star v)(t) = (u \cdot v)(t) - (v \cdot u)(t) = v(t) \int_0^t u(\tau)d\tau - u(t) \int_0^t v(\tau)d\tau. \quad (8)$$

Using the chronological product (7) and the product (8) one is able to compute the expansion $\sum_{i=1}^{\infty} \varepsilon^i \Lambda^{(i)}$ of the logarithm Λ_ε of the diffeomorphism $M_\varepsilon = \overrightarrow{\exp} \int_0^1 \varepsilon X_\theta d\theta$. The first terms of this series (expressed in terms of Lie brackets) are

$$\Lambda^1(X_\cdot) = \int_0^1 X_t dt, \qquad \Lambda^2(X_\cdot) = \frac{1}{2} \int_0^1 \left[\int_0^{t_1} X_{t_2} dt_2, X_{t_1} \right] dt_1,$$

$$\Lambda^3(X_\cdot) = -\frac{1}{12} \left[\int_0^1 X_{t_1} dt_1, \int_0^1 \left[\int_0^{t_1} X_{t_2} dt_2, X_{t_1} \right] dt_1 \right] + \quad (9)$$

$$\frac{1}{6} \left(\int_0^1 \left[\int_0^{t_1} \left[\int_0^{t_2} X_{t_3} dt_3, X_{t_2} \right] dt_2, X_{t_1} \right] dt_1 + \int_0^1 ad^2 \left(\int_0^{t_1} X_{t_2} dt_2 \right) X_{t_1} dt_1 \right)$$

Using integration by parts and the Jacobi identity one can transform the expression for $\Lambda^3(X_\cdot)$ into

$$\Lambda^3(X_\cdot) = -\frac{1}{2} \left[\Lambda^1(X_\cdot), \Lambda^2(X_\cdot) \right] + \frac{1}{3} \int_0^1 ad^2 \left(\int_0^{t_1} X_{t_2} dt_2 \right) X_{t_1} dt_1;$$

here $\mathrm{ad}X\ Y = [X, Y],\ \mathrm{ad}^i X\ Y = [X, \mathrm{ad}^{i-1} X\ Y],\ i \geq 2$.

Let us comment on a particular case of linear vector fields $X_t(x) = A_t x$. For linear vector fields Ax and Bx the Lie bracket is again a linear vector field equal to $[Ax, Bx] = BAx - ABx = -[A, B]x$, where $[A, B]$ is the commutator of matrices A and B. Suppressing x we may treat the terms $\Lambda^{(i)}(A_\cdot)$ in the expansion (6) as matrices. To compute these matrices one has to substitute in the previous expressions $X_\cdot$ by $A_\cdot$ and multiply each Lie monomial of order k by $(-1)^{k-1}$.

We will now sketch the method of computation of the expansion for the logarithm.

Consider the right-chronological exponent $\Phi_\varepsilon^\tau = \overrightarrow{\exp} \int_0^\tau \varepsilon X_t dt$. Obviously $M_\varepsilon = \Phi_\varepsilon^1$. Differentiating Φ_ε^τ with respect to ε we obtain (see [3]):

$$(\partial/\partial\varepsilon)\Phi_\varepsilon^\tau = Z_\varepsilon^\tau \circ \Phi_\varepsilon^\tau,$$

where the Z^τ_ε is time-varying vector field depending absolutely continuously on time τ; $Z^\tau_0 = \int_0^\tau X_t dt$. This vector field has been called in [3] 'angular velocity' of the flow. The angular velocity Z^τ_ε satisfies the equation

$$(\partial/\partial\varepsilon)Z^\tau_\varepsilon = (Z^\cdot_\varepsilon * Z^\cdot_\varepsilon)^\tau = \int_0^\tau \left[Z^\theta_\varepsilon, \frac{\partial}{\partial\theta} Z^\theta_\varepsilon \right] d\theta, \tag{10}$$

where $*$ is the chronological product introduced in the Definition 5.

The equation (10) allows us to find the expansion $Z^\tau_\varepsilon = Z_0 + \sum_{k=1}^\infty Z^\tau_{(k)}(\varepsilon^k/k!)$. Various methods of practical computation of $Z^\tau_{(k)}$ were introduced in [3]. For example one can consider the free chronological algebra with a generator λ and a derivation D defined by: $D\lambda = \lambda * \lambda$. Iterating this derivation we obtain a sequence of polynomials with respect to the product $*$:

$$p_1(\lambda) = \lambda,\ p_2(\lambda) = \lambda * \lambda,\ , p_3(\lambda) = (\lambda * \lambda) * \lambda + \lambda * (\lambda * \lambda), \ldots. \tag{11}$$

Then $Z^\tau_{(k)} = p_{k+1}\left(\int_0^\cdot X_t dt\right)$, where $*$ stays for the chronological product.

We yet need another construction to compute the expansion of the logarithm Λ_ε. Let us differentiate the equality $e^{\Lambda_\varepsilon} = M_\varepsilon = \overrightarrow{\exp} \int_0^1 \varepsilon X_t dt$ with respect to ε. We obtain (see [1] for the details):

$$\int_0^1 e^{\tau \mathrm{ad}_{\Lambda_\varepsilon}} d\tau \frac{\partial \Lambda}{\partial \varepsilon} \circ e^{\Lambda_\varepsilon} = Z^1_\varepsilon \circ \overrightarrow{\exp} \int_0^1 \varepsilon X_t dt.$$

and consequently

$$\int_0^1 e^{\tau \mathrm{ad}_{\Lambda_\varepsilon}} d\tau \frac{\partial \Lambda}{\partial \varepsilon} = Z^1_\varepsilon, \tag{12}$$

Let us consider the function $\int_0^1 e^{\tau\xi} d\tau = (e^\xi - 1)/\xi$ and denote by $\phi(\xi) = \xi/(e^\xi - 1)$. Then (12) is equivalent to the formal equation

$$\frac{\partial \Lambda}{\partial \varepsilon} = \phi(\mathrm{ad}\ (\Lambda_\varepsilon)) Z^1_\varepsilon.$$

To obtain the expansion for Λ_ε from this latter equation we need Taylor expansion for $\phi(\xi)$ It is known to be $\phi(\xi) = \sum_{i=1}^n B_i(\xi^i/i!)$, where B_i are Bernoulli numbers. The first Bernoulli numbers are

$$B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_{2k+1} = 0,\ k > 0,\ B_4 = -1/30, B_6 = 1/42, \ldots.$$

From (11), (12) we derive the expressions for $\Lambda^{(i)}$ and in particular the expressions (9).

6 Stability for linear systems and equations with fast-oscillating coefficients

We are going to illustrate how the Theorem 3.1 together with the expressions (9) can be applied to some particular cases - study of stability of linear systems and of second and third order linear differential equations with periodic fast oscillating coefficients.

6.1 Asymptotic stability for linear systems: some examples

Here we arrange several simple examples of linear systems, whose stability properties are determined by the high-order averaging(s).

Example 1. Consider linear system

$$\dot{x}_1 = -x_1 \pm x_2 \sin 2\pi t/\varepsilon,\ \dot{x}_2 = x_1 \cos 2\pi t/\varepsilon - x_2/3.$$

Its first-order averaging equals $\Lambda^{(1)} = \mathrm{diag}\{-1, -1/3\}$. The first-order averaging is asymptotically stable and hence the system possesses property $\mathbf{S}_{as}$ $\square$

Example 2. Consider (a pair of) linear system(s)

$$\dot{x}_1 = -x_1 \pm x_2 \sin 2\pi t/\varepsilon,\ \dot{x}_2 = x_1 \cos 2\pi t/\varepsilon. \tag{13}$$

Its first-order averaging $\Lambda^{(1)} = \mathrm{diag}\{-1, 0\}$ is 'critical' - one of its eigenvalues vanishes. Computing $\Lambda^{(2)}$ we obtain

$$\Lambda_\varepsilon = \varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} + o(\varepsilon^2) = \begin{pmatrix} -\varepsilon \pm \frac{\varepsilon^2}{4\pi} & \mp\varepsilon^2/2\pi \\ 0 & \mp\frac{\varepsilon^2}{4\pi} \end{pmatrix} + o(\varepsilon^2).$$

Its trace equals $-\varepsilon$ (all the matrices $\Lambda^{(i)},\ i \geq 2$ have zero trace), while its determinant equals $\pm\varepsilon^3/2 + o(\varepsilon^3)$. The system (13) possesses property $\mathbf{S}_{as}$ provided that the sign '+' is chosen in the first equation and possesses property $\mathbf{S}_{us}$ otherwise. $\square$

If ε not only appears as rate of oscillation but also affects the coefficients of the system, i.e. $A = A(\varepsilon, t/\varepsilon)$, then one has to invoke also the expansion of A with respect to ε.

Example 3. Consider the system

$$\dot{x}_1 = -x_1 + \alpha x_2 \sin 2\pi t/\varepsilon,\ \dot{x}_2 = x_1 \cos 2\pi t/\varepsilon + \varepsilon x_2/3. \tag{14}$$

The first-order averaging equals to $\Lambda^{(1)} = \mathrm{diag}\{-1, \varepsilon/3\}$ and is unstable. Taking the second-order truncation, we obtain

$$\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} = \begin{pmatrix} -\varepsilon + \alpha\frac{\varepsilon^2}{4\pi} & -\alpha\varepsilon^2/2\pi \\ 0 & \varepsilon^2/3 - \alpha\frac{\varepsilon^2}{4\pi} \end{pmatrix}.$$

Its trace equals $-\varepsilon + o(\varepsilon)$, while its determinant equals $\varepsilon(-\varepsilon^2/3 + \alpha\frac{\varepsilon^2}{4\pi}) + o(\varepsilon^3)$.

This means that the original system (14) does possess the property $\mathbf{S}_{as}$, provided that $\alpha > 4\pi/3$, and possesses property $\mathbf{S}_{us}$, if $\alpha < 4\pi/3$. □

6.2 Second-order linear differential equation

Let us consider the second-order ODE

$$\ddot{x}(t) + q(t/\varepsilon)\dot{x}(t) + p(t/\varepsilon)x(t) = 0,\ p(\tau) = p(\tau+1), q(\tau+1) = q(\tau). \tag{15}$$

Let $\bar{P} = \int_0^1 p(\tau)d\tau,\ \bar{Q} = \int_0^1 q(\tau)d\tau$. The following proposition is a direct application of standard averaging result.

Proposition 6.2.1 *If $\bar{P} > 0 \bigwedge \bar{Q} > 0$ then the property $\mathbf{S}_{as}$ holds; if $\bar{P} < 0 \bigvee \bar{Q} < 0$ then the property $\mathbf{S}_{us}$ holds.*

Proof. Write down the second-order equation as a second-order system and proceed with a time substitution $\tau = t/\varepsilon$. The matrix εA_τ of the coefficients and its averaging $\Lambda_\varepsilon^{(1)}$ equal

$$A_\tau = \begin{pmatrix} 0 & 1 \\ -p(\tau) & -q(\tau) \end{pmatrix},\ \varepsilon\Lambda^{(1)} = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon\bar{P} & -\varepsilon\bar{Q} \end{pmatrix},\ \Lambda_\varepsilon = \varepsilon\Lambda^{(1)} + o(\varepsilon)$$

respectively. If $\bar{P} > 0 \bigwedge \bar{Q} > 0$ then the eigenvalues of $\Lambda^{(1)}$ and of Λ_ε have negative real parts, if $\bar{P} < 0 \bigvee \bar{Q} < 0$ then at least one of the eigenvalues of Λ_ε has positive real part. □

We are interested in the *critical cases*. To proceed with them we will need the second order term of the expansion of Λ_ε. As we know

$$\Lambda^{(2)} = 2^{-1} \int_0^1 \left[A_t, \int_0^t A_\tau d\tau \right] dt.$$

Hence

$$\varepsilon^2 \Lambda^{(2)} = \frac{\varepsilon^2}{2} \begin{pmatrix} \int_0^1 (1 \star p)(\tau) d\tau & \int_0^1 (1 \star q)(\tau) d\tau \\ \int_0^1 (p \star q)(\tau) d\tau & -\int_0^1 (1 \star p)(\tau) d\tau \end{pmatrix};$$

recall (see (8)) that $(u \star v)(t) = v(t) \int_0^t u(\tau) d\tau - u(t) \int_0^t v(\tau) d\tau$.

Assume first $\bar{P} = 0 \wedge \bar{Q} > 0$.

Proposition 6.2.2 *Let $\bar{P} = 0 \wedge \bar{Q} > 0$. If*

$$\int_0^1 \left((p - \bar{P}) \star (q - \bar{Q}) \right)(t) dt < 0, \tag{16}$$

then the property $\mathbf{S}_{as}$ holds. If the sign in (16) is opposite, then the property $\mathbf{S}_{us}$ holds.

(Although $\bar{P}$ vanishes, we introduce it in the latter condition for the sake of symmetry.)

To derive the condition we have to analyze the truncated logarithm $\varepsilon \Lambda^{(1)} + \varepsilon^2 \Lambda^{(2)}$ which in this case equals

$$\begin{pmatrix} 2^{-1} \varepsilon^2 \int_0^1 (1 \star p)(\tau) d\tau & \varepsilon + 2^{-1} \varepsilon^2 \int_0^1 (1 \star q)(\tau) d\tau \\ 2^{-1} \varepsilon^2 \int_0^1 (p \star q)(t) dt & -\varepsilon \bar{Q} - 2^{-1} \varepsilon^2 \int_0^1 (1 \star p)(\tau) d\tau. \end{pmatrix}$$

The trace of the latter matrix equals $-\varepsilon \bar{Q} < 0$, while its determinant differs from $\det \Lambda_\varepsilon$ by $o(\varepsilon^3)$ and equals

$$-\frac{\varepsilon^3}{2} \left(\bar{Q} \int_0^1 (1 \star p)(\tau) d\tau + \int_0^1 (p \star q)(t) dt \right).$$

The condition of positiveness of the latter determinant can be easily transformed into (16).

If $\bar{P}$ is arbitrary and $\bar{Q} = 0$, the trace of the matrix $\Lambda^{(1)}$ vanishes and according to Liouville theorem then for all ε the determinant of the monodromy matrix M_ε equals 1 and the trace of Λ_ε vanishes. Hence there can be no asymptotic stability in this case. To study stability (property $\mathbf{S}_0$) we have to analyze

again the truncation $\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)}$. If $\bar{P} \neq 0$, then the determinant of the latter matrix equals $\varepsilon^2\bar{P} + o(\varepsilon^2)$, while its trace vanishes. We come to the

Proposition 6.2.3 *If $\bar{Q} = 0$ then (15) is not asymptotically stable. If $\bar{P}$ is positive, then the property $\mathbf{S}_0$ holds, if $\bar{P}$ is negative, then the property $\mathbf{S}_{us}$ holds.*

The next important critical case is where $\bar{Q} = 0 \wedge \bar{P} = 0$. Again the system can not be asymptotically stable. To arrive to stability condition we analyze the matrix $\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)}$ coming to the following conclusion.

Proposition 6.2.4 *If $\bar{P} = \bar{Q} = 0$ then (15) is not asymptotically stable. If the inequality (16) holds then the property $\mathbf{S}_0$ holds for (15), if the value in the left-hand side of (16)is positive, then the property $\mathbf{S}_{us}$ holds for (15).*

We summarize results concerning second-order equation (15) in the following

Theorem 6.1 *For the second-order linear equation (15) let $\bar{P} = \int_0^1 p(\tau)d\tau$, $\bar{Q} = \int_0^1 q(\tau)d\tau$. If $\bar{Q} \leq 0$, then the equation (15) is not asymptotically stable. Besides*

1) If $\bar{P} > 0 \bigwedge \bar{Q} > 0$ then the property $\mathbf{S}_{as}$ holds; if $\bar{P} < 0 \bigvee \bar{Q} < 0$ then the property $\mathbf{S}_{us}$ holds;

2) If $\bar{Q} = 0 \bigwedge \bar{P} > 0$ then the property $\mathbf{S}_{as}$ holds, if $\bar{Q} = 0 \bigwedge \bar{P} < 0$ then the property $\mathbf{S}_{us}$ holds;

3)Let $\bar{P} = 0 \wedge \bar{Q} \geq 0$. If $\int_0^1 \left(p(t) - \bar{P}\right) \star \left(q(t) - \bar{Q}\right) dt$ is positive then the property $\mathbf{S}_{us}$ holds. If this value is negative, then property $\mathbf{S}_{as}$ holds provided that $\bar{Q} > 0$, and property $\mathbf{S}_0$ holds provided that $\bar{Q} = 0$. □

6.3 Reduced second-order differential equation

An important case in which $\bar{Q} = 0$ is the one of reduced second-order equation

$$\ddot{x} + p(t/\varepsilon)x = 0. \tag{17}$$

Here $q(t) \equiv 0$ and zero solution is never asymptotically stable. From a standard averaging result we deduce that the property S_0 holds provided that $\bar{P} > 0$ and the property $\mathbf{S}_{us}$ holds provided that $\bar{P} < 0$. [3]

[3] The famous Lyapunov's integral criterion for stability of the second-order periodic equation must be mentioned. It regards coefficient $p(\cdot)$ of *arbitrary* period but under additional assumption: $p(\cdot) \geq 0$. We deal with $p(\cdot)$ of *arbitrary* sign and of small

If $\bar{P} = 0$ then in contrast to the previous treatment one should analyze also the third term $\varepsilon^3\Lambda^{(3)}$ of the expansion for the logarithm in order to come to a conclusion about stability. Indeed

$$\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} = \begin{pmatrix} \varepsilon^2 P^{(1)} & \varepsilon \\ 0 & -\varepsilon^2 P^{(1)} \end{pmatrix},$$

where $P^{(1)} = \int_0^1 tp(t)dt$. The trace of the latter matrix vanishes while its determinant is negative. Still one is not able to conclude the instability property $\mathbf{S}_{us}$ because the determinant has order of smallness $O(\varepsilon^4)$ and the term of order ε^3 in the lower-left corner may affect the sign of the determinant. It does affect it in a radical way! As we will see the determinant of the matrix $\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} + \varepsilon^3\Lambda^{(3)}$ is always positive provided that $p(\tau) \not\equiv 0$.

Indeed a direct computation involving the last of the formulae (9) gives us the matrix

$$\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} + \varepsilon^3\Lambda^{(3)} = \begin{pmatrix} \varepsilon^2 P^{(1)} & \varepsilon + O(\varepsilon^3) \\ -\varepsilon^3 \int_0^1 (\int_0^t p(\tau)d\tau)^2 dt & -\varepsilon^2 P^{(1)} \end{pmatrix}. \qquad (18)$$

The determinant of this matrix equals $\varepsilon^4 \left(-(P^{(1)})^2 + \int_0^1 (\int_0^t p(\tau)d\tau)^2 dt\right) + o(\varepsilon^4)$. Integration by parts transforms $P^{(1)} = \int_0^1 tp(t)dt$ into $-\int_0^1 (\int_0^t p(\tau)d\tau)dt$ (recall that $\int_0^1 p(t)dt = 0$). Hence by virtue of Cauchy-Schwarz inequality

$$(P^{(1)})^2 = \left(\int_0^1 (\int_0^t p(\tau)d\tau)dt\right)^2 \le \int_0^1 (\int_0^t p(\tau)d\tau)^2 dt \int_0^1 dt,$$

what means that the determinant of (18) is positive provided that $p(\tau) \not\equiv 0$. We established the following property.

Theorem 6.2 *If $\bar{P} = 0$, then the reduced second order equation (17) is not asymptotically stable. If $p(\tau) \not\equiv 0$, then the property $\mathbf{S}_0$ holds for it.*

6.4 Stability properties for third-order linear differential equation

Let us consider the third-order ODE

$$\dddot{x}(t) + p_3(t/\varepsilon)\ddot{x}(t) + p_2(t/\varepsilon)\dot{x}(t) + p_1(t/\varepsilon)x(t) = 0, \qquad (19)$$

period. For nonnegative $p(\cdot)$ of period ε the Lyapunov's criterion requires $0 < \varepsilon\bar{P} \le 4$ for the property $\mathbf{S}_0$ to hold. It holds trivially for sufficiently small $\varepsilon > 0$.

where $p_j(\tau) = p_j(\tau + 1),\ j = 1, 2, 3.$

Let us denote

$$\bar{P}_j = \int_0^1 p_j(\tau)d\tau,\ j = 1, 2, 3;\ P_j^{(i)} = \int_0^1 \tau^i p_j(\tau)d\tau,\ j = 1, 2, 3;\ i \geq 1.$$

The equation (19) can be transformed into a third-order system $\dot{z} = \varepsilon A_\tau z;\ \tau = t/\varepsilon$. The matrix A_τ and its averaging - the term $\varepsilon\Lambda^{(1)}$ of the corresponding logarithm equal to

$$A_\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_1(\tau) & -p_2(\tau) & -p_3(\tau) \end{pmatrix},\ \varepsilon\Lambda^{(1)} = \begin{pmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \\ -\varepsilon\bar{P}_1 & -\varepsilon\bar{P}_2 & -\varepsilon\bar{P}_3 \end{pmatrix}$$

respectively. The characteristic polynomial of the matrix $\varepsilon\Lambda^{(1)}$ is $\lambda^3 + \varepsilon\bar{P}_3\lambda^2 + \varepsilon^2\bar{P}_2\lambda + \varepsilon^3\bar{P}_1$. If the roots of this polynomial are located in the open left half-plane, then the property $\mathbf{S}_{as}$ holds for the equation (19). The Routh-Hurwitz criterion provides the conditions for this to occur:

$$\bar{P}_1 > 0 \bigwedge \bar{P}_3 > 0 \bigwedge \bar{P}_3\bar{P}_2 - \bar{P}_1 > 0.$$

If $\bar{P}_1 < 0 \bigvee \bar{P}_3 < 0 \bigvee \bar{P}_3\bar{P}_2 - \bar{P}_1 < 0$ then the property $\mathbf{S}_{us}$ holds for the equation (19).

Again we are interested in critical cases, where some of the roots of the characteristic polynomial are located on the imaginary axis. To study stability in the critical cases the second-order term of the logarithm is needed and we compute it at once:

$$\Lambda^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

where

$$b_i = \frac{1}{2}\int_0^1 (1 \star p_i)(t)dt, i = 1, 2, 3;\ c_1 = \frac{1}{2}\int_0^1 (p_1 \star p_3)(t)dt,$$

$$c_2 = \frac{1}{2}\left(-b_1 + \int_0^1 (p_2 \star p_3)(t)dt\right),\ c_3 = -b_2.$$

The characteristic polynomial of the matrix $\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)}$ equals

$$\lambda^3 + (\varepsilon\bar{P}_3 + o(\varepsilon^2))\lambda^2 + \left(\varepsilon^2\bar{P}_2 + (\varepsilon^3/2)(b_3\bar{P}_2 - b_2\bar{P}_3 - c_2 - b_1) + o(\varepsilon^3)\right) + \left(\varepsilon^3\bar{P}_1 + (\varepsilon^4/2)(b_3\bar{P}_1 - b_1\bar{P}_3 - c_1) + o(\varepsilon^4)\right) \quad (20)$$

Let us proceed with the analysis of particular cases.

Simple zero eigenvalue Let the matrix $\Lambda^{(1)}$ have a simple zero eigenvalue and two eigenvalues with negative real part(s). This implies $\bar{P}_1 = 0,\ \bar{P}_2, \bar{P}_3 > 0$. In this case asymptotic stability of the zero solution is determined by the sign of the expression $b_1\bar{P}_3 + c_1 = \int_0^1 \left(p_1 \star \left(p_3(\cdot) - \bar{P}_3\right)\right)(t)dt$.

Proposition 6.4.1 *If $\bar{P}_1 = 0$, then the property S_{as} holds for (19), provided that*

$$\int_0^1 \left(p_1(\cdot) \star \left(p_3(\cdot) - \bar{P}_3\right)\right)(t)dt < 0, \quad (21)$$

and the property $\mathbf{S}_{us}$ holds, if the sign in (21) is opposite.

This result can be extended onto the case of n-th order equation

$$x^{(n)}(t) + p_n(t/\varepsilon)x^{(n-1)}(t) + \cdots + p_1(t/\varepsilon)x(t) = 0. \quad (22)$$

Again if the characteristic polynomial [4] of the first-order averaging is Hurwitzian, then the equation possesses property $\mathbf{S}_{as}$. Consider the first critical case where all roots but one are located in the open left half-plane and there is a simple zero root. This implies $\bar{P}_1 = 0$.

Theorem 6.3 *If $\bar{P}_1 = 0$, and all but one eigenvalues of the first-order averaging are located in the open left half-plane, then the property $\mathbf{S}_{as}$ holds for the equation (22) if $\int_0^1 \left(p_1(\cdot) \star \left(p_n(\cdot) - \bar{P}_n\right)\right)(t)dt$ is negative and the property $\mathbf{S}_{us}$ holds if this value is positive.*

Now we come back to the equation (19) and consider the case where the matrix $\Lambda^{(1)}$ corresponding to the first-order averaging has zero eigenvalue of multiplicity 2.

Zero eigenvalue of multiplicity 2 Assume that the nonzero eigenvalue has negative real part. This implies $\bar{P}_1 = \bar{P}_2 = 0 \bigwedge \bar{P}_3 > 0$. Analyzing the characteristic polynomial (20) we come to a conclusion that the property $\mathbf{S}_{as}$ holds for (19) provided that

$$b_1\bar{P}_3 + c_1 < 0 \bigwedge b_2\bar{P}_3^2 + c_2\bar{P}_3 - c_1 < 0. \quad (23)$$

[4] equal to $\sum_{i=(0)}^{n} \varepsilon^i \bar{P}_{n-i+1}\lambda^{n-i}$, where $\bar{P}_{n+1} = 1,\ \bar{P}_i = \int_0^1 p_i(\tau)d\tau, i = 1, \ldots, n$,

If either of the inequalities holds with opposite sign, then the property $\mathbf{S}_{us}$ holds for (19).

The first of the inequalities (23) coincides with (21) which appeared in the previous case. The second inequality can be transformed into

$$\int_0^1 \left(\left(\bar{P}_3 p_2(\cdot) - p_1(\cdot)\right) \star \left(p_3(\cdot) - \bar{P}_3\right)\right)(t)dt < 0. \tag{24}$$

Proposition 6.4.2 *If $\bar{P}_1 = \bar{P}_2 = 0 \bigwedge \bar{P}_3 > 0$, then the property S_{as} holds for the third-order equation (19) provided that both of the inequalities (21) and (24) hold. If the sign is opposite in any of these inequalities, then the property $\mathbf{S}_{us}$ holds.*

Pair of imaginary eigenvalues If the first-order averaging possesses a pair of imaginary eigenvalues, this means: $\bar{P}_1, \bar{P}_2, \bar{P}_3 > 0,\ \bar{P}_1 = \bar{P}_2\bar{P}_3$.

Proposition 6.4.3 *If $\bar{P}_1, \bar{P}_2, \bar{P}_3 > 0, \bar{P}_1\bar{P}_2 = \bar{P}_3$, then the property $\mathbf{S}_{as}$ (respectively the property $\mathbf{S}_{us}$) holds provided that the inequality (24) (respectively the opposite to (24)) holds.*

Theorem 6.4 (stability for the third-order equation) *1)If $\bar{P}_1 = 0$, then the property S_{as} holds for the third-order equation (19), provided that the inequality (21) holds. The property $\mathbf{S}_{us}$ holds for (19) provided that the opposite to (21) inequality is true;*

2) If $\bar{P}_1 = \bar{P}_2 = 0,\ \bar{P}_3 > 0$, then the property $\mathbf{S}_{as}$ holds for (19) provided that both of the inequalities (21) and (24) are true; if any of these inequalities holds with the opposite sign, then the property $\mathbf{S}_{us}$ holds for (19);

3)If $\bar{P}_1, \bar{P}_2, \bar{P}_3 > 0,\ \bar{P}_1 = \bar{P}_2\bar{P}_3$, then the property $\mathbf{S}_{as}$ holds for (19), provided that the inequality (24) is true; if the inequality (24) holds with the opposite sign, then the property $\mathbf{S}_{us}$ holds for (19).

7 Time-varying stabilizers for bilinear systems: R.Brockett's example

In the context of time-variant output feedback stabilization R.Brockett considered following example (see [4]).

Given a bilinear system $\dot{x} = (A + B_1 u_1 + B_2 u_2)x,\ x \in \mathbb{R}^3$, where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

find controls $u_1(t), u_2(t)$ such that the resulting system is asymptotically stable.

It is easy to see that for each given t the matrix $A+B_1u_1(t)+B_2u_2(t)$ has vanishing determinant (in fact it has always a zero eigenvalue and a double eigenvalue equal -1). Still it is known that the system can be made asymptotically stable by a proper choice of time-variant parameters (controls) $u_1(t), u_2(t)$.

We assume the controls to be fast oscillating functions $k_1(t/\varepsilon), k_2(t/\varepsilon)$ with $k_1(\tau), k_2(\tau)$ being 1-periodic. Our goal is to establish the condition under which the system

$$\dot{x} = (A + B_1 k_1(t/\varepsilon) + B_2 k_2(t/\varepsilon))\, x \tag{25}$$

possesses the property $\mathbf{S}_{as}$?

Computing the first two terms of the corresponding logarithm we obtain $\Lambda^{(1)} = A + B_1\bar{K}_1 + B_2\bar{K}_2$, where $\bar{K}_i = \int_0^1 k_i(\tau)d\tau, i = 1, 2$;

$$\Lambda^{(2)} = 2^{-1} \begin{pmatrix} \int_0^1 (k_2 \star k_1)(t)dt & 0 & \int_0^1 (1 \star k_1)(t)dt \\ 0 & \int_0^1 (k_2 \star k_1)(t)dt & \int_0^1 (1 \star k_2)(t)dt \\ -\int_0^1 (1 \star k_2)(t)dt & \int_0^1 (1 \star k_1)(t)dt & 2\int_0^1 (k_1 \star k_2)(t)dt \end{pmatrix}.$$

It is obvious that the matrix $\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)}$ has two eigenvalues with the real part close to -1 and hence for the third eigenvalue to have negative real part it is necessary and sufficient that the determinant of $\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)}$ is negative. After a computation we come to the following

Theorem 7.1 *If* $\int_0^1 \left((k_1(\cdot) - \bar{K}_1) \star (k_2(\cdot) - \bar{K}_2)\right) dt$ *is negative then the property* $\mathbf{S}_{as}$ *holds for the system (25). If this value is positive then the property* $\mathbf{S}_{us}$ *holds.*

8 High-order averaging for time-varying nonlinear systems and stability

Let us show how the technique works in periodic nonlinear non-fast-oscillating case.

Assume $X(t,x)$ to be nonlinear time-periodic vector field in $\mathbb{R}^n$, continuous together with all its partial derivatives with respect to x_i's; $X(t,0)\equiv 0$.

We will need to introduce some homogeneity. To this end let us consider a dilation

$$\delta^r_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n : \delta^r_\varepsilon(x_1,\dots,x_n) = (\varepsilon^{r_1}x_1,\dots,\varepsilon^{r_n}x_n),\ \varepsilon>0,\ r_i\geq 0,\ i=1,\dots,n.$$

The dilation gives rise to filtrations in the algebras of smooth functions in $\mathbb{R}^n$, of differential operators and of vector fields (see [3] for details). For example the weight of a monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ equals $\sum_{k=1}^n \alpha_k r_k$; the weight of a monomial vector field $x_1^{\alpha_1}\cdots x_n^{\alpha_n}\partial/\partial x_\beta$ equals $-r_\beta+\sum_{k=1}^n \alpha_k r_k$. For the sake of brevity let us say that vector field is of weight $\geq s$ if all the monomials in its Taylor expansion with respect to x are of weights $\geq s$ (see [3] for 'more invariant' definition). For the rest of this section assume $X(t,x)$ to be of weight $\geq s$ for some positive s.

Consider *Fourier expansion*

$$X^0(x)+\sum_{n=1}^{\infty}(X^n\cos 2\pi nt + Y^n \sin 2\pi nt)$$

of $X(t,x)$. Obviously X^0 coincides with the first-order averaging of X : $X^0=\int_0^1 X(t,x)dt$ and all the vector fields X^n, Y^n are of weight $\geq s$. In addition we will *assume* [5] *X^0 to be of weight $\geq 2s$*. This includes the case $X^0=0$, we assume that arbitrary weight can be assigned to vanishing vector field. Let us represent $X^i, Y^i,\ i\geq 1$, as $X^i=\tilde{X}^i+\cdots, Y^i=\tilde{Y}^i+\cdots$, where $\tilde{X}^i,\tilde{Y}^i$ are homogeneous of weight s and the rest terms are of weights $>s$. Let us represent X^0 as $X^0=\tilde{X}^0+\cdots$, where $\tilde{X}^0$ is homogeneous of weight $2s$, and the rest is of weight $>2s$.

Theorem 8.1 *If the equilibrium is locally asymptotically stable for the time-invariant homogeneous vector field*

$$\tilde{X}^0(x)+\sum_{n=1}^{\infty}(4\pi n)^{-1}[\tilde{X}^n,\tilde{Y}^n] \tag{26}$$

[5] This assumption for X^0 can be weakened. We will return to this problem in a forthcoming publication

then it is asymptotically stable for the system $\dot{x} = X(t, x)$.

Remark. This result can be compared with the results of R.M'Closkey and P.Morin (see [12]) where they studied the system

$$\dot{x} = X^0(x) + X^1(x)\cos 2\pi t + Y^1(x)\sin 2\pi t$$

under some homogeneity assumptions for the vector fields X^0, X^1, Y^1 having common equilibrium point.

We sketch the proof of the theorem 8.1. Take the time-1 map for $X(t, x)$. Calculating the first two terms of the expansion of its logarithm (the logarithm may not exist, but the expansion exists as a formal series!) we obtain $\Lambda^{(1)} = X^0(x)$, $\Lambda^{(2)} = \sum_{n=1}^{\infty}(4\pi n)^{-1}[X^n, Y^n]$. Denote by $\tilde{\Lambda}^{(1)} = \tilde{X}^0$, $\tilde{\Lambda}^{(2)} = \sum_{n=1}^{\infty}(4\pi n)^{-1}[\tilde{X}^n, \tilde{Y}^n]$. Then $\Lambda = \Lambda^{(1)} + \tilde{\Lambda}^{(2)} + \cdots$, where the omitted terms are of weights $> 2s$. Therefore the homogeneous of weight $2s$ vector field $\Lambda^{(1)} + \tilde{\Lambda}^{(2)}$, appearing in (26), is the principal part of Λ. If the origin is asymptotically stable for (26) and Λ exists (i.e. the respective formal series converges) then the origin is asymptotically stable for Λ by virtue of Massera-Hermes theorem ([7,6]) and hence is asymptotically stable for $X(t, x)$. If Λ only exists as a formal series still we are able to prove that the vector field (26) determines the asymptotics and stability of the trajectories of the system $\dot{x} = X(t, x)$.

9 Time-variant stabilization for nonholonomic systems

9.1 Unicycle example

In this section we deal with time-variant stabilization of so-called nonholonomic control-linear driftless systems:

$$\dot{x} = \sum_{i=1}^{r} X^i(x)u_i, \tag{27}$$

where the dimension r of the control is strictly less than the dimension of the state x. It is known that this system is not stabilizable by means of continuous time-invariant feedback controls: $u(x) = (u_1(x), \ldots, u_r(x))$, since it fails to satisfy R.Brockett's stabilizability criterion. Still it is known that if the Lie rank of the system (27) is complete, then it can be stabilized by means of time-variant feedback ([5]).

One can utilize the method, developed above, to construct time-varying feedbacks for such systems. We illustrate it with a simple example of a unicycle.

The kinematics of the unicycle is described by a simplest kind of so-called chained system

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_2 u_1. \tag{28}$$

Different types of stabilizing feedback controls can be and have been constructed for this system ([20,18]). We will utilize the following time-variant smooth feedback law

$$u_1 = -x_1 + \varepsilon^{-1/2} x_3^2 \sin t/\varepsilon,\ u_2 = -x_2 - \varepsilon^{-1/2} x_3 \cos t/\varepsilon. \tag{29}$$

We will use averaging technique to prove that this feedback stabilizes the system (28) provided that $\varepsilon > 0$ is small enough. [6] The achieved asymptotic stability is *not* exponential.

Proposition 9.1 *The time-varying smooth feedback (29) stabilizes asymptotically (nonexponentially) the system (28).*

(We will assume $\cos t$ and $\sin t$ to be 1-periodic in order to avoid a multitude of 2π factors in computations.) We start with the substitution of the time variable $\tau = t/\varepsilon$ obtaining the system

$$\frac{dx_1}{d\tau} = -\varepsilon x_1 + \varepsilon^{1/2} x_3^2 \sin\tau,\ \frac{dx_2}{d\tau} = -\varepsilon x_2 - \varepsilon^{1/2} x_3 \cos\tau,$$
$$\frac{dx_3}{d\tau} = -\varepsilon x_1 x_2 + \varepsilon^{1/2} x_2 x_3^2 \sin\tau \tag{30}$$

Let X_τ be the right-hand side of (30):

$$X_\tau = (-\varepsilon x_1 + \varepsilon^{1/2} x_3^2 \sin\tau)\partial/\partial x_1 + (-\varepsilon x_2 - \varepsilon^{1/2} x_3 \cos\tau)\partial/\partial x_2 +$$
$$(-\varepsilon x_1 x_2 + \varepsilon^{1/2} x_2 x_3^2 \sin\tau)\partial/\partial x_3.$$

A tempting idea would be considering X_τ as a sum $X^0 + X^1 \cos\tau + Y^1 \sin\tau$ and apply the result of the previous section. Unfortunately we do not have needed homogeneity and special arrangements must be done in this and similar cases.

Computing the first two terms of the corresponding logarithm we obtain

[6] A similar stabilizing feedback law has been utilized by R.T.M'Closkey and R.Murray in [13]. Their proof of stability is based on a construction of time-variant Lyuapunov function

$$\Lambda^{(1)}(X.) = \int_0^1 X_\tau d\tau = -\varepsilon x_1 \partial/\partial x_1 - \varepsilon x_2 \partial/\partial x_2 - \varepsilon x_1 x_2 \partial/\partial x_3,$$

$$\Lambda^{(2)}(X.) = 2^{-1} \int_0^1 \left[\int_0^\tau X_\xi d\xi, X_\tau \right] d\tau = -2^{-1} \varepsilon x_3^3 \partial/\partial x_3 +$$

$$A_3 \varepsilon^{3/2} x_1 x_2 x_3 \partial/\partial x_1 + (A_4 \varepsilon^{3/2} x_3 x_1 + A_5 \varepsilon^{3/2} x_1 x_2 + A_6 \varepsilon x_2 x_3^2) \partial/\partial x_2 +$$

$$(A_1 \varepsilon x_1 x_2 + A_2 \varepsilon^{3/2} x_3^2 x_2) \partial/\partial x_3).$$

The principal or the *diagonal* part of the sum $\Lambda^{(1)} + \Lambda^{(2)}$ is formed by the summands $-\varepsilon x_1 \partial/\partial x_1 - \varepsilon x_2 \partial/\partial x_2 - 2^{-1} \varepsilon x_3^3 \partial/\partial x_3$. Obviously the origin is asymptotically (nonexponentially) stable equilibrium for the diagonal part.

It suffices to take the Lyapunov function $V = x_1^2 + x_2^4 + x_3^2$ to prove that (for sufficiently small $\varepsilon > 0$) the origin is asymptotically stable equilibrium for the vector field $\Lambda^{(1)} + \Lambda^{(2)}$. What is more difficult is proving that the higher-order terms do not destroy this property.

A standard tool to prove the latter fact would be utilizing homogeneity argument. The difficulty which arises is that the principal part $-\varepsilon x_1 \partial/\partial x_1 - \varepsilon x_2 \partial/\partial x_2 - \varepsilon x_3^3 \partial/\partial x_3$ *ceases to be homogeneous with respect to any dilation with any positive weights* (r_1, r_2, r_3), or better to say with any weight $r_3 \neq 0$.

To overcome this difficulty we suggest a less universal way of assigning weights to the vector fields. This assignment does not suit to any filtration of the Lie algebra of vector fields but still possess some properties with respect to the Lie multiplication. These properties will be utilized for establishing asymptotic stability.

9.2 Homogeneity argument

Consider the space $\mathbb{R}^n$ with a fixed coordinate system $x_1, \ldots, x_n$. To define the weights of the variables let us assign to each x_i two numbers - a positive number a_i and a natural number k_i. The weight of the variable x_i is $r_i = 1/(a_i + k_i)$.

Let $\bar{a} = (a_1, \ldots, a_n), \bar{k} = (k_1, \ldots, k_n)$. We define the $(\bar{a}, \bar{k})-$*weight of a monomial vector field* in a following (non-canonical) way

$$\nu\left(x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m} \partial/\partial x_i\right) = \sum_{s=1} \alpha_s \nu(x_{i_s}) + \frac{a_i}{a_i + k_i} = \sum_{s=1} \alpha_s \nu(x_{i_s}) - \nu(x_i) + \frac{a_i + 1}{a_i + k_i}.$$

The difference with the 'usual' definition is in appearance of the last summand. This summand spoils the consistency with a filtration but makes the construction proper for dealing with the vector fields like $x_s^{k_s} \partial/\partial x_s$. Indeed under this

definition the weight of the latter vector field equals $k_s\nu(x_s)+a_s/(a_s+k_s)$ i.e. is 1 *for any choice of* a_s.

Lemma 9.2.1 *Assume an analytic vector field* X, $X(0)=0$, *to be of the form* $X_d+(X-X_d)$ *where* X_d *is a 'diagonal' vector field*

$$X_d=-\sum_{s=1}^{n} b_s \text{ sign } x_s|x_s|^{k_s}\partial/\partial x_s,\ k_s\geq 1, b_s>0,\ s=1,\dots,n,$$

$\bar{k}=(k_1,\dots,k_n)$. *Assume that there exist positive* $\bar{a}=(a_1,\dots,a_n)$ *such that all the terms of the Taylor expansion of the vector field* $X-X_d$ *have* $(\bar{a},\bar{k})-$ *weights* >1. *Then the origin is asymptotically stable equilibrium for* X.

Proof is pretty simple. Consider Lyapunov function $V=\sum_{s=1}^{n}|x_s|^{1+a_s}$. Then $L_{X_d}V=-\sum_{s=1}^{n}(1+a_s)b_s|x_s|^{k_s+a_s}$, is a homogeneous function of weight 1 taking negative values outside the origin. At the same time $L_{(X-X_d)}V$ has weight >1. Hence V is local Lyapunov function for $X=X_d+(X-X_d)$.□

As we told, the way, the weights of vector fields have been introduced, does not respect the structure of the Lie algebra of vector fields. For example the weight of a Lie bracket of two monomial fields is not equal to the sum of weights of the factors. Still some weaker property holds.

Lemma 9.2.2 *If* X_1,X_2 *are polynomial vector fields, homogeneous of weights* w_1 *and* w_2 *correspondingly, then the weights of the summands of the polynomial vector field* $[X_1,X_2]$ *(which may cease to be homogeneous) are* $\geq w_1+w_2-1$. *Therefore the polynomial vector fields of weight* ≥ 1 *form a Lie subalgebra; for each* $r\geq 1$ *the vector fields of weight* $\geq r$ *form an ideal of this subalgebra.*

Proof is a direct computation. Weights of the monomial vector fields $x^{\alpha_i}\partial/\partial x_{\beta_i}$ equal

$$\nu\left(x^{\alpha_i}\partial/\partial x_{\beta_i}\right)=\nu\left(x^{\alpha_i}\right)-\nu\left(x_{\beta_i}\right)+\frac{a_{\beta_i}+1}{a_{\beta_i}+k_{\beta_i}},\ i=1,2.$$

Their Lie bracket equals

$$\left(\partial x^{\alpha_2}/\partial x_{\beta_1}\right)\partial/\partial x_{\beta_2}-\left(\partial x^{\alpha_1}/\partial x_{\beta_2}\right)\partial/\partial x_{\beta_1}.$$

Its second and first summands have weights

$$\nu(x^{\alpha_1})+\nu(x^{\alpha_2})-\nu(x_{\beta_1})-\nu(x_{\beta_2})+\frac{a_{\beta_i}+1}{a_{\beta_i}+k_{\beta_i}},\ i=1,2,$$

correspondingly. Both of these weights differ from the sum of the weights $\nu\left(x^{\alpha_1}\partial/\partial x_{\beta_1}\right)+\nu\left(x^{\alpha_2}\partial/\partial x_{\beta_2}\right)$ by a value, whose module is ≤ 1.

9.3 Proof of stability for the unicycle example

Coming back to the system (30) let us assign the weights to the variables x_1, x_2, x_3 choosing the parameters $\bar{k} = (1, 1, 3)$, $\bar{a} = (1, 3, 1)$. Then $\nu(x_1) = 1/2, \nu(x_2) = 1/4, \nu(x_3) = 1/4$. Computing the weights of the addends of the right-hand side of (30) we get

$$\begin{aligned} \nu(x_1\partial/\partial x_1) = \nu(x_3^2\partial/\partial x_1) = \nu(x_2\partial/\partial x_2) = 1, \\ \nu(x_3\partial/\partial x_2) = \nu(x_1x_2\partial/\partial x_3) = \nu(x_2x_3^2\partial/\partial x_3) = 1, \end{aligned} \tag{31}$$

i.e. the right-hand side of the system is homogeneous. According to the Lemma 9.2.2 all the addends of the logarithm Λ_ε of the corresponding flow have weight ≥ 1, as long as they are computed via the Lie brackets of the vector fields of weight ≥ 1 appearing in (31). The principal part $\varepsilon X_d = \varepsilon(-x_1\partial/\partial x_1 - x_2\partial/\partial x_2 - 2^{-1}x_3^3\partial/\partial x_3)$ of the truncation $\Lambda^{(1)} + \Lambda^{(2)}$ of the corresponding logarithm is homogeneous of weight 1; it is also homogeneous (of order 1) with respect to ε. It is easy to see that the difference $\Lambda_\varepsilon - X_d$ between the logarithm (the complete averaging) and the diagonal part contains only terms of weight ≥ 1. Besides the terms of $\Lambda_\varepsilon - X_d$ of weight 1 have order $O(\varepsilon^{3/2})$ with respect to ε. This proves the property $\mathbf{S}_{as}$ for the vector field $\Lambda_\varepsilon = \varepsilon X_d + (\Lambda_\varepsilon - \varepsilon X_d)$, provided that $\varepsilon > 0$ is small enough.

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