A Pontryagin Maximum Principle for systems of flows

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We present a generalization of the Pontryagin Maximum Principle, in which the usual adjoint equation, which contains derivatives of the system vector fields with respect to the state, is replaced by an integrated form, containing only differentials of the reference flow maps. In this form, the conditions of the maximum principle make sense for a number of control dynamical laws whose right-hand side can be nonsmooth, nonlipschitz, and even discontinuous. The "adjoint vectors" that are solutions of the "adjoint equation" no longer need to be absolutely continuous, and may be discontinuous and unbounded. We illustrate this with two examples: the "reflected brachistochrone problem" (RBP), and the derivation of Snell's law of refraction from Fermat's minimum time principle. In the RBP, where the dynamical law is Hölder continuous with exponent 1/2, the adjoint vector turns out to have a singularity, in which one of the components goes to infinity from both sides, at an interior point of the interval of definition of the reference trajectory. In the refraction problem, where the dynamical law is discontinuous, the adjoint vector is bounded but has a jump discontinuity.

1 Introduction

It is well known that the minimum time problem whose solution is Snell's law of refraction was the first link of a long chain of mathematical developments that eventually led to the Pontryagin Maximum Principle (PMP) of optimal control theory: Snell's law was used by Johann Bernoulli's in his solution of the brachistochrone problem; this in turn was a decisive step towards the formulation of the general necessary condition of Euler and Lagrange for the classical Calculus of Variations; the Euler-Lagrange conditions were then strengthened by Legendre, whose second-order condition was later strengthened by Weierstrass; and, finally, Weierstrass' excess function condition led to the Pontryagin Maximum Principle (PMP), stated and proved in [1].

In the approximately 50 years since the formulation of the PMP, the result has been generalized in many directions, incorporating high-order conditions(cf. [8], [9]) and various types of nonsmoothness (cf. [2], [3], [4], [5], [6], [7], [10], [16]), and producing intrinsic coordinate-free formulations on manifolds (cf. [12]). It is remarkable and somewhat disappointing, however, that the refraction problem that leads to Snell's law does not fit within the framework of any of these generalizations, because even the non-smooth versions of the PMP require Lipschitz conditions on the system vector fields, and for the refraction problem the vector fields are actually discontinuous. A similar phenomenon occurs with the "reflected brachistochrone problem" (RBP), a very natural optimization problem with a Hölder continuous right-hand side.

The purpose of this note is to present a generalization of the PMP that applies to problems such as refraction[1] and the RBP. This result—of which a preliminary announcement was made in 2004 in [14]—is a special case of several far-reaching extensions of the PMP proved by us in other papers (cf. [11, 13, 14, 15]) that are much longer and more technical. We choose to isolate this particular aspect of the general results and present it separately because it lends itself to a relatively simple and self-contained treatment.

In our version of the PMP, the usual adjoint equation, which contains derivatives with respect to the state, is replaced by an integrated form, containing only differentials of the reference flow maps. In this form, the conditions of the maximum principle make sense for a number of control dynamical laws whose right-hand side can be nonsmooth, nonlipschitz, and even discontinuous. The "adjoint vectors" that are solutions of the "adjoint equation" no longer need to be absolutely continuous, and could even be discontinuous and unbounded. In both the refraction problem and the RBP, the state space is $\mathbb{R}^2$, and the system vector fields are smooth everywhere, except along the x axis. For the refraction problem, the system vector fields are discontinuous, and the adjoint vector turns out to be discontinuous as well, but bounded, having a jump discontinuity at the point where the trajectory crosses the x axis. axis. For the RBP, the system vector fields are Hölder continuous with exponent $1/2$, and—somehwat surprisingly, considering that the RBP vector fields are less irregular than those of the refraction problem—the adjoint vector turns out to be discontinuous with a worse singularity: at the point where the trajectory crosses the x axis, the adjoint vector

[1] Some readers may object to our inclusion of the refraction example here, on the grounds that the solution can easily be found by elementary means. Our motivation is identical to that of many authors of calculus of variations textbooks, who choose to include, as an application of the Euler-Lagrange equations, the derivation of the fact that the shortest path joining two points is a straight line segment, even though this can also be proved by completely trivial arguments that do not use any calculus of variations at all. In both cases, the purpose is to show that the new general necessary condition for a minimum does apply to the very old problem that played a role in the history of the subject.

becomes infinite. For both problems, the adjoint vector cannot possibly be characterized in terms of a differential equation, because for solutions of such an equation there is no natural way to connect two different solutions and say that they are part of the same solution unless the function obtained by connecting the two solutions is continuous.

2 Preliminaries on sets, maps, and flows

Sets and maps. If S is a set, then $\mathbb{I}_S$ will denote the identity map of S. If A, B are sets, then the notations $f : A \hookrightarrow B$, $f : A \mapsto B$ will indicate, respectively, that f is a possibly partially defined (abbr. "ppd") map from A to B and that f is an everywhere defined map from A to B. If $A \subseteq B$ and $f : B \hookrightarrow C$, then $f\lceil A$ denotes the restriction of f to A, so $f\lceil A : A \hookrightarrow C$, and $f\lceil A : A \mapsto C$ if $f : B \mapsto C$.

Totally ordered sets. If E is a totally ordered set, with ordering $\preceq$, we use $E^{\preceq,2}$ to denote the set of all ordered pairs $(s,t) \in E \times E$ such that $s \preceq t$, and write $E^{\preceq,3}$ to denote the set of all ordered triples $(r,s,t) \in E \times E \times E$ such that $r \preceq s \preceq t$. A *subinterval* of E is a subset I of E such that, whenever $x \preceq y \preceq z$, $x \in I$, $z \in I$, and $y \in E$, it follows that $y \in I$. If $a \in E$, $b \in E$, and $a \preceq b$, then the *E-interval from a to b* is the set $[a,b]_E \stackrel{\text{def}}{=} \{x \in E : a \preceq x \preceq b\}$.

Manifolds, tangent and cotangent spaces. In this paper, "manifold" means "smooth manifold", and "smooth" means "of class C^∞." We use T_xM, T_x^*M to denote, respectively, the tangent and cotangent spaces to a manifold M at a point x of M.

Set separation. Let S_1 and S_2 be subsets of a Hausdorff topological space T, and let p be a point of T. We say that S_1 and S_2 are *separated at p* if $S_1 \cap S_2 \subseteq \{p\}$, i.e. if S_1 and S_2 have no common point other than p. We say that S_1 and S_2 are *locally separated at p* if there exists a neighborhood V of p such that $S_1 \cap V$ and $S_2 \cap V$ are separated.

Flows and their trajectories. Every sufficiently well-behaved vector field gives rise to a flow, but flows are typically less well-behaved than the vector fields that generate them. This is a reason for studying flows independently from their generators, as we now do.

Definition 1. Let E be a totally ordered set with ordering $\preceq$, and let Ω be a set. A *flow* on Ω with time set E (or, more simply, a *flow on* (Ω, E)) is a family $\Phi = \{\Phi_{t,s}\}_{(s,t)\in E^{\preceq,2}}$ of ppd maps[2] from Ω to Ω such that

(F1) $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$ *whenever* $(r,s,t) \in E^{\preceq,3}$,

[2] It is also possible to consider more general flows, in which the maps $\Phi_{t,s}$ are allowed to be set-valued. One can then study flow systems and augmented flow systems as in §3, but with set-valued flow maps. This would, however, require the use of differentials of set-valued maps, which are beyond the scope of this paper.

(F2) $\Phi_{t,t} = \mathbb{I}_\Omega$ *whenever* $t \in E$.

A *trajectory* of a flow Φ on (Ω, E) is a map $\xi : I \mapsto \Omega$, defined on a subinterval I of E, such that $\xi(t) = \Phi_{t,s}(\xi(s))$ whenever $(s,t) \in I^{\preceq,2}$. ♢

Real augmentaition of sets. If Ω is a set, then we will write $\Omega^\# = \mathbb{R} \times \Omega$. If Ω is a smooth manifold, then $\Omega^\#$ is obviously a smooth manifold as well. In that case, if $x^\# = (x_0, x) \in \Omega^\#$, the tangent space $T_{x^\#}\Omega^\#$ and the cotangent space $T^*_{x^\#}\Omega^\#$ will be identified with the products $\mathbb{R} \times T_x\Omega$ and $\mathbb{R} \times T^*_x\Omega$ using the canonical identification maps.

Augmented flows and their trajectories. In optimal control theory, it is often customary to "add the cost variable to the state of a system," thus transforming the optimization problem into a set separation problem in one higher dimension. This augmentation procedure can be carried out directly for flows.

Definition 2. If Φ is a flow on (Ω, E), a *real augmentation* of Φ is a family $c = \{c_{t,s}\}_{(s,t)\in E^{\preceq,2}}$ of ppd functions from Ω to $\mathbb{R}$ such that

(RA) $c_{t,r}(x) = c_{s,r}(x) + c_{t,s}(\Phi_{s,r}(x))$ *whenever* $x \in \Omega$ *and* $(r,s,t) \in E^{\preceq,3}$.

A *flow-augmentation pair* (abbr. F-A pair) on (Ω, E) is a pair (Φ, c) such that Φ is a flow on Ω with time set E and c is a real augmentation of Φ. ♢

(Notice that (RA) implies, in particular, that $c_{t,t}(x) = 0$, since we can always take $r = s = t$ and use the fact that $\Phi_{t,t}(x) = x$.)

To any F-A pair (Φ, c) on (Ω, E) we can associate a family of mappings $\Phi^{\#,c}_{t,s} : \Omega^\# \hookrightarrow \Omega^\#$, by letting

$$\Phi^{\#,c}_{t,s}(x_0, x) = (x_0 + c_{t,s}(x), \Phi_{t,s}(x)) \quad \text{for each} \quad (s,t) \in E^{\preceq,2}\,.$$

It is then clear that $\Phi^{\#,c} = \{\Phi^{\#,c}_{t,s}\}_{(s,t)\in E^{\preceq,2}}$ is a flow on $\Omega^\#$. A flow Ψ such that $\Psi = \Phi^{\#,c}$ for some Φ, c is called a *real-augmented flow*. It is easy to see that a flow $\Psi = \{\Psi_{t,s}\}_{(s,t)\in E^{\preceq,2}}$ on $\mathbb{R} \times \Omega$ is a real-augmented flow if and only if—if we write $\Psi_{t,s}(x_0, x) = (\psi_{0,t,s}(x_0, x), \psi_{t,s}(x_0, x))$—the maps $\psi_{0,t,s}$, $\psi_{t,s}$ are such that the point $\psi_{t,s}(x_0, x) \in \Omega$ and the number $\psi_{0,t,s}(x_0, x) - x_0 \in \mathbb{R}$ do not depend on x_0. In that case, the pair (Φ, c) such that $\Psi = \Phi^{\#,c}$ is uniquely determined by Ψ as follows: $\Phi_{t,s}(x) = \psi_{t,s}(x_0, x)$ and $c_{t,s}(x) = \psi_{0,t,s}(x_0, x) - x_0$ if $(x_0, x) \in \Omega^\#$.

Definition 3. An *augmented trajectory* of a flow-augmentation pair (Φ, c) on a pair (Ω, E) is a trajectory of the flow $\Phi^{\#,c}$, i. e., a map

$$I \ni t \mapsto \xi^\#(t) = (\xi_0(t), \xi(t)) \in \Omega^\#\,,$$

defined on a subinterval I of E, such that $\xi(t) = \Phi_{t,s}(\xi(s))$ and $\xi_0(t) = \xi_0(s) + c_{t,s}(\xi(s))$ whenever $(s,t) \in I^{\preceq,2}$. ♢

Differentiability of flows. Given a trajectory ξ of a flow Φ, it makes sense to talk about differentiability of Φ along ξ.

Definition 4. Assume that Ω is a manifold, $\Phi = \{\Phi_{t,s}\}_{(s,t)\in E^{\preceq,2}}$ is a flow on (Ω, E), I is a subinterval of E, and $\xi : I \mapsto \Omega$ is a trajectory of Φ. We say that Φ is

(1) *continuous near* ξ if for each $(s,t) \in I^{\preceq,2}$ the map $\Phi_{t,s}$ is continuous on a neighbourhood of $\xi(s)$.
(2) *differentiable along* ξ if for each $(s,t) \in I^{\preceq,2}$ the map $\Phi_{t,s}$ is differentiable at $\xi(s)$. ♢

The above definition can be applied to an augmented flow $\Phi^{\#,c}$. If $\Phi^{\#,c}$ is differentiable along an augmented trajectory $\xi^{\#} = (\xi_0, \xi)$, then the differentials $D\Phi^{\#,c}_{t,s}(\xi^{\#}(s))$ have a special structure, reflecting the special structure of the maps $\Phi^{\#,c}_{t,s}$. Indeed, if we write $M^{\#}_{t,s} = D\Phi^{\#,c}_{t,s}(\xi^{\#}(s))$ (so that $M^{\#}_{t,s}$ is a linear map from $T_{\xi^{\#}(s)}\Omega^{\#}$ to $T_{\xi^{\#}(t)}\Omega^{\#}$), then it is easy to see that the result $M^{\#}_{t,s|} \cdot (v_0, v)$ of applying the linear map $D\Phi^{\#,c}_{t,s}(\xi^{\#}(s))$ to a tangent vector $(v_0, v) \in T_{\xi^{\#}(s)}\Omega^{\#} \sim \mathbb{R} \times T_{\xi(s)}\Omega$ is the vector $(v_0 + m_{t,s} \cdot v, M_{t,s} \cdot v)$, which belongs to $T_{\xi^{\#}(t)}\Omega^{\#} \sim \mathbb{R} \times T_{\xi(t)}\Omega$, where

$$m_{t,s} = Dc_{t,s}(\xi(s))\,, \quad M_{t,s} = D\Phi_{t,s}(\xi(s))\,, \tag{1}$$

so that $m_{t,s} \in T^*_{\xi(s)}\Omega$ and $M_{t,s}$ is a linear map from $T_{\xi(s)}\Omega$ to $T_{\xi(t)}\Omega$.

Variational fields. The differentials $D\Phi_{t,s}(\xi(s))$, $D\Phi^{\#}_{t,s}(\xi^{\#}(s))$, can be used to propagate tangent vectors forwards and cotangent vectors backwards.

Definition 5. A field of tangent vectors $I \ni s \mapsto v(s) \in T_{\xi(s)}\Omega$ such that $v(t) = M_{t,s} \cdot v(s)$ whenever $(s,t) \in I^{\preceq,2}$—where $M_{t,s}$ is defined by (1)—is called a *variational vector field of* (Φ, c) *along* ξ. ♢

Definition 6. A field of tangent vectors

$$I \ni s \mapsto v^{\#}(s) = (v_0(s), v(s)) \in T_{\xi^{\#}(s)}\Omega^{\#}$$

such that

$$v_0(t) = v_0(s) + m_{t,s} \cdot v(s) \quad \text{and} \quad v(t) = M_{t,s} \cdot v(s) \tag{2}$$

whenever $(s,t) \in I^{\preceq,2}$—where $m_{t,s}$, $M_{t,s}$ are defined by (1)—is called an *augmented variational vector field of* (Φ, c) *along* ξ. ♢

Adjoint fields. The dual maps $\Lambda^{\#}_{t,s} \stackrel{\text{def}}{=} \Big(D\Phi^{\#,c}_{t,s}(\xi(s))\Big)^{\dagger} : T^*_{\xi^{\#}(t)}\Omega^{\#} \mapsto T^*_{\xi^{\#}(s)}\Omega^{\#}$ (where, naturally, we use the canonical identification $T^*_{\xi^{\#}(r)}\Omega^{\#} \sim \mathbb{R} \times T^*_{\xi(r)}\Omega$ for every r) are given (if we write the maps as acting on the right on augmented covectors) by

$$(\omega^{\#}(t) \cdot \Lambda^{\#}_{t,s}) \cdot v^{\#}(s) = \omega^{\#}(t) \cdot (M^{\#}_{t,s} v^{\#}(s)) \,,$$

so that, if we write $\omega^{\#}(t) = (\omega_0(t), \omega(t))$, we see that

$$\begin{aligned}(\omega_0(s), \omega(s)) \cdot (v_0(s), v(s)) &= \Big((\omega_0(t), \omega(t)) \cdot \Lambda^{\#}_{t,s}\Big) \cdot (v_0(s), v(s)) \\ &= (\omega_0(t), \omega(t)) \cdot (v_0(s) + m_{t,s} \cdot v(s), M_{t,s} \cdot v(s)) \\ &= \omega_0(t) v_0(s) + \omega_0(t) m_{t,s} \cdot v(s) + (\omega(t) \circ M_{t,s}) \cdot v(s) \,,\end{aligned}$$

and then

$$\begin{aligned}\omega_0(s) &= \omega_0(t) \,, \\ \omega(s) &= \omega_0 m_{t,s} + \omega(t) \circ M_{t,s} \,.\end{aligned}$$

Definition 7. A field of covectors

$$I \ni s \mapsto \omega^{\#}(s) = (\omega_0(s), \omega(s)) \in T^*_{\xi^{\#}(s)} \Omega^{\#}$$

such that ω_0 is a constant function, and ω satisfies

$$\omega(s) = \omega_0 m_{t,s} + \omega(t) \circ M_{t,s} \quad \text{whenever } (s,t) \in I^{\preceq,2} \tag{3}$$

is called an *augmented adjoint field of covectors* (or *augmented adjoint vector*) of (Φ, c) along ξ. ♢

The constant $-\omega_0$ is the *abnormal multiplier*, and the identity (3) is the *adjoint equation.*

Remark 1. It might be even better to call (3) the "integrated adjoint equation." Indeed, in the classical case, when I is a subinterval of $\mathbb{R}$, Ω is an open subset of $\mathbb{R}^n$—so that each map $M_{t,s}$ can simply be represented by an $n \times n$ matrix, and each $m_{t,s}$ by a $1 \times n$ matrix—and the augmented flow $\Phi^{\#,c}$ is obtained by solving a pair of differential equations

$$\dot{x}_0 = f_{*,0}(x,t) \,, \qquad \dot{x} = f_*(x,t) \tag{4}$$

(where the reference vector field $(x,t) \mapsto f_*(x,t) \stackrel{\text{def}}{=} f(x, \eta_*(t), t)$ is a time-varying vector field, the reference Lagrangian $(x,t) \mapsto f_{*,0}(x,t) \stackrel{\text{def}}{=} f_0(x, \eta_*(t), t)$ is a time-varying scalar function, and η_* is the reference control), Equation (3) is equivalent to the differential equation

$$\dot{\omega}(s) = -\omega(s) \cdot \frac{\partial f_*}{\partial x}(\xi_*(s), s) - \omega_0 \cdot \frac{\partial f_{*,0}}{\partial x} \,, \tag{5}$$

which is the usual adjoint equation[3] ♢

[3] As will be seen below, in the Maximum Principle the number ω_0 is non-positive. Equation (5) is often written in terms of the abnormal multiplier $\check{\omega}_0 \stackrel{\text{def}}{=} -\omega_0$ rather than its negative ω_0. It then takes the much more familiar form $\dot{\omega}(s) = -\omega(s) \cdot \frac{\partial f}{\partial x}(\xi_*(s), s) + \check{\omega}_0 \cdot \frac{\partial f_0}{\partial x}$, i.e., $\dot{\omega}(s) = -\frac{\partial H_{\check{\omega}_0}}{\partial x}(\xi_*(s), \omega(s), \eta_*(s), s)$, where the Hamiltonian H is given by the familiar optimal control theory formula $H_{p_0}(x,p,u,t) = p \cdot f(x,u,t) - p_0 f_0(x,u,t)$.

Approximating cones. A *cone* in a real linear space X is a nonempty subset of X which is closed under multiplication by nonnegative scalars, i.e., such that if $c \in C$, $r \in \mathbb{R}$, and $r \geq 0$, it follows that $rc \in C$. (It then follows automatically that $0 \in C$.) The *polar* of a cone C in X is the set $C^\dagger = \{\lambda \in X^\dagger : \lambda(c) \leq 0 \text{ whenever } c \in C\}$, where $X^\dagger$ is the dual space[4]

Definition 8. If M is a smooth manifold, $S \subseteq M$, and $s \in S$, a *Boltyanskii approximating cone to S at s* is a convex cone C in T_sM having the property that there exist m, U, D,F, L, such that

(1) $m \in \mathbb{N}$, U is a neighborhood of 0 in $\mathbb{R}^m$, and D is a convex cone in $\mathbb{R}^m$,
(2) $F : U \cap D \mapsto M$ is a continuous map such that $F(U \cap D) \subseteq S$,
(3) $L : \mathbb{R}^m \mapsto T_sM$ is a linear map,
(4) $F(v) - s - L \cdot v = o(\|v\|)$ as $v \to 0$ via values in $U \cap D$,
(5) $L \cdot D = C$.

3 The optimal control problem

We consider optimal control problems arising from an *augmented flow system* $\Psi = \{{}_\eta\Psi\}_{\eta\in\mathcal{U}}$, indexed[5] by a "class of admissible controls" $\mathcal{U}$. We assume that a fixed totally ordered set E is specified, such that the time set E_η of each ${}_\eta\Psi$ is equal to E.

The state space of the system is a smooth manifold Ω. Each ${}_\eta\Psi$ is a flow on the real-augmented space $\Omega^\# = \mathbb{R} \times \Omega$, given by ${}_\eta\Psi = {}_\eta\Phi^{\#,\,{}_\eta c}$, where the pair $({}_\eta\Phi,\, {}_\eta c)$ is a real-augmented flow on Ω with time set E. We use $\preceq$ to denote the ordering of E. We assume we are given an initial state $\hat{x} \in \Omega$, a terminal set S, which is a subset of Ω, and initial and terminal times $\hat{a} \in E$, $\hat{b} \in E$, such that $\hat{a} \preceq \hat{b}$.

The objective is to minimize the cost ${}_\eta c_{\hat{b},\hat{a}}(\hat{x})$ in the class $\mathcal{A}$ of all $\eta \in \mathcal{U}$ such that the terminal point ${}_\eta\Phi_{\hat{b},\hat{a}}(\hat{x})$ belongs to S. Equivalently, we want to minimize the cost $\xi_0(\hat{b}) - \xi_0(\hat{a})$ in the class $\tilde{\mathcal{A}}$ of all pairs $(\eta, \xi^\#)$ such that $\eta \in \mathcal{U}$, $\xi^\# = (\xi_0, \xi)$ is an augmented trajectory of $({}_\eta\Phi,\, {}_\eta c)$, $\xi(\hat{a}) = \hat{x}$,.and $\xi(\hat{b}) \in S$.

4 The main theorem

To state our generalization of the maximum principle, we need some additions definitions and notational conventions. We assume that we are given data

[4] In all cases occuring in this paper, X is finite-dimensional, so we do not need to distinguish between algebraic and topological duals.

[5] We put the subscript η on the left because we will want to write formulas such as ${}_\eta\Psi = {}_\eta\Phi^{\#,{}_\eta c}$, ${}_\eta\Psi = \{{}_\eta\Psi_{t,s}\}_{(s,t)\in E^{\preceq,2}}$ and ${}_\eta\Phi^{\#,{}_\eta c} = \{{}_\eta\Phi^{\#,{}_\eta c}_{t,s}\}_{s,t\in E_\eta}$.

$$\mathcal{D} = (n, \Omega, E, \preceq, \mathcal{U}, \Phi, c, \Psi, \hat{a}, \hat{b}, \hat{x}, S) \tag{6}$$

as in the previous section, so that $\Psi = \{_\eta\Psi\}_{\eta\in\mathcal{U}}$, $\Phi = \{_\eta\Phi\}_{\eta\in\mathcal{U}}$, $c = \{_\eta c\}_{\eta\in\mathcal{U}}$, and $_\eta\Psi = {}_\eta\Phi^{\#,\,_\eta c}$ for every $\eta \in \mathcal{U}$. We define $\hat{I} = \{t \in E : \hat{a} \preceq t \preceq \hat{b}\}$.

Precisely, we will assume that $\mathcal{D}$ satisfies

(A1) *$n \in \mathbb{N}$, and Ω is a smooth manifold of dimension n;*
(A2) *E is a totally ordered set, with partial ordering $\preceq$;*
(A3) *$\mathcal{U}$ is a set;*
(A4) *$\Psi = \{_\eta\Psi\}_{\eta\in\mathcal{U}}$ is an augmented flow system on Ω with time set E;*
(A5) *$_\eta\Psi = {}_\eta\Phi^{\#,\,_\eta c}$, where $(_\eta\Phi,\, _\eta c)$ is a flow-augmentation pair on Ω with time set E;*
(A6) *$\hat{x} \in \Omega$, $\hat{a} \in E$, $\hat{b} \in E$, and $\hat{a} \preceq \hat{b}$;*
(A7) *$\hat{a} \in E$, $\hat{b} \in E$, and $\hat{a} \preceq \hat{b}$.*

We let $\tilde{\mathcal{A}}$ be the class of all pairs $(\eta, \xi^\#)$ such that $\eta \in \mathcal{U}$, $\xi^\# = (\xi_0, \xi)$ is an augmented trajectory of $(_\eta\Phi,\, _\eta c)$, $\xi(\hat{a}) = \hat{x}$, and $\xi(\hat{b}) \in S$.

We assume that we are given a candidate control η_* and candidate augmented trajectory $\xi_*^\# = (\xi_{*,0}, \xi_*)$, such that

(A8) *$(\eta_*, (\xi_{*,0}, \xi_*)) = (\eta_*, \xi_*^\#) \in \tilde{\mathcal{A}}$.*

Clearly, then, the three maps $\xi_* : \hat{I} \mapsto \Omega$ (the "reference trajectory"), $\xi_{0,*} : \hat{I} \mapsto \mathbb{R}$ (the "reference running cost"), $\xi_*^\# : \hat{I} \mapsto \Omega^\#$ (the "reference augmented trajectory"), satisfy, for all $t \in \hat{I}$,

$$\xi_*(t) = {}_\eta\Phi_{t,\hat{a}}(\hat{x}_*)\,, \qquad \xi_{0,*}(t) = {}_\eta c_{t,\hat{a}}(\hat{x}_*)\,, \qquad \xi_*^\#(t) = {}_\eta\Phi_{t,a}^{\#,\,\eta c}(0, \hat{x}_*)\,,$$

as well as $\xi_*^\#(t) = (\xi_{0,*}(t), \xi_*(t))$.

Our key assumption is that the pair $(\eta_*, \xi_*^\#)$ is a solution of our optimal control problem, that is, that

(A9) *$\xi_{*,0}(\hat{b}) - \xi_{*,0}(\hat{a}) \le \xi_0(\hat{b}) - \xi_0(\hat{a})$ for all $(\eta, (\xi_0, \xi)) \in \tilde{\mathcal{A}}$.*

In addition, we make the crucial technical assumption that

(A10) *The reference flow $_{\eta_*}\Psi$ is continuous near the reference trajectory $\xi_*^\#$, and differentiable along $\xi_*^\#$.*

We define an *impulse vector* for the data 12-tuple $\mathcal{D}$ and the reference control-augmented trajectory pair $(\eta_*, \xi_*^\#)$ to be a pair $(v^\#, t)$ such that $t \in \hat{I}$ and $v^\# \in T_{\xi_*^\#(t)}\Omega^\#$. We use $\mathcal{V}_{max}(\mathcal{D}, (\eta_*, \xi_*^\#))$ to denote the set of all impulse vectors for $\mathcal{D}$, $(\eta_*, \xi_*^\#)$.

Let $\mathbf{v}^\# = ((v_1^\#, t_1), \ldots, (v_m^\#, t_m))$ be a finite sequence of members of $\mathcal{V}_{max}(\mathcal{D}, (\eta_*, \xi_*^\#))$, and assume that (A10) holds. We then define linear maps $L_0^{\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#} : \mathbb{R}^m \times T_{\hat{x}}\Omega \mapsto \mathbb{R}$, $L^{\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#} : \mathbb{R}^m \times T_{\hat{x}}\Omega \mapsto T_{\xi_*(\hat{b})}\Omega$, and $L^{\#,\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#} : \mathbb{R}^m \times T_{\hat{x}}\Omega \mapsto T_{\xi_*^\#(\hat{b})}\Omega^\#$, by first writing $v_j^\# = (v_{0,j}, v_j)$, with $v_{0,j} \in \mathbb{R}$, $v_j \in T_{\xi_*(t)}\Omega$, and then letting

$$L_0^{\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}(\varepsilon_1,\ldots,\varepsilon_m,w) = m_{\hat{b},\hat{a}}\cdot w + \sum_{j=1}^{m}\varepsilon_j(v_{0,j} + m_{\hat{b},t_j}\cdot v_j)\,,$$

$$L^{\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}(\varepsilon_1,\ldots,\varepsilon_m,w) = M_{\hat{b},\hat{a}}\cdot w + \sum_{j=1}^{m}\varepsilon_j M_{\hat{b},t_j}\cdot v_j\,,$$

$$L^{\#,\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}(\varepsilon_1,\ldots,\varepsilon_m,w) = \\ (L_0^{\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}(\varepsilon_1,\ldots,\varepsilon_m,w), L^{\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}}(\varepsilon_1,\ldots,\varepsilon_m,w))\,.$$

In the following definition, $\mathbb{R}_+^m$ denotes the nonnegative orthant of $\mathbb{R}^m$, that is, the set $\{(h_1,\ldots,h_m)\in\mathbb{R}^m : h_1\geq 0,\ldots,h_m\geq 0\}$. Furthermore, for any $a,b\in E$ such that $a\preceq b$, and any $x\in\Omega$, $\mathcal{R}_{a,b}^\#(x)$ denotes the "reachable set from x for the augmented system over the interval from a to b," so that

$$\mathcal{R}_{a,b}^\#(x) \stackrel{\text{def}}{=} \Big\{(r,y) : (\exists\eta\in\mathcal{U})\Big(y =_\eta \Phi_{b,a}(x),\ \text{and}\ r =_\eta c_{b,a}(x)\Big)\Big\}\,.$$

Definition 9. A set $\mathcal{V}$ of impulse vectors is *variational* for $\mathcal{D}$, η_*, $\xi_*^\#$ if for every finite sequence $\mathbf{v}^\# = ((v_1^\#,t_1),\ldots,(v_m^\#,t_m))$ of members of $\mathcal{V}$ it follows that there exist neighborhoods P, Q of 0, $\hat{x}$, in $\mathbb{R}_+^m$, Ω, respectively, and a continuous map $F : P\times Q\mapsto\Omega^\#$, such that

(1) *F is differentiable at $(0,\hat{x})$ with differential $L^{\#,\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}$ (in the precise sense of Remark 2 below).* ◊
(2) $F(P\times\{x\})\subseteq\mathcal{R}_{\hat{a},\hat{b}}^\#(x)$ for every $x\in Q$.

Remark 2. The precise meaning of the assertion that "F is differentiable at $(0,\hat{x})$ with differential $L^{\#,\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}$" is as follows. Let $\hat{y}=\xi_*(\hat{b})$, $\hat{y}_0=\xi_{0,*}(\hat{b})$, $\hat{y}^\# = (\hat{y}_0,\hat{y})$, so $\hat{y}^\# = F(0,\hat{x})$. Let $\tilde{P}$, $\tilde{Q}$, R, J be open neighborhoods of 0, $\hat{x}$, $\hat{y}_0$, $\hat{y}$, in $\mathbb{R}^m$, Ω, $\mathbb{R}$, Ω, respectively, such that $\tilde{P}\subseteq P$, $\tilde{Q}\subseteq Q$, $F((\tilde{P}\cap\mathbb{R}_+^m)\times\tilde{Q})\subseteq J\times R$, $\tilde{Q}$ is the domain of a coordinate chart $\kappa:\tilde{Q}\mapsto\mathbb{R}^n$ for which $\kappa(\hat{x})=0$, and R is the domain of a coordinate chart $\zeta:R\mapsto\mathbb{R}^n$ for which $\zeta(\hat{y})=0$. Use κ and ζ to identify the sets $\tilde{Q}$ and R with their images $\kappa(\tilde{Q})$, $\zeta(R)$, so $\tilde{Q}$ and R are now open subsets of $\mathbb{R}^n$. Then

$$F(\varepsilon_1,\ldots,\varepsilon_m,w) = L^{\#,\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}(\varepsilon_1,\ldots,\varepsilon_m,w) + o(\varepsilon_1+\ldots+\varepsilon_m+\|w\|) \quad (7)$$

as $(\varepsilon_1,\ldots,\varepsilon_m,w)$ goes to 0 via values in $(\tilde{P}\cap\mathbb{R}_+^m)\times\tilde{Q}$.

Our last two assumptions are

(A11) *$\mathcal{V}$ is a variational set of impulse vectors for $\mathcal{D}$, η_*, $\xi_*^\#$.*
(A12) *C is a Boltyanskii approximating cone to S at $\xi_*(\hat{b})$.*

The following is then our main result.

Theorem 1. *Assume that we are given a data* 12*-tuple* $\mathcal{D}$ *as in (6), as well as* η_*, $\xi_*^\#$, $\mathcal{V}$, C, *such that Assumptions (A1) to (A12) hold. Write*

$$\Phi^*_{t,s} = {}_{\eta_*}\Phi_{t,s}\,,\ \ c^*_{t,s} = {}_{\eta_*}c_{t,s}\,,\ \ M_{t,s} = D\Phi^*_{t,s}(\xi_*(s))\,,\ \ m_{t,s} = Dc^*_{t,s}(\xi_*(s))\,,$$

(so that $m_{t,s} = \nabla c^*_{t,s}(\xi_*(s))$*). Then there exist a map* $\hat{I} \ni t \mapsto \omega(t) \in \mathbb{R}_n$ *and a real constant* ω_0 *such that*

(1) $\omega_0 \geq 0$,
(2) $(\omega_0, \omega(t)) \neq (0,0)$ *for all* $t \in I$,
(3) $\omega(s) = \omega(t) \cdot M_{t,s} - \omega_0 m_{t,s}$ *whenever* $s, t \in E$ *and* $s \leq t$,
(4) $\langle \omega(t), v\rangle - \omega_0 v_0 \leq 0$ *whenever* $(v^\#, t) = ((v_0, v), t) \in \mathcal{V}$,
(5) the transversality condition $-\omega(\hat{b}) \in C^\dagger$ *holds.*

Proof. Fix a norm $\|\cdot\|$ on the tangent space $T_{\xi_*(\hat{b})}\Omega$. Let $\mathcal{K}$ be the set of all pairs $(\tilde{\omega}_0, \bar{\omega}) \in \mathbb{R} \times T_{\xi_*(\hat{b})}\Omega$ such that $\tilde{\omega}_0 \leq 0$, $|\tilde{\omega}_0| + \|\bar{\omega}\| = 1$, and $-\bar{\omega} \in C^\dagger$. Then $\mathcal{K}$ is clearly compact. For each subset $\mathcal{W}$ of $\mathcal{V}$, let $\mathcal{K}(\mathcal{W})$ be the subset of $\mathcal{K}$ consisting of all $(\tilde{\omega}_0, \bar{\omega}) \in \mathcal{K}$ such that

$$\Big\langle \bar{\omega} \cdot D_{\eta_*}\Phi_{\hat{b},t}(\xi_*(t)) + \tilde{\omega}_0 \nabla c_{\hat{b},t}(\xi_*(t)), v \Big\rangle + \tilde{\omega}_0 v_0 \leq 0$$

for all $((v_0, v), t) \in \mathcal{W}$.

It clearly suffices to prove that the set $\mathcal{K}(\mathcal{V})$ is nonempty. Indeed, if a pair $(\tilde{\omega}_0, \bar{\omega})$ belongs to $\in \mathcal{K}(\mathcal{V})$, we may define $\omega_0 = -\tilde{\omega}_0$ and then let

$$\omega(t) = \bar{\omega} \cdot M_{\hat{b},t} + \tilde{\omega}_0 m_{\hat{b},t}$$

(that is, $\omega(t) = \bar{\omega} \cdot M_{\hat{b},t} - \omega_0 m_{\hat{b},t}$) for $t \in I$, and then set $\omega^\# = (\omega_0, \omega)$. A simple calculation shows that $\omega^\#$ is an augmented adjoint vector that satisfies all our conclusions.

Furthermore, it is evident from the definition of the sets $\mathcal{K}(\mathcal{W})$ that if a subset $\mathcal{W}$ of $\mathcal{V}$ is the union $\cup_{\lambda \in \Lambda} \mathcal{W}_\lambda$ of a family of subsets $\mathcal{V}$, then

$$\mathcal{K}(\mathcal{W}) = \cap_{\lambda \in \Lambda} \mathcal{K}(\mathcal{W}_\lambda)\,.$$

Hence it suffices to prove that $\mathcal{K}(\mathcal{W})$ is nonempty whenever $\mathcal{W}$ is a finite subset of $\mathcal{V}$.

So let $\mathcal{W}$ be a finite subset of $\mathcal{V}$. Let $\mathbf{v}^\# = ((v_1^\#, t_1), \ldots, (v_m^\#, t_m))$ be a finite sequence that contains all the members of $\mathcal{W}$, and write $v_j^\# = (v_{0,j}, v_j)$ for $j = 1, \ldots, m$. Since $\mathcal{V}$ is variational, Definition 4 enables us to pick neighborhoods P, Q, R of 0, $\hat{x}(= \xi_*(\hat{a}))$, $\xi_*^\#(\hat{b})$, in $\mathbb{R}^m_+$, Ω, $\Omega^\#$, respectively, and a continuous map $F : P \times Q \mapsto R$, which is differentiable at $(0, \hat{x})$ with differential $L^{\#,\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}$, so that F satisfies—relative to coordinate charts near $\hat{x}$, $\xi_*(\hat{b})$ for which $\hat{x} = 0$ and $\xi_*(\hat{b}) = 0$—the condition

$$F(\varepsilon_1, \ldots, \varepsilon_m, w) = L^{\#,\mathcal{D},\eta_*,\xi_*^\#,\mathbf{v}^\#}(\varepsilon_1, \ldots, \varepsilon_m, w) + o(\varepsilon_1, \ldots, \varepsilon_m + \|w\|) \quad (8)$$

as $(\varepsilon_1, \ldots, \varepsilon_m, w)$ goes to $(0,0)$ via values in $\mathbb{R}^m_+ \times Q$, as well as the property that $F(P \times \{x\}) \subseteq \mathcal{R}^{\#}_{\hat{a},\hat{b}}(x)$ for every $x \in Q$.

In particular, if we let $G : P \mapsto \Omega^{\#}$ be the map given by

$$G(\varepsilon_1, \ldots, \varepsilon_m) = F(\varepsilon_1, \ldots, \varepsilon_m; 0)\,,$$

then G is a continuous map into $\mathcal{R}^{\#}_{\hat{a},\hat{b}}(\hat{x})$ which is differentiable at 0 with differential $\check{L}^{\#,\mathcal{D},\eta_*,\xi^{\#}_*,\mathbf{v}^{\#}}$, where $\check{L}^{\#,\mathcal{D},\eta_*,\xi^{\#}_*,\mathbf{v}^{\#}}$ is the map

$$(\varepsilon_1, \ldots, \varepsilon_m) \mapsto \Big(\sum_{j=1}^{m} \varepsilon_j (v_{0,j} + m_{\hat{b},t_j} \cdot v_j)\,, \sum_{j=1}^{m} \varepsilon_j M_{\hat{b},t_j} \cdot v_j \Big)\,,$$

and $M_{t,s}$, $m_{t,s}$, are defined by

$$M_{t,s} = D\Phi_{t,s}(\xi_*(s))\,, \quad m_{t,s} = \nabla c_{t,s}(\xi_*(s)) \text{ for } s,t \in I,\ s \preceq t\,.$$

Let $w^{\#}_j = \check{L}^{\#,\mathcal{D},\eta_*,\xi^{\#}_*,\mathbf{v}^{\#}} \cdot e^m_j$, where $e^m_j = (\delta^1_j, \ldots, \delta^m_j)$, and the δ^k_j are the Kronecker symbols. Then

$$w^{\#}_j = (v_{0,j} + m_{\hat{b},t_j} \cdot v_j\,,\, M_{\hat{b},t_j} \cdot v_j)\,,$$

and

$$\check{L}^{\#,\mathcal{D},\eta_*,\xi^{\#}_*,\mathbf{v}^{\#}}(\varepsilon_1, \ldots, \varepsilon_m) = \sum_{j=1}^{m} \varepsilon_j w^{\#}_j\,.$$

It is then clear (by applying Definition 8, with $D = \mathbb{R}^m_+$, $L = \check{L}^{\#,\mathcal{D},\eta_*,\xi^{\#}_*,\mathbf{v}^{\#}}$) that, if we write $\mathcal{C}$ to denote the convex cone $\mathcal{C}$ generated by the vectors $w^{\#}_j$ (so that $\mathcal{C} = \check{L}^{\#,\mathcal{D},\xi^{\#}_*,\eta_*,\mathbf{v}^{\#}}(\mathbb{R}^m_+)$), then $\mathcal{C}$ is a Boltyanskii approximating cone to the augmented reachable set $\mathcal{R}^{\#}_{\hat{a},\hat{b}}(\hat{x})$ at $\xi^{\#}_*(\hat{b})$.

Now, let $S^{\#} = \{(x_0, x) \in \Omega^{\#} : x \in S \text{ and } x_0 \leq \xi_{0,*}(\hat{b}) - \xi_{0,*}(\hat{a}) - \psi(x)\}$, where ψ is a smooth function on Ω that vanishes at $\xi_*(\hat{b})$ and is strictly positive everywhere else. Let $C^{\#} =]-\infty, 0] \times C$. Then $C^{\#}$ is a Boltyanskii approximating cone to $S^{\#}$ at $\xi^{\#}_*(\hat{b})$. Furthermore, it is easy to see that the optimality of $(\eta_*, \xi^{\#}_*)$ implies that $\mathcal{R}^{\#}_{\hat{a},\hat{b}}(\hat{x})$ at $\xi^{\#}_*(\hat{b})$ and $S^{\#}$ are separated at $\xi^{\#}_*(\hat{b})$. Then standard set separation theorems tells us that the cones $\mathcal{C}$ and $C^{\#}$ are not strongly transversal. Since $C^{\#}$ is not a linear subspace, the cones $\mathcal{C}$ and $C^{\#}$ are in fact not transversal. This implies that there exists a nonzero covector $\bar{\omega}^{\#} = (\tilde{\omega}_0, \bar{\omega}) \in T_{\xi^{\#}_*(\hat{b})}\Omega^{\#}$ such that $\langle \bar{\omega}^{\#}, z \rangle \geq 0$ whenever $z \in C^{\#}$, and $\langle \bar{\omega}^{\#}, z \rangle \leq 0$ whenever $z \in \mathcal{C}$.

It follows that $-\bar{\omega} \in C^{\dagger}$, and also that $\tilde{\omega}_0 \leq 0$. Define

$$\omega(t) = \bar{\omega} \cdot D_{\eta_*}\Phi_{\hat{b},t}(\xi_*(t)) + \tilde{\omega}_0 \nabla c_{\hat{b},t}(\xi_*(t))$$

for $t \in I$. If $j = 1, \ldots, m$, then

$$\begin{aligned} 0 \geq \langle \bar{\omega}^{\#}, w_j^{\#} \rangle &= \Big\langle \Big((\tilde{\omega}_0, \bar{\omega}), (v_{0,j} + m_{\hat{b},t_j} \cdot v_j \,,\, M_{\hat{b},t_j} \cdot v_j) \Big) \Big\rangle \\ &= \tilde{\omega}_0 v_{0,j} + \tilde{\omega}_0\, m_{\hat{b},t_j} \cdot v_j + \bar{\omega} \cdot M_{\hat{b},t_j} \cdot v_j \\ &= \tilde{\omega}_0 v_{0,j} + \Big(\tilde{\omega}_0\, m_{\hat{b},t_j} + \bar{\omega} \cdot M_{\hat{b},t_j} \Big) \cdot v_j \\ &= \tilde{\omega}_0 v_{0,j} + \omega(t_j) \cdot \cdot v_j \,. \end{aligned}$$

This shows that $\bar{\omega}^{\#} \in \mathcal{K}(\mathcal{W})$, establishing that $\mathcal{K}(\mathcal{W}) \neq \emptyset$, and completing our proof. $\diamondsuit$

5 Variable time problems

A minimum time problem is, by its very nature, a variable time-interval problem. Hence such a problem does not fit the framework of our main theorem, if we require that the time set E be a subset of $\mathbb{R}$, and that the time from s to t be precisely $t - s$. It is possible, however, to apply Theorem 1 to minimum time problems, and to more general variable time-interval problems, by means of a simple device. Assume that we start with a situation in which E is a subset of $\mathbb{R}$ and our flow-augmentation pairs $({}_{\eta}\Phi,\ {}_{\eta}c)$ are such that that ${}_{\eta}c_{t,s}(x) = t - s$ whenever $(s,t) \in E^{\preceq,2}$. We want to change our point and think of E as representing a "pseudotime" which is no longer physical time, although it will correspond to physical time along the reference trajectory—for example, in the form of a clock that displays at each $t \in E$ the value t. For this purpose, we allow "insertion variations" in which the reference augmented flow map ${}_{\eta_*}\Phi^{\#,\,{}_{\eta_*}c}_{\hat{b},\hat{a}}$ is replaced by the map ${}_{\eta_*}\Phi^{\#,\,{}_{\eta_*}c}_{\hat{b},t}\ {}_{\eta}\Phi^{\#,\,{}_{\eta}c}_{t+\varepsilon,t}\ {}_{\eta_*}\Phi^{\#,\,{}_{\eta_*}c}_{t,\hat{a}}$ *still regarded as a transition map from "time" $\hat{a}$ to "time" $\hat{b}$*, even though the true physical time $\xi_0(\hat{b}) - \xi_0(\hat{a})$ during which this transition occurs is $\hat{b} - \hat{a} + \varepsilon$. We also allow "deletion variations" in which the reference augmented flow map ${}_{\eta_*}\Phi^{\#,\,{}_{\eta_*}c}_{\hat{b},\hat{a}}$ is replaced by the map ${}_{\eta_*}\Phi^{\#,\,{}_{\eta_*}c}_{\hat{b},t}\ {}_{\eta}\Phi^{\#,\,{}_{\eta_*}c}_{t-\varepsilon,\hat{a}}$, again regarded as a transition map from "time" $\hat{a}$ to "time" $\hat{b}$, even though the true physical time $\xi_0(\hat{b}) - \xi_0(\hat{a})$ of this transition is $\hat{b} - \hat{a} - \varepsilon$. (Naturally, for this to be possible, we need, for example, to be able to regard ${}_{\eta_*}\Phi^{\#,\,{}_{\eta_*}c}_{\hat{b},t}$ as a "time $t + \varepsilon$ to time $\hat{b} + \varepsilon$" map. The key condition needed for all this to work is to have a *time-translation invariant* system, that is, a system for which ${}_{\eta}\Phi_{t,s} = {}_{\eta}\Phi_{t+\alpha,s+\alpha}$ for all $\alpha \in \mathbb{R}$.)

The variatonal impulses $(v^{\#}, t)$ that occur in our main theorem are, in general, of a special form. First of all, for each $t \in E$ that occurs in one of the members $(v^{\#}, t) \in \mathcal{V}$, there exists a vector $v^{\#}_{del}(t)$, depending on t but not on $v^{\#}$, that corresponds to the "deletion of the reference control on intervals of legth ε." Second, for each $v^{\#} \in \mathcal{V}[t]$—where $\mathcal{V}[t] = \{v^{\#} : (v^{\#}, t) \in \mathcal{V}\}$—the vector $v^{\#}$ corresponds to the "deletion of the reference control on intervals of legth ε followed by an insertion of some other control on an interval of lentgh ε, " so that $v^{\#}_{ins} \stackrel{\text{def}}{=} v^{\#} - v^{\#}_{del}(t)$ corresponds to an insertion without deletion, and then $v^\# = v^\#_{ins} + v^\#_{del}(t)$. If we allow the insertions to be carried out without a corresponding deletion we get, in addition to the inequalities $\langle \omega^\#(t), v^\# \rangle \le 0$ that occur in (4) of the statement of Theorem 1, the new inequalities $\langle \omega^\#(t), v^\#_{ins} \rangle \le 0$. If we also allow deletions to be carried out without a corresponding insertion, we get the inequalities $\langle \omega^\#(t), v^\#_{del}(t) \rangle \le 0$. On the other hand, one of the controls that can be used in an insertion is the reference control itself, and this insertion corresponds to the vector $-v^\#_{del}(t)$, yielding the inequality $\langle \omega^\#(t), -v^\#_{del}(t) \rangle \le 0$. So $\langle \omega^\#(t), v^\#_{del}(t) \rangle = 0$. and $\langle \omega^\#(t), -v^\#_{del}(t) \rangle = 0$. In other words,

(%) *for a variable time interval problem where the impulses* $(v^\#, t)$ *admit the decomposition* $v^\# = v^\#_{ins} + v^\#_{del}(t)$ *as above, the conclusion of Theorem 1, that* $\langle \omega^\#(t), v^\# \rangle \le 0$*—which is equivalent to the inequality* $\langle \omega^\#(t), v^\#_{ins} \rangle \le \langle \omega^\#(t), -v^\#_{del}(t) \rangle$*—can be strengthened to*

$$\langle \omega^\#(t), v^\#_{ins} \rangle \le \langle \omega^\#(t), -v^\#_{del}(t) \rangle = 0\,. \tag{9}$$

6 The reflected brachistochrone

As an example of a nontrivial application of Theorem 1, we study the "reflected brachistochrone problem" (RBP), that is, the minimum time problem for the dynamical law

$$\dot{x} = u\sqrt{|y|}\,, \qquad \dot{y} = v\sqrt{|y|}\,,$$

with state $(x, y) \in \mathbb{R}^2$ and control $(u, v) \in \mathbb{R}^2$ subject to the control constraint $u^2 + v^2 \le 1$. Given points $A, B \in \mathbb{R}^2$, we want to characterize the minimum-time trajectory from A to B.

To solve the RBP, we use Theorem 1, together with the remarks of §5 and the classical (1696-7) results about the solutions of the *brachistochrone problem* (BP) of Johann Bernoulli. Define closed half-planes H^+, H^-, by

$$H^+ = \{(x, y) : y \ge 0\}\,, \qquad H^- = \{(x, y) : y \le 0\}\,.$$

Let $\mathcal{P}^+$, resp. $\mathcal{P}^-$, be the minimum time problems for curves entirely contained in H^+ (resp. H^-) with endpoints in H^+ (resp. H^-). Define a "v-cycloid" to be an arc which is entirely contained in H^+ or H^- and is either (a) a vertical line segment or (b) a smooth subarc[6] of a maximal cycloid generated by a point P on a circle $\mathit{\Gamma}$ that is tangent to the x axis and rolls without slipping. (In particular, if $H = H^+$ or $H = H^-$, and $\xi_* : [0, T] \mapsto H$ is a v-cycloid, then $\xi_*(t) \notin H^+ \cap H^-$ whenever $0 < t < T$.) Then it is well known that the solutions of $\mathcal{P}^+$ and $\mathcal{P}^-$ are v-cycloids.

[6] Recall that a maximal cycloid has cusp points, located on the x axis at points of the form $a + 2k\pi R$, $k \in \mathbb{Z}$, where R is the radius of the rolling circle $\mathit{\Gamma}$. A "smooth subarc" is one that does not contain any cusp points other than the endpoints.

We now solve the RBP. Let $\xi_* : [0,T] \mapsto \mathbb{R}^2$ be a solution of the RBP with endpoints A, B. If ξ_* is entirely contained in H^+ or H^-, then ξ_* is a solution of $\mathcal{P}^+$ or of $\mathcal{P}^-$, so ξ_* is a v-cycloid. So all we need is to determine the minimum-time trajectories ξ_* that are not entirely contained in H^+ or H^-. Fix one such ξ_*. Then there must exist a time τ such that $0 \le \tau \le T$ and $\xi_*(\tau) \in H^+ \cap H^-$. It is then easy to show that τ is unique. (If τ was not unique, let τ_1 be the smallest t such that $\xi_*(t) \in H^+ \cap H^-$, and let τ_2 be the largest. Then $0 \le \tau_1 < \tau_2 \le T$, $\xi_*(\tau_i) \in H^+ \cap H^-$ for $i = 1, 2$, and $\xi_*(t) \notin H^+ \cap H^-$ for $0 \le t < \tau_1$ or $\tau_2 < t \le T$. Assume, without loss of generality, that $\xi_*(t) \in H^+$ for $0 \le t \le \tau_1$. Then the set $S = \{t \in [\tau_1, \tau_2] : \xi_*(t) \notin H^+ \cap H^-\}$ is open, so it is a union of a finite or countable set $\mathcal{I}$ of pairwise disjoint open intervals, each one of which is of the form $]\,\alpha, \beta\,[$, with $\tau_1 \le \alpha < \beta \le \tau_2$, $\xi_*(\alpha) \in H^+ \cap H^-$, and $\xi_*(\beta) \in H^+ \cap H^-$. If I is one of those intervals, then either $\xi_*(t) \in H^+ \backslash H^-$ for all $t \in I$ or $\xi_*(t) \in H^- \backslash H^+$ for all $t \in I$. In the latter case, we may replace the restriction of ξ_* to I by its reflection with respect to the x axis without changing the time. If we do this for all $I \in \mathcal{I}$, we obtain a new trajectory $\tilde{\xi}_*$ that goes from A to B in the same time as ξ_* and is such that $\tilde{\xi}_*(t) \in H^+ \backslash H^-$ for all $t \in I$ for all $I \in \mathcal{I}$. Then the restriction $\hat{\xi}_*$ of $\tilde{\xi}_*$ to the interval $[0, \tau_2]$ is a time-optimal trajectory that goes from A to $\xi_*(\tau_2)$ and is entirely contained in H^+. Hence $\hat{\xi}_*$ is a v-cycloid, and $\hat{\xi}_*(t)$ can only belong to the x axis when t is one of the endpoints of $[0, \tau_2]$. Since $\hat{\xi}_*(\tau_1) \in H^+ \cap H^-$, and $\tau_1 < \tau_2$, it follows that $\tau_1 = 0$. A similar argument shows that $\tau_2 = T$. Hence both A and B belong to $H^+ \cap H^-$. It then follows that $\tilde{\xi}$ is a solution of $\mathcal{P}^+$ with endpoints A, B. So $\tilde{\xi}(t) \notin H^+ \cap H^-$ whenever $0 < t < T$, and this implies, given our construction of $\tilde{\xi}$ from ξ by reflections, that $\tilde{\xi}$ is either ξ itself or its reflection with respect to the x axis. In either case, ξ is entirely contained in one of the half-planes H^+, H^-, which is a contradiction.)

Let $\bar{\tau}$ be the unique τ such that $0 \le \tau \le T$ and $\xi_*(\tau) \in H^+ \cap H^-$. Then $0 < \bar{\tau} < T$, and the points A and B belong to different sides of the x axis. (Indeed, if $\bar{\tau} = 0$ then $\xi_*(t)$ would belong to one of H^+, H^- whenever $0 < t \le T$, so ξ_* would be entirely contained in H^+ or H^-. A similar contradiction would arise if $\bar{\tau} = T$. So $0 < \bar{\tau} < T$. If A and B were both in H^+, then $\xi_*(t) \in H^+$ for $0 \le t < \bar{\tau}$ and also for $\bar{\tau} < t \le T$, so once again ξ_* would be entirely contained in H^+. A similar contradiction arises if $A \in H^-$ and $B \in H^-$.) So without loss of generality we may assume that $A \in H^+ \backslash H^-$ and $B \in H^- \backslash H^+$. Then $\xi_*(t) \in H^+ \backslash H^-$ whenever $0 \le t < \bar{\tau}$ and $\xi_*(t) \in H^- \backslash H^+$ whenever $\bar{\tau} < t \le T$. So ξ_* is the concatenation of two time-optimal curves $\xi_*^+ : [0, \bar{\tau}] \mapsto H^+$, $\xi_*^- : [\bar{\tau}, T] \mapsto H^-$. Then ξ_*^+ and ξ_*^- are v-cycloids contained in H^+ and H^-.

Let us assume that ξ_*^+ and ξ_*^-, are both arcs of cycloids. Let C_* be the point where ξ_* crosses the x axis, so $C_* = \xi_*(\bar{\tau})$. Then the necessary conditions of the classical maximum principle do not determine C_*, because they only apply on the intervals $\{t : 0 \le t < \bar{\tau}\}$, $\{t : \bar{\tau} < t \le T\}$, and say nothing about what happens at time $\bar{\tau}$, where our controlled dynamics is not of class C^1. We will now show how Theorem 1 yields an extra condition that determines C_*.

Our first step is to embed ξ_* in a flow arising from a feedback control law. The arcs ξ_*^+, ξ_*^-, are parts of full cycloid arcs Ξ_*^+, Ξ_*^-, such that Ξ_*^+ goes from a point Q^+ on the x axis to the point C_* and has the property that all the other points of Ξ_*^+ belong to $H^+\backslash H^-$, while Ξ_*^- goes from C_* to a point Q^- on the x axis and is such that all the other points of Ξ_*^- belong to $H^-\backslash H^+$. Write $Q^+ = (\alpha^+, 0)\,,\ Q^- = (\alpha^-, 0)\,,\ C_* = (\alpha^0, 0)\,.$

The arcs Ξ_*^+, Ξ_*^-, are the loci of points P^+, P^-, attached to rolling circles Γ^+, Γ^-, of radii R^+, R^-, and then $|\alpha^0-\alpha^+|=2\pi R^+$ and $|\alpha^0-\alpha^-|=2\pi R^-$. Parametric equations for Ξ_*^+ can be written using as parameter the abscissa α of the point where the rolling circle Γ^+ intersects the x axis $H^+ \cap H^-$. Then α takes values in the interval $I^+ = [\min(\alpha^0, \alpha^+), \max(\alpha^0, \alpha^+)]$, which has length $2\pi R^+$. If we let $\theta \stackrel{\text{def}}{=} (R^+)^{-1}(\alpha - \alpha_+)$, then the position of P^+ for a given value of α is $\Xi_*^+(\alpha) = (\alpha - R^+ \sin\theta\,,\, R^+(1 - \cos\theta))$. (The circle Γ^+ rolls from left to right if $\alpha^+ < \alpha^0$, and from right to left if $\alpha^0 < \alpha^+$.)

The midpoint μ^+ of the interval I^+ is given by $\mu^+ = \frac{1}{2}(\alpha^+ + \alpha^0)$. We let $\hat{Q}^+$ be the point where Γ^+ intersects the x axis when $\alpha = \mu^+$, so that $\hat{Q}^+ = (\mu^+, 0)$. We define parametrized trajectories $\Xi_*^{+,\sigma}$, for each σ in a neighborhood $N^+ = [1 - \varepsilon_1^+, 1 + \varepsilon_2^+]$ of 1 (where $\varepsilon_1^+, \varepsilon_2^+$ are chosen so that $0 < \varepsilon_1^+ < 1$ and $0 < \varepsilon_2^+$), by letting $\Xi_*^{+,\sigma}(\alpha) = \hat{Q}^+ + \sigma(\Xi_*^+(\alpha) - \hat{Q}^+)$ whenever $\alpha \in I^+$. Then each $\Xi_*^{+,\sigma}$ is an arc of cycloid, generated exactly like Ξ_*^+, with R^+ replaced by σR^+, and having contact points $Q^{+,\sigma}$, $C^{+,\sigma}$ with the x axis, where $Q^{+,\sigma} = \hat{Q}^+ + \sigma(\Xi_*^+(\alpha^+) - \hat{Q}^+)$ and $C^{+,\sigma} = \hat{Q}^+ + \sigma(\Xi_*^+(\alpha^0) - \hat{Q}^+)$, so that $Q^{+,\sigma}$ and $C^{+,\sigma}$ are given by $Q^{+,\sigma} = \Big((1 - \sigma)\mu^+ + \sigma\alpha^+, 0\Big)$,

$$C^{+,\sigma} = \Big((1 - \sigma)\mu^+ + \sigma\alpha^0, 0\Big)\,. \tag{10}$$

Then, if we let $\mathcal{S}^+ = \{\Xi_*^{+,\sigma}(\alpha) : \sigma \in N^+\,,\, \alpha \in I^+\}$, the set $\mathcal{S}^+$ is clearly the homeomorphic image of the rectangle $\mathcal{R}^+ \stackrel{\text{def}}{=} N^+ \times I^+$ under the map $\Psi^+ : \mathcal{R}^+ \mapsto H^+$ given by $\Psi^+(\sigma, \alpha) \stackrel{\text{def}}{=} \Xi_*^{+,\sigma}(\alpha)$. Furthermore, the two images $\Psi^+(N^+ \times \{\alpha^+\})$, $\Psi^+(N^+ \times \{\alpha^0\})$, are subintervals of the x axis, while the images of all the points of $\mathcal{R}^+$ that do not belong to $N^+ \times \{\alpha^+, \alpha^0\}$ lie in the open half-plane $H^+\backslash H^-$. The map Ψ^+ is real analytic, and the partial derivatives $\frac{\partial\Psi^+}{\partial\alpha}$, $\frac{\partial\Psi^+}{\partial\sigma}$, are given by the formulas $\frac{\partial\Psi^+}{\partial\alpha} = \sigma{\Xi_*^+}'(\alpha)\,,$ $\frac{\partial\Psi^+}{\partial\sigma} = \Xi_*^+(\alpha) - \hat{Q}^+$, where ${\Xi_*^+}'(\alpha) = (1 - \cos\theta, \sin\theta)\,.$

If J^+ is the Jacobian determinant of Ψ^+ with respect to σ and α then if we write $\tilde{\theta} = \frac{\theta}{2}$, we have

$$\begin{aligned} J^+ &= \sigma\sin\theta\Big(\alpha - \mu^+ - R^+\sin\theta\Big) - \sigma\Big(1 - \cos\theta\Big)R^+\Big(1 - \cos\theta\Big) \\ &= \sigma(\alpha - \mu^+)\sin\theta - \sigma R^+\Big(\sin^2\theta + (1 - \cos\theta)^2\Big) \end{aligned}$$

$$= \sigma R^+\Big(\frac{\alpha-\mu^+}{R^+}\sin\theta - (2-2\cos\theta)\Big)$$
$$= \sigma R^+\Big(2\frac{\alpha-\mu^+}{R^+}\sin\tilde\theta\cos\tilde\theta - 4\sin^2\tilde\theta\Big)$$
$$= 2\sigma R^+\sin\tilde\theta\Big(\frac{\alpha-\mu^+}{R^+}\cos\tilde\theta - 2\sin\tilde\theta\Big)$$

If $\alpha^+ < \alpha^0$, then $\tilde\theta$ varies in the interval $[0,\pi]$ when α varies in I^+. Hence $\sin\tilde\theta = 0$ iff $\tilde\theta = 0$ or $\tilde\theta = \pi$, i.e., iff $\alpha = \alpha^+$ or $\alpha = \alpha^0$. If $\alpha^+ < \alpha \le \mu^+$, then $0 < \tilde\theta \le \frac{\pi}{2}$, so $\cos\tilde\theta \ge 0$, $\frac{\alpha-\mu^+}{R^+} \le 0$, and $\sin\tilde\theta > 0$, from which it follows that $J^+ \neq 0$. A similar argument shows that $J^+ \neq 0$ when $\mu^+ \le \alpha < \alpha^0$). a simple calculation shows that $J^+ = 0$ iff $\alpha = \alpha^+$ or $\alpha = \alpha^0$. So

$(*)$ *Ψ^+ is a real analytic diffeomorphism on the set $N^+ \times \mathrm{Interior}(I^+)$.*

We now analyze the time parameter along the curves $\Xi_*^{+,\sigma}$. Let $\delta^+ = +1$ if $\alpha^+ < \alpha^0$ (i.e., if Γ^+ rolls from left to right, so time increases as α increases, i.e., $dt/d\alpha > 0$), and $\delta^+ = -1$ if $\alpha^+ > \alpha^0$ (i.e., if Γ^+ rolls from right to left, in which case $dt/d\alpha < 0$). If $\Xi_*^{+,\sigma}(\alpha) = (x(\alpha), y(\alpha))$, then $dx = (\sigma - \sigma\cos\theta)\,d\alpha$, $dy = \sigma\sin\theta\,d\alpha$,

$$dx^2 + dy^2 = \sigma^2\Big((1-\cos\theta)^2 + \sin^2\theta\Big)\,d\alpha^2 = 2\sigma^2(1-\cos\theta)\,d\alpha^2\,,$$

and $dx^2 + dy^2 = y\,dt^2$, so $y\,dt^2 = 2\sigma^2(1-\cos\theta)\,d\alpha^2$, while on the other hand $y = R^+(1-\cos\theta)$, so $dt^2 = \frac{4\sigma^2}{R^+}\,d\alpha^2$ and then it is easy to see that $dt = 2\sigma(R^+)^{-\frac{1}{2}}\delta^+\,d\alpha$. It follows that

(#) *the time along the curve $\Xi_*^{+,\sigma}$ from $\Xi_*^{+,\sigma}(\alpha_1)$ to $\Xi_*^{+,\sigma}(\alpha_2)$ is equal to $2\sigma\delta^+(R^+)^{-\frac{1}{2}}(\alpha_2-\alpha_1)$.*

A similar construction works for Ξ_*^-. In this case, the parametric equations turn out to be $\Xi_*^-(\alpha) = (\alpha - R^-\sin\theta, -R^-(1-\cos\theta))$, where the variable α now takes values in the interval $I^- = [\min(\alpha^0,\alpha^-), \max(\alpha^0,\alpha^-)]$ (which has length $2\pi R^-$), and $\theta = \frac{\alpha-\alpha^-}{R^-}$.

The circle Γ^- rolls from left to right if $\alpha^0 < \alpha^-$, and from right to left if $\alpha^- < \alpha^0$. (Notice that Γ^- rolls from left to right iff it rotates *counterclockwise*, whereas Γ^+ rolls from left to right iff it rotates clockwise.)

The midpoint of I^- is $\mu^- = \frac{1}{2}(\alpha^- + \alpha^0)$. We let $\hat{Q}^- = (\mu^-, 0)$. We then define parametrized arcs $\Xi_*^{-,\sigma}$, for σ in a neighborhood $N^- = [1-\varepsilon_1^-, 1+\varepsilon_2^-]$ of 1 (where $0 < \varepsilon_1^- < 1$ and $0 < \varepsilon_2^-$), by letting $\Xi_*^{-,\sigma}(\alpha) = \hat{Q}^- + \sigma(\Xi_*^-(\alpha) - \hat{Q}^-)$ for $\alpha \in I^-$. Then each $\Xi_*^{-,\sigma}$ is an arc of cycloid, having contact points $Q^{-,\sigma}$, $C^{-,\sigma}$ with the x axis, where $Q^{-,\sigma} = \hat{Q}^- + \sigma(\Xi_*^-(\alpha^-) - \hat{Q}^-)$ and $C^{-,\sigma} = \hat{Q}^- + \sigma(\Xi_*^-(\alpha^0) - \hat{Q}^-)$, so that $Q^{-,\sigma}$ and $C^{-,\sigma}$ are given by

$$Q^{-,\sigma} = \Big((1-\sigma)\mu^- + \sigma\alpha^-, 0\Big)\,, \qquad C^{-,\sigma} = \Big((1-\sigma)\mu^- + \sigma\alpha^0, 0\Big)\,. \quad (11)$$

Then, if we let $\mathcal{S}^- = \{\Xi_*^{-,\sigma}(\alpha) : \sigma \in N^-, \alpha \in I^-\}$, it is clear that the set $\mathcal{S}^-$ is the homeomorphic image of $\mathcal{R}^- \stackrel{\text{def}}{=} N^- \times I^-$ under the smooth map $\Psi^- : \mathcal{R}^- \mapsto H^-$ given by $\Psi^-(\sigma, \alpha) \stackrel{\text{def}}{=} \Xi_*^{-,\sigma}(\alpha)$. The Jacobian determinant of Ψ^- vanishes iff $\alpha = \alpha^-$ or $\alpha = \alpha^0$. Hence

(∗∗) *Ψ^- is a diffeomorphism on $N^- \times \text{Interior}(I^-)$.*

If we let $\delta^- = +1$ if $\alpha^0 < \alpha^-$ (i.e., if Γ^- rolls from left to right, in which case $dt/d\alpha > 0$), and $\delta^- = -1$ if $\alpha^- < \alpha^0$ (i.e., if Γ^- rolls from right to left, in which case $dt/d\alpha < 0$), then $dt = 2\sigma\delta^-(R^-)^{-\frac{1}{2}}\, d\alpha$, from which it follows that

(##) *the time along $\Xi_*^{-,\sigma}$ from $\Xi_*^{-,\sigma}(\alpha_1)$ to $\Xi_*^{-,\sigma}(\alpha_2)$ is equal to $2\sigma\delta^-(R^-)^{-\frac{1}{2}}(\alpha_2 - \alpha_1)$.*

We now combine the two constructions by letting Ξ_*^σ be, for each $\sigma \in I^+$, the concatenation of $\Xi_*^{+,\sigma}$ and $\Xi_*^{-,\hat{\sigma}}$ where $\hat{\sigma}$ is chosen so that $C^{-,\hat{\sigma}} = C^{+,\sigma}$. In view of (10) and (11), it follows that $\hat{\sigma}$ is given in terms of σ by $\hat{\sigma} = \zeta(\sigma)$, where

$$\zeta(\sigma) \stackrel{\text{def}}{=} (\alpha^0 - \mu^-)^{-1}(\mu^+ - \mu^- + \sigma(\alpha^0 - \mu^+)) . \tag{12}$$

(We guarantee that the map $I^+ \ni \sigma \mapsto \hat{\sigma} \in I^-$ is bijective by choosing the $\varepsilon_j^\pm$ so that $\zeta(I^+) = I^-$.)

We now study the flow maps $\Phi_{t,s}$ associated to this family of trajectories. Let $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$. Given any point $q \in \mathcal{S}$, q belongs to the curve Ξ_*^σ for a unique $\sigma \in I^+$. If $s, t \in \mathbb{R}$, and $t \geq s$, then we can follow Ξ_*^σ in the direction of increasing time, starting at q at time s, until we exit $\mathcal{S}$. If t does not exceed the exiting time from $\mathcal{S}$, then $\Phi_{t,s}(q)$ is defined, and equal to the point of Ξ_*^σ attained in this way at time t. We also define the augmentations $c_{t,s}$ by letting $c_{t,s}(q) = t - s$.

In order to apply Theorem 1, we take E to be the set $[0, T] \backslash \{\bar{\tau}\}$. In addition, it will also be convenient to embed our reference trajectory ξ_* in the "extended reference trajectory" $\Xi_* = \Xi_*^1$, that we parametrize by time in such a way that $\Xi_*(\bar{\tau}) = C_*$, so that $\Xi_*(t) = \xi_*(t)$ for $t \in [0, T]$, and Ξ_* is defined on the interval $[\tau_1, \tau_2]$, where $\tau_1 = \bar{\tau} - 4\pi\sqrt{R^+}$ and $\tau_2 = \bar{\tau} + 4\pi\sqrt{R^-}$. Then

(&) *If $\tau_1 < s \leq t < \tau_2$, and $s \neq \bar{\tau} \neq t$, then $\Phi_{t,s}$ is a real analytic diffeomorphism near $\Xi_*(s)$.*

To prove (&), we will consider separately three cases, namely, (i) $t < \bar{\tau}$, (ii) $s > \bar{\tau}$, and (iii) $s < \bar{\tau} < t$. In Case (i), $\Xi_*(s)$ clearly belongs to the set $\Psi^+(N^+ \times \text{Interior}(I^+))$, so for q near $\Xi_*(s)$ we can find $\Phi_{t,s}(q)$ by inverting the diffeomorphism Ψ^+, letting $(\sigma, \alpha) = (\Psi^+)^{-1}(q)$, defining $\tilde{\alpha}$ by $\tilde{\alpha} = \alpha + \frac{1}{2}\delta^+\sqrt{R^+}(t - s)$, and, finally, writing $\Phi_{t,s}(q) = \Psi^+(\sigma, \tilde{\alpha})$. The conclusion then follows because the map $(\sigma, \alpha) \mapsto (\sigma, \alpha + \frac{\delta^+}{2\sigma}\sqrt{R^+}(t - s))$ is a diffeomorphism. The proof for Case (ii) is similar. Finally, in Case (iii) we can find $\Phi_{s,t}(q)$ by first inverting Ψ^+ near q to find (σ, α), as in Case (i), and then going from time s to time t by letting $\nu^+ = \delta^+\sqrt{R^+}$, $\nu^- = \delta^-\sqrt{R^-}$, writing

$$\tilde{\sigma}=\zeta(\sigma)\,,\ \ \tilde{\alpha}=\alpha_0+\frac{\nu^-}{2\tilde{\sigma}}(t-s)-\frac{\sigma\nu^-}{\tilde{\sigma}\nu^+}(\alpha_0-\alpha)\,, \tag{13}$$

and then defining $\Phi_{t,s}(q) = \Psi^-(\tilde{\sigma},\tilde{\alpha})$. The map $(\sigma,\alpha)\mapsto(\tilde{\sigma},\tilde{\alpha})$ defined by (13) is a diffeomorphism, since $\frac{\partial\tilde{\sigma}}{\partial\sigma} = \frac{d\zeta}{d\sigma} \neq 0$ (in view of (12)), $\frac{\partial\tilde{\sigma}}{\partial\alpha} = 0$, and $\frac{\partial\tilde{\alpha}}{\partial\alpha} = \frac{\sigma\nu^-}{\tilde{\sigma}\nu^+} \neq 0$ (because of (13)). So the conclusion follows in Case (iii) as well, and (&) is proved.

For $\tau_1 < s \le t < \tau_2$, $s \neq \bar{\tau} \neq t$, let $D_{t,s}$ be the Jacobian matrix of $\Phi_{t,s}$ at $\Xi_*(s)$. Then we can apply Theorem 1 to each of the three curves $\Xi_{*,i} : J^i \mapsto \mathbb{R}^2$, $i = 1, 2, 3$, where $\Xi_{*,1}$ and $\Xi_{*,2}$ are the restrictions of Ξ_* to intervals J_1, J_2 of the form $[\tau_1+\delta,\bar{\tau}-\delta]$ and $[\bar{\tau}+\delta,\tau_2-\delta]$, for some small δ, and $\Xi_{*,3} = \xi_*$, so $J_3 = [0,T]$. (We are assuming that $\Xi_{*,3}$ is time-optimal, and the curves $\Xi_{*,1}$ and $\Xi_{*,2}$ are also optimal because they are solutions of the classical BP.) If we let $\hat{J}_1 = J_1$, $\hat{J}_2 = J_2$, $\hat{J}_3 = E$, then our theorem implies that there exists nontrivial solutions $\hat{J}_i \ni t \mapsto \tilde{\pi}_i(t) \in \mathbb{R}_2$, of the IAE $\pi(s) = \pi(t).D_{t,s}$ such that the HMC holds for every $t \in \hat{J}^i$ and the values of the maximized Hamiltonian are nonnegative constants $\tilde{\pi}_{0,i}$. The HMC at $\Xi_{*,i}(t)$ says that (a) the control $\eta_*(t)=(u_*(t),v_*(t))$ must maximize the product $\tilde{\pi}_i(t)\cdot(u,v)^\dagger$ (where "†" denotes transpose) for (u,v) in the unit disc of $\mathbb{R}_2$, from which it follows that $\|\tilde{\pi}_i(t)\|\eta_*(t) = \tilde{\pi}_i(t)$, and in addition (b) the maximum value $\tilde{\pi}_{0,i}$ of the Hamiltonian is $\sqrt{|y_*(t)|}\,\|\tilde{\pi}_i(t)\|$, if we write $\Xi_{*,i}(t) = (x_*(t),y_*(t))$. This implies, in particular, that $\tilde{\pi}_{0,i}>0$, because if $\tilde{\pi}_{0,i}=0$ then $\|\tilde{\pi}_i(t)\|=0$ (since $\Xi_{*,i}(t) \notin H_+\cap H_-$ whenever $t \in \hat{J}_i$), and then $\tilde{\pi}_i(t) = 0$, contradicting the NTC. It then follows that $\dot{\Xi}_{*,i}(t)^\dagger = \sqrt{|y_*(t)|}\frac{\tilde{\pi}_i(t)}{\|\tilde{\pi}_i(t)\|}$ for $t\in\hat{J}_i$, so $\langle\tilde{\pi}(t)\cdot w\rangle = 0$ whenever $w \in \mathbb{R}^2$ is orthogonal to $\dot{\Xi}_{*,i}(t)$. Now, if $t,s \in \hat{J}_i$, $s \le t$, $w \in \mathbb{R}^2$, and $\hat{w} = D_{t,s}w$, then $\tilde{\pi}_i(t)\cdot\hat{w} = \tilde{\pi}_i(t)\cdot D_{t,s}(w) = (\tilde{\pi}_i(t)\circ D_{t,s})(w)\,,$ so $\tilde{\pi}_i(t)\cdot\hat{w}=\tilde{\pi}_i(s)\cdot w$. Therefore $\langle\dot{\Xi}_{*,i}(t),D_{t,s}w\rangle = 0$ iff $\langle\dot{\Xi}_{*,i}(s),w\rangle = 0$, and we have obtained the geometric condition

(G1) *If $i \in \{1,2,3\}$, $s,t \in \hat{J}_i$, and $s \le t$, then the linear map $D_{t,s}$ is such that a vector $w \in \mathbb{R}^2$ is orthogonal to $\dot{\Xi}_{*,i}(s)$ if and only if $D_{t,s}w$ is orthogonal to $\dot{\Xi}_{*,i}(t)$.*

Now, for $i = 1$, we can let s_1 be the time corresponding to the midpoint of the α-interval I^+, so $s_1 = \bar{\tau} - \frac{\delta^+}{\sqrt{R^+}}(\alpha^0-\alpha^+)$. Then we can choose t_1 to be any point in $\hat{J}_1\cap\hat{J}_3$, i.e., in $J_1\cap E$. Similarly, we can choose $t_2 = \bar{\tau}+\frac{\delta^-}{\sqrt{R^-}}(\alpha^--\alpha^0)$, i.e. the time corresponding to the midpoint of the α-interval I^-. Then we can choose s_2 to be any point in $\hat{J}_2\cap\hat{J}_3$, i.e., in $J_2\cap E$. If we then apply (G1) successively with $s = s_1$ and $t = t_1$, with $s = t_1$ and $t = s_2$, and with $s = s_2$ and $t = t_2$, and observe that the vectors orthogonal to $\dot{\Xi}_{*,1}(s_1)$ and those orthogonal to $\dot{\Xi}_{*,3}(t_2)$ are just the vertical vectors, we find

(G2) *The linear map D_{t_2,s_1} sends vertical vectors to vertical vectors.*

The segments $\sigma \mapsto \Xi_*^{+,\sigma}(\mu^+)$, $\sigma\mapsto\Xi_*^{-,\zeta(\sigma)}(\mu^-)$ are vertical and go through $\Xi_*^+(\mu^+)$ and $\Xi_*^-(\mu^-)$, respectively, when $\sigma = 1$. Since it is clear that

$$\Phi_{t_2,s_1}(\Xi_*^+(\mu^+)) = \Xi_*^-(\mu^-)\,,$$

Condition (G2) holds if and only if $\rho(\sigma) = t_2 - s_1 + o(|\sigma - 1|)$ as $\sigma \to 1$, where $\rho(\sigma)$ is the time to go from $\Xi_*^{+,\sigma}(\mu^+)$ to $\Xi_*^{-,\zeta(\sigma)}(\mu^-)$ along the curve Ξ_*^σ. On the other hand, (#) and (##) easily imply that $\rho(\sigma)$ is equal to $2\sigma\delta^+(R^+)^{\frac{1}{2}}(\alpha^0 - \mu^+) + 2\zeta(\sigma)\delta^-(R^-)^{\frac{1}{2}}(\mu^- - \alpha^0)$. Then (G2) holds iff $\rho'(1) = 0$. But

$$\rho'(1) = 2\delta^+(R^+)^{\frac{1}{2}}(\alpha^0 - \mu^+) + 2\zeta'(1)\delta^-(R^-)^{\frac{1}{2}}(\mu^- - \alpha^0)\,,$$

and (12) implies $\zeta'(\sigma) = \frac{\alpha^0-\mu^+}{\alpha^0-\mu^-}$. So

$$\rho'(1) = 2\delta^+(R^+)^{\frac{1}{2}}(\alpha^0 - \mu^+) - 2\delta^-(R^-)^{\frac{1}{2}}(\alpha^0 - \mu^+)\,,$$

and then (G2) holds if and only if $\delta^+(R^+)^{\frac{1}{2}} = \delta^-(R^-)^{\frac{1}{2}}$, that is, if and only if

(G3) $\delta_+ = \delta_-$ *and* $R_+ = R_-$.

In other words,

($) *the additional necessary condition for optimality is that the rolling circles that generate the upper and lower parts of* ξ_* *should roll in the same direction (i.e., both from left to right or both from right to left) and have equal radii.*

Remark 3. The above result has been proved, of course, under the assumption that both ξ_*^+ and ξ_*^- are cycloid arcs. There remain to consider the degenerate cases when one or both are vertical segments. If both are vertical segments, then it is easy to see that ξ_* is optimal. Finally, if one of ξ_*^+, ξ_*^- is a cycloid arc, and the other one is a vertical segment, then an argument similar to the one we used for the case of two cycloid arcs (but much simpler) shows that ξ_* is not optimal, concluding the analysis of all possible cases. ◊

7 Snell's law of refraction

We consider the minimum time problem for the two-dimensional system

$$\dot{x} = c(x,y)u\,, \qquad \dot{y} = c(x,y)v\,,$$

where the control (u,v) takes values in the unit circle

$$\mathbb{S}^1 = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 = 1\}\,,$$

and the function c (the "speed of light") is given by

$$c(x,y) = c_+ \text{ if } y \geq 0\,, \qquad c(x,y) = c_- \text{ if } y < 0\,.$$

Here c_+ and c_- are two fixed positive constants such that $c_+ > c_-$. We will focus on the problem of finding a time-minimizing arc from a point $A = (x_A, y_A)$ such that $y_A > 0$ to a point $B = (x_B, y_B)$ such that $y_B < 0$.

The solution of this problem—Snell's law of refraction—is well known, and can be derived by very elementary means: first, one shows that the solution must consist of a straight segment from A to a point C lying on the x axis, followed by the segment from C to B; finding C then becomes a rather simple first-year calculus exercise. Here we will show how our version of the Maximum Principle applies to this problem, and leads to Snell's law.

We take our control set U to be the product $\mathbb{S}^1 \times \mathbb{S}^1$, and then define, for each $z = (u_+, v_+, u_-.v_-) \in U$, a discontinuous vector field X_z by letting $X_z(x, y) = (c_+u_+, c_+v_+)$ if $y > 0$, and $X_z(x, y) = (c_-u_-, c_-v_-)$ if $y \leq 0$. We let G be the subset of U consisting of those $(u_+, v_+, u_-.v_-) \in U$ such that $v_+ < 0$ and $v_- < 0$. We use L to denote the x axis.

An elementary argument shows that an optimal trajectory ξ_* must consist of a segment from A to C followed by a segment from C to B, where $C \in L$. That is, we can confine ourselves to a trajectory $\xi_* : [0, T] \mapsto \mathbb{R}^2$ such that (i) $\xi_*(0) = A$, (ii) $\xi_*T) = B$, (iii) $\xi_*(\tau) \in L$ for some τ such that $0 < \tau < T$, (iv) if $\xi_*(t) = (x_*(t), y_*(t))$ for $t \in [0, T]$, then $y_*(t) > 0$ for $0 \leq t < \tau$ and $y_*(t) < 0$ for $\tau < t \leq T$, and (v) the curve ξ_* is a trajectory of a *constant* control $z_* \in U$.

All that is left now is to find a condition that will determine C. With our choice of U, constant controls have two degrees of freedom, but one is removed when we stipulate that ξ_*, starting at A, has to go through B, so we need to find an extra constraint on z_*.

Let us compute the flow of X_z for a $z \in G$. It suffices to compute the maps $\Phi^{X_z}_{t,0}$, since $\Phi^{X_z}_{t,s} = \Phi^{X_z}_{t-s,0}$.

If $y > 0$, $t > 0$, and we let $(\tilde{x}, \tilde{y}) = \Phi^{X_z}_{t,0}(x, y)$, then $y > \tilde{y}$ and, in addition, $\tilde{y} > 0$ as long as $t < \tau_z(x, y)$, where $\tau_z(x, y)$ is the time for which $\Phi^{X_z}_{\tau_z(x,y),0}(x, y) \in L$. It is clear that

$$\begin{aligned} \tau_z(x, y) &= -\frac{y}{c_+v_+}\,, \\ \Phi^{X_z}_{t,0}(x, y) &= (x + tc_+u_+\,,\, y + tc_+v_+) \ \text{ if } \ 0 < t < \tau_z(x, y)\,, \\ \Phi^{X_z}_{\tau_z(x,y)+t,0}(x, y) &= (x + \tau_z(x, y)c_+u_+ + tc_-u_-\,,\, tc_-v_-) \ \text{ if } \ t > 0\,. \end{aligned}$$

In particular, given a t such that $t \neq \tau_z(x_A, y_A)$, the flow map $\Phi^{X_z}_{t,0}$ is of class C^1 near A, and is given, for (x, y) in some neighborhood $N(t)$ of A, by

$$\Phi^{X_z}_{t,0}(x, y) = (x + tc_+u_+, y + tc_+v_+) \ \text{ if } \ 0 < t < \tau_z(x_A, y_A)\,,$$

and

$$\Phi^{X_z}_{t,0}(x, y) = (x + \tau_z(x, y)c_+u_+ + (t - \tau_z(x, y))c_-u_-\,,\, (t - \tau_z(x, y))c_-v_-)$$

$$= \Big(x - \frac{y}{v_+}u_+ + (t + \frac{y}{c_+v_+})c_-u_- \,,\, (t + \frac{y}{c_+v_+})c_-v_-\Big)$$
$$= \Big(x + \frac{c_-u_- - c_+u_+}{c_+v_+}y + tc_-u_- \,,\, \frac{c_-v_-}{c_+v_+}y + tc_-v_-\Big)$$

if $t > \tau_z(x_A, y_A)$.

It follows from this that the differential $D^z(t)$ of $\Phi^{X_z}_{t,0}$ at A is given by

$$D^z(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } \ t < -\frac{y_A}{c_+v_+} \\ \\ \begin{bmatrix} 1 & \frac{c_-u_- - c_+u_+}{c_+v_+} \\ 0 & \frac{c_-v_-}{c_+v_+} \end{bmatrix} & \text{if } \ t > -\frac{y_A}{c_+v_+} \end{cases} . \tag{14}$$

We now let $\hat{D}(t)$ denote the differential of the flow map $\Phi^{X_{z_*}}_{T,t}$ at $\xi_*(t)$, where $\xi_* : [0, T] \mapsto\in \mathbb{R}^2$ is our reference trajectory $z_* = (u_{*,+}, v_{*,+}, u_{*,-}.v_{*,-})$ is our constant reference control, and It is then clear that

$$\hat{D}(t) = D^{z_*}(T)D^{z_*}(t)^{-1} \,,$$

and then (14) implies

$$\hat{D}(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } \ t > -\frac{y_A}{c_+v_{*,+}} \\ \\ \begin{bmatrix} 1 & \frac{c_-u_{*,-} - c_+u_{*,+}}{c_+v_{*,+}} \\ 0 & \frac{c_-v_{*,-}}{c_+v_{*,+}} \end{bmatrix} & \text{if } \ t < -\frac{y_A}{c_+v_{*,+}} \end{cases} .$$

Notice that we have not written explicit formulas for the flow maps $\Phi^{X_z}_{t,0}$ near A, and we have not given formulas for the differentials $D^{z_*}(t)$, $\hat{D}(t)$, when $t = \tau_{z_*}(A)$. This is so for the simple reason that the map $\Phi^{X_z}_{t,0}$ is not differentiable at A when $t = \tau_{z_*}$. So, in order to apply our flow version of the Maximum Principle, we take the time set E to be $[0, T]\backslash\{\tau_{z_*}(A)\}$. Then each flow map $\Phi^{X_{z_*}}_{t,s}$ is of class C^1 (and, in fact, real analytic) on a neighborhood of $\xi_*(s)$, as long as $s, t \in E$ and $s \leq t$.

Let ω be the adjoint vector given by the Maximum Principle. Then

$$\omega(t) = \begin{cases} \omega(T) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } \ t > -\frac{y_A}{c_+v_{*,+}} \\ \\ \omega(T) \cdot \begin{bmatrix} 1 & \frac{c_-u_{*,-} - c_+u_{*,+}}{c_+v_{*,+}} \\ 0 & \frac{c_-v_{*,-}}{c_+v_{*,+}} \end{bmatrix} & \text{if } \ t < -\frac{y_A}{c_+v_{*,+}} \end{cases} .$$

If we write $\omega(T) = (\bar{\omega}_x, \bar{\omega}_y)$, we see that

$$\omega(t) = \begin{cases} \bar{\omega}^- & \text{if } \ t > -\frac{y_A}{c_+ v_{*,+}} \\ \bar{\omega}^+ & \text{if } \ t < -\frac{y_A}{c_+ v_{*,+}} \ , \end{cases}$$

where $\bar{\omega}^- = (\bar{\omega}_x, \bar{\omega}_y)$, $\bar{\omega}^+ = (\bar{\omega}_x, \hat{\omega}_y)$, and

$$\hat{\omega}_y = \frac{1}{c_+ v_{*,+}} \Big((c_- u_{*,-} - c_+ u_{*,+})\bar{\omega}_x + c_- v_{*,-}\bar{\omega}_y \Big) \, .$$

The Hamiltonian maximization condition of the Maximum Principle implies that ω^- must be a scalar multiple of $(u_{*,-}, v_{*,-})$, and ω^+ has to be a scalar multiple of $(u_{*,+}, v_{*,+})$. This means that

$$\bar{\omega}_x = k_- u_{*,-} = k_+ u_{*,+} \, , \qquad \bar{\omega}_y = k_- v_{*,-} \, , \ \text{ and } \hat{\omega}_y = k_+ v_{*,+}$$

for some positive constants k_-, k_+.

It follows that

$$\frac{k_-}{k_+} = \frac{u_{*,+}}{u_{*,-}} \, . \tag{15}$$

Let ω_0 be the abnormal multiplier. Then

$$0 = \langle \bar{\omega}^-, c_-(u_{*,-}, v_{*,-}) \rangle - \omega_0 = \langle \bar{\omega}^+, c_+(u_{*,+}, v_{*,+}) \rangle - \omega_0 \, ,$$

so $\langle \bar{\omega}^-, c_-(u_{*,-}, v_{*,-}) \rangle = \langle \bar{\omega}^+, c_+(u_{*,+}, v_{*,+}) \rangle$.

Furthermore, $\bar{\omega}^- = k_-(u_{*,-}, v_{*,-})$, $\bar{\omega}^+ = k_+(u_{*,+}, v_{*,+})$, and both $(u_{*,-}, v_{*,-})$, and $(u_{*,+}, v_{*,+})$ are unit vectors. It follows that $k_- c_- = k_+ c_+$. Hence

$$\frac{k_-}{k_+} = \frac{c_+}{c_-} \, . \tag{16}$$

Comnining (15) and (16), we get

$$\frac{u_{*,+}}{u_{*,-}} = \frac{c_+}{c_-} \, . \tag{17}$$

Let θ_i be the "angle of incidence," that is, the angle between the line AC and the y axis. Let θ_r be the "angle of refraction," that is, the angle between the line CB and the y axis. It is then clear that $u_{*,+} = \sin\theta_i$ and $u_{*,-} = \sin\theta_r$. Then (17) says that

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{c_+}{c_-} \, , \tag{18}$$

which is precisely Snell's law.

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