On the Noether Theorem for Optimal Control*

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Abstract

We obtain a generalization of E. Noether's theorem for the optimal control problems. The generalization involves a one-parameter family of smooth maps which may depend also on the control and a Lagrangian which is invariant up to an addition of an exact differential.

Keywords: optimal control, calculus of variations, Noether's theorem, Pontryagin extremals, first integrals.

1 Introduction

E. Noether's celebrated theorem is the result of major importance for various areas of modern physics, such as classical and quantum mechanics, classical and quantum field theory, geometric optics and general theory of relativity. It relates the invariance properties of an integral functional $\int_a^b L(t,\, x(t),\, \dot{x}(t))\, dt$ with the *conservation laws*, i.e., with the integrals of the corresponding Euler–Lagrange or Hamiltonian differential equations. Thus, conservation of momentum and angular momentum in mechanics correspond, respectively, to translational and rotational invariance of the *action*, while time invariance conduces to energy conservation. For an account of the relevance of Noether's theorem to physics, and for an historical survey, see [7].

Typical formulation of the Noether's theorem in the calculus of variations considers a one-parameter family of differentiable maps $h^s(t, x)$, $h^s : [a,\, b] \times \mathbb{R}^n \to$

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$\mathbb{R}\times\mathbb{R}^n$ (see e.g. [12]). The following generalizations of the theorem of E. Noether are widely known: invariant integrands of the calculus of variations defined on a manifold M (see e.g. [4]); problems of the calculus of variations with multiple integrals (see e.g. [5]); invariance with respect to r-parameter families of maps from finite- or infinite-dimensional Lie group ([15, 16]); variational problems of (higher order) supermechanics (see [8]). A generalization of the Noether's theorem which involves the maps depending on $\dot{x}$ has been considered in the textbook by I. M. Gelfand and S. V. Fomin [11]. Another interesting generalization can be found in the book of H. Rund [18]: invariance of the Lagrangian up to addition of an exact differential $d\Phi^s(t,\, x)$, with Φ linear on the parameter s. H. Rund remarks however (cf. [18, Remark 2, p. 297]) that this generalization is incompatible with the one of Gelfand and Fomin. Below we provide a formulation which involves both generalizations. Furthermore, the assumption on linear dependence of Φ with respect to the parameter is eliminated.

We provide the formulation of the Noether's theorem in the optimal control setting. This means in particular the Hamiltonian rather the Lagrangian formulation. Versions of the Noether's theorem in the optimal control setting appeared earlier in the papers of A. J. van der Schaft [24] and H. J. Sussmann [20], and in the book of V. Jurdjevic [13, Ch. 13]. Recently, a version of the Noether's theorem for optimal control problems appeared in the work of G. Blankenstein and A. van der Schaft [6]. Here we extend the ideas of Gelfand and Fomin to provide a new version of E. Noether's theorem for the h^s-invariant optimal control problems, where h^s can depend also on the control variable (Theorem 1). Afterwards we use in addition a time reparameterization to obtain the E. Noether's theorem in more general form (Theorem 2). We conclude providing concrete examples of application of the main results to situations not covered by the previous methods of [24, 20, 13, 6].

2 Lagrange Problem of Optimal Control and Preliminaries

Given $a,\, b \in \mathbb{R}$, $a < b$; an arbitrary set $U \subseteq \mathbb{R}^r$; $L(\cdot, \cdot, \cdot) \in C^1\left([a,\, b] \times \mathbb{R}^n \times U;\, \mathbb{R}\right)$ and $\varphi(\cdot, \cdot, \cdot) \in C^1\left([a,\, b] \times \mathbb{R}^n \times U;\, \mathbb{R}^n\right)$; we define the following Lagrange problem of optimal control.

Problem (P).

$$J\left[x(\cdot),\, u(\cdot)\right] = \int_a^b L\left(t,\, x(t),\, u(t)\right)\, dt \longrightarrow \min$$

$$\dot{x}(t) = \varphi\left(t,\, x(t),\, u(t)\right) \quad a.e. \;\; on \;\; [a,\, b],$$

$$x(\cdot) \in W_{1,\,1}\left([a,\, b];\, \mathbb{R}^n\right),\; u(\cdot) \in L_\infty\left([a,\, b];\, U\right).$$

Notation. The notation $W_{1,\,1}$ is used for the class of absolutely continuous functions; while $W_{1,\,\infty}$ is the class of Lipschitzian functions; and L_∞ is the class of measurable and essentially bounded functions.

Remark 1. We treat optimal control problems in the situation when we do not know the initial and terminal conditions. The results obtained are valid for arbitrary boundary constraints.

Definition 1. A pair $(x(\cdot),\, u(\cdot))$ is said to be *admissible* if it satisfies the dynamics equation $\dot{x}(t) = \varphi\,(t,\, x(t),\, u(t))$.

Definition 2. We say that $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$, with admissible $(x(\cdot),\, u(\cdot))$; $\psi_0 \in \mathbb{R}_0^-$; and $\psi(\cdot) \in W_{1,\,1}\,([a,b];\, \mathbb{R}^n)$ ($\psi(t)$ is a covector $1 \times n$); is a *Pontryagin extremal* of (P), if the following two conditions are satisfied for almost all $t \in [a,\, b]$:

the adjoint system

$$\dot{\psi}(t) = -\frac{\partial H}{\partial x}\,(t,\, x(t),\, u(t),\, \psi_0,\, \psi(t))\,; \tag{1}$$

the maximality condition

$$H\,(t, x(t), u(t), \psi_0, \psi(t)) = \max_{u \in U} H\,(t, x(t), u, \psi_0, \psi(t))\,; \tag{2}$$

where the Hamiltonian equals

$$H\,(t,\, x,\, u,\, \psi_0,\, \psi) = \psi_0\, L(t,\, x,\, u) + \psi \cdot \varphi(t,\, x,\, u). \tag{3}$$

Remark 2. A Pontryagin extremal $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ is called normal if $\psi_0 \neq 0$ and abnormal if $\psi_0 = 0$. For normal extremals we can always consider, by scaling, that ψ_0 takes value -1. Although there are no abnormal extremals in the basic problem of the calculus of variations, the existence of abnormal minimizers in the calculus of variations and optimal control is a common phenomena (see [21, 2]). Our results are able to treat both normal and abnormal cases. Example 2, considered in §4, admits abnormal minimizers.

Remark 3. If $(x(\cdot),\, u(\cdot))$ is a minimizer (or a maximizer) of (P) then there exists a nonzero pair $(\psi_0,\, \psi(\cdot))$ such that $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ is a Pontryagin extremal of (P). Moreover, the function $H\,(t, x(t), u(t), \psi_0, \psi(t))$ is an absolutely continuous function of t and satisfies the equality $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ (see [17]). This is called the *Maximum Principle.*

Remark 4. Here we use the Maximum Principle as it was first presented in the book [17]. This version, however, does not exhaust the full power of the principle: work of proving it in other contexts is still very much in progress (see e.g. [9, 22]). The results obtained in the present work are general for all optimal control problems applying the Maximum Principle. For example, the results can be stated for control systems on arbitrary manifolds.

Definition 3. A quantity $C\left(t,\, x(t),\, u(t),\, \psi_0,\, \psi(t)\right)$ which is constant in t along every Pontryagin extremal $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ of (P), is called a *first integral.*

3 Main Results

We prove in §3.1 a particular case and in §3.2 more general form of Noether's theorem for the Lagrange problem of optimal control.

3.1 Noether theorem with no transformation of time-variable

Definition 4. Let $h^s : [a,b] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ $(s \in (-\varepsilon,\, \varepsilon))$ be a one-parameter family of C^1 maps and $h^0(t,x,u) = x$, $\forall (t,\, x,\, u) \in [a,\, b] \times \mathbb{R}^n \times U$. If there exists a function $\Phi^s(t,x,u) \in C^1\left([a,\, b],\, \mathbb{R}^n,\, U;\, \mathbb{R}\right)$ and for all $s \in (-\varepsilon,\, \varepsilon)$ and admissible $(x(\cdot),\, u(\cdot))$ there exists a control $u^s(\cdot) \in L_\infty\left([a,\, b];\, U\right)$ such that:

i)

$$\int_a^\beta L\left(t,\, h^s(t,x(t),u(t)),\, u^s(t)\right) dt \\ = \int_a^\beta \left(L(t,\, x(t),\, u(t)) + \frac{d}{dt}\Phi^s(t,\, x(t),\, u(t)) \right) dt \quad (4)$$

for all $\beta \in [a,\, b]$;

ii)

$$\frac{d}{dt}h^s(t,x(t),u(t)) = \varphi(t,\, h^s(t,x(t),u(t)),\, u^s(t)); \quad (5)$$

then the problem (P) is said to be *invariant* under the transformations $h^s(t,\, x,\, u)$ up to $\Phi^s(t,\, x,\, u)$.

Remark 5. We assume $u^0(\cdot) = u(\cdot)$ and that the derivative $\frac{\partial}{\partial s}u^s(\cdot)$ exist and belong to L_∞.

Remark 6. Clearly, condition (4) can be satisfied if and only if

$$L\left(t,\, h^s(t,x(t),u(t)),\, u^s(t)\right) = L(t,\, x(t),\, u(t)) + \frac{d}{dt}\Phi^s(t,\, x(t),\, u(t))\,. \quad (6)$$

For historical reasons (cf. e.g. [11, 14, 12]), we prefer to maintain the integral signs in Definition 4.

Remark 7. Although many of the parameter-transformations that appear in applications are groups of transformations (e.g. the Galilean group, the Lorentz group, or the conformal group), the group concept is not required for the formulation of the Noether's theorem (see e.g. [11, 12]). In [24, 20, 13, 6] Lie groups of transformations are considered. Here these stringent assumptions on the transformations are not imposed.

Theorem 1. *If (P) is invariant under the transformations $h^s(t,\, x,\, u)$ up to $\Phi^s(t,\, x,\, u)$, in the sense of the Definition 4, then*

$$\psi(t) \cdot \frac{\partial}{\partial s}\, h^s(t,\, x(t),\, u(t))|_{s=0} + \psi_0 \frac{\partial}{\partial s}\, \Phi^s(t,\, x(t),\, u(t))|_{s=0}$$

is a first integral.

Remark 8. The quantity $\psi(t) \cdot \frac{\partial}{\partial s}\, h^s(t,\, x(t),\, u(t))|_{s=0}$ is the classical *momentum map*, corresponding to the symmetry action.

Proof. Due to (4) we have, for all $s \in (-\varepsilon,\, \varepsilon)$ and all $\beta \in [a,\, b]$, that

$$\begin{aligned} &\frac{d}{ds} \int_a^\beta \frac{d}{dt} \Phi^s(t,\, x(t),\, u(t))\, dt \\ &\quad = \frac{d}{ds} \int_a^\beta L\left(t,\, h^s(t,\, x(t),\, u(t)),\, u^s(t)\right) dt \\ &\quad = \int_a^\beta \left(\frac{\partial L}{\partial x} \cdot \frac{\partial}{\partial s} h^s(t,\, x(t),\, u(t)) + \frac{\partial L}{\partial u} \cdot \frac{\partial}{\partial s} u^s(t) \right) dt \end{aligned}$$

holds with the partial derivatives of L evaluated at $(t,\, h^s(t, x(t), u(t)),\, u^s(t))$. Hence for $s = 0$

$$0 = \int_a^\beta \left(\frac{\partial L}{\partial x} \cdot \frac{\partial}{\partial s}\, h^s|_{s=0} + \frac{\partial L}{\partial u} \cdot \frac{\partial}{\partial s}\, u^s(t)|_{s=0} - \frac{d}{dt} \frac{\partial}{\partial s}\, \Phi^s|_{s=0} \right) dt, \tag{7}$$

where now, and up to the end of the proof, all the functions are evaluated at $(t,\, x(t),\, u(t))$. If $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ is a Pontryagin extremal of (P) then, by virtue of (1),

$$\dot{\psi}(t) = -\psi_0 \frac{\partial L}{\partial x} - \psi(t) \cdot \frac{\partial \varphi}{\partial x} \Leftrightarrow -\psi_0 \frac{\partial L}{\partial x} = \dot{\psi}(t) + \psi(t) \cdot \frac{\partial \varphi}{\partial x}.$$

Multiplying (7) by $-\psi_0$ one gets

$$\begin{aligned} 0 = \int_a^\beta \Bigg(&\dot{\psi}(t) \cdot \frac{\partial}{\partial s}\, h^s|_{s=0} + \psi(t) \cdot \frac{\partial \varphi}{\partial x} \cdot \frac{\partial}{\partial s}\, h^s|_{s=0} \\ &- \psi_0 \frac{\partial L}{\partial u} \cdot \frac{\partial}{\partial s}\, u^s(t)|_{s=0} + \psi_0 \frac{d}{dt} \frac{\partial}{\partial s}\, \Phi^s|_{s=0} \Bigg) dt. \end{aligned} \tag{8}$$

By the maximality condition (2) it follows that the expression

$$M_t(s) = \psi_0\, L(t,\, x(t),\, u^s(t)) + \psi(t) \cdot \varphi(t,\, x(t),\, u^s(t))$$

attains its maximum for $s = 0$. Therefore $\frac{dM_t(s)}{ds}$ is equal to zero at $s = 0$, that is,

$$\psi_0 \frac{\partial L}{\partial u} \cdot \frac{\partial}{\partial s}\, u^s(t)|_{s=0} + \psi(t) \cdot \frac{\partial \varphi}{\partial u} \cdot \frac{\partial}{\partial s}\, u^s(t)|_{s=0} = 0. \tag{9}$$

From (8) and (9) one obtains

$$0 = \int_a^\beta \left(\dot{\psi}(t) \cdot \frac{\partial}{\partial s}\, h^s|_{s=0} + \psi(t) \cdot \frac{\partial \varphi}{\partial x} \cdot \frac{\partial}{\partial s}\, h^s|_{s=0} \right. \\ \left. + \psi(t) \cdot \frac{\partial \varphi}{\partial u} \cdot \frac{\partial}{\partial s}\, u^s(t)|_{s=0} + \psi_0 \frac{d}{dt}\frac{\partial}{\partial s}\, \Phi^s|_{s=0}\right) dt. \quad (10)$$

On the other hand differentiating (5) with respect to s at $s = 0$ we get

$$\frac{d}{dt}\frac{\partial}{\partial s}\, h^s|_{s=0} = \frac{\partial \varphi}{\partial x} \cdot \frac{\partial}{\partial s}\, h^s|_{s=0} + \frac{\partial \varphi}{\partial u} \cdot \frac{\partial}{\partial s}\, u^s(t)|_{s=0}\,,$$

and therefore we can write (10) as

$$\int_a^\beta \left(\dot{\psi}(t) \cdot \frac{\partial}{\partial s}\, h^s|_{s=0} + \psi(t) \cdot \frac{d}{dt}\frac{\partial}{\partial s}\, h^s|_{s=0} + \psi_0 \frac{d}{dt}\frac{\partial}{\partial s}\, \Phi^s|_{s=0}\right) dt \\ = \int_a^\beta \frac{d}{dt}\left(\psi(t) \cdot \frac{\partial}{\partial s}\, h^s|_{s=0} + \psi_0 \frac{\partial}{\partial s}\, \Phi^s|_{s=0}\right) dt = 0.$$

This means that

$$\psi(\beta) \cdot \frac{\partial}{\partial s}\, h^s\left(\beta,\, x(\beta),\, u(\beta)\right)|_{s=0} + \psi_0 \frac{\partial}{\partial s}\, \Phi^s\left(\beta,\, x(\beta),\, u(\beta)\right)|_{s=0} \\ = \psi(a) \cdot \frac{\partial}{\partial s}\, h^s\left(a,\, x(a),\, u(a)\right)|_{s=0} + \psi_0 \frac{\partial}{\partial s}\, \Phi^s\left(a,\, x(a),\, u(a)\right)|_{s=0}$$

$\forall\, \beta \in [a, b]$, i.e.,

$$\psi(t) \cdot \frac{\partial}{\partial s}\, h^s(t, x(t), u(t))|_{s=0} + \psi_0 \frac{\partial}{\partial s}\, \Phi^s\left(t, x(t), u(t)\right)|_{s=0}$$

is constant on $[a, b]$. ☐

Remark 9. The original proof of Emmy Noether [15, 16] is based on some deep and conceptually difficult results in the calculus of variations. A better approach, that completely avoids the use of transversality conditions, was proposed by Rund in 1973 [19]. Rund's method permit to derive Noether's theorem directly and almost effortlessly, by means of elementary techniques (see also Lovelock and Rund book [14]). It can be said that the proof of Theorem 1 extends Rund's approach from the calculus of variations to optimal control.

3.2 Noether theorem with transformation of the time-variable

We now generalize the notion of invariance given in the Definition 4, admitting the possibility of a one-parameter transformation of the independent variable t.

Definition 5. Let $h^s(t,\, x,\, u) = (h_t^s(t,\, x,\, u),\, h_x^s(t,\, x,\, u))$, $h_t^s : [a,\, b] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $h_x^s : [a,\, b] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ $(s \in (-\varepsilon,\, \varepsilon))$, be a one-parameter family of

C^1 maps and $h^0(t,x,u) = (t,\, x)$ for all $(t,\, x,\, u) \in [a,\, b] \times \mathbb{R}^n \times U$. If there exists a function $\Phi^s(t,x,u) \in C^1\left([a,\, b],\, \mathbb{R}^n,\, U;\, \mathbb{R}\right)$ and for all $s \in (-\varepsilon,\, \varepsilon)$ and admissible $(x(\cdot),\, u(\cdot))$ there exists a control $u^s(\cdot) \in L_\infty\left([a,\, b];\, U\right)$ such that:

i)

$$\int_{h_t^s(a,x(a),u(a))}^{h_t^s(\beta,x(\beta),u(\beta))} L\left(t^s, h_x^s(t^s, x(t^s), u(t^s)), u^s(t^s)\right) dt^s \\ = \int_a^\beta \left(L(t,x(t),u(t)) + \frac{d}{dt}\Phi^s(t,x(t),u(t))\right) dt, \quad (11)$$

for all $\beta \in [a,\, b]$;

ii)

$$\frac{d}{dt^s} h_x^s(t^s, x(t^s), u(t^s)) = \varphi(t^s,\, h_x^s(t^s, x(t^s), u(t^s)),\, u^s(t^s)), \quad (12)$$

for $t^s = h_t^s(t,\, x(t),\, u(t))$;

then the problem (P) is said to be *invariant* under the transformations

$$(h_t^s(t,\, x,\, u),\, h_x^s(t,\, x,\, u))$$

up to $\Phi^s(t,\, x,\, u)$.

Remark 10. It is clear, by transforming the integral on the left-hand side of (11) to the interval $[a, \beta]$, that condition (11) is equivalent to the condition

$$L\left(t^s, h_x^s\left(t^s, x(t^s), u(t^s)\right), u^s(t^s)\right) \frac{dt^s}{dt} = L\left(t, x(t), u(t)\right) + \frac{d}{dt}\Phi^s\left(t, x(t), u(t)\right),$$

for $t^s = h_t^s\left(t, x(t), u(t)\right)$. In the particular case of Definition 4 we have $h_t^s = t$ and the condition (6) is obtained.

Theorem 2. *If (P) is invariant under the transformations $(h_t^s(t,\, x,\, u),\, h_x^s(t,\, x,\, u))$ up to $\Phi^s(t,\, x,\, u)$, in the sense of Definition 5, then*

$$\psi(t) \cdot \frac{\partial}{\partial s}\, h_x^s(t,\, x(t),\, u(t))\big|_{s=0} + \psi_0 \frac{\partial}{\partial s}\, \Phi^s(t,\, x(t),\, u(t))\big|_{s=0} \\ - H\left(t,\, x(t),\, u(t),\, \psi_0,\, \psi(t)\right) \frac{\partial}{\partial s}\, h_t^s(t,\, x(t),\, u(t))\big|_{s=0}$$

is a first integral.

Remark 11. In the formulation, H is the Hamiltonian (3) associated to the problem (P).

Proof. The proof amounts to reduction to the Theorem 1 by means of time reparameterization. This idea of time reparameterization is a classical one, appearing when the basic problem of the calculus of variations is considered in the parametric form (see e.g. [11, pp. 38–40]), and proved to be useful in many different contexts of the calculus of variations (see e.g. [3, 10]). We consider t as a dependent variable and introduce a one to one Lipschitzian transformation $[a,\, b] \ni t \longmapsto \tau \in [\tau_a,\, \tau_b]$, $\frac{d}{d\tau} t(\tau) > 0$, $\tau_a \leq \tau \leq \tau_b$, such that

$$L\left(t,\, x(t),\, u(t)\right)\, dt = L\left(t(\tau),\, x(t(\tau)),\, u(t(\tau))\right) \frac{dt(\tau)}{d\tau}\, d\tau\,,$$

$$\frac{d}{d\tau} x\left(t(\tau)\right) = \frac{dx\left(t(\tau)\right)}{dt} \frac{dt(\tau)}{d\tau} = \varphi\left(t(\tau),\, x\left(t(\tau)\right),\, u\left(t(\tau)\right)\right) \frac{dt(\tau)}{d\tau}\,.$$

Then the problem (P) takes the form

$$\int_{\tau_a}^{\tau_b} L\left(t(\tau),\, x(t(\tau)),\, u(t(\tau))\right) v(\tau)\, d\tau \longrightarrow \min$$

$$\begin{cases} \dfrac{d}{d\tau} t(\tau) = v(\tau) \\[2ex] \dfrac{d}{d\tau} x(t(\tau)) = \varphi\left(t(\tau),\, x(t(\tau)),\, u(t(\tau))\right) v(\tau) \end{cases} \qquad (P_\tau)$$

$$t(\tau_a) = a,\; t(\tau_b) = b,$$
$$t(\cdot) \in W_{1,\,\infty}\left([\tau_a,\, \tau_b];\, [a,\, b]\right),\; x(t(\cdot)) \in W_{1,\,1}\left([\tau_a,\, \tau_b];\, \mathbb{R}^n\right),$$
$$v(\cdot) \in L_\infty\left([\tau_a,\, \tau_b];\, \mathbb{R}^+\right),\; u(t(\cdot)) \in L_\infty\left([\tau_a,\, \tau_b];\, U\right).$$

Here the state variables are $t(\tau)$ and $x(\tau)$ while the controls are $v(\tau)$ and $u(\tau)$. In optimal control, problem (P_τ) is useful in order to establish the regularity of minimizers for the original problem (P) (cf. [25, Sec. 11.5]).

The conclusion of the Theorem 2 is obtained from the following two propositions.

Proposition 1. *If* (P) *is invariant under*

$$\left(h_t^s(t,\, x,\, u),\, h_x^s(t,\, x,\, u)\right)$$

up to $\Phi^s(t,\, x,\, u)$ *in the sense of Definition 5, then* (P_τ) *is invariant under* $h^s\left(t,\, x,\, u\right) = \left(h_t^s(t,\, x,\, u),\, h_x^s(t,\, x,\, u)\right)$ *up to* $\Phi^s(t,\, x,\, u)$ *in the sense of Definition 4.*

Proof. The Proposition is a consequence of the Definition 5. It suffices to proceed with the substitution of the variable $t^s = h_t^s(t(\tau),\, x(t(\tau)),\, u(t(\tau)))$ in the left-hand side of the equality (11); with the change $t = t(\tau)$ in the right-hand side of (11); and with the change of variable $t = t(\tau)$ in (12). □

Proposition 2. *If* $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ *is a Pontryagin extremal of* (P)*, then*

$$(t(\cdot),\, x(t(\cdot)),\, v(\cdot),\, u(t(\cdot)),\, p_0,\, p_t(\cdot),\, p_z(\cdot))$$

is a Pontryagin extremal of (P_τ) *with*

$$v(\cdot) \in L_\infty\left([\tau_a,\, \tau_b];\, \mathbb{R}^+\right)\,, \quad \int_{\tau_a}^{\tau_b} v(\theta)\, d\theta = b - a\,,$$

$$t(\tau) = a + \int_{\tau_a}^{\tau} v(\theta)\, d\theta\,,$$

$$p_0 = \psi_0\,, p_z(\tau) = \psi(t(\tau))\,,$$

$$p_t(\tau) = -H\left(t(\tau),\, x(t(\tau)),\, u(t(\tau)),\, \psi_0,\, \psi(t(\tau))\right)\,,$$

where H *is defined by* (3).

Proof. Direct computations show that such $(t(\cdot),\, x(t(\cdot)),\, v(\cdot),\, u(t(\cdot)))$ is admissible for (P_τ):

$$\frac{d}{d\tau} t(\tau) = v(\tau),$$

$$t(\tau_a) = a\,,\; t(\tau_b) = b,$$

$$\frac{d}{d\tau} x(t(\tau)) = \varphi\left(t(\tau),\, x(t(\tau)),\, u(t(\tau))\right) v(\tau).$$

The maximality condition

$$\begin{aligned}&\mathcal{H}\left(t(\tau),\, x(t(\tau)),\, v(\tau),\, u(t(\tau)),\, p_0,\, p_t(\tau),\, p_z(\tau)\right)\\ &\qquad = \max_{\substack{v>0\\ u\in U}} \mathcal{H}\left(t(\tau),\, x(t(\tau)),\, v,\, u,\, p_0,\, p_t(\tau),\, p_z(\tau)\right),\end{aligned}$$

where the Hamiltonian $\mathcal{H}$ associated to the problem (P_τ) equals

$$\begin{aligned}&\mathcal{H}\left(t,\, x(t),\, v,\, u(t),\, p_0,\, p_t,\, p_z\right)\\ &\qquad = p_0\, L(t,\, x(t),\, u(t))\, v + p_t\, v + p_z \cdot \varphi\left(t,\, x(t),\, u(t)\right) v, \quad (13)\end{aligned}$$

is trivially satisfied since we are in the singular case: we can write (13) as

$$\mathcal{H}\left(t,\, x(t),\, v,\, u(t),\, p_0,\, p_t,\, p_z\right) = \left(H\left(t,\, x(t),\, u(t),\, p_0,\, p_z\right) + p_t\right) v \quad (14)$$

and therefore the Hamiltonian $\mathcal{H}$ vanishes for

$$p_t = -H\left(t,\, x(t),\, u(t),\, p_0,\, p_z\right).$$

It remains to prove the adjoint system:

$$\begin{cases}\dfrac{dp_t(\tau)}{d\tau}\\[2ex] \dfrac{dp_z(\tau)}{d\tau}\end{cases} = \begin{cases}-\dfrac{\partial\mathcal{H}}{\partial t}\left(t(\tau), x(t(\tau)), v(\tau), u(t(\tau)), p_0, p_t(\tau), p_z(\tau)\right)\\[2ex] -\dfrac{\partial\mathcal{H}}{\partial x}\left(t(\tau), x(t(\tau)), v(\tau), u(t(\tau)), p_0, p_t(\tau), p_z(\tau)\right).\end{cases} \quad (15)$$

Since $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ (see Remark 3) the derivative of $p_t(\tau)$ with respect to τ is given by

$$\frac{dp_t}{d\tau} = -\frac{dH}{d\tau} = -\frac{dH}{dt}\,\frac{dt}{d\tau} = -\frac{\partial H}{\partial t}\,\frac{dt}{d\tau} = -\frac{\partial H}{\partial t}\, v.$$

From the relation (14) $\frac{\partial \mathcal{H}}{\partial t} = \frac{\partial H}{\partial t}\, v$ and the first of the equalities (15) is proved:

$$\frac{dp_t}{d\tau} = -\frac{\partial \mathcal{H}}{\partial t}\,.$$

Similarly, from (14) we know that $\frac{\partial \mathcal{H}}{\partial x} = \frac{\partial H}{\partial x}\, v$ and as far as $p_z(\tau) = \psi\left(t(\tau)\right)$, by (1) $\frac{d}{dt}\psi(t) = -\frac{\partial H}{\partial x}$ and hence

$$\frac{dp_z}{d\tau} = \frac{d\psi(t)}{dt}\,\frac{dt}{d\tau} = -\frac{\partial H}{\partial x}\, v = -\frac{\partial \mathcal{H}}{\partial x}.$$

The proof of the Proposition 2 is complete. □

Let $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ be a Pontryagin extremal of (P). Under the hypothesis of Theorem 2, the Propositions 2 and 1 and the Theorem 1 imply

$$\begin{aligned} p_t(\tau)\,\frac{\partial}{\partial s}\, h_t^s\left(t(\tau),\, x(t(\tau)),\, u(t(\tau))\right)\Big|_{s=0} \\ + p_z(\tau)\cdot\frac{\partial}{\partial s}\, h_x^s\left(t(\tau),\, x(t(\tau)),\, u(t(\tau))\right)\Big|_{s=0} \\ + p_0\,\frac{\partial}{\partial s}\, \Phi^s\left(t(\tau),\, x(t(\tau)),\, u(t(\tau))\right)\Big|_{s=0} \end{aligned}$$

is constant with respect to $\tau \in [\tau_a,\, \tau_b]$, or

$$\begin{aligned} - H\left(t,\, x(t),\, u(t),\, \psi_0,\, \psi(t)\right)\frac{\partial}{\partial s}\, h_t^s(t,\, x(t),\, u(t))\Big|_{s=0} \\ + \psi(t)\cdot\frac{\partial}{\partial s}\, h_x^s(t, x(t), u(t))\Big|_{s=0} + \psi_0\,\frac{\partial}{\partial s}\, \Phi^s(t, x(t), u(t))\Big|_{s=0} \end{aligned}$$

is constant on $[a,\, b]$. □

4 Illustrative Examples

One of the novelties of the results obtained, is to consider symmetries of the system which alter the cost functional up to an exact differential. As far as Φ^s may depend on the parameter s in a non-linear way, Theorems 1 and 2 generalize known results even for the basic problem of the calculus of variations. The following example illustrate this issue. We also observe that the state-transformation h^s in the example, depend not only on the state variable x but also on time t. This is not a possibility in [24, 20, 13, 6].

Example 1. Consider the following basic problem of the calculus of variations ($n = r = 1$, $U = \mathbb{R}$):

$$\int_a^b \left(u(t)\right)^2\, dt \longrightarrow \min\,,$$

$$\dot{x}(t) = u(t)\,.$$

In this case we have $L = u^2$ and $\varphi = u$. It is easily seen that such problem is invariant, in the sense of Definition 4, under the one-parameter transformation $h^s(t,x) = x + st$ $(h^0(t,x) = x)$. Indeed, we observe that for $\Phi^s(t,x) = s^2 t + 2sx$ and $u^s(t) = u(t) + s$ $(u^0(t) = u(t))$ one obtains:

$$\begin{aligned}\int_a^\beta L\left(u^s(t)\right) dt &= \int_a^\beta \left(u(t)+s\right)^2 dt = \int_a^\beta \Big(\left(u(t)\right)^2 + s^2 + 2su(t)\Big)\, dt \\ &= \int_a^\beta \Big(L\left(u(t)\right) + \frac{d}{dt}\Phi^s\left(t,x(t)\right)\Big)\, dt\,; \\ \frac{d}{dt} h^s&\left(t,x(t)\right) = \dot{x}(t) + s = \varphi\left(u^s(t)\right) .\end{aligned}$$

From Theorem 1 it follows that $\psi(t)t + 2\psi_0 x(t)$ is constant in t along the extremals. For the basic problem of the calculus of variations is always the case $\psi(t) = \frac{\partial L}{\partial u}\left(t,x(t),u(t)\right)$ and $\psi_0 = -1$, so the conclusion is that $\dot{x}(t)t - x(t)$ is a first integral of the Euler-Lagrange differential equations.

Next example is an important problem in the context of sub-Riemannian geometry (see [23]). Although the problem is autonomous, transformation of the time-variable is necessary in order to obtain the first integrals. This situation is not considered by the results in [24, 20, 13, 6]. The obtained first integrals seem to be new.

Example 2. Consider the problem $(n = 3,\ r = 2)$

$$\int_a^b \left(u_1(t)\right)^2 + \left(u_2(t)\right)^2 dt \longrightarrow \min$$

$$\begin{cases} \dot{x_1}(t) = u_1(t) \\ \dot{x_2}(t) = \dfrac{u_2(t)}{1+\alpha x_1(t)}\,, \qquad \alpha \in \mathbb{R}\,, \\ \dot{x_3}(t) = \left(x_2(t)\right)^2 u_1(t)\,. \end{cases}$$

Here the Hamiltonian equals

$$H(x_1,x_2,u_1,u_2,\psi_0,\psi_1,\psi_2,\psi_3) = \psi_0\left(u_1^2+u_2^2\right) + \psi_1 u_1 + \frac{\psi_2 u_2}{1+\alpha x_1} + \psi_3 x_2^2 u_1\,,$$

and trivial first integrals are ψ_3 and H (H does not depend on x_3 and t, and the conclusion follows directly from the adjoint system (1) and the equality $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ of Remark 3). In the case $\alpha = 0$, that is in the Martinet flat case of sub-Riemannian geometry (see [1]), ψ_1 is also a trivial first integral for the problem. Using the Noether theorem with transformation of the time-variable, new non-trivial first integrals can be obtained. We shall consider two cases: $\alpha = 0$ (flat case) and $\alpha \neq 0$ (non-flat case).

For $\alpha = 0$, the assumptions of Theorem 2 are verifiable with $t^s = h_t^s(t) = \mathrm{e}^{2s}t$, $h_{x_1}^s\left(x_1(t^s)\right) = \mathrm{e}^s x_1(t)$, $h_{x_2}^s\left(x_2(t^s)\right) = \mathrm{e}^s x_2(t)$, $h_{x_3}^s\left(x_3(t^s)\right) = \mathrm{e}^{3s} x_3(t)$, and

$\Phi^s \equiv 0$. Indeed, the problem is invariant under these transformations, in the sense of Definition 5, as long as for $u_1^s(t^s) = \mathrm{e}^{-s}u_1(t)$ and $u_2^s(t^s) = \mathrm{e}^{-s}u_2(t)$ one obtains:

$$\begin{aligned}\int_{h_t^s(a)}^{h_t^s(\beta)} \left((u_1^s(t^s))^2 + (u_2^s(t^s))^2\right) dt^s &= \int_a^\beta \left((u_1(t))^2 + (u_2(t))^2\right) \mathrm{e}^{-2s}\frac{dh_t^s(t)}{dt}dt \\ &= \int_a^\beta \left((u_1(t))^2 + (u_2(t))^2\right) dt\,;\end{aligned}$$

$$\frac{d}{dt^s}\left(h_{x_1}^s(x_1(t^s))\right) = \frac{\frac{d}{dt}(\mathrm{e}^s x_1(t))}{\frac{dh_t^s(t)}{dt}} = \frac{\mathrm{e}^s\dot{x}_1(t)}{\mathrm{e}^{2s}} = \mathrm{e}^{-s}u_1(t) = u_1^s(t^s)\,;$$

$$\frac{d}{dt^s}\left(h_{x_2}^s(x_2(t^s))\right) = \frac{\frac{d}{dt}(\mathrm{e}^s x_2(t))}{\frac{dh_t^s(t)}{dt}} = \mathrm{e}^{-s}\dot{x}_2(t) = \mathrm{e}^{-s}u_2(t) = u_2^s(t^s)\,;$$

$$\begin{aligned}\frac{d}{dt^s}\left(h_{x_3}^s(x_3(t^s))\right) &= \frac{\frac{d}{dt}\left(\mathrm{e}^{3s}x_3(t)\right)}{\frac{dh_t^s(t)}{dt}} = \mathrm{e}^s\dot{x}_3(t) = \mathrm{e}^s(x_2(t))^2 u_1(t) \\ &= \left(h_{x_2}^s(x_2(t^s))\right)^2 u_1^s(t^s)\,.\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial}{\partial s}\left(h_{x_1}^s(x_1(t^s))\right)\Big|_{s=0} &= \frac{\partial}{\partial s}\left(\mathrm{e}^s x_1(t)\right)\Big|_{s=0} = x_1(t)\,,\\ \frac{\partial}{\partial s}\left(h_{x_2}^s(x_2(t^s))\right)\Big|_{s=0} &= \frac{\partial}{\partial s}\left(\mathrm{e}^s x_2(t)\right)\Big|_{s=0} = x_2(t)\,,\\ \frac{\partial}{\partial s}\left(h_{x_3}^s(x_3(t^s))\right)\Big|_{s=0} &= \frac{\partial}{\partial s}\left(\mathrm{e}^{3s} x_3(t)\right)\Big|_{s=0} = 3x_3(t)\,,\\ \frac{\partial}{\partial s}\left(t^s\right)\Big|_{s=0} &= \frac{\partial}{\partial s}\left(\mathrm{e}^{2s} t\right)\Big|_{s=0} = 2t\,,\end{aligned}$$

Theorem 2 guarantees, then, that

$$\psi_1 x_1(t) + \psi_2(t)x_2(t) + 3\psi_3 x_3(t) - 2Ht \equiv \text{constant}\,,$$

$t \in [a, b]$, along every Pontryagin extremal

$$(x_1(\cdot), x_2(\cdot), x_3(\cdot), u_1(\cdot), u_2(\cdot), \psi_0, \psi_1, \psi_2(\cdot), \psi_3)$$

of the flat problem.

For $\alpha \neq 0$, the problem is invariant, in the sense of Definition 5, under $t^s = h_t^s(t) = \mathrm{e}^{2s}t$, $h_{x_1}^s(x_1(t^s)) = \frac{\mathrm{e}^s(\alpha x_1(t)+1)-1}{\alpha}$, $h_{x_2}^s(x_2(t^s)) = x_2(t)$, and $h_{x_3}^s(x_3(t^s)) = \mathrm{e}^s x_3(t)$. (We leave as an exercise the verification that this is the case, again with $\Phi^s \equiv 0$, $u_1^s(t^s) = \mathrm{e}^{-s}u_1(t)$ and $u_2^s(t^s) = \mathrm{e}^{-s}u_2(t)$.) Follows from Theorem 2 that $\psi_1(t)\left(x_1(t) + \frac{1}{\alpha}\right) + \psi_3 x_3(t) - 2Ht$ is a first integral.

5 Conclusions

We provide a general approach to Noether's theorem for problems of optimal control. Even for the basic problem of the calculus of variations, one is able to cover situations which are not treated by the known results.

During the reviewing process, a Noether theorem in the optimal control setting appeared in [6]. In contrast to [6], we deal with non-autonomous problems, we consider parameter-transformations depending on time, state and control variables, and treat both normal and abnormal extremals.

Previous results in [24], [20], [13] and [6] (like Theorem 1) only consider state transformations. The first integrals we have obtained in Example 2 can neither be deduced from state transformations only nor can they be obtained by introducing an extra state variable t ($t' = 1$).

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