# REGULAR SYNTHESIS AND SUFFICIENCY CONDITIONS FOR OPTIMALITY

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**Abstract.** The aim of this paper is to propose a new definition of "regular synthesis" more general than those suggested by other authors (e.g. Boltianskii [2], Brunovsky [5], [6]) and to give a detailed proof of the corresponding sufficiency theorem, a slightly weaker version of which had been stated in an earlier article (see [27]), with only a rough outline of the proof. We illustrate the strength of our result showing that the synthesis for the famous Fuller problem [12] satisfies our hypotheses.

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**COMMENT NO. 1**: This abstract is too short and uninformative. Look at the new abstract that I wrote.

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**Keywords.** Optimal control, regular synthesis, sufficient conditions.

Ref. SISSA 68/96/M

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**COMMENT NO. 2**: "Ospitality" is wrong!

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## §1. Introduction

The aim of this paper is to propose a new definition of "regular synthesis" more general than those suggested by other authors (e.g. Boltianskii [2], Brunovsky [5], [6]) and to give a detailed

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**COMMENT NO. 3**: In the SIAM J. Control paper, the name of VGB is spelled “Boltyanskii,” so I am using the same spelling here. Also, “Brunovský” is supposed to have an accent on the “y”.

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proof of the corresponding sufficiency theorem, a slightly weaker version of which had been stated in an earlier article (see [27]), with only a rough outline of the proof. We illustrate the strength of our result showing that the synthesis for the famous Fuller problem [12] satisfies our hypotheses, but fails to obey the requirements of other sufficiency theorems such as those of [2] and [6], because of the technical problem arising from the occurrence of an infinite number of switchings. (Not surprisingly, the optimality of Fuller's synthesis was originally proved without resorting to “regular synthesis” arguments, and using instead special symmetry properties of the problem, cf. e.g. Zelikin-Borisov [28].)

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**COMMENT NO. 4**: One of the most terrible structural flaws of this paper is that we start by saying that in this paper we propose a new definition of "regular synthesis" and then we never give the definition. Anybody looking at the introduction of the paper is going to start turning the pages to look for this “new definition” that we say we are proposing. And he won't find it, because IT'S NOT THERE. **Exactly where do we give this “definition” that we say we are going to give? I didn't see it anywhere!!!!** In my new version, I have put it in. Now there are definitions clearly labeled as such, and the reader can tell what it is that we are defining. Read the paper carefully and see how much difference this change makes!!! After reading your version, NOBODY (not even ourselves) would have been able to answer the question: “what is a regular synthesis according to Piccoli and Sussmann”? After reading my new version, I think the question has a clear answer. Am I right?

Also, notice that the conceptual structure now is much clearer. We define “presynthesis,” and then we define what means for a presynthesis to be “regular.” In order to make this conceptual structure even clearer, I have separated the boundary condition from the “regularity” part. Properly speaking, the condition that in your version is called 2.1.b is not a “regularity” condition at all, so I didn't incorporate it into the regularity notion.

Incidentally, since in the new version I am explicitly labeling the definition of a regular presyntheis as a definition, numbered and everything, I thought I might as well do the same thing with other definitions. Take a look at new product and see for yourself how it's much clearer now.

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The purpose of the theory of regular synthesis is to turn into a rigorous, precise theorem the vague assertion that "if a collection $\Gamma$ of extremals covers the whole space in a sufficiently smooth way, then all the members of $\Gamma$ are optimal". Naturally, a good definition will be one that is stringent enough to lead to a correct proof of the sufficiency assertion, and broad enough to cover as large as possible a class of optimal control problem. In particular, it would be desirable for the

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**COMMENT NO. 5**: It should be "problems," not "problem."

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theory to apply to well known examples where discontinuous optimal controls occur, and even controls with infinitely many switchings, as in Fuller's problem.

There are essentially two different ways of defining a synthesis. Following the approach of [2], [5], [7], [10], [16], a synthesis is a collection of trajectories generated by a piecewise regular feedback control. The set of interest (that is the set of initial points of optimal trajectories) is partitioned into a locally finite collection of submanifolds of various dimensions. More precisely, the set is stratified and for each stratum a regular feedback control is specified. An alternative strategy was followed in [14], [27], where a "synthesis" is just a collection of trajectories not necessarily arising from a feedback control. With this definition the feedback control is not sufficient for describing a synthesis, indeed a discontinuous feedback control may generate too many trajectories, some of which could fail to be optimal, as shown in Example 1 of Section 5.

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**COMMENT NO. 6**: I don't understand the grammatical structure of the above sentence, especially the role of "indeed." You seem to to think that "indeed" is a conjunction, but it isn't. It's an **adverb**.

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This same example illustrates another way in which our definition of synthesis is more general than that of [2] and [6], because we allow the cost function associated to the synthesis to be discontinuous, whereas [2] and [6] require it to be continuous

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**COMMENT NO. 7**: The definitions of [2] and [6] are different, so we should have written "those of [2] and [6]" rather than "that of [2] and [6]."

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Another point that should be further investigated is the relation between the geometrical approaches of [5], [6], [16], [17], [18], [20], [21], [25], [27], and the theory of viscosity solutions [1], [11]. We point out that the existing results of this theory for deterministic optimal control do not appear to cover cases such as Fuller's problem, cfr. [1], where the main result is for Lagrangian that are bounded away from zero.

## §2. A sufficiency theorem

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**COMMENT NO. 8**: I changed the title section. The new one is more informative, I think.

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We start with some definitions that will enable us to state and prove a mild generalization of the theorem whose proof was outlined in [27]. The assumptions made here will be minimal, so as to achieve maximum generality. In subsequent sections we will impose more restrictive conditions and apply our result to other definitions of synthesis given in the literature only for more regular control systems.

We consider a control system:

$$\dot{x} = f(x,u)\,, \qquad x \in \Omega\,, \qquad u \in U\,, \tag{2.1}$$

and the family $\{\mathcal{P}(f,L,\mathcal{T},\tau,x)\}_{x\in\Omega}$ of minimization problems, where $\mathcal{P}(f,L,\mathcal{T},\tau,x)$ is the problem:

$$\text{minimize} \quad \int_a^b L(\gamma(t),\eta(t))dt + \tau(\gamma(b)) \qquad \text{subject to (2.1)}\,,\ \gamma(a) = x\,,\ \text{and}\ \gamma(b) \in \mathcal{T}\,. \tag{2.2}$$

(A more precise formulation of the problems $\mathcal{P}(f,L,\mathcal{T},\tau,x)$, carefully specifying the space of of curves to be considered, will be given below. Since both $f$ and $L$ have no explicit dependence on $t$, we can always take $b = 0$.)

We assume that

*(A1)* $\Omega$ *is an open subset of* $\mathbb{R}^n$*,*
*(A2)* $U$ *is a set,*
*(A3)* $f : \Omega \times U \to \mathbb{R}^n$ *and* $L : \Omega \times U \to \mathbb{R}$ *are maps,*
*(A4)* $f(\cdot,u)$ *and* $L(\cdot,u)$ *are maps of class* $C^1$ *for every fixed* $u \in U$*,*
*(A5)* $\mathcal{T}$ *is a nonempty subset of* $\Omega$*,*
*(A6)* $\tau : \Omega \to \mathbb{R}$ *is a function of class* $C^1$*.*

We use $T_x^c S$ to denote the *contingent tangent* cone to a set $S$ at a point $x \in S$, defined as follows: a vector $v$ is in $T_x^c S$ iff there exists a sequence $\{x_j\}$ of points of $S$ and a sequence $\{h_j\}$ of strictly positive numbers such that $h_j \to 0$ and $x_j = x + h_j v + o(h_j)$ as $j \to \infty$. We write

$$\tilde{T}_x S = T_x^c S \cap (-T_x^c S) = \{v : v \in T_x^c S \quad \text{and} \quad -v \in T_x^c S\}\,. \tag{2.3}$$

As in Sussmann [27], we use $\tilde{f}$ to denote the $\mathbb{R}^{n+1}$ valued map $(x,u) \to (f(x,u), L(x,u))$.

If $\mu : A \to B$ is a map, we use $\mathrm{Dom}(\mu)$ to indicate the domain of $\mu$, i.e. the set $A$.

A *control* is a $U$-valued map $\eta$ such that $\mathrm{Dom}(\eta)$ is a compact subinterval $[a,b]$ of $\mathbb{R}$. A *trajectory* $\gamma$ for a control $\eta$ is an absolutely continuous map $\gamma : \mathrm{Dom}(\eta) \to \Omega$ that satisfies $\dot{\gamma}(t) = f(\gamma(t),\eta(t))$ for almost every $t \in \mathrm{Dom}(\eta)$. If $\mathrm{Dom}(\eta) = [a,b]$ (so that $\mathrm{Dom}(\gamma) = [a,b]$ as well), and $x = \gamma(a)$, $y = \gamma(b)$, we say that $\gamma$ (or the pair $(\gamma,\eta)$) *goes from $x$ to $y$*, and that $\eta$ *steers $x$ to $y$*, and we write $\gamma^- = x$, $\gamma^+ = y$.

A control $\eta : [a,b] \to U$ is *admissible* if the map $\Omega\times[a,b] \ni (x,t) \to \tilde{f}_\eta(x,t) = \tilde{f}(x,\eta(t)) \in \mathbb{R}^{n+1}$ satisfies the following $C^1$ Carathéodory conditions:

(C.i) *$t \mapsto \tilde{f}_\eta(x,t)$ is measurable for each $x \in \Omega$,*

(C.ii) *for every compact $K \subseteq \Omega$ there exists an integrable function $\varphi_K$ such that for every $x \in K$ and $t \in [a,b]$:*

$$\|\tilde{f}_\eta(x,t)\| + \|D_x \tilde{f}_\eta(x,t)\| \le \varphi_K(t)\,. \tag{2.4}$$

If $\eta$ is an admissible control and $\gamma$ a trajectory corresponding to $\eta$, then we say that $(\gamma,\eta)$ is an *admissible pair* for $\tilde{f}$. We use $\mathrm{Adm}(\tilde{f})$ to denote the set of all admissible pairs for $\tilde{f}$, and write $\mathrm{Adm}^0(\tilde{f})$, $\mathrm{Adm}^0(\tilde{f},\mathcal{T})$, $\mathrm{Adm}^0_x(\tilde{f},\mathcal{T})$, to denote, respectively, the set of all $(\gamma,\eta) \in \mathrm{Adm}(\tilde{f})$ such that $\mathrm{Dom}(\eta)$ is an interval of the form $[T,0]$ with $T \le 0$, the set of those $(\gamma,\eta) \in \mathrm{Adm}^0(\tilde{f})$ such that $\gamma(0) \in \mathcal{T}$, and the set of those $(\gamma,\eta) \in \mathrm{Adm}^0(\tilde{f},\mathcal{T})$ such that $\gamma^- = x$. For an admissible pair $(\gamma,\eta) \in \mathrm{Adm}^0(\tilde{f},\mathcal{T})$, we use $J(\gamma,\eta)$ to denote the *cost* of $(\gamma,\eta)$, so that

$$J(\gamma,\eta) = \int_T^0 L(\gamma(t),\eta(t))\,dt + \tau(\gamma^+)\,.$$

If $\Gamma \subseteq \mathrm{Adm}^0(\tilde{f}, \mathcal{T})$ is an arbitrary set of admissible pairs, we define a function $V_\Gamma : \Omega \mapsto \mathbb{R} \cup \{+\infty\}$, called the *value function of* $\Gamma$, by letting

$$V_\Gamma(x) = \inf\{J(\gamma, \eta) : (\gamma, \eta) \in \Gamma,\ \gamma^- = x\} \tag{2.5}$$

for $x \in \Omega$. (In particular, $V_\Gamma(x) = +\infty$ iff there is no $(\gamma, \eta) \in \Gamma$ starting at $x$.) If $\Gamma = \mathrm{Adm}^0(\tilde{f}, \mathcal{T})$, then the function $V_\Gamma$ is the *value function* of our problem, and in that case we will write $V_{\tilde{f},\mathcal{T}}$ rather than $V_{\mathrm{Adm}^0(\tilde{f},\mathcal{T})}$.

We can now formulate $\mathcal{P}(f, L, \mathcal{T}, \tau, x)$ precisely: $\mathcal{P}(f, L, \mathcal{T}, \tau, x)$ is the problem of finding a $(\gamma, \eta) \in \mathrm{Adm}^0_x(\tilde{f}, \mathcal{T})$ such that $J(\gamma, \eta) = V_{\tilde{f},\mathcal{T}}(x)$. Any such pair is said to be a *solution* of $\mathcal{P}(f, L, \mathcal{T}, \tau, x)$. A pair $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f}, \mathcal{T})$ is *optimal* if it is a solution of $\mathcal{P}(f, L, \mathcal{T}, \tau, \gamma^-)$.

Given $\lambda \in \mathbb{R}_n$ (the space of row $n$ vectors), $\lambda^0 \in \mathbb{R}$, $x \in \Omega$ and $u \in U$, we define:

$$\mathcal{H}(\lambda, \lambda^0, x, u) = \langle \lambda, f(x, u) \rangle + \lambda^0 L(x, u)\,, \tag{2.6}$$

$$H(\lambda, \lambda^0, x) = \inf\{\mathcal{H}(\lambda, \lambda^0, x, u) : u \in U\}\,. \tag{2.7}$$

The functions $\mathcal{H} : \mathbb{R}_n \times \mathbb{R} \times \Omega \times U \to \mathbb{R}$ and $H : \mathbb{R}_n \times \mathbb{R} \times \Omega \to \mathbb{R} \cup \{-\infty\}$ are known as the *Hamiltonian* and the *minimized Hamiltonian* of $\tilde{f}$.

We say that the pair $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f}, \mathcal{T})$ is *extremal* if there exist an absolutely continuous map $\lambda : \mathrm{Dom}(\gamma) \mapsto \mathbb{R}_n$ and a constant $\lambda^0 \geq 0$, that satisfy:

(EX1) $(\lambda^0, \lambda(t)) \neq (0, 0)$ *for some (and hence every)* $t$.

(EX2) $\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x}(\lambda(t), \lambda^0, \gamma(t), \eta(t))$ *for* a.e. $t \in \mathrm{Dom}(\eta)$.

(EX3) $H(\gamma(t), \lambda(t), \lambda^0) = \mathcal{H}(\gamma(t), \lambda(t), \lambda^0, \eta(t)) = 0$ *for* a.e. $t \in \mathrm{Dom}(\eta)$,

(EX4) $\langle \lambda(0), v \rangle = \lambda^0 \langle \nabla \tau(\gamma(0)), v \rangle$ *for all* $v \in \tilde{T}_{\gamma(0)}\mathcal{T}$.

Conditions (EX1), (EX2), (EX3), (EX4) are, respectively, the *nontriviality condition*, the *adjoint equation*, the *Hamiltonian minimization condition*, and the *transversality condition*. A pair $(\lambda, \lambda^0)$ such that (EX2) holds is called an *adjoint vector for* $(\gamma, \eta)$. An adjoint vector for which (EX3) holds is a *minimizing adjoint vector for* $(\gamma, \eta)$.

The *maximum principle* says that, under some special conditions on the target set $\mathcal{T}$, *every optimal pair* $(\gamma, \eta)$ *is extremal.* (This is true, for example, if $\mathcal{T}$ is a submanifold of $\Omega$ of class $C^1$, or a closed convex subset of $\Omega$, or a set which is locally equivalent to a convex set via a $C^1$ diffeomorphism. More generally, it is true if the cone $\tilde{T}_{\gamma(0)}\mathcal{T}$ is convex —in which case it is of course a subspace, since $v \in \tilde{T}_{\gamma(0)}\mathcal{T} \Rightarrow -v \in \tilde{T}_{\gamma(0)}\mathcal{T}$— and is an approximating cone to $\mathcal{T}$ at $\gamma(0)$. Recall that a closed convex cone $C$ is an *approximating cone* to a set $S$ at a point $s \in S$ if there exist a neighborhood $U$ of 0, and a continuous map $F : U \cap C \to S$, such that $F(x) = s + x + o(||x||)$ as $x \to 0$ via values in $C$.)

A *presynthesis* for our family of problems is a subset $\Gamma$ of $\mathrm{Adm}^0(\tilde{f}, \mathcal{T})$ such that,

(PS) *Whenever* $(\gamma_1, \eta_1) \in \Gamma$, $(\gamma_2, \eta_2) \in \Gamma$, *and* $\gamma_1^- = \gamma_2^-$, *it follows that* $(\gamma_1, \eta_1) = (\gamma_2, \eta_2)$.

The set $\mathrm{Dom}(\Gamma) \stackrel{\mathrm{def}}{=} \{\gamma^- : (\gamma, \eta) \in \Gamma\}$ is called the *domain* of $\Gamma$. We say that $\Gamma$ is a *presynthesis on* $S$ if $\Gamma$ is a presynthesis and $S = \mathrm{Dom}(\Gamma)$. Clearly, giving a presynthesis on $S$ amounts to choosing, for each $x \in S$, a $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f}, \mathcal{T})$ such that $\gamma^- = x$.

If $\Gamma$ is a presynthesis such that $\mathrm{Dom}(\Gamma)$ consists of all points that can be steered to the origin in finite time by an admissible pair, then we say that $\Gamma$ is *total*.

Given a presynthesis $\Gamma$ and an $x \in \mathrm{Dom}(\Gamma)$ we will always use $(\gamma_x, \eta_x)$ to denote the unique pair $(\gamma, \eta) \in \Gamma$ such that $\gamma^- = x$, and will write $T_x = \min \mathrm{Dom}(\gamma_x)$, so $T_x \leq 0$ and $\mathrm{Dom}(\gamma_x) = \mathrm{Dom}(\eta_x) = [T_x, 0]$.

A presynthesis is *memoryless* if whenever $x \in S$ and $t \in \mathrm{Dom}(\eta_x)$ it follows that $y = \gamma_x(t)$ belongs to $S$ and $\eta_y$ is the restriction of $\eta_x$ to the interval $[t, 0]$. A *synthesis* is a memoryless presynthesis. If each pair of a presynthesis $\Gamma$ is optimal (resp. extremal) then we say that $\Gamma$ is *optimal* (resp. *extremal*). In particular, a presynthesis $\Gamma$ is a total optimal presynthesis if and only if $V_\Gamma \equiv V_{\tilde{f}}$.

Our goal is to prove that, under suitable regularity assumptions on $\Gamma$ and $V_\Gamma$, a total extremal presynthesis is optimal. We first describe these regularity assumptions in detail.

We say that a function $V : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *weakly upper semicontinuous* (w.u.s.c) at a point $x \in \Omega$ if

$$\liminf_{y \to x} \limsup_{z \to y} V(z) \le V(x) . \tag{2.8}$$

Equivalently, $V$ is w.u.s.c. at $x$ if there exist sequences $x_j$, $\varepsilon_j$ and $\delta_j$ such that $x_j \in \Omega$, $x_j \to x$, $\varepsilon_j > 0$, $\delta_j > 0$, $\delta_j \to 0$, having the property that $V(y) \le V(x) + \delta_j$ whenever $\|y - x_j\| < \varepsilon_j$.

Given a locally Lipschitz vector field $X$ on $\Omega$, we say that $V$ has the *NDJ ("no downward jumps") property along* $X$ if the following holds:

(NDJ) *For every $\gamma$ integral curve of $X$, if* $\mathrm{Dom}(\gamma) = [a, b]$ *then for every* $t \in ]a, b]$,

$$\limsup_{h \downarrow 0} V(\gamma(t)) - V(\gamma(t-h)) \ge 0$$

.

Finally we say that $V$ satisfies the *weak continuity conditions* for the control problem (2.1), (2.2) if $V$ is lower semicontinuous and weakly upper semicontinuous, and has the NDJ property along the vector field $x \to f(x, u)$ for every $u \in U$.

We now define the "weak differentiability conditions" for $\Gamma$. In this definition it is understood that $\tilde{f}(y, \eta_x(t)) = 0$ for every $t \notin \mathrm{Dom}(\eta_x)$. A presynthesis $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ is $(f, L)$–*differentiable* at a point $\bar{x} \in \Omega$ if the following condition is satisfied:

(DC) *there exist a neighborhood $N$ of $\bar{x}$ (in $\Omega$) such that $N \subseteq \mathrm{Dom}(\Gamma)$, and an interval $[T, 0]$ such that $\mathrm{Dom}(\eta_x) \subseteq [T, 0]$ for all $x \in N$, and an $\bar{\varepsilon} > 0$, with the property that*

(DC1) *there exist, for $\varepsilon \in ]0, \bar{\varepsilon}]$, integrable functions $\psi_\varepsilon : [T, 0] \to \mathbb{R}$ such that*

$$\lim_{\varepsilon \to 0} \int_T^0 \psi_\varepsilon(t)\, dt = 0 , \tag{2.9}$$

*and the inequality*

$$\left\| \tilde{f}(y, \eta_x(t)) - \tilde{f}(\gamma_x(t), \eta_x(t)) - \frac{\partial \tilde{f}}{\partial x}\Big(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)\Big) \cdot (y - \gamma_{\bar{x}}(t)) \right\| \le \psi_\varepsilon(t).(||y - \gamma_{\bar{x}}(t)|| + ||x - \bar{x}||) \tag{2.10}$$

*holds for every $y \in \Omega$, $x \in N$, $t \in [T, 0]$ such that $\|y - \gamma_{\bar{x}}(t)\| \le \varepsilon$ and $||x - \bar{x}|| \le \varepsilon$.*

(DC2) *the map $v \to \rho_v$, where $\rho_v$ is the integrable $\mathbb{R}^{n+1}$ valued function on $[T, 0]$ given by*

$$\rho_v(t) = \tilde{f}(\gamma_{\bar{x}}(t), \eta_{\bar{x}+v}(t)) - \tilde{f}(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t))$$

*is weak*-differentiable at $v = 0$, regarded as a map into the space of $\mathbb{R}^{n+1}$–valued Borel measures. That is: for every continuous function $\alpha : [T, 0] \to \mathbb{R}$, the map $\mathbb{R}^n \ni v \to \int_T^0 \rho_v(t)\alpha(t) \in \mathbb{R}^{n+1} dt$ is differentiable at $v = 0$.*

A subset $A$ of $\Omega$ is *thin* if there exist $A_1$, $A_2$, such that $A = A_1 \cup A_2$, $A_1$ is a finite or countable union of connected $\mathcal{C}^1$ submanifolds of positive codimension and $\mathcal{H}^{n-1}(A_2) = 0$, where $\mathcal{H}^k$ denotes $k$-dimensional Hausdorff measure.

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**COMMENT NO. 9**: I completely changed the statement of the main theorem. As you will see, now I no longer asusme that the synthesis is total. One of many reasons why I wanted to do this is the following: in our previous version, we lied, because we said that the theorem here is "slightly more general" than that of my presynthesis paper, but in fact it wasn't. It was exactly the same. Now, with the new conditions on the domain, it is true that it is "slightly more general."

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We can now state the main theorem (see [27]):

**Theorem 2.1.** *let $\Omega$, $U$, $f$, $L$ be such that Assumptions (A1) – (A6) hold. Let $\Gamma$ be a total extremal presynthesis. Assume that*

- *(2.1.a) the associated value function $V_\Gamma$ satisfies the weak continuity conditions,*
- *(2.1.b) $V_\Gamma(x) \leq \tau(x)$ for every $x \in \mathcal{T}$,*
- *(2.1.c) $\Gamma$ is $(f, L)$–differentiable at all points in the complement of a thin set $A$.*

*Then $\Gamma$ is optimal.*

*PROOF.* We let $S = \mathrm{Dom}(\Gamma)$, so $S$ is exactly the set of all $x \in \Omega$ that can be steered to the target $\mathcal{T}$ in finite time by an admissible control.

We first establish the relation between the differential of the value function $V_\Gamma$ and the adjoint covector fields along the extremal trajectories of the presynthesis. This result is important in its own right, so we state it as a separate theorem:

**Theorem 2.2.** *Let $\Omega$, $U$, $f$, $L$ be such that Assumptions (A1) – (A6) hold. Let $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ be a presynthesis. Let $\bar{x} \in S$ be such that $\Gamma$ is $(f, L)$–differentiable at $\bar{x}$. Then*

- *(2.2.a) the function $V_\Gamma$ is differentiable at $\bar{x}$;*
- *(2.2.b) if the pair $(\gamma_{\bar{x}}, \eta_{\bar{x}})$ is extremal, then every minimizing adjoint vector $(\lambda, \lambda^0)$ for $(\gamma_{\bar{x}}, \eta_{\bar{x}})$ satisfies the identity $\lambda^0 D_y V_\Gamma(\bar{x}) = \lambda(T_{\bar{x}})$. In particular, $\lambda^0 \neq 0$, and $(\lambda, \lambda^0)$ is unique up to multiplication by a positive constant.*

*PROOF OF THEOREM 2.2.* We use Theorem A.5 in Appendix A with $P = \mathbb{R}^n$, letting $f_x(y,t) = f(y, \eta_x(t))$, $L_x(y,t) = L(y, \eta_x(t))$, for $y \in \Omega$, $x \in N$, $t \in [T_x, 0]$, and defining $f_x(y,t) = 0$, $L_x(y,t) = 0$, for $T \leq t < T_x$. Let $\hat{V}_\Gamma(x) = \int_{T_x}^0 L(\gamma_x(t), \eta_x(t))\,dt$, so $\hat{V}_\Gamma$ is the "Lagrangian part" of our cost functional.

The map $p \mapsto \bar{x}_p$ of Theorem A.5 is the identity map, which is certainly differentiable at $\bar{x}$. So Theorem A.5 tells us that the "endpoint map"$x \mapsto \mathcal{E}(x) \stackrel{\mathrm{def}}{=} \gamma_x(0)$ and the function $\hat{V}_\Gamma$ are differentiable at $\bar{x}$, and the identity

$$\lambda(0).D_y\mathcal{E}(\bar{x}).v + \lambda^0 D_y \hat{V}_\Gamma(\bar{x}).v - \lambda(T_{\bar{x}}) \cdot v$$

$$= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_T^0 \Big( H(\lambda(t), \lambda^0, \gamma_{\bar{x}}(t), \eta_{\bar{x}+\varepsilon v}(t)) - H(\lambda(t), \lambda^0, \gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \Big)\, dt. \quad (2.11)$$

holds for every vector $v$ if $(\lambda, \lambda^0)$ is an adjoint vector for $(\gamma_{\bar{x}}, \eta_{\bar{x}})$. (Here the function $\lambda$ is extended to $[T, 0]$ by letting it equal zero on $[T, T_{\bar{x}}[$. The Hamiltonian $H(\lambda(t), \lambda^0, y, \eta_x(t))$ is also defined for all $t \in [T, 0]$, by letting $H((\lambda(t), \lambda^0, y, \eta_x(t)) = \lambda(t).f(y, \eta_x(t)) + \lambda^0 L(y, \eta_x(t))$, and recalling that this expression vanishes for $t \in [T, T_x[$, because $f(y, \eta_x(t))$ and $L(y, \eta_x(t))$ have been defined to be zero for $t \in [T, T_x[$.)

Now, if $(\lambda, \lambda^0)$ is minimizing, then the right-hand side of the inequality (2.11) is nonnegative, because $H(\lambda(t), \lambda^0, \gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) = 0$ and $H(\lambda(t), \lambda^0, \gamma_{\bar{x}}(t), \eta_{\bar{x}+\varepsilon v}(t)) \geq 0$ for almost all $t \in [T, 0]$. So

$$\lambda(0).D_y\mathcal{E}(\bar{x}).v + \lambda^0 D_y\hat{V}_\Gamma(\bar{x}).v - \lambda(T_{\bar{x}}) \cdot v \geq 0 \qquad \text{for all } v \in \mathbb{R}^n\,. \tag{2.12}$$

Given $v$, the curve $r \mapsto \gamma_{\bar{x}+rv}(0)$ is defined for $r$ in a neighborhood of 0 in $\mathbb{R}$, and is differentiable at 0 with derivative $D_y\mathcal{E}(\bar{x}).v$. On the other hand, this curve is contained in $\mathcal{T}$, so the vector $w = D_y\mathcal{E}(\bar{x}).v$ belongs to $\tilde{T}_{\gamma_{\bar{x}}(0)}\mathcal{T}$. The transversality condition (EX4) then says that $\langle\lambda(0), w\rangle = \lambda^0\langle\nabla\tau(\gamma(0)), w\rangle$. Therefore

$$\lambda^0\Big(\nabla\tau(\gamma(0)).D_y\mathcal{E}(\bar{x}).v + D_y\hat{V}_\Gamma(\bar{x}).v\Big) - \lambda(T_{\bar{x}}) \cdot v \geq 0 \qquad \text{for all } v \in \mathbb{R}^n\,. \tag{2.13}$$

Since the left-hand side of this inequality is linear in $v$, it has to be equal to zero. So

$$\lambda^0\Big(\nabla\tau(\gamma(0)).D_y\mathcal{E}(\bar{x}).v + D_y\hat{V}_\Gamma(\bar{x}).v\Big) = \lambda(T_{\bar{x}}) \cdot v \qquad \text{for all } v \in \mathbb{R}^n\,. \tag{2.14}$$

Clearly, the left-hand side of this equation is $\lambda^0 D_y V_\Gamma(\bar{x}).v$. Since the equality holds for all $v$, we have $\lambda^0 D_y V_\Gamma(\bar{x}) = \lambda(T_{\bar{x}})$.

In particular, $\lambda^0$ cannot vanish, because $\lambda^0 = 0$ would imply $\lambda(T_{\bar{x}}) = 0$, contradicting (EX1). So we may, after multiplication by a positive constant, assume that $\lambda^0 = 1$, and then

$$D_y V_\Gamma(\bar{x}) = \lambda(T_{\bar{x}}). \tag{2.15}$$

This proves, in particular, the uniqueness of $(\lambda(\cdot), \lambda^0)$ up to multiplication by a positive constant, since the Cauchy problem for the adjoint equation has unique solutions. ∎

*CONTINUATION OF THE PROOF OF THEOREM 2.1.* The minimization condition (EX4) of the Pontryagin Maximum Principle implies that, for a given $x \in S \setminus A$, $u \in U$, the inequality

$$\langle\lambda(t), f(\gamma_x(t), u)\rangle + L(\gamma_x(t), u) \geq 0$$

holds for almost all $t \in [T_x, 0]$, where $(\lambda, \lambda^0)$ is a minimizing adjoint vector for $(\gamma_x, \eta_x)$, normalized so that $\lambda^0 = 1$. Since $t \mapsto \langle\lambda(t), f(\gamma_x(t), u)\rangle + L(\gamma_x(t), u)$ is continuous on $[T_x, 0]$, the inequality is valid at time $T_x$. In view of Theorem 2.2, $\lambda(T_x) = dV_\Gamma(x)$, so

$$\langle dV_\Gamma(x), f(x, u)\rangle + L(x, u) \geq 0\,. \tag{2.16}$$

Now consider an arbitrary $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f}, \mathcal{T})$, with domain $[T, 0]$. We want to prove that $J(\gamma, \eta) \geq V_\Gamma(x)$, if $x = \gamma^-$. It follows from Lemma 4.1 of [26] that there exists a sequence of admissible pairs $(\gamma_j, \eta_j)$, with domain $[T, 0]$, such that $\gamma_j(0) = \gamma(0)$, $\eta_j$ is piecewise constant, $\gamma_j \to \gamma$ uniformly on $[T, 0]$, and $J(\gamma_j, \eta_j) \to J(\gamma, \eta)$. Since the function $V_\Gamma$ is lower semicontinuous, it follows that, if $V_\Gamma(\gamma_j(T)) \leq J(\gamma_j, \eta_j)$, then $V_\Gamma(x) \leq J(\gamma, \eta)$. Hence it suffices to assume that the control $\eta$ is piecewise constant, and we make this assumption from now on.

Let $\eta : [T, 0] \to U$ be a piecewise constant control, and let $t_0, \ldots, t_m$, $u_1, \ldots, u_m$ be such that $T = t_0 < t_1 < \ldots < t_{m-1} < t_m = 0$ and $\eta(t) = u_i$ if $t_{i-1} < t < t_i$ for $i = 1, \ldots, m$. For each $y \in \Omega$, let $t \to \zeta^y(t)$ be the trajectory of $\eta$ that satisfies the terminal condition $\zeta^y(0) = y$. Let $W$ be an open subset of $\Omega$ such that $\zeta^y(T)$ is defined for all $y \in W$ and $\gamma(0) \in W$. We will show that

($\#$) *for almost every $y \in W$ the set $B^y = \{t : \zeta^y(t) \in A\}$ is finite or countable.*

Since $A$ is thin, we can write $A = A_1 \cup A_2$, where $A_1 = \cup_j M_j$ and $\{M_j\}_{j \in J}$ is a finite or countable family of connected submanifolds of $\Omega$ of codimension $d_j > 0$, while $\mathcal{H}^{n-1}(A_2) = 0$. After replacing each $M_j$ by a finite or countable family of open submanifolds of $M_j$, we may assume that the $M_j$ are embedded.

For $i = 1, \ldots, m$, let $\Phi_i : W \times [t_{i-1}, t_i] \to \Omega$ be the map $(y,t) \to \zeta^y(t)$. Then $\Phi_i$ is a $\mathcal{C}^1$ submersion, because for each fixed $t$ the map $y \to \Phi_i(y,t)$ is a $\mathcal{C}^1$ diffeomorphism. Therefore each set $\tilde{M}_{j,i} = \Phi_i^{-1}(M_j)$ is an embedded submanifold of $W \times [t_{i-1}, t_i]$ of codimension $d_j$, and $\mathcal{H}^n(\Phi_i^{-1}(A_2)) = 0$.

Let $\pi_i : W \times [t_{i-1}, t_i] \to W$ be the projection $(y,t) \to y$. Let $S_{j,i}$ be the set of points $s \in \tilde{M}_{j,i}$ such that the restriction $\pi_i \lceil \tilde{M}_{j,i}$ of $\pi$ to $\tilde{M}_{j,i}$ is not regular at $s$. (Recall that, if $S_1$, $S_2$ are $\mathcal{C}^1$ manifolds, then a $\mathcal{C}^1$ map $F : S_1 \to S_2$ is *regular* at $s \in S_1$ if the differential $d\pi(s)$ maps the tangent space $T_s S_1$ surjectively onto $T_{F(s)} S_2$.)

Then $\mathcal{H}^n(\pi_i(\Phi_i^{-1}(A_2))) = 0$ (see for example [8]), and $\mathcal{H}^n(\pi_i(S_{j,i})) = 0$ for each $i, j$, by Sard's Theorem. So the "bad" set

$$\mathcal{B} = \Big(\bigcup_{i=1}^{m} \pi_i(\Phi_i^{-1}(A_2))\Big) \cup \Big(\bigcup_{j,i} \pi_i(S_{j,i})\Big)$$

has measure zero.

Now let $y \in W \backslash \mathcal{B}$. Then $\zeta^y(t) \notin A_2$ if $T \le t \le 0$. (Otherwise, if $\zeta^y(t) \in A_2$, $t_{i-1} \le t \le t_i$, it would follow that $(y,t) \in \Phi_i^{-1}(A_2)$, so $y \in \pi_i(\Phi_i^{-1}(A_2))$, and then $y \in \mathcal{B}$.) Also, we claim that the set $\{t : \zeta^y(t) \in A_1\}$ is at most countable. To see this, it suffices to show that, for each $j, i$, the set $\{t : t_{i-1} \le t \le t_i\, ,\, \zeta^y(t) \in M_j\}$ is finite. Fix $i, j$, and suppose $\zeta^y(t) \in M_j$. The manifold $\tilde{M}_{j,i}$ has codimension $d_j$ in $W \times [t_{i-1}, t_i]$. Since $d_j > 0$, the dimension $\nu_{j,i}$ of $\tilde{M}_{j,i}$ is $\le n$. Moreover, $d\pi(y,t)$ maps $T_{y,t}\tilde{M}_{j,i}$ surjectively onto $\mathbb{R}^n$ (the tangent space to $\Omega$ at $y$), because $y \notin \mathcal{B}$. So $\nu_{j,i} = n$, and $d\pi(y,t)$ is injective on $T_{y,t}\tilde{M}_{j,i}$. Since $\left\langle d\pi(y,t), \frac{\partial}{\partial t} \right\rangle$ is obviously equal to 0, it follows that $\frac{\partial}{\partial t} \notin T_{y,t}\tilde{M}_{j,i}$. Since $\tilde{M}_{j,i}$ is embedded, $(y, \tau) \notin \tilde{M}_{j,i}$ if $0 < |\tau - t| \le \varepsilon$, provided that $\varepsilon$ is sufficiently small. Therefore $t$ is an isolated point of $\{\tau : t_{i-1} \le \tau \le t_i\, ,\, \zeta^y(\tau) \in M_j\}$. So $\{t : t_{i-1} \le t \le t_i\, ,\, \zeta^y(t) \in M_j\}$ is finite.

---

**COMMENT NO. 10**: In other words, you are saying that, if every point of a subset of $\mathbb{R}$ is isolated, then the set is finite!!!! How about the set $\{\frac{1}{n} : n = 1, 2, 3 \ldots\}$? I am amazed that the referee allowed this to get by. I guess he must not have been a very competent referee. But obviously we cannot say that to Henk Nijmeijer!

---

We have therefore proved that, if $y \in W \backslash \mathcal{B}$, then $\zeta^y(t) \notin A_2$ for all $t \in [T, 0]$, and the set $\{t : \zeta^y(t) \in A_1\}$ is at most countable. Since $A = A_1 \cup A_2$, and $\mathcal{B}$ has measure zero, ($\#$) follows.

From the weak upper semicontinuity of $V_\Gamma$, there exist $x_i$, $\varepsilon_i > 0$, $\delta_i > 0$ such that $x_i \to \gamma(0)$, $\delta_i \to 0$ and $V_\Gamma(y) \le V_\Gamma(\gamma(0)) + \delta_i$ whenever $\|y - x_i\| \le \varepsilon_i$.

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**COMMENT NO. 11**: I don't understand the above construction. How can something "exist" *from* the weak semicontinuity of something?

---

Choose the $\varepsilon_i$ so that $\|y - x_i\| \leq \varepsilon_i$ implies $y \in W$. Then, by (#), we can choose for each $i$ a $y_i$ such that $\|y_i - x_i\| \leq \varepsilon_i$ and the bad set $B^{y_i}$ is at most finite or countable. Since $\Gamma$ is total and $V_\Gamma(\zeta^{y_i}(0)) < +\infty$, it is clear that, for every $i$, $\zeta^{y_i}(t) \in S$ for every $t \in [T, 0]$. Hence (2.16) implies that the function:

$$\varphi_i(t) = V_\Gamma(\zeta^{y_i}(t)) + \int_T^t L(\zeta^{y_i}(s), \eta(s))ds$$

is differentiable at every $t$, except for a finite or countable number of values (which may include the switching times of $\eta$), with nonnegative derivative. Thus, $\varphi_i$ verifies assumption (B.1.a) of Lemma B.1 (Appendix B).

---

**COMMENT NO. 12**: I don't understand how a mathematical object can "verify" an assumption. It can SATISFY it, but not VERIFY it. Can you say in Italian, foe example, that a function "verifies the assumptions of the mean value theorem"? I bet you the full value of a Luiggi di Biaggio that you cannot say it. I am sure that in Italian, as in English, things "satisfy" conditions, and we humans "verify" that certain things "satisfy" certain properties!!!!!

---

Moreover, the lower semicontinuity of $V_\Gamma$ ensures assumption (B.1.b) of Lemma B.1 for $\varphi_i$, and the (NDJ) condition ensures assumption (B.1.c). Hence we can apply Lemma B.1 to $\varphi_i$, and conclude that $\varphi_i(T) \leq \varphi_i(0)$. So

$$V_\Gamma(\zeta^{y_i}(T)) \leq \int_T^0 L(\zeta^{y_i}(t), \eta(t))dt + V_\Gamma(y_i) \leq \int_T^0 L(\zeta^{y_i}(t), \eta(t))dt + V_\Gamma(\gamma(0)) + \delta_i.$$

Since $V_\Gamma(\gamma(0)) \leq \tau(\gamma(0))$ and $\zeta^{y_i} \to \gamma$ uniformly (and in particular $\zeta^{y_i}(T) \to \gamma(T) = x$), we conclude, using the lower semicontinuity of $V_\Gamma$, that $V_\Gamma(x) \leq J(\gamma, \eta)$.

So the desired inequality holds when $\eta$ is piecewise constant. As explained earlier, this implies that $\Gamma$ is optimal.

**Remark 1.** Notice that the weak upper semicontinuity of $V_\Gamma$ everywhere is equivalent to its weak upper semicontinuity at the points of the target. Indeed, let $x \in \Omega$. If $x \notin S$, then $V_\Gamma(x) = +\infty$, and (2.8) is clearly true. Next, suppose $x \in S$, and let $(\gamma_x, \eta_x)$ be its associated admissible pair. Assume that $V_\Gamma$ is weakly upper semicontinuous at $\gamma_x(0)$. Let $y_j$, $\varepsilon_j$, $\delta_j$ be such that $y_j \to \gamma_x(0)$, $\varepsilon_j > 0$, $\delta_j > 0$, $\delta_j \to 0$, and $V_\Gamma(y) \leq V_\Gamma(\gamma_x(0)) + \delta_j$ whenever $\|y - y_j\| \leq \varepsilon_j$. Let $\zeta^{y_j}$ be the trajectory corresponding to the control $\eta_x$ and such that $\zeta^{y_j}(0) = y_j$.

Since $y_j \to \gamma_x(0)$, we may assume, by picking $j$ large enough, that $\zeta^{y_j}$ is defined on $[T_x, 0]$ for all $j$, and let $x_j = \zeta^{y_j}(T_x)$. Let $\tilde{\varepsilon}_j$ be such that $0 < \tilde{\varepsilon}_j < \frac{1}{j}$ and, if $\|z - x_j\| \leq \tilde{\varepsilon}_j$, then the trajectory $\theta^z$ of $\eta_x$ with initial condition $\theta^z(T_x) = z$ is defined on $[T_x, 0]$ and satisfies $\|\theta^z(0) - y_j\| < \varepsilon_j$. Then, using the same reasoning as in the proof of Theorem 2.2, we can show that

---

**COMMENT NO. 13**: If you had used the label for this theorem, rather than 2.2, then it would have been less work for me to renumber things!!!!!

In addition, it seems to me that what you really meant was Theorem 2.1, not 2.2. Right? For example, Statement (#) is in the proof of Theorem 2.1, not the proof of Theorem 2.2. Right?

---

$$V_\Gamma(z) \le \int_{T_x}^0 L(\theta^z(t), \eta_x(t))\,dt + V_\Gamma(\gamma_x(0)) + \delta_j \tag{2.17}$$

whenever $\|z - x_j\| \le \tilde{\varepsilon}_j$. (First approximate $\eta_x$ by piecewise constant controls, $\eta^k$, and let $\theta^{z,k}$ be the corresponding trajectories with terminal condition $\theta^{z,k}(0) = \theta^z(0)$. Then for each $k$ use Statement (#) of the proof of Theorem 2.2 to approximate the terminal point

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**COMMENT NO. 14**: As I said above, I think you meant 2.1. And it would have been better if you had used the labels. That's what they are for.

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$\theta^z(0)$ by points $w^{z,k,i}$ with the property that, if $\tilde{\theta}^{z,k,i}$ is the trajectory for $\eta^k$ for which $\tilde{\theta}^{z,k,i}(0) = w^{z,k,i}$, it follows that $V_\Gamma(\tilde{\theta}^{z,k,i}(T_x)) \le \int_{T_x}^0 L(\theta^{z,k,i}(t), \eta^k(t))\,dt + V_\Gamma(w^{z,k,i})$. Since $\|\theta^z(0) - y_j\| < \varepsilon_j$, the $w^{z,k,i}$ satisfy $\|w^{z,k,i} - y_j\| < \varepsilon_j$ if $i$ is large enough, so $V_\Gamma(\tilde{\theta}^{z,k,i}(T_x)) \le V_\Gamma(\gamma_x(0)) + \delta_j$ when $i$ is large enough. If we let $i \to \infty$ and then let $k \to \infty$, and use the lower semicontinuity of $V_\Gamma$, we find that (2.17) holds.)

Let $\tilde{\delta}_j = \sup\Big\{\| \int_{T_x}^0 L(\theta^z(t), \eta_x(t))\,dt - \int_{T_x}^0 L(\gamma_x(t), \eta_x(t))\,dt\| : \|z - x_j\| \le \tilde{\varepsilon}_j\Big\}$. Then $\tilde{\delta}_j \to 0$, because $x_j \to x$ and $\tilde{\varepsilon}_j \to 0$. On the other hand, $V_\Gamma(z) \le V_\Gamma(x) + \delta_j + \tilde{\delta}_j$ whenever $\|z - x_j\| \le \tilde{\varepsilon}_j$. So $V_\Gamma$ is weakly upper semicontinuous at $x$.

## §3. The Fuller phenomenon

The aim of this section is to prove that the famous control system considered by Fuller in [12]—and extensively studied by other authors, e.g. Zelikin-Borisov [28]—satisfies the hypotheses of Theorem 2.1.

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**COMMENT NO. 15**: Once again, it would have been better if you had used the labels.

---

Consider the system:

$$\begin{cases} \dot{x}_1 = x_2 & , \\ \dot{x}_2 = u & , \\ |u| \le 1 & , \end{cases} \tag{3.1}$$

and the minimization problem:

$$\min \int_0^{+\infty} x_1^2(t)dt. \tag{3.2}$$

We now describe the optimal synthesis for this problem, determined by Fuller. First define the "switching locus" $\zeta = \{(x_1, x_2) : |x_1| = \xi x_2^2, x_1 x_2 \le 0\}$, where $\xi$ is the unique positive root of the polynomial $x^4 + x^2/12 - 1/18$, and write $\zeta = \zeta_+ \cup \zeta_-$, where $\zeta_\pm = \{(x_1, x_2) \in \zeta : \pm x_1 \ge 0\}$. Then the piecewise smooth curve $\zeta$ divides the plane into two regions $A^-$ and $A^+$ consisting, respectively, of the points that lie above and below $\zeta$. For every $x \in \mathbb{R}^2$ there exist a unique optimal trajectory $\hat{\gamma}_x$ and corresponding control $\hat{\eta}_x$, related by $\hat{\eta}_x(t) = +1$ when $\hat{\gamma}_x(t) \in A^+$, and $\hat{\eta}_x(t) = -1$ when $\hat{\gamma}_x(t) \in A^-$. The switchings occur along $\zeta$. The curve $\hat{\gamma}_x$ reaches the origin in a finite time $\tau_x$, and then stays there. Therefore $\hat{\eta}_x(t) = 0$ when $\hat{\gamma}_x(t) = 0$, which happens iff $t \ge \tau_x$ (see Fig. 1.)

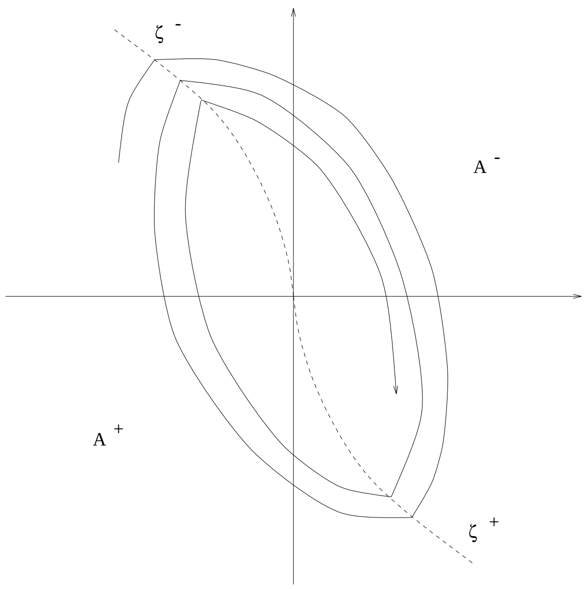


Figure 1

Therefore the admissible pairs $(\hat{\gamma}_x, \hat{\eta}_x)$ actually solve the problem of reaching the origin in finite time, subject to the dynamical constraints (3.1), and minimizing the cost (3.2). However, the sufficiency theorems of [2] and [6] do not apply, since the controls $\hat{\eta}_x$ have an infinite number of switchings. For this reason, the optimality of the synthesis obtained by Fuller is usually established by other means.

For example, one can prove optimality using the *symmetry properties* of the problem. Precisely, there exists a one parameter family $\Delta = \{\Delta_\rho\}_{\rho>0}$ of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that, if $(\gamma, \eta)$ is an admissible pair having cost $c$, and we let $\gamma_\rho(t) = \Delta_\rho(\gamma(\frac{t}{\rho}))$, $\eta_\rho(t) = \eta(\frac{t}{\rho})$, then $(\gamma_\rho, \eta_\rho)$ is an admissible pair having cost $\rho^5 c$. It then follows that the $\Delta_\rho$ map optimal trajectories to optimal trajectories. The proof of optimality using the dilation symmetry is given, for example, in [28], and will not be repeated here. It will turn out, however, that the dilations $\Delta_\rho$ will play a role in our arguments, so we recall their definition and one simple property:

(1) By definition, $\Delta_\rho(x_1, x_2) = (\rho^2 x_1, \rho x_2)$.
(2) It follows from (1), in particular, that $\Delta_\rho(x_1, x_2) - x = (\rho - 1)(2x_1, x_2) + (\rho - 1)^2 (x_1, 0)$ for all $\rho, x_1, x_2$. This identity will be useful later.

---

**COMMENT NO. 16**: You say that we are going to "recall their definition and one simple property," and then you have items (1) and (2). So the reader is going to assume that (1) is the "definition" and (2) is the "simple property" that you are "recalling." Now: is the statement that "This identity will be useful later" part of the "simple property" you are "recalling"? Or is it a comment you make about the property and about the reason for mentioning it here?

---

In order to apply our Theorem 2.1, we first make a time translation and define $\gamma_x(t) = \hat{\gamma}_x(t + \tau_x)$, $\eta_x(t) = \hat{\eta}_x(t + \tau_x)$, for $-\tau_x \leq t \leq 0$. Then $\Gamma = \{(\gamma_x, \eta_x)\}_{x \in \mathbb{R}^2}$ is a memoryless family of admissible pairs such that $\gamma_x$ is defined on $[-\tau_x, 0]$, $\gamma_x(-\tau_x) = x$, and $\gamma_x(0) = 0$. We want to prove that $\Gamma$ is an optimal synthesis. For this purpose, we will verify the hypotheses of our Theorem 2.1, and in particular the $(f, L)$–differentiability of $\Gamma$ at every point $x \in \mathbb{R}^2 \backslash \zeta$. Since it is very easy to check Property (DC1), we will only verify (DC2).

Fix a point $\bar{x} \in \mathbb{R}^2 \backslash \{(0,0)\}$. Then there exist $s_0, s_1$ such that $s_0 < s_1 < 0$, $s_1 \leq \theta s_0$,

$$\eta_{\bar{x}} = \pm \sum_{k=0}^{+\infty} (-1)^k \chi_{I_k}$$

where $I_0 = [s_0, s_1]$ and, for $k = 1, 2, \ldots$,

$$I_k = [\theta^{k-1} s_1, \theta^k s_1], \qquad \theta = \sqrt{\frac{\frac{1}{2} - \xi}{\frac{1}{2} + \xi}},$$

$\chi_S$ is the indicator function of a set $S$, and the sign $\pm$ depends on $\bar{x}$ as follows: it is $+$ if $\bar{x} \in A^+ \cup \zeta^+$, and $-$ if $\bar{x} \in A^- \cup \zeta^-$.

Let $[T, 0]$ be an interval containing $\mathrm{Dom}(\eta_x)$ for $x$ near $\bar{x}$. Fix a continuous function $\alpha : [T, 0] \to \mathbb{R}$. To prove (DC2) we must study the dependence on $x$ of the integral

$$J^\alpha(x) = \int_T^0 \alpha(t).\tilde{f}(\gamma_{\bar{x}}(t), \eta_x(t))\, dt\,, \tag{3.3}$$

and show that $J^\alpha$ is differentiable at $x = \bar{x}$ for every $\bar{x} \in \mathbb{R}^2 \setminus \zeta$. We will do this by showing the existence, and continuity with respect to $x$, of the directional derivatives $D_{v_1(x)} J^\alpha$, $D_{v_2(x)} J^\alpha$, where $v_1(x)$, $v_2(x)$ are two linearly independent directions depending smoothly on $x$. We choose $v_1(x)$ to be the direction of the curve $\rho \to \Delta_\rho(x)$ at $\rho = 1$, and take $v_2(x) = f(x, \eta_x(-\tau_x))$. We will just determine the directional derivatives at $\bar{x}$, and the result will make it obvious that they depend continuously on $\bar{x}$, so differentiability will follow.

Let us assume, for simplicity, that $\bar{x} \in A^+$. (The case when $\bar{x} \in A^-$ is similar.) We first differentiate $J^\alpha$ in the direction of $v_1(\bar{x})$. If $\rho > 0$, we have (letting $s_k = \theta^{k-1} s_1$ for $k = 2, 3, \ldots$):

$$\tau_{\Delta_\rho(\bar{x})} = \rho \tau_{\bar{x}}\,, \quad \eta_{\Delta_\rho(\bar{x})} = \sum_{k=0}^{+\infty} (-1)^k \chi_{I_k^\rho}\,, \quad I_k^\rho = [\rho s_k, \rho s_{k+1}]$$

and

$$\tilde{f}(\gamma_{\bar{x}}(t), \eta_{\Delta_\rho(\bar{x})}(t)) - \tilde{f}(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) = \begin{pmatrix} 0 \\ \eta_{\Delta_\rho(\bar{x})}(t) - \eta_{\bar{x}}(t) \\ 0 \end{pmatrix} + R_\rho(t), \tag{3.4}$$

where $R_\rho(t)$ is a term arising because of the possibility that $\tau_{\bar{x}} \neq \tau_{\Delta_\rho(\bar{x})}$.

Then, if $\rho \geq 1$ and $\rho - 1$ is sufficiently small,

$$\begin{aligned}
\int_T^0 (\eta_{\Delta_\rho(\bar{x})}(t) - \eta_{\bar{x}}(t))\alpha(t)dt &= \int_{\rho s_0}^0 \sum_{k=0}^{+\infty} (-1)^k (\chi_{I_k^\rho}(t) - \chi_{I_k}(t))\alpha(t)dt \\
&= \sum_{k=0}^{+\infty} (-1)^k \left( \int_{\rho s_k}^{\rho s_{k+1}} \alpha(t)dt - \int_{s_k}^{s_{k+1}} \alpha(t)dt \right) \\
&= \sum_{k=0}^{+\infty} (-1)^k \left( \int_{\rho s_k}^{s_k} \alpha(t)dt - \int_{\rho s_k}^{s_{k+1}} \alpha(t)dt \right) \\
&= \int_{\rho s_0}^{s_0} \alpha(t)dt + 2 \sum_{k=1}^{+\infty} (-1)^k \left( \int_{\rho s_k}^{s_k} \alpha(t)dt \right).
\end{aligned}$$

The numbers $(\rho-1)^{-1}\Big\|\int_{\rho s_k}^{s_k}\alpha(t)dt\|$ are bounded by $\theta^{k-1}|s_1|.\|\alpha\|_{L^\infty}$, which is the general term of a convergent series, since $\theta<1$. So we can divide by $\rho-1$ and let $\rho\to 1$ taking the limit of each term separately. We then conclude that

$$\lim_{\rho\downarrow 1}\frac{1}{\rho-1}\int_T^0(\eta_{x_\rho}(t)-\eta_{\bar{x}}(t))\alpha(t)dt=-s_0\,\alpha(s_0)-2s_1\sum_{k=1}^{+\infty}(-1)^k\theta^{k-1}\alpha(\theta^{k-1}s_1)\,. \tag{3.5}$$

We now find the limit of $(\rho-1)^{-1}\int_T^0\alpha(t).R_\rho(t)\,dt$ as $\rho\downarrow 1$. Then $-\tau_{\Delta_\rho(\bar{x})}=-\rho\tau_{\bar{x}}<-\tau_{\bar{x}}$, so

$$\int_T^0\alpha(t).R_\rho(t)\,dt=\int_{-\rho\tau_{\bar{x}}}^{-\tau_{\bar{x}}}\alpha(t).\begin{pmatrix}\gamma^2_{\Delta_\rho(\bar{x})}(t)\\ 0\\ \left(\gamma^1_{\Delta_\rho(\bar{x})}(t)\right)^2\end{pmatrix}dt\,,$$

where $\gamma^i_{\Delta_\rho(\bar{x})}(t)$, for $i=1,2$, is the $i$-th component of $\gamma_{\Delta_\rho(\bar{x})}(t)$. Clearly, then, if $\bar{x}=(\bar{x}_1,\bar{x}_2)$, we have

$$\lim_{\rho\downarrow 1}(\rho-1)^{-1}\int_T^0\alpha(t).R_\rho(t)\,dt=\tau_{\bar{x}}\alpha(\tau_{\bar{x}}).\begin{pmatrix}\bar{x}_2\\ 0\\ \bar{x}_1^2\end{pmatrix}\,. \tag{3.6}$$

---

**COMMENT NO. 17**: This formula does not look right. According to your choices, $\tau_{\bar{x}}$ is positive. But $\alpha$ is a function defined on $[T,0]$. So you cannot have $\alpha(\tau_{\bar{x}})$. It has to be $\alpha(-\tau_{\bar{x}})$.

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Similar computations shows that (3.5) and (3.6) hold as well

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**COMMENT NO. 18**: It should be “Similar computations show”, or “A similar computation shows”, but not “Similar computations shows”.

---

for the limits as $\rho\uparrow 1$. So the directional derivative of $J^\alpha$ at $x=\bar{x}$ in the direction of $v_1(\bar{x})$ exists and is given by the sums of the right-hand sides of (3.5) and (3.6), which depend continuously on $\bar{x}$, since $s_0$, $s_1$ and $\tau_{\bar{x}}$ (which is none other than $-s_0$) depend continuously on $\bar{x}$ as long as $\bar{x}\in A^+$.

A much simpler argument verifies the existence of the derivative of $J^\alpha$ in the direction of $v_2(\bar{x})$, i.e. of $f(\bar{x},1)$, whose value turns out to be the vector with components $\alpha(\tau_{\bar{x}})\bar{x}_2$, $\alpha(\tau_{\bar{x}})$, and $\alpha(\tau_{\bar{x}})\bar{x}_1^2$. Once again, this expression depends continuously on $\bar{x}$, so the proof of (DC2) is complete.

The other assumptions of Theorem 2.1 are easily checked. So the theorem applies, and the optimality of the synthesis described above follows.

## §4. The Boltianskii and Brunovsky definitions of synthesis

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**COMMENT NO. 19**: I changed the section title to “Comparison with other definitions of synthesis,” because we also discuss a definition proposed in some of my papers.

---

We start by reviewing the definition of synthesis proposed by Brunovsky [6], and then we explain how it differs from the more restrictive definition given by Boltianskii [2].

---

**COMMENT NO. 20**: It is not completely true that the Boltyanskii definition is "more restrictive" than that of Brunovský (or, as you said in the August 97 version, "then we recall some further restrictions existing in the definition given by Boltyanskii [2]"). For example: Boltianskii does not require that the partition into cells be a stratification, but Brunovský does.

Even more dramatically: **Boltyanskii does not require the set of cells to be locally finite.** Of course, he has a local finiteness condition in his definition of "piecewise smooth set," but this does not imply that the set of cells, as he defines the notion of "cell" (a connected components of $P^i \setminus (P^{i-1} \cup N)$ for some $i$), is locally finite. My own guess is that he probably intended to make that requirement, and he thought that his conditions implied that the set of cells is locally finite. But *they don't.* You can see by looking at the Boltyanskii paper that he is very imprecise, so it's perfectly possible that he may have made a mistake here. In the new version of the paper, I put in an example showing that for a Boltyanskii regular synthesis the set of cells need not be locally finite.

Also, Boltyanskii allows a set $N$ where things can be quite bad (e.g. the "marked trajectory" there need not be unique) but Brunovský does not. So in some ways the Boltianskii definition is **less restrictive** rather that more restrictive than that of Brunovský. Of course, in some other much more important ways the Boltianskii definition is indeed "more restrictive," but the only true statement we can make is that the two definitions are not comparable.

Here again we see an example of how I made a mistake by trusting you and expecting that you would carefully do the scholarly work involved in checking the references and making sure that when we quoted them we did so accurately. **I had to go and read the Boltyanskii and Brunovský papers carefully by myself in order to get a precise understanding of the comparison between the two notions.**

---

**COMMENT NO. 21**: In this section, you seem to have completely forgotten that we are now dealing with a new version in which the target is no longer the origin. For that reason, you talk throughout the section about "the problem with target the origin." To fix this, I had two choices: (a) rewriting everything in the section with the general target, or (b) say that, "for simplicity," in this section we take the target to be the origin. The first choice was too much work for me.

It turns out, however, that both Boltianskii and Brunovsky work with a one-point target anyhow. So the best thing to do is to tell the reader that, since they both work a one-point target, we will do the same. That gives a totally different impression: we are not limiting ourselves to a point target because we are lazy or we don't know how to discuss the general case. We do it because the others did it. On the other hand, OUR theorem is for a general target. So here is another reason why our theorem is better. But we don't say so. (After all, the generalization from point target to general target is sort of trivial.) But the reader will clearly see the point, and find on his own another reson in favor of our paper, and in addition respect us for our modesty in not saying so openly.

---

We consider the system (2.1), (2.2), with the restriction that $f$ is $C^1$ as a function of $(x, u)$.

---

**COMMENT NO. 22**: What on Earth does it mean for $f$ to be a $C^1$ function of $x$ and $u$ if $u \in U$ and $U$ is just a set? In addition: we also need to require that $L$ be $C^1$, right?

---

Given a $k \in \{1, 2, \ldots\} \cup \{+\infty, \omega\}$, and a manifold $M$ of class $\mathcal{C}^k$, a $\mathcal{C}^k$ *stratification in* $M$ of a

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**COMMENT NO. 23**: Now, suddenly, in the middle of a discussion of systems and optimal control, you start talking about stratifications without giving any indication to the reader of why this is relevant at this point! **You should never switch from one subject to another without a continuous transition, or an explanation of why this digression is necessary!!!!**

---

subset $S$ of $M$ is a partition $\mathcal{P}$ of $S$ into connected embedded $\mathcal{C}^k$ submanifolds of $M$ such that $\mathcal{P}$ is locally finite in $M$ (i.e. every compact subset of $M$ intersects finitely many members of $\mathcal{P}$) and the following "frontier axiom" holds:

(FA) *if* $P_1, P_2 \in \mathcal{P}$, *and* $P_1 \cap clos_M(P_2) \neq \emptyset$, *then* $P_1 \subset clos_M(P_2)$ *and* $dim(P_1) < dim(P_2)$.

---

**COMMENT NO. 24**: The Frontier axiom, stated this way, is **wrong**, because $P_1$ could be equal to $P_2$. In fact, the way you stated it the FA is ALWAYS false, because you can take any member $P_1$ of $\mathcal{P}$ and take $P_2$ to be $P_1$, and then the conditions $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap clos_M(P_2) \neq \emptyset$ will hold, but $dim(P_1) < dim(P_2)$ will not be true.

---

A $\mathcal{C}^k$*-stratified subset of* $M$ is a pair $(S, \mathcal{P})$ such that $S \subseteq M$ and $\mathcal{P}$ is a $C^k$ stratification of $S$ in $M$.

Let $S \subseteq \Omega$ be an open subset such that the origin belongs to $S$. Brunovsky's definition (somewhat modified by us to correct what we thought were some minor imprecisions in the original formulation), says that a *regular synthesis in $S$ for the control problem (2.1), (2.2) with target the origin*, is a 6–tuple $\Gamma = (\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \Pi, \Sigma, v)$ such that

1) *$\mathcal{P}$ is the union of $\{\{0\}\}$ and a locally finite (in $S$) partition $\mathcal{P}\backslash\{\{0\}\}$ of $S \setminus \{0\}$ into connected embedded $\mathcal{C}^1$ submanifolds (called "cells"),*

2) *$\mathcal{P}\backslash\{\{0\}\}$ is the disjoint union of $\mathcal{P}_1$ (the set of "Type I cells") and $\mathcal{P}_2$ (the set of "Type II cells"),*

3) *$v : S \to U$ (the "closed loop control"), $\Pi : \mathcal{P}_1 \to \mathcal{P}$, and $\Sigma : \mathcal{P}_2 \to \mathcal{P}_1$ are maps such that the following properties are satisfied:*

   3.A *If $\mathcal{P}' = \{P \in \mathcal{P} : \dim P < n\,,\ P \neq \{0\}\}$, and $S' = \bigcup\{P \in \mathcal{P}'\}$, then $(S', P')$ is a stratified subset of $S$ of dimension $< n$.*

---

**COMMENT NO. 25**: This statement is O.K. but it is extremely weird. Look at the comment I added, with a detailed proof that I put in the long version. Of course, this is Brunovský's fault, not yours or mine, but I think that whenever one can make things clearer one should.

---

*3.B The function $v$ is $\mathcal{C}^1$ on each cell.*

---

**COMMENT NO. 26**: Again, what could it possibly mean to say that a function with values in an abstract set $U$ is $C^1$?

---

*3.C If $P_1 \in \mathcal{P}_1$ and $x \in P_1$, then $f(x, v(x)) \in T_x P_1$ (the tangent space to $P_1$ at $x$) and there exists a time $t_x$ such that the maximally defined solution $\xi_x$ of the initial value problem*

$$\dot{\xi} = f(\xi, v(\xi))\,,\ \ \xi(0) = x\,,\ \ \xi \in P_1$$

*is such that $[0, t_x[\ \subseteq Dom(\xi_x)$ and $\lim_{t \uparrow t_x} \xi_x(t)$ exists (in $\Omega$) and belongs to $\Pi(P_1)$.*

---

**COMMENT NO. 27**: Compare my rewording of (3.C) with your original wording. Make sure you understand why mine is better. The way you had it, there is ONE universal quantifier "for all $x \in P_1$" covering all occurrences of $x$. Within this quantifier, you don't know that $f(., v(.))$ is a vector field tangent to $P_1$ everywhere. So you cannot talk about $\xi_x$ the way you do. See?

Another problem with your version of 3.C is that it would allow for the possibility that $\Pi(P_1)$ is just equal to $P_1$ and the time $t_x$ is always chosen to be smaller than the true exit time from $P_1$. You don't want that. Look at the way I rewrote this. (You NEVER said that $\Pi(P_1)$ cannot be equal to $P_1$. And you NEVER said that $t_x$ is the exit time. All you say is that the interval $[0, t_x[$ is a subset of $Dom(\xi_x)$. This does not imply that $t_x$ is the sup of $Dom(\xi_x)$, i.e. that $t_x$ is the exit time.)

Finally, notice that Brunovský made a mistake in talking about $\xi_x$ being the solution of the IVP $\dot{x} = f(x, v(x))$ with initial condition $\xi_x(0) = x$. Once you fix $x$ to be the initial condition, you cannot use $x$ as a variable. I have corrected that.

---

*3.D If $P_2 \in \mathcal{P}_2$, then the control $v$ is continuous on $P_2 \cup \Sigma(P_2)$, and for each $x \in P_2$ there exists a unique curve $\xi_x : [0, t_x[\to \Omega$ such that $\xi_x \lceil ]0, t_x[$ is a maximally defined integral curve of the vector field $f(\cdot, v(\cdot))$ on $\Sigma(P_2)$, and $\xi_x(0) = x$.*

*3.E On every cell $P$, $x \to t_x$ is a continuously differentiable function, and $(t, x) \to \xi_x(t)$, $(t, x) \to u_x(t) \stackrel{\text{def}}{=} v(\xi_x(t))$ are continuously differentiable functions of $t, x$ on the set*

$$E(P) \stackrel{\text{def}}{=} \{(t, x) : x \in P\,,\ t \in [0, t_x]\}$$

*in the sense that they can be prolonged to $\mathcal{C}^1$ functions on some relatively open subset of $P \times [0, \infty[$ containing $E(P)$.*

---

**COMMENT NO. 28**: It should be $[0,\infty[\times P$, not $P\times[0,\infty[$. And, in any case, it seems to me that what we really want is $\mathbb{R}\times P$.

---

*3.F For every $x\in S\backslash\{0\}$, if we let $\tilde{\xi}_x$ denote the curve obtained in an obvious way by piecing together the trajectories on every single cell, and write $\tilde{\eta}_x(t)=v(\tilde{\xi}_x(t))$, then the admissible pair $(\tilde{\xi}_x,\tilde{\eta}_x)$ ends at the origin after passing through a finite number of cells, and is extremal.*

*3.G The value function $V_\Gamma$ is continuous.*

Boltianskii's definition, as given in [2], imposes the following additional requirements:

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**COMMENT NO. 29**: The way you phrased it, you are saying that Boltyanskii imposes all the requirements that Brunovsky does, plus a few "additional" ones. That is not true. For example, Boltyanskii never talks about stratifications, so in particular Boltyanskii does not require that anything be a stratification. Moreover, as I explained earlier, Boltyanskii never says that the collection of cells has to be locally finite. Also, Boltyanskii has the set $N$, and you completely forgot about it. Finally, Boltyanskii's definition is not phrased in terms of the same object as Brunovský's—a 6–tuple $\Gamma=(\mathcal{P},\mathcal{P}_1,\mathcal{P}_2,\Pi,\Sigma,v)$—but in terms of a sequence $P^0,P^1,\ldots,P^n$ of "piecewise smooth sets," from which he gets the "$k$-dimensional cells" as the connecetd components of $P^k\setminus(P^{k-1}\cup N)$. I don't see any mention of all this in what you wrote.

---

*4.A $dim(\Pi(P))=\dim(P)-1$ whenever $P\in\mathcal{P}_1$.*

*4.B The only zero dimensional cell is $\{0\}$.*

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**COMMENT NO. 30**: Expressions such as "zero-dimensional" or "one-dimensional" should be hyphenated.

---

*4.C Every trajectory $\xi_x$ must enter a new cell "at a nonzero angle". (That is, more precisely: whenever $P\in\mathcal{P}_1$ and $x\in P$, the tangent vector to $\xi_x$ at time $t_x$ is not tangent to $\Pi(P)$ at $\xi_x(t_x)$.)*

**Remark 2.** Another definition —closely related to Brunovsky's— was proposed in [16], and used to prove an existence theorem for stabilizing synthesis. The definition of Sussmann [27] —which is the one used in this paper— is more general in a number of ways, e.g. since it is not based on the existence of any feedback control generating the trajectories.

The examples of the following section show optimal syntheses that do not match the definitions above. Let $\Gamma$ be the

**COMMENT NO. 31**: To "match a definition"? What's that? I have never seen it anywhere. Where did you get it from?

synthesis, in Brunovsky's sense, of Example 1. We have that $V_\Gamma$ is discontinuous, hence $\Gamma$ does not satisfy the point 3.$G$ of the Brunovsky definition.

**COMMENT NO. 32**: The previous paragraph is very bad English. See how I rewrote it. In particular, "we have that" is truly awful. And I don't see how something can "satisfy a point." Can you talk about "satisfying a point" in Italian? I doubt it. It seems to me that, in all the Western languages that I know, the kinds of things that can be "satisfied" in mathematics are properties, conditions, requirements, assumptions, but not "points."

The same example show another important feature of Brunovsky's definition.

**COMMENT NO. 33**: "The same example show." It should be "shows."?

It shows the necessity

**COMMENT NO. 34**: The verb "show" is repeated just one line later. That's very poor writing.

of specifying, for each cell $P_2 \in \mathcal{P}_2$, a cell $\Sigma(P_2) \in \mathcal{P}_1$ entered by every trajectory $\xi_x$ (of 3.D) for $x \in P_2$. Indeed, if we do not define the map $\Sigma$ for Example 1 then some not optimal trajectories would be admissible solutions of the corresponding discontinuous feedback $v$. The hypothesis of the uniqueness of the $\xi_x$ entering $\Sigma(P_2)$ can be relaxed after adding a specification of $\xi_x$ on an initial interval of time. This was done in [16], [25], where for every $x \in P_2 \in \mathcal{P}_2$ we have an assigned trajectory $\xi_x$, defined for an interval of time of length $\varepsilon_x$, that enters $\Sigma(P_2)$.

**COMMENT NO. 35**: I found this whole paragraph completely incomprehensible as written. After reading it twice, I still wasn't able to figure out what you were trying to say, even though presumably I am in a better position than most readers to understand what you are talking about. I am sure no reader would be able to figure it out either. It took me a lot of effort to decipher it, and I think I have now succeeded in a way that can actually be understood. Please compare what I have written with what you had written, and make sure that you understand the difference. Actually, I found the whole text, from here up to the end of the section, disorganized, incoherent, bordering on the incomprehensible. You first announce that we are going to give "examples of syntheses that do not match the definitions", but then what you actually do is meander back and forth between this and several other quite different things, without much of a pattern. You start by talking about the Brunosvky definition and referring to Example 1 of the following section, but the only way in which this example does not "match the definition" is that the cost function is discontinuous. You then venture into the quite different issue of explaining why in Brunovsky's definition one needs to specify $\Sigma$. This is indeed important, but it doesn't have a damn thing to do with "not matching" the brunosvky definition: quite the contrary, it shows why the Brunovsky definition is good. You then make a cryptic reference to "relaxing" the uniqueness as I did in my papers, and then you go back to the discussion of Example 1 and how it shows the importance of specifying $\Sigma$. You explanantion of how it does so is incomprehensible. You make a mysterious mention of the fact that at the point $(-1,0)$ both controls 1 and $-1$ determine the same velocity vector, but you don't give any clue to the reader about why this observation matters. If at this point you had told the reader that Italy did not get to the finals of the World Cup, this observation would not have appeared any more irrelevant to the rest of the discussion than your remark about the two controls giving rise to the same velocity. (There is even inconsistent notation: in this this discussion you refer to the feedback of Example 1 as "$v$", but when you actually present Example 1 in the following section you call it "$u$." Is it the same object?) You then move, in a seemeingly randon way, to repeating the observation made earlier that the synthesis for Fuller's problem does not "match" the Brunovsky definition. You then go in the other direction, by loooking at the flows of the Boltyianskii definition. In all this, there is nodiscernible oreder, no progression, no structure to help the reader make sense of what he is reading. Please take a careful look at what you had written, compare it with my rewriting, and try to understand the difference. I know you are not going to take seriously what I am saying, and you are going to think that I am just complaining about your English, and I am being intolerant because you are not a native English speaker. This is NOT my problem. I think all the points I am raising are language-independent, and would apply equally well if you had written this in Italian. They are matters of logic, of organization and intellectual discipline, the things that I would call "pre-mathematics", meaning things that are not yet truly mathematical but are a precondition for mathematical thinking and communication.

---

In any case it is important to specify, for every $P_2 \in \mathcal{P}_2$, the cell the trajectory $\xi_x$, $x \in P_2$, is supposed to enter. If we allow all the trajectories of the discontinuous feedback $v$ then we obtain some not optimal trajectories. For example, starting from the Type II cell $R_3 = \{(-1,0)\}$ we can use the trajectory corresponding to the control $+1$ for time $2\pi$, reaching again the same point, and then go to the origin using the control $-1$. The curve $\gamma_1$ defined in this way is an admissible trajectory of $v$, indeed at the

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**COMMENT NO. 36**: Here you have two complete sentences, not connected by a conjunction, separated by a comma. That is contrary not only to grammr, but to logic and common sense, and I don't think this is just a problem of English. I bet you cannot write like that in Italian either. Can you? In addition, I really cannot figure out what this sentence means. Are you sure you understand the word "indeed"?

---

point $(-1,0)$ the two controls $\pm 1$ determine the same velocity vector. However, $\gamma_1$ is clearly not time optimal. We can also define the trajectory $\gamma_2$ of $v$, with domain $[0,+\infty[$, corresponding to the constant control $+1$ and starting from $(-1,0)$. This trajectory neither reaches the origin in finite time nor approaches it as time goes to infinity. Notice that the trajectories $\gamma_1$ and $\gamma_2$ satisfy the equation $\dot{\gamma}(t) = f(\gamma(t), v(\gamma(t)))$ for every $t$ in their domain. Hence they satisfy the equation corresponding to the discontinuous feedback $v$ in any reasonable *weak* sense. For a classical definition of solutions to a discontinuous ODE we refer to [9].

---

**COMMENT NO. 37**: I don't understand why you mention Filippov in this context. It is NOT true that, if a curve $\xi$ satisfies $\dot{\xi}(t) = f(\xi(t))$ for almost all $t$ (or even for all $t$), then $\xi$ is a solution of the equation in Filippov's sense. (For example, in 1 dimension, let $f(x) = 1$ if $x \neq 0$, $f(0) = 0$. Let $\xi(t) = 0$. Then $\xi$ is a classical solution but not a Filippov solution.) Yet, you say "Notice that the trajectories $\gamma_1$ and $\gamma_2$ satisfy the equation $\dot{\gamma}(t) = f(\gamma(t), v(\gamma(t)))$ for every $t$ in their domain. Hence they satisfy the equation corresponding to the discontinuous feedback $v$ in any reasonable *weak* sense." And you even add "For a classical definition of solutions to a discontinuous ODE we refer to [9]." This clearly tells the reader that you think that the Filippov sense is one of these "reasonable senses" that you had been talking about.

In my new version, I removed the Filippov reference at this point. But I gave a much more detailed discussion of the precise relationship between classical solutions, Filippov solutions, and the "feedback solutions" of Eduardo Sontag and his collaborators. Please read this carefully (it's **your** paper, after all!) and see how things now make sense.

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The synthesis of Fuller example does not match the definitions above

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**COMMENT NO. 38**: To "match" definitions? What's that? I have never seen it anywhere. Where did you get it from?

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despite its evident *regularity*. Indeed since every trajectory has an infinite number of switchings the synthesis does not satisfy assumption 3.F.

More dramatic is the situation for the definition given by Boltianskii. The synthesis of Example 1 does not match the requirements 4.A and 4.B: the cells $R_7$ and $R_8$ are one

---

**COMMENT NO. 39**: To "match a requirement"?

---

dimensional Type I cells and $\Pi(R_8) = R_7$, the cell $R_3 = \{(-1, 0)\}$ is a zero dimensional cell different from the origin.

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**COMMENT NO. 40**: I think it is silly to use this example as one where Boltianskii's condition 4.A fails. Of course it fails with this particular choice of the partition into cells, but one can eliminate this problem by refining the partition, adding new 0-dimensional cells. This is a natural thing to do, because when one does it one gets a true stratification, so all the Brunovsky conditions are satisfied, except for the continuity of the cost function. In my new version I use this partition instead of yours.

---

The requirements 4.A and 4.B are not satisfied even by very simple synthesis $\Gamma$ for which $V_\Gamma$ is continuous. In [3], [4], [14], [15], [19], [20], [23], [24], [25], it was studied the time optimal synthesis and the time optimal trajectories properties for control affine single–input two dimensional systems. There are some singular trajectories called turnpike. In general these turnpikes

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**COMMENT NO. 41**: "turnpikes", not "turnpike"!

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are one dimensional cells of the time optimal synthesis with target the origin and they are not necessarily ending at the origin but enter other one dimensional or zero dimensional cells. For some explicit examples see [15].

The synthesis of Example 2 does not match the assumption 4.A as it is

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**COMMENT NO. 42**: Now it's "match an assumption"!!!! You really seem to have liked this "match" stuff, because you use it over and over and over. Where did you get it from?

---

easily shown, since some trajectories leave a three dimensional cell entering a one dimensional cell.

Finally, in [6] it was shown an example not satisfying hypothesis 4.C.

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**COMMENT NO. 43**: "It was shown an example" is not English!!!!

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## §5. Examples

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**COMMENT NO. 44**: As you will notice, I have expanded and improved the "Examples" section, so that we now have examples for lots of other things that we didn't even mention before. In addition, I want you to pay attention to the **structure**: if someone reads this section, he will be able to know what an example is about before reading the details of the example. The previous version was written in such a way that nobody would have been able to figure out what was the point of Example 1.

---

**Example 1.**
We study the time-optimal synthesis for the planar system:

$$\dot{q} = F(q) + u\,G(q)\,, \qquad |u| \le 1\,, \tag{5.1}$$

where

$$q \doteq \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2\,, \qquad F(q) \doteq \begin{pmatrix} 1 - \frac{y}{2} \\ \frac{x+1}{2} \end{pmatrix}\,, \qquad G(q) \doteq \begin{pmatrix} -\frac{y}{2} \\ \frac{x+1}{2} \end{pmatrix}\,, \tag{5.2}$$

and the target is the origin of $\mathbb{R}^2$.

The trajectories of (5.1) corresponding to the constant control $u \equiv -1$ are straight horizontal lines going from left to right, while those corresponding to $u \equiv +1$ are circles centered at the point (-1,1), running counterclockwise. The optimal synthesis, to be determined below, is shown in Fig.2.

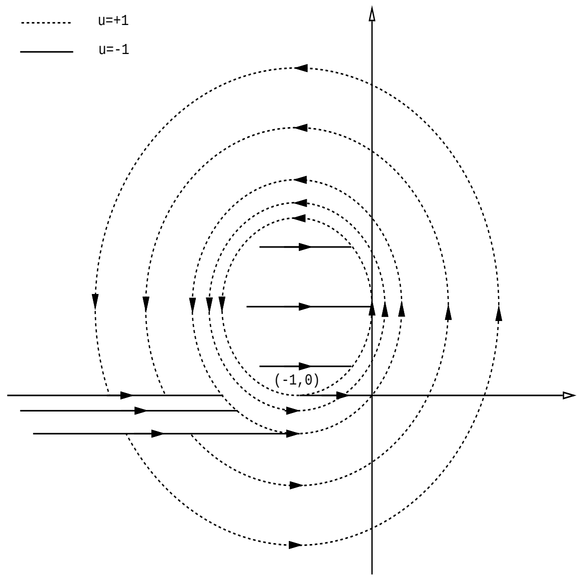


Figure 2

An arc $\gamma : [a, b] \to \mathbb{R}^2$ is *simple* if $\gamma$ is an injective map. It is clear that an optimal trajectory is simple.

Since $G(q) = 0$ only for $q = (-1, 0)$, it is clear that a trajectory $\gamma : [a, b] \to \mathbb{R}^2$ uniquely determines the corresponding control $u(\cdot)$, unless the set $\{t : \gamma(t) = (-1, 0)\}$ is of positive measure. In particular, for simple trajectories $\gamma$ —the only ones that will be of interest to us— the control is uniquely determined by $\gamma$.

We use $\mathbb{R}^k_*$ to denote the space of real row vectors with $k$ entries,

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**COMMENT NO. 45**: Inconsistent notation. In earlier sections we used $\mathbb{R}_k$ to denote the space of row $k$-vectors!

---

while $\mathbb{R}^k$ denotes the space of column vectors. If $\Omega$ is an open subset of $\mathbb{R}^k$ and $X : \Omega \to \mathbb{R}^k$ is a vector field of class $C^1$, then $\nabla X$ will denote the Jacobian matrix of $X$, i.e. the $k \times k$-matrix-valued function on $\Omega$ whose rows are the gradients

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**COMMENT NO. 46**: Inconsistent notation again. We used $DX$ earlier to denote the Jacobian of $X$!!!!

---

of the components of $X$. If $X, Y$ are vector fields of class $C^1$, then $[X, Y]$ will denote the Lie bracket of $X$ and $Y$, so $[X, Y] = (\nabla Y).X - (\nabla X).Y$.

If $X, Y$ are $\mathbb{R}^2$-valued functions on a subset of $\mathbb{R}^2$, then $(X, Y)$ will denote the $2 \times 2$ matrix with columns $X, Y$. In particular, we can define the functions, see [23]:

$$\Delta_A = det\,(F, G) \qquad \Delta_B = det\,(G, [F, G]) \tag{5.3}$$

where "*det*" stands for determinant.

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**COMMENT NO. 47**: Can you explain to me the grammatical structure of the sentence "we can define the functions, see [23]: $\Delta_A = det\,(F,G) \qquad \Delta_B = det\,(G,[F,G])$"? I don't see it. It seems to me that "see [23]" is an entirely separate sentence, not connected to the rest by any connective (conjunction or preposition or relative pronoun) that might make it an admissible part of a larger sentence. If you don't see my problem, I suggest that you translate this word by word into Italian and then, once you got it in Italian, you try to parse it using the techniques for analyzing sentences that I am sure you learned in your Italian high school. Use your techniques to parse the sentence, and then let me know what you found. My guess is that you will not be able to do it, because this sentence is grammatically wrong in Italian as well as in English. Please compare with the way I rewrote this, and you will see that now it DOES make sense, grammatically as well as logically.

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from the results of [23], we know that every extremal trajectory of (5.1) is a finite concatenation of "bang" and "singular" arcs. A bang arc is a trajectory for a constant control with values in the set $\{-1, 1\}$ of endpoints of the control interval. Singular arcs lie in the set of zeros of the function $\Delta_B$. In our case,

$$\Delta_A = \frac{x+1}{2} \quad \text{and} \quad \Delta_B = -\frac{y}{4} \tag{5.4}$$

so the only singular arcs are those entirely contained in the $x$ axis, and these correspond to the control $-1$, so they actually are bang-bang. Therefore *all the extremals of our problem are bang-bang.*

If $\Delta_A(q) \neq 0$ then the vectors $F(q)$ and $G(q)$ are linearly independent, and we can define the numbers $f(q)$ and $g(q)$ as the coefficients of the linear combination:

$$[F,G](q) = f(q)F(q) + g(q)G(q). \tag{5.5}$$

Let $\Omega_A$, $\Omega_B$ be, respectively, the sets $\{q : \Delta_A(q) \neq 0\}$, $\{q : \Delta_B(q) \neq 0\}$, and define $\Omega_{AB} = \Omega_A \cap \Omega_B$. As was proved in [23], if $\gamma$ is an extremal trajectory and $\gamma(t) \in \Omega_{AB}$ for every $t$, then $\gamma$ is bang-bang with at most one switching, and this switching can only be from the control value $-1$ to the value $+1$ if $f > 0$, and from $+1$ to $-1$ if $f < 0$. In our case, it follows from (5.4) that

$$\Omega_A = \{(x,y) : x \neq -1\}\,, \quad \Omega_B = \{(x,y) : y \neq 0\}\,, \tag{5.6}$$

and

$$[F,G] = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} = \frac{y}{2(x+1)}F + \frac{2-y}{2(x+1)}G\,, \tag{5.7}$$

so $\Omega_{AB}$ is the union of four open quadrants $\Omega_{AB,i}$, $i = 1,2,3,4$, defined by

$$\begin{aligned} \Omega_{AB,1} &= \{(x,y) : x > -1\,,\ y > 0\}\,, \\ \Omega_{AB,2} &= \{(x,y) : x < -1\,,\ y > 0\}\,, \\ \Omega_{AB,3} &= \{(x,y) : (x,-y) \in \Omega_{AB,2}\}\,, \\ \Omega_{AB,4} &= \{(x,y) : (x,-y) \in \Omega_{AB,1}\}\,. \end{aligned} \tag{5.8}$$

---

**COMMENT NO. 48**: See how, in my version, I reordered the three preceding formulas. I think now it makes more logical sense. Agreed?

---

Clearly, $f > 0$ on $\Omega_{AB,1} \cup \Omega_{AB,3}$ and $f < 0$ on $\Omega_{AB,2} \cup \Omega_{AB,4}$. So the control switchings are from $-1$ to 1 on $\Omega_{AB,1} \cup \Omega_{AB,3}$ and from 1 to $-1$ on $\Omega_{AB,2} \cup \Omega_{AB,4}$.

To construct an optimal synthesis, we will actually construct a collection $\mathcal{F}$ of simple trajectories such that every optimal trajectory ending at the target is in $\mathcal{F}$, and for every initial point $q$ we will find a $\gamma_q \in \mathcal{F}$ such that if $\delta \in \mathcal{F}$ steers $Q$ to 0 and $\delta \neq \gamma_q$

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**COMMENT NO. 49**: According to our definitions, which make perfect logical sense, a CONTROL "steers" a point to another point. A TRAJECTORY does not.

In addition, you must have meant $q$ rather than $Q$ in the previous line.

---

then $\delta$ is not optimal. Since the existence of optimal trajectories from any initial point follows by elementary arguments, we will be able to conclude that $\gamma_q$ is optimal.

Suppose that $\gamma : [a, b] \to \mathbb{R}^2$ is time-optimal, $\gamma(b) = (0, 0)$, $a < b$, and $\gamma$ arises from an open-loop control $t \to u(t)$. Then $\gamma$is simple, and is a concatenation of finitely many maximal bang arcs. Since $\gamma$ is an extremal, we can fix a nontrivial minimizing adjoint vector $\lambda$ along $\gamma$ and a constant $\lambda_0 \geq 0$ for which (EX4) holds. Define the *switching function*

$$\varphi(t) = \lambda(t) \cdot G(\gamma(t)) .$$

The minimization condition implies that

$$\varphi(t) \neq 0 \implies u(t) = -sgn(\varphi(t)) . \tag{5.9}$$

Let $\delta : [c, d] \to \mathbb{R}^2$ be a maximal $u = -1$ piece of $\gamma$, so

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**COMMENT NO. 50**: This business of "maximal $u = -1$ piece" sounds bad to me. See what I wrote instead.

---

$c < d$ and $\delta$ is of the form:

$$\delta(t) = (x_0 + t, y_0) . \tag{5.10}$$

For a.e. $t \in [c, d]$, we have $\dot{\lambda} = -\lambda \cdot \nabla(F - G) = 0$. So $\lambda = (\lambda_1, \lambda_2)$ is constant and:

$$\varphi(t) = -\lambda_1 \frac{y_0}{2} + \lambda_2 \frac{x_0 + t + 1}{2} \tag{5.11}$$

is a linear function of $t$. Since a linear function cannot have more than one zero unless it vanishes identically, we conclude that $a < c < d < b$ can only be true if $y_0 = 0$. If $d = b$, then of course $y_0 = 0$, since $\gamma(b) = (0, 0)$. So one of the following two possibilities occurs:

(I) $\delta$ is contained in the $x$ axis,
(II) $a = c$.

This shows that all the maximal $u = -1$ pieces of $\gamma$ are entirely contained in the $x$ axis, with the only possible exception that $\gamma$ may contain an initial horizontal segment $[a, d]$ for which $y \neq 0$.

---

**COMMENT NO. 51**: $\gamma$ is a curve in the plane. How can it contain a segment $[a, d]$? And what can it possibly mean for a subinterval $[a, d]$ of the real line to be "horizontal"? I guess what you meant was that THE DOMAIN OF $\gamma$ contains $[a, d]$. See how I rewrote this.

---

In particular there are extremal trajectories that switch from control $+1$ to control $-1$ at a point $(-L, 0)$ with $L < 0$, and then reach the origin with control $-1$.

---

**COMMENT NO. 52**: I deleted the previous sentence because I don't think it's needed.

---

Let us now analyze the adjoint equation for a maximal $u = +1$ piece $\delta : [c, d] \to \mathbb{R}^2$ of $\gamma$, such that $c < d$. Clearly, $\delta$ is of the form:

$$x(t) = -1 + r\cos(t + \theta)\,, \qquad y(t) = 1 + r\sin(t + \theta) \qquad c \le t \le d\,, \tag{5.12}$$

where $r > 0$ is a constant and $\theta \in [0, 2\pi]$. (The possibility that $r = 0$ is obviously excluded, because $\gamma$ is time-optimal and $c < d$.) The components $\lambda_1, \lambda_2$ of $\lambda$ satisfy $\dot\lambda_1 = -\lambda_2$ and $\dot\lambda_2 = -\lambda_1$,

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**COMMENT NO. 53**: The equations cannot be $\dot\lambda_1 = -\lambda_2$ and $\dot\lambda_2 = -\lambda_1$. One of the minus signs must be wrong!

---

so there exist constants $A > 0$, $\theta_1 \in [0, 2\pi]$, such that

$$\lambda_1(t) = A\cos(t + \theta_1)\,, \qquad \lambda_2(t) = A\sin(t + \theta_1) \qquad c \le t \le d\,. \tag{5.13}$$

Therefore, up to multiplication by a positive constant,

$$\varphi(t) = -\cos(t + \theta_1) + r\sin(\theta_1 - \theta)\,. \tag{5.14}$$

Now suppose that one of the two endpoints $c, d$ of the domain of $\delta$ is a switching time and the corresponding point in $\mathbb{R}^2$ lies in the $x$ axis. That is, we assume that $h \in \{c, d\}$ is such that $\varphi(h) = 0$ and $y(h) = 0$. Then

$$r\sin(h + \theta) = -1\,, \tag{5.15}$$

so $\varphi(t) = -\cos(t + \theta_1) + r\Big(\sin(h + \theta_1)\cos(h + \theta) - \cos(h + \theta_1)\sin(h + \theta)\Big)$, and then

---

**COMMENT NO. 54**: Are you saying that the previous equation follows from (5.15)? I don't see it. It seems to me that this equation just follows from (5.14) by the formula for the sine of a difference. (5.15) has nothing to with this!

---

$$\varphi(t) = -\cos(t+\theta_1) + \cos(h+\theta_1) + r\sin(h+\theta_1)\cos(h+\theta)\,. \tag{5.16}$$

In particular,

$$0 = \varphi(h) = r\sin(h+\theta_1)\cos(h+\theta)\,, \tag{5.17}$$

so $\varphi(t) = -\cos(t+\theta_1) + \cos(h+\theta)$. If $r \neq 1$ then (5.15) implies $0 \geq \sin(h+\theta) \neq -1$, so $\cos(h+\theta) \neq 0$. But then (5.17) implies $\sin(h+\theta_1) = 0$, so $\cos(h+\theta_1) = \pm 1$. The possibility that $\cos(h+\theta_1) = 1$ is excluded, for in that case $\varphi$ would be $> 0$ throughout $(c,d)$, contradicting the minimization condition. So $\varphi(t) = -1 - \cos(t+\theta_1)$. This shows that $\varphi(t) < 0$ for $t \in [c,d] - \{h\}$, unless $d - c = 2\pi$. But if $d - c = 2\pi$ then $\gamma$ would not be simple. So $d - c < 2\pi$, and we have shown that the other endpoint $h'$ of $[c,d]$ cannot be a switching point, and therefore $h' \in \{a,b\}$.

It then follows that one of the following possibilities must occur:

(i) $d = b$;
(ii) $a = c < d < b$ and $\gamma(d)$ lies in the $x$ axis;
(iii) $a < c < d < b$ and $\gamma(d) = (-1,0)$.

Indeed, if $d < b$ but $\gamma(d)$ is not in the $x$ axis, then we could let $\delta'$ be the maximal $u = -1$ piece of $\gamma$ that starts at time $d$, and our analysis of the $u = -1$ pieces of $\gamma$, applied to $\delta'$, would show that $\delta'$ cannot in fact occur, since $\delta'$ is not an initial segment of $\gamma$ and is not contained in the $x$ axis. So, if $d < b$ then $\gamma(d)$ lies in the $x$ axis, and $d$ is a switching time of $\gamma$. Our previous argument shows that $c = a$, unless $r = 1$. But if $r = 1$, and $\gamma(d)$ is in the $x$ axis, then $\gamma(d) = (-1,0)$. So $d < b$ implies that either (ii) or (iii) must hold.

We now combine the results about the two types of pieces and provide a complete description of the optimal trajectories ending at the origin. Let us use $L$, $L_0$, $C$ to denote, respectively, the trajectory types "$u = -1$ and not contained in the $x$ axis," "$u = -1$ and contained in the $x$ axis," and "$u = 1$." (The symbols $L$ and $C$ stand for "line" and "circle.") Let us use $T_1T_2\ldots T_m$ to denote the trajectory type "$T_1$ followed by $T_2$ ... followed by $T_m$".

With this notation, we have established that a nontrivial optimal

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**COMMENT NO. 55**: What do you mean, "we have established?" We haven't, yet!!! In fact, we are going to establish it in the next 13 lines or so.

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trajectory $\gamma$ ending at the origin is of one of the following seven types: $L_0$, $C$, $L_0C$, $LC$, $CL_0$, $LCL_0$, $CL_0C$. (Concatenations of six or more types are ruled out because any such concatenation would have to contain two different $C$ pieces for which (iii) above holds, so $\gamma$ would go at least twice through the point $(-1,0)$, and would not be simple. excluded for the same reason. The type $L_0CL_0$ is excluded

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**COMMENT NO. 56**: I don't understand the previous line! Something must be missing! This seems to be an old mistake. I went back to the file you sent to me on August 20 1997, and the same line is still there. So that means that you haven't reread the paper carefully after that date. And it also means that the referee didn't read it carefully either.

---

because the last switching would have to happen at $(-1,0)$, so $C$ would have to be the circle with center $(-1,1)$ and radius 1, which intersects the $x$ axis only at $(-1,0)$. So the other switching would also happen at $(-1,0)$, and $\gamma$ would not be simple. The exclusion of $L_0CL_0$ implies in turn that the types $CL_0CL_0$, $LCL_0CL_0$ and $L_0CL_0CL_0$ are ruled out as well. The types $L_0CL_0C$ and $LCL_0C$ are excluded because the switching from $C$ to $L_0$ would have to happen at $(-1,0)$, so the $L_0$ piece following the switching has to be a segment going from $(-1,0)$ to a point $(\alpha,0)$ and $\alpha$ must satisfy $-1<\alpha<0$, for if $\alpha$ was $\geq 0$ then the $L_0$ piece would already reach the origin before switching to the last $C$. But if $-1<\alpha<0$ then the $C$ arc starting at $(\alpha,0)$ does not go through the origin. Finally, having excluded $L_0CL_0C$ we are also forced to exclude $CL_0CL_0C$.) The $LCL_0$ trajectories are further restricted by the condition that the switching from $C$ to $L_0$ must happen at the point $(-1,0)$. Let us call those $LCL_0$ trajectories that satisfy this extra condition "good."

It turns out that the optimal synthesis involves all seven types listed above, but not all trajectories of one of these types are optimal, so we need a finer analysis. We let $\mathcal{F}$ be the set of all simple trajectories that end at the origin and are either of one of the six types $L_0$, $C$, $L_0C$, $LC$, $CL_0$, $CL_0C$, or of type $LCL_0$ and good.

We now observe that $\dot{x}=1-y$ along a $u=1$ arc, so $\dot{x}>1$ as long as $y<0$, $u=-1$. Therefore, if $q_1=(x_1,y_1)$ and $q_2=(x_2,y_2)$ satisfy

---

**COMMENT NO. 57**: I dont' understand the two lines above. Shouldn't it be $u=1$ before "Therefore"?

---

$y_1=y_2\leq 0$, $x_1<x_2$, $x_1+x_2=-2$, so that $q_1$ and $q_2$ can be joined both by a $C$ arc and an $L$ (or $L_0$) arc, then the $C$ arc is contained in the lower half-plane, and is faster than the $L$ or $L_0$ arc. In particular, if we let $L_0(\beta)$ denote, for $\beta>0$, the $L_0$ arc going from $(-\beta,0)$ to $(0,0)$, we see that $L_0(\beta)$ *is not optimal if* $\beta>1$, because the piece of $L_0$ going from $(-\beta,0)$ to $(-2+\beta,0)$ can be replaced by a faster type $C$ arc.

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**COMMENT NO. 58**: It seems to me that this argument is only valid if $\beta\leq 2$. Right?

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Next we show that $L_0(1)$ *is optimal.* It suffices to notice that there is no trajectory from $(-1,0)$ to $(0,0)$ of any of the types $C$, $LC$, $LCL_0$, and exactly one trajectory of each of the types $L_0$, $L_0C$, $CL_0$, and $CL_0C$, but the last three are not simple, so $L_0(1)$ is left as the only possible candidate.

We now study the optimal trajectories that are of one of the types $C$, $L_0C$, $LC$, and $CL_0C$. Let $\gamma:[a,b]\to\mathbb{R}^2$ be such a trajectory, and take $b=0$, so $\gamma(0)=(0,0)$.

If $\gamma$ is a $C$ arc, then $a>-2\pi$, for otherwise $\gamma$ would contain a full loop. For $0<a<2\pi$, we let $C(a)$ be the $C$ arc $\gamma:[-a,0]\to\mathbb{R}^2$ such that $\gamma(0)=(0,0)$. It will be shown below that the arcs $C(a)$ are all optimal.

Next suppose that $\gamma$ is $LC$. Since the $C$ piece satisfies (5.12), we must have $r=\sqrt{2}$ and $\theta=-\frac{\pi}{4}$. Let

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**COMMENT NO. 59**: I thought you had said earlier that $\theta\in[0,2\pi]$! So how can you have $\theta=-\frac{\pi}{4}$?

---

$c$ be the time when the switching from $L$ to $C$ occurs. On $[c, 0]$, the switching function $\varphi$ is given, up to multiplication by a positive constant, by:

$$\varphi(t) = \sqrt{2}\sin\left(\theta_1 - \frac{\pi}{4}\right) - \cos(t + \theta_1) = \sin\theta_1 - \cos\theta_1 - \cos(t + \theta_1).$$

---

**COMMENT NO. 60**: The first equality of the previous formula is wrong. In (5.14) you had $\sin(\theta_1 - \theta)$. So you should have had $\sin\left(\theta_1 + \frac{\pi}{4}\right)$ rather than $\sin\left(\theta_1 - \frac{\pi}{4}\right)$.

This mistake in the sign of one term in the formula for $\varphi$ changes the whole analysis. I have had to redo the whole thing, using the new formula. Please read what I have done carefully. As you will see, the analysis using the correct expression is much simpler and more elegant than the wrong analysis we had in the original version. This is in fact how I discovered the error. It seemed wrong to me that we would prove that $c$ has to be $> -\pi$ and then use a different, ad hoc argument (based on the fact that certain switchings are not permitted in certain quadrants) to exclude the possibility that $c < -\frac{\pi}{2}$. seemed to tells us that one can construct an $LC$ arc with a switching function having the correct sign required by the MP, and then we pull this other thing out of the hat to say that some of these $LC$ arcs are not allowed, even though one can clearly construct them. The two proofs that certain switchings are not allowed are essentially the same, so they cannot give different results. I spent several hours thinking about this, until I found the mistake. Luckily, once we correct the mistake and do things correctly we end up with the same synthesis, the proof becomes simpler and nicer, and the apparent contradiction that we appeared to have reached goes away, as it should have.

This should teach you a lesson: WHEN I TELL YOU TO BE CAREFUL AND PAY ATTENTION TO THE DETAILS I KNOW WHAT I AM TALKING ABOUT. I know that you think that I am taking this whole thing too seriosly, and it doesn 't matter too much if there are lots of mistakes as long as the referees don't catch them, and that just one little sign in one formula isn't important. I think it is, though. Thank God that this time the mistake in the proof did not affect the result. But it could have.

Finally, in a sense this whole thing started with the "ospitality" misspelling. It was this that gave me the clue that you had not really done your job carefully and thoroughly. Armed with this basic intuition, I went on reading what you had written expecting to find lots of mistakes, and it seems that this was the right thing for me to do.

Just one more thing: If I had been a referee, I would have acted the same way. The "ospitality" spelling would have alerted me to the fact that the authors of this paper were not serious. Then I would have looked for mistakes all over the place, and they are not that hard to find, as I am sure you see by now if you have read all my comments. So I would have recommended that the paper not be accepted as it was. I think it is a good indication of the poor quality of current journals, and the lack of truly competent refereeing, that a paper like ours got accepted.

Unfortunately, it would not be a good strategy on our part to use the above considerations as arguments to prove to Henk Nijmeijer that the reviewer didn't really read the paper carefully, and for that reason when he recommends all those shortenings he should not be taken seriously. We'd better keep our opinion of the referee to ourselves.

In any case, I am sure that in the future you will spell "hospitality" correctly. And I hope you will spell other words correctly too. As Aesopus could have said, "people who make small mistakes usually make big ones as well."

So

$$\varphi(t) = (1 + \sin t)\sin\theta_1 - (1 + \cos t).\cos\theta_1$$

In addition, $c$ cannot be $-\pi$, because $c = -\pi$ would correspond to $\gamma(c) = (-2, 2) \in Q_{AB,2}$,

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**COMMENT NO. 61**: I don't understand what you mean by the notation $Q_{AB,2}$. Could it be the same object that you called $\Omega_{AB,2}$ before? If so, then we would have another case of inconsistent notation.

---

and we know that a switching from $u = -1$ to $u = 1$ is not permitted at points in $Q_{AB,2}$.

If $\sin\theta_1 = 0$, then $\cos\theta_1 = \pm 1$, so $\cos\theta_1 = 1$, because $\varphi(0) \le 0$. So $\varphi(t) = 1 + \cos t$, and $\varphi(c) = 0$ would imply $c = -\pi$. Since this possibility has been ruled out, we conclude that $\sin\theta_1 \neq 0$. We can then define

$$\psi(t) = \frac{1 + \sin t}{1 + \cos t}, \quad \nu = \frac{\cos\theta_1}{\sin\theta_1}.$$

Then $\psi$ is well defined and smooth except at values of $t$ such that $t = \pi + 2k\pi$, $k$ an integer. In our case, $\psi$ is defined on the whole interval $[c, 0]$, except at $t = -\pi$ (if $-\pi \in [c, 0]$). Since $c \neq -\pi$, $\psi(c)$ is defined, and the switching condition $\varphi(c) = 0$ says that $\psi(c) = \nu$. If $\sin\theta_1 > 0$, then $\varphi(t) < 0$ iff $\psi(t) < \nu$ and $\varphi(t) = 0$ iff $\psi(t) = \nu$. Then $\psi(0) \le \nu$ and, since $\lim_{t \to -\pi}\psi(t) = +\infty$, we see that $c$ must satisfy $-\pi < c < 0$. If $\sin\theta_1 < 0$, then $\varphi(t) < 0$ iff $\psi(t) > \nu$. So $\frac{1}{2} = \psi(0) \ge \nu$. A straightforward computation shows that $\frac{d\psi}{dt}(t) > 0$ for $-2\pi \le t < -\pi$. Since $\psi(-2\pi) = \psi(0) \ge \nu$, we conclude that $\psi(t) > \nu$ for $-2\pi \le t < -\pi$. So, once again, the switching cannot occur at a point in $[-2\pi, -\pi)$, and then $c \in (-\pi, 0)$.

So we have shown in all possible cases that $-\pi < c < 0$. If $-\pi < c < -\frac{\pi}{2}$, then $\gamma(c) \in Q_{AB,2}$, so a switching from $L$ to $C$ is not permitted. If $c = -\frac{\pi}{2}$, then $\gamma(c)$ lies in the $x$ axis, so the previous piece is $L_0$ rather than $L$. So $-\frac{\pi}{2} < c < 0$. If $\frac{\pi}{4} < c$, then $\gamma(c) \in Q_{AB,4}$, and the switching is not permitted either. So the only possibility left to us is $-\frac{\pi}{2} < c \le -\frac{\pi}{4}$.

For $0 < \sigma < 2\pi$, $s > 0$, let $LC(s, \sigma)$ denote the trajectory that consists of a $u = -1$ arc during time $s$, followed by a $u = 1$ arc during time $\sigma$, and ending at the origin. We have shown that *$LC(s, \sigma)$ is not optimal unless $\frac{\pi}{4} \le \sigma < \frac{\pi}{2}$.*

We now show that *when $\frac{\pi}{4} \le \sigma < \frac{\pi}{2}$ then $LC(s, \sigma)$ is optimal.* To see this, let $q$ be the starting point of $LC(s, \sigma)$, for $s > 0$, $\frac{\pi}{4} \le \sigma < \frac{\pi}{2}$. It is clear that $LC(s, \sigma)$ is the only $LC$ trajectory from $q$ to $(0, 0)$. Also, it is easy to see that there are no trajectories from $q$ to $(0, 0)$ of the types $L_0$, $C$, $L_0C$, or good $LCL_0$. There is exactly one $CL_0$ trajectory, but it is easy to see that the time $\hat{\tau}$ along this trajectory is larger than the time $\tau$ along $LC(s, \sigma)$. (Actually, $\tau < |x| < \hat{\tau} - \frac{3\pi}{2}$.) Finally, there is exactly one $CL_0C$ trajectory $\gamma$, which can also be ruled out. (If $\bar{x} = -1 - \sqrt{2 - (y + 1)^2}$, so

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**COMMENT NO. 62**: I dont' understand this calculation. Shouldn't it be $(y - 1)^2$ rather than $(y + 1)^2$?

---

$\bar{q} = (\bar{x}, y)$ is the point where $LC(s, \sigma)$switches from $L$ to $C$, then both $LC(s, \sigma)$ and $\gamma$ go through $\bar{q}$ and coincide from that point on. So it suffices to compare the parts of $LC(s, \sigma)$ and $\gamma$ up to $\bar{q}$. The time along $LC(s, \sigma)$ is $\bar{x} - x$, and that along $\gamma$ is $> \sqrt{(1 + x)^2 + (1 - y)^2 - 1} - 1 + \frac{3\pi}{2}$, so it suffices to notice

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**COMMENT NO. 63**: I don't understand this. I made the calculation, based on the fact that the time along the $L_0$ piece must be more than the difference of the radii of the larger circle (with radius $\sqrt{(1+x)^2+(1-y)^2}$) and the smaller one (with radius $\sqrt{2}$), and I got that the time along $\gamma$ is $> \sqrt{(1+x)^2+(1-y)^2} - \sqrt{2} + \frac{3\pi}{2}$, which isn't exactly the same thing. Can you explain to me how you got your formulas?

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that $\sqrt{(1+x)^2+(1-y)^2-1} + \frac{3\pi}{2} + x - \sqrt{2-(y+1)^2} > 0$.)

---

**COMMENT NO. 64**: Take a look at my own calculation. It came out quite different, although the final conclusion is the same. I think that my calculation is right, an in addition it's cleaner than yours. Am I right? **PLEASE CHECK MY NEW CALCULATION CAREFULLY!**

---

for $s > 0$, we let $L_0C(s)$ denote the trajectory that consists of a $u = -1$ arc during time $s$ followed by a $u = 1$ arc during time $\frac{\pi}{2}$, and ending at the origin. Then is is easily shown that *$L_0C(s)$ is optimal for every $s > 0$.* (Indeed, if $q = (x, 0)$ is the starting point of $L_0C(s)$, for $s > 0$, then $L_0C(s)$ is the only $L_0C$ trajectory from $q$ to $(0, 0)$, and there are no simple trajectories from $q$ to $(0, 0)$ of the types $C$, $LC$, $CL_0$, $LCL_0$ or $CL_0C$. There is an $L_0$ arc, namely, $L_0(|x|)$, but we know that this arc is not optimal, because $|x| > 1$.)

Now define $R_1 = \{(0, 0)\}$, $R_2 = (-1, 0) \times \{0\}$, $R_3 = \{(-1, 0)\}$.

---

**COMMENT NO. 65**: In the above line, the expression $(-1, 0)$ occurs twice, once meaning interval and once meaning ordered pair. This proves the wisdom of Bourbaki, when they chose to use $]a, b[$ rather than $(a, b)$ as a notation for open intervals. I have followed Bourbaki throughout the paper. This has required quite a bit of work!!!!!

---

For $q = (0, 0)$, let $\Gamma_q$ be the trajectory $\gamma$ with domain $\{0\}$

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**COMMENT NO. 66**: Weren't we using $\gamma_q$ (with LOWER CASE $\gamma$) to denote the trajectories of a synthesis $\Gamma$?

---

such that $\gamma(0) = (0, 0)$. For $q = (x, 0) \in R_2 \cup R_3$, let $\Gamma_q = L_0(|x|)$. Then the $\Gamma_q$, $q \in R_1 \cup R_3 \cup R_3$ are optimal.

For $r > 0$, let $D(r)$ be the open disc with center $(-1, 1)$ and radius $r$, and let $\partial D(r)$ denote the boundary of $D(r)$. Let $R_4 = D(1)$, $R_5 = \partial D(1) \backslash \{(-1, 0)\}$. Then a point $q \in \mathbb{R}^2$ can be joined to $(0, 0)$ by a good $LCL_0$ trajectory if and only if $q \in R_4$,

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**COMMENT NO. 67**: This is not true. There are lots of points that are not in $R_4$ and can be joined to 0 by a good $LCL_0$ trajectory!!!

and in that case the good trajectory is unique, and we will use $LCL_0(q)$ to denote it. Then *the arcs* $LCL_0(q)$*, for* $q \in R_4$*, are optimal.* To see this, observe that if $q \in R_4$ then the only trajectories in $\mathcal{F}$ other than $LCL_0(q)$ that go from $q$ to the origin are either $LCL_0$ but not good, or $LC$ with a switching from $L$

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**COMMENT NO. 68**: Something is wrong here. The way we defined $\mathcal{F}$, an $LCL_0$ trajectory can only b in $\mathcal{F}$ if it is good, so how can we talk about "trajectories in $\mathcal{F}$ ... that ... are ... $LCL_0$ but not good?

---

to $C$ at a time $c$ that violates the requirement that $-\frac{\pi}{2} < c$. So all the alternative candidates are excluded, and $LCL_0(q)$ must be the optimal trajectory.

A similar argument shows that if $q \in R_5$ then the unique $CL_0$ trajectory going from $q$ to $(0,0)$ —henceforth labeled $CL_0(q)$— is optimal.

Next, let $R_6 = D(\sqrt{2})\backslash(R_2 \cup r_3 \cup R_4 \cup R_5)$.

---

**COMMENT NO. 69**: In the above line it should be $R_3$, not $r_3$.

The analysis for $R_6$ is incomplete. You failed to discuss the possibility that $y = 2$. This case actually requires some extra work to be excluded! I have done it in my version.

---

If $q \in R_6$, then the obvious candidate for optimality is the arc $\Gamma_q$ of type $CL_0$ obtained by following the $C$ arc through $q$ until $R_2$ is reached, and then continuing along $R_2$ towards the origin. We now prove that *the arcs* $\Gamma_q$*,* $q \in R_6$*, are optimal.* To see this, observe that if $q = (x,y) \in R_6$ and $y \neq 0$ then there are no simple arcs from $q$ to $(0,0)$ of the types $L_0$, $C$, $L_0C$, $CL_0C$. There is an $LC$ arc, but (i) if $y < 0$ then this arc switches from $L$ to $C$ in $Q_{AB,4}$, so it is not optimal; (ii) if $y > 0$, then the arc is $LC(s,\sigma)$ for some $\sigma \in \pi, 2\pi)$, so it is not optimal either. If $y < 0$ or $y > 2$ or

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**COMMENT NO. 70**: Even if you use the standard $(a,b)$ notation for open intervals rather than Bourbaki's, there is a parenthesis missing in the previous line.

---

$x > -1$, then there is no good $LCL_0$ arc. If $0 < y < 2$ and $x < -1$ then there is a good $LCL_0$ arc, but it switches from $L$ to $C$ at a point in $Q_{AB,2}$, so this arc is not optimal. Finally, if $y < 0$ then $\Gamma_q$ is the only $CL_0$ arc, and if $y > 0$ then there are two such arcs, but the other one is not optimal, because it contains $L_0(\beta)$ for some $\beta > 1$. This proves the optimality of the arcs $\Gamma_q$ for $q \in R_6$, $y \neq 0$. A similar analysis works for $y = 0$. (Alternatively, if $q = (x,0)$ then $\Gamma_q$ is a cofinal piece of $\Gamma_{q'}$ for some $q' = (x',y') \in R_6$ such that $y' \neq 0$, so $\Gamma_q$ is optimal.)

Next, we let $R_7 = \partial D(\sqrt{2})\backslash\{(0,0)\}$. For $q \in R_7$ there is a unique trajectory $\Gamma_q$ of type $C$ going from $q$ to $(0,0)$. The optimality of $\Gamma_q$ for $q \in R_7$ is proved by considerations similar to those used for $R_6$.

We let $R_8 = (-\infty,-2) \times \{0\}$, $R_9 = \{(x,y) : 1-\sqrt{2} < y < 0\,,\, x < -1\,,\, (x+1)^2 + (1-y)^2 > 2\}$, $R_{10} = (-\infty,-1) \times \{1-\sqrt{2}\}$. Then $R_9 \cup R_{10}$ is exactly the set of starting points $q$ of the trajectories $LC(s,\sigma)$, for $s > 0$, $\frac{\pi}{4} \leq \sigma < \frac{\pi}{2}$, and we know that for each $q \in R_9 \cup R_{10}$ the $LC(s,\sigma)$ trajectory is unique and optimal. We define $\Gamma_q$ to be this trajectory. A similar fact is true for $R_8$, with $L_0C(s)$ rather than $LC(s,\sigma)$, and $\Gamma_q$ is then defined to be the optimal trajectory.

Finally, we let $R_{11} = \mathbb{R}^2 \backslash (R_1 \cup \ldots \cup R_{10})$. For $q \in R_{11}$, we let $\Gamma_q$ be the unique simple $CL_0C$ trajectory from $q$ to $(0,0)$. We show that $\Gamma_q$ *is optimal for all* $q \in R_{11}$. Let $\gamma \in \mathcal{F}$ be an extremal trajectory starting at $q$ and verifying $\gamma(0) = 0$. There is no such trajectory of type $L_0$,$C$ or $L_0C$. The case $LC$ is impossible, otherwise $\gamma$ would have a switching at time $c \in [-5\pi/4, -\pi/2[$.

---

**COMMENT NO. 71**: I don't see why the switching time $c$ would have to satisfy $c \geq -5\pi/4$. It doesn't matter, anyhow, because the thing that suffices to rule this case out is $c < -\pi/2$.

---

If $\gamma$ is of type $CL_0$ then it contains a not optimal

---

**COMMENT NO. 72**: "a not optimal trajectory" is not acceptable in English.

---

trajectory $L_0(\beta)$, $\beta > 1$.

---

**COMMENT NO. 73**: I don't understand the grammar of the above sentence. It seems to me that "$\beta > 1$" is a separate sentence, since you didn't connect it to anything else with words like "such that" or "with" of "for which." But then you cannot just adjoin it to the rest with a comma. (You cannot do this in Italian either, I bet. Can you? Please tell me.) Look at the way I rewrote this sentence.

---

Finally, if $\gamma$ is of type $LCL_0$ good then $\gamma(t) = p \in R_6$ for some $t$, but $\Gamma_p$ is not the restriction of $\gamma$

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**COMMENT NO. 74**: I rewrote this. Please check.

---

to $[t, 0]$. Hence, if $\gamma$ is optimal then it is of type $CL_0C$.

Since we have considered all the extremal trajectories

---

**COMMENT NO. 75**: The sentence "we have considered all the extremal trajectories" is extremely vague. Look at how I rewrote this.

---

this concludes the construction.

---

**COMMENT NO. 76**: The construction of WHAT? The discussion of this example begins with the words "we study the optimal synthesis for the problem ..." So you did not ANNOUNCE at the beginning that you were going to "construct" the optimal synthesis. I have fixed that, by changing both the beginning and the ending of the discussion.

---

This ends the analysis and proves that the synthesis of Fig.1 is the optimal one.

---

**COMMENT NO. 77**: Isn't it Figure 2 rather than Figure 1?

---

Let define the feedback control $u(x, y)$ to be equal to 0 on

---

**COMMENT NO. 78**: "Let define" should be "Let us define."

---

---

**COMMENT NO. 79**: Is this feedback $u$ the same one that you called $v$ in the previous section? If so, why change the letter?

---

$R_1$, to $-1$ on the set $R_2 \cup R_3 \cup R_4 \cup R_8 \cup R_9 \cup R_{10}$, and to $+1$ on $R_5 \cup R_6 \cup R_7 \cup R_{11}$. Then the $+1$ trajectory starting from the point $(-1, 0)$ is an admissible trajectory for this feedback control but it is not optimal.

**Example 2.**
The following example provides a synthesis that is optimal but does not match the restriction 4.A of Boltianskii's definition.

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**COMMENT NO. 80**: "Matching a restriction is awful." See how I rewrote this.

---

**COMMENT NO. 81**: Example 2 is obviously wrong. I didn't even attempt to follow your calculations, but it's clear to me that the answer that you get cannot be right.

Let me explain. As you state it, the problem that you are looking at is one of finding the distance function to the origin and the optimal arcs for lenght minimization for a perfectly ordinary smooth Riemannian metric $g$ on $\mathbb{R}^3$. Moreover, it's a fairly simple metric, given by

$$ds^2 = Adx^2 + Bdy^2 + Cdz^2 \qquad 1)$$

where $A$, $B$, $C$ are smooth strictly positive functions. (I'll show you below what $A$, $B$, $C$ are in this case.) When you have a Riemannian metric given by (1), then an arc $t \to \xi(t) = (x(t), y(t), z(t))$ is parametrized by arc-length if $ds$ is just $dt$, i.e. if

$$A\frac{dx}{dt}^2 + B\frac{dy}{dt}^2 + C\frac{dz}{dt}^2 = 1 \qquad (2)$$

If you write the evolution equations as

$$\dot{x} = \frac{u}{\sqrt{A}}, \dot{x} = \frac{v}{\sqrt{B}}, \dot{y} = \frac{w}{\sqrt{C}},$$

then (2) corresponds exactly to $u^2+v^2+w^2 = 1$. Naturally, if you use instead the control constraint $u^2 + v^2 + w^2 \leq 1$, you get the same minimizers.

Your example 2 arises if we take

$$A = B = \frac{1 + x^2 + y^2}{x^2 + y^2}, \quad C = 1 + x^2 + y^2$$

So the optimal control problem of your Example 2 is just a nice ordinary Riemannian problem. For such a problem the minimizers are the GEODESICS. And the geodesics are ALL SMOOTH. So I don't see how you could possibly get the solutions that you say you do.

---

Consider the system:

$$\begin{cases} \dot{x} = u\frac{\sqrt{x^2+y^2}}{\sqrt{1+x^2+y^2}} \\ \dot{y} = v\frac{\sqrt{x^2+y^2}}{\sqrt{1+x^2+y^2}} \\ \dot{z} = w\frac{1}{\sqrt{1+x^2+y^2}} \end{cases} \quad u^2 + v^2 + w^2 \leq 1 \quad \cdot$$

and the problem of stabilization to the origin in minimum time. Consider the following regions:

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**COMMENT NO. 82**: Too many "consider".

---

$$R_0 = \{(0,0,0)\}, \qquad R_1 = \{(x,y,z) : x = y = 0,\ z > 0\}, \qquad R_2 = \{(x,y,z) : x = y = 0,\ z < 0\},$$
$$R_3 = \{(x,y,z) : x^2 + y^2 > 0,\ z = 0\}, \qquad R_4 = \{(x,y,z) : x^2 + y^2 > 0,\ z > 0\},$$
$$R_5 = \{(x,y,z) : x^2 + y^2 > 0,\ z < 0\}.$$

---

**COMMENT NO. 83**: The above formula looks ugly. I realigned it.

---

We assign a feedback control $(u_i(x,y,z), v_i(x,y,z), w_i(x,y,z))$ for every region $R_i$ in the following way:

$$u_0 = v_0 = w_0 = 0, \qquad u_1 = v_1 = 0,\ w_1 = -1 \qquad u_2 = v_2 = 0,\ w_2 = 1,$$
$$u_3 = u_4 = u_5 = -\frac{x}{\sqrt{x^2+y^2}},\ v_3 = v_4 = v_5 = -\frac{y}{\sqrt{x^2+y^2}},\ w_3 = w_4 = w_5 = 0.$$

---

**COMMENT NO. 84**: Again, the above formula looks ugly. I realigned it.

---

Hence starting from a point $(x,y,z) \in R_3 \cup R_4 \cup R_5$ we first reach

---

**COMMENT NO. 85**: I don't understand this "hence." "Hence" means "for this reason." I don't see what "reason" you have in mind.

---

the $z$ axis using control $(u_3, v_3, w_3)$, and then, if we are not yet at the origin, we reach the origin running on the $z$ axis. In this way we have constructed a synthesis $\Gamma$. It is possible to prove that this synthesis $\Gamma$ is optimal applying

---

**COMMENT NO. 86**: "we have constructed a synthesis $\Gamma$" immediately followed by "It is possible to prove that this synthesis $\Gamma$ ..." !!!!! That's very poor writing. Repetitive!!!

---

Theorem 2.1. However, in this case it is immediate to verify that the

---

**COMMENT NO. 87**: I wish you had used the labels rather that the number 2.1. That's what labels are for!!!!!

---

associated value function

$$V_\Gamma = |z| + \sqrt{x^2 + y^2}$$

satisfies the Hamilton–Jacobi equation on the set $\{(x, y, z) : x^2 + y^2 > 0, z \neq 0\}$. Indeed:

$$\inf_{u^2+v^2+z^2 \leq 1} \left( \frac{\partial V}{\partial x}\, u\, \frac{\sqrt{x^2+y^2}}{\sqrt{1+x^2+y^2}} + \frac{\partial V}{\partial y}\, v\, \frac{\sqrt{x^2+y^2}}{\sqrt{1+x^2+y^2}} + \frac{\partial V}{\partial z}\, w\, \frac{1}{\sqrt{1+x^2+y^2}} \right) =$$

$$= - \left\| \left( \frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}} \right) \right\| = -1.$$

We can deduce that equation (2.16) holds at every point of $R_4 \cup R_5$.

---

**COMMENT NO. 88**: My God! Why didn't you use labels? How am I going to figure out, after all the changes, how exactly which equation is (2.16)? Do you realize that by being lazy and not doing the job properly you end up making this much worse and making me work double the time just to figure out what these stupid numbers stand for?

---

The other assumptions of Theorem 2.1, necessary to conclude the proof, are easily checked.

---

**COMMENT NO. 89**: I don't understand the grammatical structure of the above sentence, and I don't understand the meaning. Are you trying to say that one can check ALL the other assumptions, or only those assumptions that are needed to conclude the proof? (In English, no matter which one was the case, you would NOT say it the way you did. And I bet you cannot say it that way in Italian either!!!)

Even more seriously, I don't understand the LOGICAL structure of the above paragraph, and in fact I cannot figure out what you are saying: you seem to say first that we could use Theorem 2.1 but it's easier to do it without using it, via the HJB equation; but then you go ahead and talk —very confusingly— about "veryfying the assumptions of Theorem 2.1". So, are we applying Theorem 2.1, yes or no? I cannot tell? How do you expect the reader to understand this?

---

Notice that every trajectory starting from a point of the three dimensional cell $R_4$, respectively $R_5$, reaches the one dimensional cell $R_1$, respectively $R_2$. Every trajectory starting from the two dimensional cell $R_3$ reaches the zero dimensional cell $R_0$. Hence, the synthesis $\Gamma$ does not match the restriction 4.A of section 4.

---

**COMMENT NO. 90**: “Match”?

---

**Appendix A . Differentiation of trajectories with respect to a parameter .**

The purpose of this appendix is to prove a general theorem on differentiation with respect to a parameter $p$, taking values in a normed space $P$, of a family of trajectories $x_p : [a,b] \mapsto \mathbb{R}^n$ of vector fields $f_p$. We will state and prove the theorem under minimal hypotheses, much weaker than what is actually needed for the main results of this paper, because the theorem has other applications, such as the theory of envelopes, where the more general statement is useful.

If $\dot{x}_p(t) = f_p(x_p(t), t)$, then formal differentiation with respect to $p$ at $p = p_0$ in some direction $v$ yields the variational equation

$$\dot{y}_{p_0,v}(t) = \frac{\partial f_{p_0}}{\partial x}\Big(x^{p_0}(t), t\Big).y_{p_0,v}(t) + \lim_{\varepsilon \downarrow 0} \frac{w_{p_0+\varepsilon v}(t) - w_{p_0}(t)}{\varepsilon}\,, \tag{A.1}$$

where $y_{p_0,v}(t)$ is the directional derivative of $x_p(t)$ at $p_0$ in the $v$-direction, and $w_p(t) = f_p(x_{p_0}(t), t)$. The technical problem that will concern us here is to make (A.1) rigorous even when the limit in the right-hand side only exists in a weak sense.

The variational equation that arises when the limit of (A.1) exists in a weak sense is a linear equation $\dot{y}(t) = A(t)y(t) + w(t)$ where the “input” $w$ is the formal derivative of a function $W$ which is not necessarily absolutely continuous or even continuous. So our first task will be to study the solutions of linear systems with “generalized inputs” $w$.

Let $-\infty < a < b < \infty$, and let $n$ be a positive integer. If $A \in L^1([a,b];\mathbb{R}^{n\times n})$, and $\bar{Y} \in \mathbb{R}^n$, then it is well known that the initial value problem

$$\dot{Y}(t) = A(t)\cdot Y(t) + w(t)\,, \quad Y(a) = \bar{Y} \tag{A.2}$$

has a unique solution for every initial condition and every map $w \in L^1([a,b],\mathbb{R}^n)$. Because (A.2) can also be written formally as

$$dY(t) = A(t)\cdot Y(t)\,dt + w(t)\,dt\,, \quad Y(a) = \bar{Y} \tag{A.3}$$

we will use $Y_{A,w\,dt,\bar{Y}}$ to denote the unique solution of (A.2).

As explained before, it will be important for us to be able to solve (A.2) for “inputs” $w$ more general than integrable functions. The appropriate class of inputs turns out to be that of *formal derivatives of bounded measurable functions.* Precisely, we let $BM([a,b],\mathbb{R}^n)$ denote the space of all bounded Lebesgue measurable functions $W : [a,b] \to \mathbb{R}^n$, and use $BM_0([a,b],\mathbb{R}^n)$ to denote the set of those $W \in BM([a,b],\mathbb{R}^n)$ such that $W(a) = 0$. (In $BM([a,b],\mathbb{R}^n)$ we do *not* identify two functions that are equal almost everywhere.) We endow $BM([a,b],\mathbb{R}^n)$ with the norm

$$||W||_{BM} = \sup\{||W(t)|| : t \in [a,b]\}\,. \tag{A.4}$$

Then $BM([a,b],\mathbb{R}^n)$ is a Banach space, and $BM_0([a,b],\mathbb{R}^n)$ is a closed subspace of $BM([a,b],\mathbb{R}^n)$. The space $C^0([a,b],\mathbb{R}^n)$ of continuous maps $W:[a,b]\mapsto\mathbb{R}^n$ is of course a closed subspace of $BM([a,b],\mathbb{R}^n)$, but $L^\infty([a,b],\mathbb{R}^n)$ is *not* a subspace of $BM([a,b],\mathbb{R}^n)$, because in $L^\infty([a,b],\mathbb{R}^n)$ we identify functions that are equal a.e., but in $BM([a,b],\mathbb{R}^n)$ we do not.

If $W\in BM_0([a,b],\mathbb{R}^n)$ is absolutely continuous, and $w=\dot W$, so that, formally, $dW=w\,dt$, then we can rewrite (A.2) formally as

$$dY(t)=A(t)\cdot Y(t)\,dt+dW(t)\qquad Y(a)=\bar Y\,. \tag{A.5}$$

A solution of (A.5) is then a continuous map $t\to Y(t)$ such that

$$Y(t)=\bar Y+\int_a^t A(s)\cdot Y(s)\,ds+W(t)\qquad \text{for}\quad \text{all}\quad t\in[a,b]\,. \tag{A.6}$$

Now (A.6) makes sense for arbitrary $Y\in BM([a,b],\mathbb{R}^n)$, $W\in BM_0([a,b],\mathbb{R}^n)$, even if $W$ is not absolutely continuous. So we turn it into a definition: for a $W\in BM_0([a,b],\mathbb{R}^n)$, a *solution* of (A.5) is a map $Y\in BM([a,b],R^n)$ such that (A.6) holds. It then follows that

(EU) *for every* $W\in BM_0([a,b],\mathbb{R}^n)$ *and every* $\bar Y\in\mathbb{R}^n$ *there exists a unique solution* $Y$ *of* (A.5).

Indeed, uniqueness is trivial, since the difference $Y^1-Y^2$ of two solutions is a solution of the homogeneous problem (i.e. of (A.2) with $w\equiv0$ and $\bar Y=0$), so $Y^1-Y^2\equiv0$. To prove existence, we observe that (A.6) is equivalent to

$$Y(t)-W(t)=\bar Y+\int_a^t A(s)\cdot(Y(s)-W(s))\,ds+\int_a^t A(s)\cdot W(s)\,ds\qquad \text{for}\quad \text{all}\quad t\in[a,b]\,, \tag{A.7}$$

i.e. to $dZ=AZ\,dt+AW\,dt$, $Z(a)=\bar Y$, with $Z=Y-W$. Since $AW\in L^1$, because $A\in L^1$ and $W$ is bounded and measurable, the equation $\dot Z(t)=A(t)\cdot Z(t)+A(t)W(t)$ has a unique solution $Z^*$ for which $Z^*(a)=\bar Y$. If we let $Y(t)=Z^*(t)+W(t)$, and recall that $W(a)=0$, we see that $Y$ is the desired solution of (A.5).

We will use $Y_{A,dW,\bar Y}$ to denote the solution of (A.5), whose existence and uniqueness has just been established. In addition to proving (EU), the preceding argument also gives an explicit formula for $Y_{A,dW,\bar Y}$, namely:

$$Y_{A,dW,\bar Y}=W+Y_{A,AW\,dt,\bar Y}\,. \tag{A.8}$$

Clearly, the function $Y_{A,AW\,dt,\bar Y}$ is absolutely continuous, so the regularity properties of $Y_{A,dW,\bar Y}$ —modulo absolutely continuous functions— are exactly the same as those of $W$. (For example: $Y$ is continuous, right-continuous, or left-continuous at a point $t\in[a,b]$ if and only if $W$ is; $Y$ is of bounded variation iff $W$ is; $Y$ is absolutely continuous iff $W$ is.)

Gronwall's inequality, applied to (A.7), tells us that

$$||Y_{A,dW,\bar Y}(t)-W(t)||\le e^{||A||_{L^1}}\Big(||\bar Y||+||A||_{L^1}||W||_{BM}\Big)\,. \tag{A.9}$$

Therefore

$$||Y_{A,dW,\bar Y}(t)||_{BM}\le ||W||_{BM}+e^{||A||_{L^1}}\Big(||\bar Y||+||A||_{L^1}||W||_{BM}\Big)\,. \tag{A.10}$$

On the other hand, (A.6) also implies that

$$||W||_{BM}\le||\bar Y||+||Y_{A,dW,\bar Y}||_{BM}\Big(1+||A||_{L^1}\Big)\,. \tag{A.11}$$

**Theorem A.1.** *Let $\{W_j\}$, $\{\bar{Y}_j\}$, $\{A_j\}$ be sequences in $BM_0([a,b],\mathbb{R}^n)$, $\mathbb{R}^n$, $L^1([a,b],\mathbb{R}^{n\times n})$, respectively. Assume that $\{A_j\}$ is bounded in $L^1([a,b],\mathbb{R}^{n\times n})$. Let $Y_j = Y_{A_j,dW_j,\bar{Y}_j}$. Then*

(A.1.1) *$\{Y_j\}$ is bounded in $BM([a,b],\mathbb{R}^n)$ if and only if $\{W_j\}$ is bounded in $BM([a,b],\mathbb{R}^n)$ and $\{\bar{Y}_j\}$ is bounded in $\mathbb{R}^n$.*

(A.1.2) *if $\{A_j\}$ converges in $L^1([a,b],\mathbb{R}^n)$ to a limit $A$, $\bar{Y}\in\mathbb{R}^n$, $\bar{Y}_j\to\bar{Y}$, $\{W_j\}$ is bounded in $BM([a,b],\mathbb{R}^n)$, $W\in BM_0([a,b],\mathbb{R}^n)$, and $Y = Y_{A,dW,\bar{Y}}$, then $Y_j\to Y$ almost everywhere iff $W_j\to W$ almost everywhere. In that case, $Y_j - W_j$ converges to $Y - W$ uniformly on $[a,b]$. In particular, for every $t\in[a,b]$, $Y_j(t)\to Y(t)$ iff $W_j(t)\to W(t)$.*

*PROOF*. Statement (A.1.1) follows from the bounds (A.10) and (A.11). To prove (A.1.2), we first observe that under the hypotheses of (A.1.2) we can apply (A.1.1) and conclude that $\{Y_j\}$ is bounded in $BM([a,b],\mathbb{R}^n)$.

Let $\tilde{Y}_j = Y_j - W_j = Y_{A_j,A_jW_j\,dt,\bar{Y}_j}$, $\tilde{Y} = Y - W = Y_{A,AW\,dt,\bar{Y}_j}$. Then

$$\tilde{Y}_j(t) = \bar{Y}_j + \int_a^t A_j(s)Y_j(s)\,ds = \bar{Y}_j + \int_a^t A_j(s)\tilde{Y}_j(s)\,ds + \int_a^t A_j(s)W_j(s)\,ds\,, \tag{A.12}$$

so

$$\tilde{Y}_j(t) - \tilde{Y}(t) = \int_a^t A(s)\Big(\tilde{Y}_j(s) - \tilde{Y}(s)\big)\,ds + R_j(t)\,, \tag{A.13}$$

where

$$R_j(t) = \bar{Y}_j - \bar{Y} + \int_a^t \Big(A_j(s) - A(s)\Big)\tilde{Y}_j(s)\,ds - \int_a^t A(s)\Big(W_j(s) - W(s)\Big)\,ds\,, \tag{A.14}$$

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**COMMENT NO. 91**: This formula seems to be wrong! I think the correct formula is

$$R_j(t) = \bar{Y}_j - \bar{Y} + \int_a^t \Big(A_j(s) - A(s)\Big)Y_j(s)\,ds + \int_a^t A(s)\Big(W_j(s) - W(s)\Big)\,ds\,.$$

Please check. I know that you are going to say that this is the part that I wrote, so you are not responsible for my mistakes. But I cannot accept that! As you may have noticed by now, I did check carefully everything you wrote. So you should have checked carefully everything I wrote.

---

and also

$$\tilde{Y}_j(t) - \tilde{Y}(t) = \bar{Y}_j - \bar{Y} + \int_a^t \Big(A_j(s) - A(s)\Big)Y_j(s)\,ds + \int_a^t A(s)\Big(Y_j(s) - Y(s)\Big)\,ds\,. \tag{A.15}$$

If $W_j - W \to 0$ a.e., then, since the sequences $\{\tilde{Y}_j\}$, $\{Y_j\}$ and $\{W_j\}$ are uniformly bounded, and $A_j\to A$ in $L^1$, it is clear that the first two terms of the sum defining $R_j(t)$ go to zero uniformly as $j\to\infty$. Also,

$$\left\|\int_a^t A(s)\Big(W_j(s) - W(s)\big)\,ds\right\| \le \int_a^b ||A(s)||\cdot||W_j(s) - W(s)||\,ds\,, \tag{A.16}$$

which goes to zero because $s\to||A(s)||$ is integrable and the sequence $\{W_j - W\}$ is uniformly bounded and goes to zero almost everywhere. So, if we let $K_j = \sup\{||R_j(t)|| : a\le t\le b\}$, we see that $K_j\to 0$ as $j\to\infty$. Then Gronwall's inequality, applied to (A.13), gives $||\tilde{Y}_j(t) - \tilde{Y}(t)|| \le K_j e^{||A||_{L^1}}$. So $\tilde{Y}_j - \tilde{Y}\to 0$ uniformly.

If $Y_j - Y\to 0$ a.e., then the first two terms of the right-hand side of (A.15) go to zero uniformly, and the third one is bounded by $\int_a^b ||A(s)||\cdot||Y_j(s) - Y(s)||\,ds$, which also goes to zero. So $\tilde{Y}_j - \tilde{Y}\to 0$ uniformly.

We have thus shown that $\tilde{Y}_j - \tilde{Y}\to 0$ uniformly if either $W_j\to W$ a.e. or $Y_j\to Y$ a.e. This clearly implies all the conclusions of (A.1.2). ∎

A very important subspace of $BM([a,b],\mathbb{R}^n)$ is $BV([a,b],\mathbb{R}^n)$, the space of all functions $W : [a,b] \mapsto \mathbb{R}^n$ such that (a) $W$ is of bounded variation, (b) $W$ is right-continuous at every point of $]a,b]$, and (c) $W(a) = 0$. There is a canonical correspondence between $BV([a,b],\mathbb{R}^n)$ and the dual space $C^0([a,b],\mathbb{R}_n)^*$ of the Banach space $C^0([a,b],\mathbb{R}_n)$ of continuous functions $[a,b] \mapsto \mathbb{R}_n$. (We write $R_n$ rather than $R^n$ because we want to think of the members of $C^0([a,b],\mathbb{R}_n)$ as *row-vector-valued functions*, cf. below.)

The members of $C^0([a,b],\mathbb{R}_n)^*$ are the *$\mathbb{R}^n$-valued Borel measures on* $[a,b]$. When $n = 1$, they are the *finite signed Borel measures on* $[a,b]$. The identification map from $BV([a,b],\mathbb{R}^n)$ to $C^0([a,b],\mathbb{R}_n)^*$ is the one that assigns to a function $W \in BV([a,b],\mathbb{R}^n)$ the unique Borel measure $\mu_W$ such that $\mu_W([a,t]) = W(t)$ for $a < t \le b$.

From now on we identify a function $W \in BV([a,b],\mathbb{R}^n)$ with its corresponding measure $\mu_W$, and write $\int \alpha \cdot dW$ for $\int \alpha \cdot d\mu_W$. Clearly, $\int \alpha \cdot dW$ is defined, more generally, for an arbitrary bounded Borel measurable $\mathbb{R}_n$-valued map $\alpha$ on $[a,b]$.

Naturally, the integral $\int \alpha \, dW$ of a scalar bounded Borel measurable function $\alpha : [a,b] \mapsto \mathbb{R}$ is also well defined, and is a vector in $\mathbb{R}^n$.

On $BV([a,b],R^n)$ there are at least three important topologies , namely, (a) the one induced by the norm $||\ldots||_{BM}$, (b) the strong topology of $BV([a,b],\mathbb{R}^n)$ as the dual of the Banach space $C^0([a,b],\mathbb{R}_n)$, (c) the weak$^*$ topology, also corresponding to the duality with $C^0([a,b],\mathbb{R}_n)$. The second one is induced by the *total variation norm*:

$$||W||_{TV} = \sup\left\{\left|\int \alpha \cdot dW\right| : \alpha \in C^0([a,b],\mathbb{R}_n)\,,\ ||\alpha||_{sup} \le 1\right\}. \tag{A.17}$$

It is clear that $||W||_{BM} \le ||W||_{TV}$ for all $W$ in $BV([a,b],\mathbb{R}^n)$. Therefore, *a subset of $BV([a,b],\mathbb{R}^n)$ which is bounded in $TV$ norm is also bounded in $BM$ norm.*

We will need the following

**Lemma A.1.** *If a sequence $\{W_j\}$ in $BV([a,b],\mathbb{R}^n)$ weak$^*$-converges to a $W \in BV([a,b],\mathbb{R}^n)$, then (a) $\{W_j\}$ is bounded in $TV$ norm, (b) $W_j(a) \to W(a)$ and $W_j(b) \to W(b)$, (c) every subsequence of $\{W_j\}$ has a subsequence that converges to $W$ at all but at most countably many points of $[a,b]$, (d) $\int_a^b \beta(s)||W_j(t) - W(t)||\,dt \to 0$ for every function $\beta \in L^1([a,b],\mathbb{R})$.*

*PROOF*. Statement (a) follows from the uniform boundedness theorem, and Statement (b) is trivial, since $W_j(a) = 0 = W(a)$, $W_j(b) = \int 1\, dW_j$, and $W(b) = \int 1\, dW$. Statement (d) clearly follows from (c), using the dominated convergence theorem together with the fact that boundedness in $TV$ norm implies boundedness in $BM$ norm.

So all we need to do is prove (c). If $Z^i$ is the $i$-th component of $Z$ for $Z \in BM([a,b],\mathbb{R}^n)$, $i = 1,\ldots,n$, then $\{W_j\}$ weak$^*$-converges to $W$ iff $\{W_j^i\}$ weak$^*$-converges to $W^i$ for each $i$. From this it clearly follows that it suffices to prove (c) for $n = 1$.

If $n = 1$, then every finite signed Borel measure $\mu$ on $[a,b]$ admits a unique decomposition $\mu = \mu^+ - \mu^-$, where $\mu^+$ and $\mu^-$ are nonnegative Borel measures that are mutually singular (that is: there exist two $\mu$-measurable sets $E$ and $F$, such that $[a,b] = E \cup F$, $E \cap F = \emptyset$, $\mu^+(E) = \mu^-(F) = 0$). This gives, using the identification of $BV([a,b],R)$ with $C^0([a,b],R)^*$, a decomposition $Z = Z^+ - Z^-$ of every $Z \in BV([a,b],\mathbb{R})$ into a difference of two members $Z^+$, $Z^-$ of $\mathcal{M}(a,b)$, the space of all monotonically nondecreasing functions $W : [a,b] \mapsto \mathbb{R}$ such that $W(a) = 0$ and $W$ is right-continuous at every $t \in ]a,b]$. It is well known that $||Z||_{TV} = Z^+(b) + Z^-(b)$.

Now let $\{W_{j(k)}\}_{k=1}^\infty$ be an arbitrary subsequence of $\{W_j\}$, and write $Z_k = W_{j(k)}$. Letting $Z_k = Z_k^+ - Z_k^-$, we have $||Z_k^+||_{TV} = Z_k^+(b) \le ||Z_k||_{TV}$, so the sequence $\{Z_k^+\}$ is bounded in $BV([a,b],\mathbb{R})$. Similarly, $\{Z_k^-\}$ is bounded in $BV([a,b],\mathbb{R})$. Since the closed balls of $BV([a,b],R)$ are weak$^*$-compact and metrizable, there exists a subsequence $\{Z_{k(\ell)}\}$ of $\{Z_k\}$ such that $Z_{k(\ell)}^+$ and

$Z^-_{k(\ell)}$ weak*-converge to limits $U$, $V$. Then of course $Z_{k(\ell)} = Z^+_{k(\ell)} - Z^-_{k(\ell)}$ weak*-converges to $U - V$, and also to $W$, so $U - V = W$. (But this does *not* mean that $U = W^+$ and $V = W^-$.)

Since the $Z^+_{k(\ell)}$ are in $\mathcal{M}(a,b)$, it is well known (and very easy to prove directly) that $\{Z^+_{k(\ell)}(t)\}$ converges to $U(t)$ at every point of continuity of $U$. Since $U$ is monotonically nondecreasing, $Z^+_{k(\ell)}(t) \to U(t)$ at all but countably many $t$'s in $[a,b]$. A similar conclusion holds for $Z^-_{k(\ell)}$ and $V$. So $Z_{k(\ell)}(t) \to W(t)$ for all but countably many $t$'s, concluding our proof. ∎

As a corollary of Lemma A.1, we get the integration by parts formula:

**Lemma A.2.** *If $\lambda : [a,b] \mapsto \mathbb{R}_n$ is absolutely continuous, and $W \in BV([a,b],\mathbb{R}^n)$, then*

$$\lambda(b) \cdot W(b) = \int \lambda \cdot dW + \int_a^b \dot\lambda(t) \cdot W(t)\, dt\,. \tag{A.18}$$

*PROOF.* The result is obviously true, from the standard product rule for derivatives, if $W$ is also absolutely continuous. In the general case, we use the fact that an arbitrary $W \in BV([a,b],\mathbb{R}^n)$ is weak*-limit of a sequence $\{W_j\}$ of absolutely continuous functions. Then (i) $\lambda(b) \cdot W_j(b) \to \lambda(b) \cdot W(b)$, because $W_j(b) \to W(b)$, (ii) $\int \lambda \cdot dW_j \to \int \lambda \cdot dW$, because $\{W_j\}$ weak*-converges to $W$, and $\lambda$ is continuous, and (iii) $\int_a^b \dot\lambda(t) \cdot W_j(t)\, dt \to \int_a^b \dot\lambda(t) \cdot W(t)\, dt$, because Part (d) of Lemma A.1 implies that $\int_a^b ||\dot\lambda(t)||.||W_j(t) - W(t)||\, dt \to 0$, since $\dot\lambda \in L^1$. ∎

An important consequence of Lemma A.2 is the following

**Lemma A.3.** *Let $W \in BV([a,b],\mathbb{R}^n)$, $A \in L^1([a,b],R^{n\times n})$, $\bar Y \in \mathbb{R}^n$, and write $Y = Y_{A,dW,\bar Y}$. Suppose $\lambda : [a,b] \to \mathbb{R}_n$ is absolutely continuous and satisfies $\dot\lambda(t) = -\lambda(t)A(t)$ for almost all $t$. Then*

$$\lambda(b) \cdot Y(b) - \lambda(a) \cdot Y(a) = \int \lambda \cdot dW\,. \tag{A.19}$$

*PROOF.* Write $Y(t) = V(t) + W(t)$, where $V(t) = Y(a) + \int_a^t A(s).Y(s)\, ds$. Then

$$\lambda(b) \cdot Y(b) - \lambda(a) \cdot Y(a) = \lambda(b) \cdot V(b) - \lambda(a) \cdot V(a) + \lambda(b) \cdot W(b)\,. \tag{A.20}$$

The function $t \to \lambda(t) \cdot V(t)$ is absolutely continuous and its derivative is $\dot\lambda(t) \cdot V(t) + \lambda(t) \cdot \dot V(t)$, which equals $-\lambda(t)A(t)Y(t) + \lambda(t)A(t)Y(t) - \dot\lambda(t) \cdot W(t)$, i.e. $-\dot\lambda(t) \cdot W(t)$. So

$$\lambda(b) \cdot Y(b) - \lambda(a) \cdot Y(a) = \lambda(b) \cdot W(b) - \int_a^b \dot\lambda(t) \cdot W(t)\, dt\,. \tag{A.21}$$

Equation (A.19) follows from this and the integration by parts formula (A.18). ∎

It is clear that, if $A \in L^1([a,b],\mathbb{R}^{n\times n})$, $\bar Y \in \mathbb{R}^n$, and $W \in BM_0([a,b],\mathbb{R}^n)$, then $Y_{A,dW,\bar Y} - W$ is absolutely continuous, so $W \in BV([a,b],\mathbb{R}^n)$ iff $Y_{A,dW,\bar Y}$ is of bounded variation, which happens iff $Y_{A,dW,\bar Y} - \bar Y \in BV([a,b],\mathbb{R}^n)$.

**Theorem A.2.** *Let $\{W_j\}$, $\{\bar Y_j\}$, $\{A_j\}$ be sequences in $BV([a,b],\mathbb{R}^n)$, $\mathbb{R}^n$, $L^1([a,b],\mathbb{R}^{n\times n})$, respectively. Assume that $\{A_j\}$ is bounded in $L^1([a,b],\mathbb{R}^{n\times n})$ and $\{\bar Y_j\}$ is bounded in $\mathbb{R}^n$. Let $Y_j = Y_{A_j,dW_j,\bar Y_j}$ . Then*

(A.2.1) $\{Y_j - \bar{Y}_j\}$ *is bounded in* $BV([a,b],\mathbb{R}^n)$ *if and only if* $\{W_j\}$ *is bounded in* $BV([a,b],\mathbb{R}^n)$.

(A.2.2) *If* $\{A_j\}$ *converges in* $L^1([a,b],\mathbb{R}^n)$ *to a limit* $A$, $\bar{Y}_j \to \bar{Y}$, $W \in BV([a,b],\mathbb{R}^n)$, *and we write* $Y = Y_{A,dW,\bar{Y}}$, *then* $\{W_j\}$ *weak*-converges to* $W$ *iff* $\{Y_j - \bar{Y}_j\}$ *weak*-converges to* $Y - \bar{Y}$. *In that case,* $Y_j - W_j - \bar{Y}_j$ *converges to* $Y - W - \bar{Y}$ *strongly in* $BV([a,b],\mathbb{R}^n)$. *In particular, (a)* $W_j(b) \to W(b)$, *(b)* $Y_j(b) \to Y(b)$, *(c) for every* $t \in [a,b]$, $Y_j(t) \to Y(t)$ *iff* $W_j(t) \to W(t)$.

*PROOF.* We know from theorem A.1 that $\{Y_j - \bar{Y}_j\}$ is bounded in $BM$ norm iff $\{W_j\}$ is. Since boundedness in $BV$ norm implies boundedness in $BM$ norm, we see that boundedness of one of the two sequences in $BV$ norm implies that there is a constant $C$ such that $||Y_j(t)|| \le C$ for all $j,t$, and then the integral $I_j = \int_a^b ||A_j(t)Y_j(t)||\,dt$ is bounded by a fixed constant $K$. But $I_j = ||Y_j - \bar{Y}_j - W_j||_{BV}$. So boundedness in $BV$ of one of the sequences $\{W_j\}$, $\{Y_j\}$ implies boundedness in $BV$ of the other one.

If either $W_j$ weak*-converges to $W$, or $Y_j - \bar{Y}_j$ weak*-converges to $Y - \bar{Y}$, then the uniform boundedness theorem implies that either $\{W_j\}$ or $\{Y_j - \bar{Y}_j\}$ is bounded in $BV$ norm, so both sequences are bounded in $BV$ norm —and hence also in $BM$— norm by a constant $C$. Writing $Z_j = Y_j - \bar{Y}_j - W_j$, $Z = Y - \bar{Y} - W$, we have then $Z_j - Z = R_j^1 + R_j^2$, where

$$R_j^1(t) = \int_a^t \Big(A_j(s) - A(s)\Big)Y_j(s)\,ds\,, \qquad R_j^2(t) = \int_a^t A(s)\Big(Y_j(s) - Y(s)\Big)\,ds\,. \tag{A.22}$$

Then $||R_j^1||_{BV} \le C||A_j - A||_{L^1}$, so $||R_j^1||_{BV} \to 0$. Also,

$$||R_j^2||_{BV} \le ||A||_{L^1}.||\bar{Y}_j - \bar{Y}|| + \int_a^b ||A(t)||.||Y_j(t) - \bar{Y}_j - (Y(t) - \bar{Y})||\,dt\,. \tag{A.23}$$

If $\{Y_j - \bar{Y}_j\}$ weak*-converges to $Y - \bar{Y}$, then Lemma A.1 implies that $||R_j^2||_{BV} \to 0$, so $Z_j \to Z$ in $BV([a,b],\mathbb{R}^n)$.

If $\{W_j\}$ weak*-converges to $W$, then we rewrite $R_j^2$ as

$$R_j^2(t) = \int_a^t A(s)\Big(Z_j(s) - Z(s)\Big)\,ds + R_j^3(t) + R_j^4(t)\,, \tag{A.24}$$

where $R_j^3(t) = \int_a^t A(s)\Big(\bar{Y}_j - \bar{Y}\Big)\,ds$ and $R_j^4(t) = \int_a^t A(s)\Big(W_j(s) - W(s)\Big)\,ds$. We then have $||R_j^3||_{BV} \le ||A||_{L^1}.||\bar{Y}_j - \bar{Y}||$ and $||R_j^4||_{BV} \le \int_a^b ||A(s)||.||W_j(s) - W(s)||\,ds$. Lemma A.1 then implies that $||R_j^4||_{BV} \to 0$. So $Z_j(t) - Z(t) = \int_a^t A(s).(Z_j(s) - Z(s))\,ds + R_j(t)$, where $R_j = R_j^1 + R_j^3 + R_j^4$, so $||R_j||_{BV} \to 0$. Then Gronwall's inequality implies that $||Z_j - Z||_{sup} \le e^{||A||_{L^1}}||R_j||_{BV}$. If $Q_j(t) = \int_a^t A(s).(Z_j(s) - Z(s))\,ds$, then $||Q_j||_{BV} \le ||A||_{L^1}||Z_j - Z||_{sup}$, so $||Q_j||_{BV} \to 0$. Since $Z_j - Z = Q_j + R_j$, we see that $||Z_j - Z||_{BV} \to 0$.

So we have shown that if one of the two sequences $\{W_j - W\}$, $\{Y_j - \bar{Y}_j - (Y - \bar{Y})\}$ weak*-converges to 0, then $Y_j - \bar{Y}_j - W_j \to Y - \bar{Y} - W$ strongly in $BV([a,b],\mathbb{R}^n)$. This implies all the conclusions of (A.2.2). ∎

Recall that, if $P, Q$ are normed spaces, and $F : \Omega \to Q$ is a map defined on a neighborhood $\Omega$ of a point $p$ of $P$, then $F$ is said to be *Fréchet differentiable* at $p$ if there exists a bounded linear map $L$ —called the *Fréchet differential* of $F$ at $p$— such that $F(p') = F(p) + L(p' - p) + o(||p' - p||)$ as $p' \to p$. If $X$ is a Banach space and $Q = X^*$, then we call $F$ *weak* Fréchet differentiable* at $p$ if for every $x \in X$ the map $\Omega \ni p' \mapsto \langle F(p'), x\rangle \in \mathbb{R}$ is Fréchet differentiable at $p$. (We use $\langle x^*, x\rangle$, for $x \in X$, $x^* \in X^*$, as an alternative notation for $x^*(x)$.)

Suppose $P$ is a normed space, $X$ is a Banach space, $p \in P$, $F : \Omega \to X^*$ is a map defined on a neighborhood $\Omega$ of $p$, and $F$ is weak* Fréchet differentiable at $p$. Then for each $x \in X$ there is a bounded linear functional $\theta_x$ on $P$ such that $\langle F(p+v), x\rangle = \langle F(p), x\rangle + \theta_x(v) + o(||v||)$ as $v \to 0$ in $P$. Clearly, $\theta_x$ is unique, and is given by the formula

$$\theta_x(v) = \lim_{h\to 0} h^{-1}\langle F(p+hv) - F(p), x\rangle\,. \tag{A.25}$$

For any given $v \in P$, use $\theta^v$ to denote the map $x \to \theta_x(v)$. Then $\theta^v$ is a limit of linear maps, so it is linear. If $\{h_j\}$ is any sequence going to 0, then the sequence of linear functionals $\nu_j$ on $X$ given by $\nu_j(x) = h_j^{-1}\langle F(p+h_jv) - F(p), x\rangle$ converges pointwise to $\theta^v$. Since each $\nu_j$ is a bounded linear functional on $X$, it follows for the uniform boundedness theorem that $\theta^v$ is bounded. Since $\theta^v(x) = \theta_x(v)$, which is linear with respect to $v$, we can conclude that $v \mapsto \theta_v$ is a linear map from $P$ to $X^*$. We use $D^wF(p)$ to denote this map, and $D^wF(p).v$ to denote its value for a particular $v$, so $D^wF(p).v = \theta^v$, and

$$\langle D^wF(p).v, x\rangle = \lim_{h\to 0} h^{-1}\langle F(p+hv) - F(p), x\rangle\,. \tag{A.26}$$

**Lemma A.4.** *If $P$ is a normed space, $X$ is a Banach space, $p \in P$, $F : \Omega \to X^*$ is a map defined on a neighborhood $\Omega$ of $p$, and $F$ is weak* Fréchet differentiable at $p$. Then:*

(A.4.i) *there is a $C > 0$ such that*

$$||F(p+v) - F(p)|| \le C||v|| \qquad \text{for all sufficiently small} \qquad v \in P\,, \tag{A.27}$$

(A.4.ii) *the linear map $D^wF(p) : P \mapsto X^*$ defined by (A.26) is bounded.*

*PROOF*. Suppose (A.4.i) is not true. Then there exists a sequence $\{v_j\}$ in $P$, converging to 0 and such that $||F(p+v_j) - F(p)|| > j||v_j||$. Define, for each $j$, a member $\sigma_j$ of $X^*$, by

$$\sigma_j = \frac{F(p+v_j) - F(p) - \theta^{v_j}}{||v_j||}\,. \tag{A.28}$$

Then $\sigma_j(x) \to 0$ for each $x$, so the sequence $\{\sigma_j\}$ is pointwise bounded and hence uniformly bounded.

On the other hand, for each $x \in X$, the numbers $\frac{\theta^{v_j}(x)}{||v_j||}$ are bounded, because $\theta^{v_j}(x) = \theta_x(v_j)$, and $\theta_x$ is a bounded linear functional on $P$, so $|\theta^{v_j}(x)| \le ||\theta_x||.||v_j||$. So, using the uniform boundedness theorem once again, we conclude that the sequence $\left\{\frac{\theta^{v_j}}{||v_j||}\right\}$ is bounded in $X^*$. Since $\{\sigma_j\}$ is bounded, we conclude that the sequence $\left\{\frac{F(p+v_j)-F(p)}{||v_j||}\right\}$ is bounded, contradicting the fact that $||F(p+v_j) - F(p)|| > j||v_j||$. So (A.27) holds for some $C > 0$, and we have proved (A.4.i).

It follows from (A.4.i) that $|\langle F(p+v) - F(p), x\rangle| \le C||x||.||v||$ for all $x \in X$, if $v \in P$ is small enough. Applying this with $hv$ in the role of $v$, and letting $h \downarrow 0$, we find that

$$|\langle D^wF(p).v, x\rangle| \le C||x||.||v|| \qquad \text{for all} \qquad x \in X\,,\ v \in P\,. \tag{A.29}$$

Therefore $||D^wF(p).v|| \le C||v||$, so $D^wF(p)$ is a bounded linear operator from $P$ to $X^*$. ∎

We are now ready to study the differentiability properties with respect to a parameter $p$, belonging to a normed space $P$, of a family of trajectories $x_p : [a,b] \to \mathbb{R}^n$ of time-varying vector fields $f_p$. We will prove, under minimal technical hypotheses, that if the initial condition map $p \mapsto x_p(a)$ is differentiable at $p = 0$, and the map $p \mapsto f_p$ satisfies a "weak differentiability condition" at the trajectory $x_0$, then the map $p \mapsto x_p$, is weak*- differentiable

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**COMMENT NO. 92**: There should not be a comma after $x_p$.

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at $p = 0$, as a map into $BV([a,b],\mathbb{R}^n)$, and in particular the endpoint map $p \mapsto x_p(b)$ is differentiable at $p = 0$.

A *curve in* $\mathbb{R}^n$ is a continuous map $\xi : I \mapsto \mathbb{R}^n$, whose domain $I = \mathrm{Dom}(\xi)$ is a nonempty interval. A curve $\xi$ is an *arc* if $\mathrm{Dom}(\xi)$ is compact. The *graph* $G(\xi)$ of a curve $\xi$ is the set $G(\xi) = \{(\xi(t),t) : t \in \mathrm{Dom}(\xi)\}$. If $\xi : [a,b] \mapsto \mathbb{R}^n$ is an arc in $\mathbb{R}^n$, and $\varepsilon > 0$, then the *$\varepsilon$-tube about* $\xi$ is the set $\mathcal{T}(\xi,\varepsilon) = \{(x,t) : x \in \mathbb{R}^n, a \le t \le b, ||x - \xi(t)|| \le \varepsilon\}$.

A *time-varying vector field* on $\mathbb{R}^n$ is an $\mathbb{R}^n$-valued map $f$ whose domain $\mathrm{Dom}(f)$ is a —possibly empty— subset $S$ of the product $\mathbb{R}^n \times \mathbb{R}$. We will use $TVVF(\mathbb{R}^n)$ to denote the set of all time-varying vector fields on $\mathbb{R}^n$. A *trajectory*, or *integral curve*, of an $f \in TVVF(\mathbb{R}^n)$ is a locally absolutely continuous curve $\xi$ in $\mathbb{R}^n$ such that $G(\xi) \subseteq \mathrm{Dom}(f)$ and $\dot\xi(t) = f(\xi(t),t)$ for almost all $t \in \mathrm{Dom}(\xi)$. We use $\mathrm{Traj}(f)$ to denote the set of all trajectories of $f$, and $\mathrm{Traj}_c(f)$ to denote the set of all $\xi \in \mathrm{Traj}(f)$ such that $\mathrm{Dom}(\xi)$ is compact.

If $f \in TVVF(\mathbb{R}^n)$ and $\xi \in \mathrm{Traj}(f)$, we say that $\xi$ is a *maximal* trajectory of $f$ if it cannot be extended to a trajectory $\tilde\xi : \tilde I \mapsto \mathbb{R}^n$ of $f$ defined on an interval $\tilde I$ such that $I \subseteq \tilde I$ but $I \neq \tilde I$. We use $\mathrm{MTraj}(f)$ to denote the set of all maximal trajectories of $f$. Given $\bar x \in \mathbb{R}^n$, $\bar t \in \mathbb{R}$, we use $\mathrm{Traj}(f,\bar x,\bar t)$, $\mathrm{MTraj}(f,\bar x,\bar t)$ to denote, respectively, the set of all $\xi \in \mathrm{Traj}(f)$ such that $\xi(\bar t) = \bar x$, and the set $\mathrm{Traj}(f,\bar x,\bar t) \cap \mathrm{MTraj}(f)$. Zorn's lemma implies that every $\xi \in \mathrm{Traj}(f,\bar x,\bar t)$ can be extended to a $\tilde\xi \in \mathrm{MTraj}(f,\bar x,\bar t)$, so $\mathrm{Traj}(f,\bar x,\bar t) \neq \emptyset$ iff $\mathrm{MTraj}(f,\bar x,\bar t) \neq \emptyset$. Clearly, $\mathrm{Traj}(f,\bar x,\bar t) \neq \emptyset$ iff $(\bar x,\bar t) \in \mathrm{Dom}(f)$. (Indeed, if $(\bar x,\bar t) \in \mathrm{Dom}(f)$, then the map $\xi$ with domain $\{\bar t\}$ such that $\xi(\bar t) = \bar x$ is in $\mathrm{Traj}(f)$.)

We say that an $f \in TVVF(\mathbb{R}^n)$ has the *forward existence property* at a point $(\bar x,\bar t)$ if for some $\varepsilon > 0$ there exists a $\xi \in \mathrm{Traj}(f,\bar x,\bar t)$ with domain $[\bar t,\bar t + \varepsilon]$. We say that $f$ has the *forward limit property* on a subset $S$ of $\mathrm{Dom}(f)$ if, whenever $\xi : [a,b[ \mapsto \mathbb{R}^n$ is a trajectory of $f$ which is contained in a compact subset of $S$, it follows that $\lim_{t\uparrow b}\xi(t)$ exists.

Suppose we are given the following data:

(D.1) a normed space $P$,
(D.2) a family $\mathbf{f} = \{f_p\}_{p\in P}$ of time-varying vector fields on $\mathbb{R}^n$, depending on a parameter $p \in P$,
(D.3) a $p_0 \in P$,
(D.4) a trajectory $\xi : [a,b] \mapsto \mathbb{R}^n$ of $f_{p_0}$.

We then define functions $w_p : [a,b] \mapsto \mathbb{R}^n$ by

$$w_p(t) = f_p(\xi(t),t) - f_{p_0}(\xi(t),t) \quad \text{for} \quad t \in [a,b]\,, \tag{A.30}$$

so $w_p$ is defined on $[a,b]$ whenever $G(\xi) \subseteq \mathrm{Dom}(f_p)$. If $w_p$ is integrable, we let

$$W_p(t) = \int_a^t w_p(s)\,ds \qquad \text{for} \qquad t \in [a,b]\,. \tag{A.31}$$

We then let $\mathcal{W}$ be the map that sends $p$ to $\mathcal{W}(p) = W_p$, with domain

$$\mathrm{Dom}(\mathcal{W}) = \{p \in P : G(\xi) \subseteq \mathrm{Dom}(f_p),\ w_p \in L^1([a,b],\mathbb{R}^n)\}\,. \tag{A.32}$$

We regard $\mathcal{W}$ as a map from $\mathrm{Dom}(\mathcal{W})$ to $BV([a,b],\mathbb{R}^n)$.

We then say that $\mathbf{f}$ is *weakly differentiable at* $p_0$ *along* $\xi$ if there exists $\bar{\varepsilon} > 0$ such that

(WD1) $\mathcal{T}(\xi, \bar{\varepsilon}) \subseteq \mathrm{Dom}(f_p)$ *for every* $p$ *such that* $||p - p_0|| \leq \bar{\varepsilon}$.

(WD2) *For every* $p$ *such that* $||p - p_0|| \leq \bar{\varepsilon}$, $f_p$ *has the forward limit property on* $\mathcal{T}(\xi, \bar{\varepsilon})$ *and the forward existence property at every* $(x, t)$ *such that* $a \leq t < b$ *and* $||x - \xi(t)|| < \bar{\varepsilon}$.

(WD3) *There exist* $A \in L^1([a, b], \mathbb{R}^{n\times n})$, *and functions* $\psi_\varepsilon \in L^1([a, b], \mathbb{R})$, *for* $\varepsilon \in ]0, \bar{\varepsilon}]$, *such that*

$$||f_p(x, t) - f_p(\xi(t), t) - A(t).(x - \xi(t))|| \leq \psi_\varepsilon(t)(||x - \xi(t)|| + ||p - p_0||) \qquad \text{(A.33)}$$

*whenever* $a \leq t \leq b$, $||x - \xi(t)|| \leq \varepsilon$, $||p - p_0|| \leq \varepsilon$, *and*

$$\lim_{\varepsilon \downarrow 0} \int_a^b \psi_\varepsilon(t)\, dt = 0\,. \qquad \text{(A.34)}$$

(WD4) *The function* $w_p$ *defined by (A.30) is integrable for every* $p$ *such that* $||p - p_0|| \leq \bar{\varepsilon}$ *and, if* $\mathcal{W}$ *is defined as above, then* $\mathcal{W}$ *is weak* Fréchet differentiable at* $p_0$.

The second part of condition (WD4) says that for each $\alpha \in C^0([a, b], \mathbb{R}_n)$ the map $p \to \int \alpha \cdot dW_p$ is Fréchet differentiable at $p_0$. Lemma A.4 then implies —since $W_{p_0} = 0$— that there exists a bounded linear map $D^w\mathcal{W}(p_0) : P \mapsto BV([a, b], \mathbb{R}^n)$ having the property that

$$\int \alpha \cdot dW_p = \langle \mathcal{W}(p) - \mathcal{W}(p_0), \alpha \rangle = \int \alpha \cdot \Big(D^w\mathcal{W}(p_0).(p - p_0)\Big) + o(||p - p_0||) \qquad \text{as} \qquad p \to p_0 \quad \text{(A.35)}$$

for each $\alpha \in C^0([a, b], \mathbb{R}_n)$, and also that there is a constant $C$ such that

$$||W_p||_{TV} \leq C||p - p_0|| \qquad \text{for all } p \in P \text{ such that } ||p - p_0|| \text{ is sufficiently small.} \qquad \text{(A.36)}$$

**Theorem A.3.** *Let* $P$, $\mathbf{f}$, $p_0$, $\xi$ *be data as in (D.1,2,3,4). Assume that* $\mathbf{f}$ *is weakly differentiable at* $p_0$ *along* $\xi$. *Let* $\bar{\varepsilon} > 0$ *be such that (WD1) to (WD4) hold. Then*

(A.3.a) *For every* $\varepsilon \in ]0, \bar{\varepsilon}]$ *there exists an* $\delta \in ]0, \varepsilon[$ *such that, whenever* $||p - p_0|| \leq \delta$, $||\bar{x} - \xi(a)|| \leq \delta$, *and* $\zeta \in \mathrm{MTraj}(f_p, \bar{x}, a)$, *it follows that* $[a, b] \subseteq \mathrm{Dom}(\zeta)$ *and* $||\zeta(t) - \xi(t)|| \leq \varepsilon$ *for all* $t \in [a, b]$.

(A.3.b) *If* $\{\bar{x}_p\}_{||p - p_0|| \leq \bar{\varepsilon}}$ *is a family of points of* $\mathbb{R}^n$ *such that the map* $p \mapsto \bar{x}_p$ *is Fréchet differentiable at* $p_0$ *and* $\bar{x}_{p_0} = \xi(a)$, *then, if* $\{\xi_p\}_{||p - p_0|| \leq \bar{\varepsilon}}$ *is a family of curves such that* $\xi_p \in \mathrm{MTraj}(f_p, \bar{x}_p, a)$, *and* $x_p$ *denotes the restriction of* $\xi_p$ *to* $[a, b]$ *—which is well defined for small enough* $||p - p_0||$ *by (A.3.a)— then the map* $\mathcal{X}$ *that sends each* $p$ *to the function* $x_p$ *is weak* Fréchet differentiable at* $p_0$ *as a map from* $P$ *to* $BV([a, b], \mathbb{R}^n)$. *Moreover,* $D^w\mathcal{X}(p_0)$ *can be computed as follows. For* $v \in P$,

$$D^w\mathcal{X}(p_0).v = Y_{A,\, d(D^w\mathcal{W}(0).v),\, L.(p - p_0)}\,, \qquad \text{(A.37)}$$

*where* $L : P \to \mathbb{R}^n$ *is the differential at* $p_0$ *of the map* $p \to \bar{x}_p$.

*PROOF*. We will assume, as we clearly may without loss of generality, that $p_0 = 0$. Also, we know that there is a $C > 0$ such that the bound (A.36) holds. So we can assume, by making $\bar{\varepsilon}$ smaller, if necessary, that $||W_p||_{TV} \leq C||p||$ whenever $||p|| \leq \bar{\varepsilon}$. By making $C$ larger and $\bar{\varepsilon}$ smaller, if necessary, we may also assume that $C \geq 1$, and

$$8e^{||A||_{L^1}}||\psi_\varepsilon||_{L^1} < 1 \qquad \text{for} \qquad \varepsilon \in ]0, \bar{\varepsilon}]\,. \qquad \text{(A.38)}$$

Pick $\varepsilon \in ]0, \bar\varepsilon[$, and choose $\delta > 0$ such that

$$C\delta < \varepsilon\,. \tag{A.39}$$

Notice that $\delta < \varepsilon$, since $C \geq 1$.

Let $p$, $\bar x$ be such that $||p|| \leq \delta$ and $||\bar x - \xi(a)|| \leq \delta$. Let $\zeta \in \mathrm{MTraj}(f_p, \bar x, a)$, and let $I = \mathrm{Dom}(\zeta)$. Then $a \in I$. Let $c$ be the supremum of those $t$ such that $a \leq t \leq b$ and $||\zeta(s) - \xi(s)|| \leq \varepsilon$ for $a \leq s \leq t$. Then $a < c \leq b$, $c \in I$, and either $c = b$ or $||\zeta(c) - \xi(c)|| = \varepsilon$. (The fact that $a < c$ follows from the local existence property, since $||\bar x - \xi(a)|| \leq \delta < \varepsilon < \bar\varepsilon$, and $\zeta$ is continuous. The fact that $c \in I$ follows from the forward limit property, since the graph of the restriction of $\zeta$ to $[a, c[$ is contained in $\mathcal{T}(\xi, \varepsilon)$. If $c < b$ and $||\zeta(c) - \xi(c)|| < \varepsilon$, then the forward existence property implies that $\zeta$ is not maximal.)

We will estimate $||\zeta(t) - \xi(t)||$ for $t \in [a, c]$, and show that —if $\varepsilon$ is small enough, and $\delta$ is suitably chosen— $||\zeta(t) - \xi(t)|| < \varepsilon$ for all such $t$, including $t = c$. This will imply that $c = b$,

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**COMMENT NO. 93**: The paragraph should end with a period, not a comma.

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From now on, we work on the interval $[a, c]$. We have

$$\zeta(t) - \xi(t) = \bar x - \xi(a) + \int_a^t \Big(f_p(\zeta(s), s) - f_0(\xi(s), s)\Big)\, ds\,, \tag{A.40}$$

so

$$\zeta(t) - \xi(t) = \bar x - \xi(a) + W_p(t) + \int_a^t \Big(f_p(\zeta(s), s) - f_p(\xi(s), s)\Big)\, ds\,. \tag{A.41}$$

Let $r_p(t) = f_p(\zeta(t), t) - f_p(\xi(t), t) - A(t).(\zeta(t) - \xi(t))$. Then

$$\zeta(t) - \xi(t) = \bar x - \xi(a) + W_p(t) + \int_a^t A(s).(\zeta(s) - \xi(s))\, ds + \int_a^t r_p(s)\, ds\,, \tag{A.42}$$

so

$$||\zeta(t) - \xi(t)|| \leq ||\bar x - \xi(a)|| + ||W_p||_{sup} + \int_a^t ||A(s)||.||\zeta(s) - \xi(s)||\, ds + \int_a^t r_p(s)\, ds\,, \tag{A.43}$$

and then

$$||\zeta(t) - \xi(t)|| \leq ||\bar x - \xi(a)|| + ||W_p||_{sup} + 2\varepsilon||\psi_\varepsilon||_{L^1} + \int_a^t ||A(s)||.||\zeta(s) - \xi(s)||\, ds\,, \tag{A.44}$$

because $||r_p(s)|| \leq 2\varepsilon\psi_\varepsilon(s)$ for $s \in [a, c]$, since $||\zeta(s) - \xi(s)|| \leq \varepsilon$ and $||p|| \leq \varepsilon$.

Since $||\bar x - \xi(a)|| \leq \delta$, and $||W_p||_{sup} \leq ||W||_{TV} \leq C||p|| \leq C\delta$, Gronwall's inequality implies

$$||\zeta(t) - \xi(t)|| \leq e^{||A||_{L^1}}\Big((C+1)\delta + 2\varepsilon||\psi_\varepsilon||_{L^1}\Big)\,. \tag{A.45}$$

In view of (A.38),we have

$$2\varepsilon e^{||A||_{L^1}}||\psi_\varepsilon||_{L^1} < \frac{\varepsilon}{4}\,. \tag{A.46}$$

Now suppose that

$$\delta \leq \kappa\varepsilon\,, \qquad \text{where} \qquad \kappa = (4C+4)^{-1}e^{-||A||_{L^1}}\,. \tag{A.47}$$

Then

$$||\zeta(t) - \xi(t)|| \leq \frac{\varepsilon}{2} \qquad \text{for} \qquad \text{all} \qquad t \in [a,c]\,. \tag{A.48}$$

This is our desired estimate, from which it follows, as explained above, that $c = b$, and then $||\zeta(t) - \xi(t)|| \leq \varepsilon$ for all $t \in [a,b]$. This completes the proof of (A.3.a).

Besides proving (A.3.a), we have also shown that, if $\kappa$ is the constant given by (A.47), then the conclusion of (A.3.a) holds for every $\delta$ such that $0 < \delta \leq \kappa\varepsilon$. (Notice that if $\delta \leq \kappa\varepsilon$ then (A.39) holds.)

Now suppose we are given a map $P \ni p \mapsto \bar{x}_p \in \mathbb{R}^n$ which is defined for $||p|| \leq \bar{\delta}$ for some $\bar{\delta} > 0$. Assume that this map is Fréchet differentiable at 0, with differential $L$, and is such that $\bar{x}_0 = \xi(a)$. Write $z_p = Y_{A,dV_p,L.p}$, where $V_p = D^w\mathcal{W}(0).p$.

It then follows that there exist a constant $C' \geq 1$ and a $\bar{\delta}' \in ]0,\bar{\delta}]$ such that (i) $C'\bar{\delta}' \leq \kappa\bar{\varepsilon}$, and (ii) $||\bar{x}_p - \xi(a)|| \leq C'||p||$ for $||p|| \leq \bar{\delta}'$.

Pick in an arbitrary fashion $\xi_p \in \mathrm{MTraj}(f_p, \bar{x}_p, a)$ for $||p|| \leq \bar{\delta}'$. Then, if $||p|| \leq \bar{\delta}'$, it follows that (i) $\xi_p$ is defined, (ii) $\xi_p \in \mathrm{MTraj}(f_p, \bar{x}, a)$ for an $\bar{x}$ —namely, $\bar{x}_p$— such that $||\bar{x} - \xi(a)|| \leq C'\bar{\delta}' \leq \kappa\bar{\varepsilon}$. Since $C' \geq 1$, we also have $||p|| \leq \kappa\bar{\varepsilon}$. Therefore we can conclude —using $\varepsilon = \bar{\varepsilon}$ and $\delta = \kappa\bar{\varepsilon}$, that $[a,b] \subseteq \mathrm{Dom}(\xi_p)$ and $(\xi_p(t), t) \in \mathcal{T}(\xi, \bar{\varepsilon})$ for $t \in [a,b]$.

Let $x_p$ be the restriction of $\xi_p$ to $[a,b]$. For $||p|| \leq \bar{\delta}'$, we have

$$x_p(t)-\xi(t)-z_p(t) = \bar{x}_p-\xi(a)-L.p+\int_a^t A(s).(x_p(s)-\xi(s)-z_p(s))\,ds+W_p(t)-V_p(t)+R_p(t)\,, \tag{A.49}$$

where $R_p(t) = \int_a^t r_p(s)\,ds$. If $p \neq 0$, let

$$Y_p = ||p||^{-1}(x_p - x_0 - z_p)\,, \qquad U_p = ||p||^{-1}(W_p - V_p + R_p)\,, \qquad \bar{Y}_p = ||p||^{-1}(\bar{x}_p - x_0(a) - L.p)\,. \tag{A.50}$$

Then $Y_p = Y_{A,dU_p,\bar{Y}_p}$.

We have to prove that $Y_p$ weak* converges to 0 as $p \to 0$. Since $\bar{Y}_p \to 0$, our conclusion will follow if we show that $U_p$ weak* converges to 0. Since $V_p = D^w\mathcal{W}(0).p$, it is clear that $||p||^{-1}(W_p - V_p)$ weak*-converges to 0. So all we need is to prove that $||p||^{-1}R_p$ weak*-converges to 0. If $||p||$ is sufficiently small, then we can apply the previous estimates using $\delta = ||p||$, $\varepsilon = \frac{\delta}{\kappa}$. We conclude that $||r_p(s)|| \leq 2\varepsilon\psi_\varepsilon(s)$ for all $s \in [a,b]$. This implies that

$$||r_p||_{L^1} \leq \frac{2||p||}{K}||\psi_{\kappa^{-1}||p||}||_{L^1}\,. \tag{A.51}$$

This shows that $||R_p||_{TV} = ||r_p||_{L^1} = o(||p||)$ as $p \to 0$. So $||p||^{-1}R_p$ converges to 0 strongly —and *a fortiori* weakly*— in $BV([a,b],\mathbb{R}^n)$. This completes our proof. ∎

If $f \in TVVF(\mathbb{R}^n)$ and $\xi : [a,b] \mapsto \mathbb{R}^n$ is a trajectory of $f$, we say that $f$ *is differentiable along* $\xi$ if the map $x \mapsto f(x,t)$ is differentiable at $\xi(t)$ for almost every $t$. In that case, we write $D_\xi f(t) \stackrel{\mathrm{def}}{=} \frac{\partial f}{\partial x}(\xi(t),t)$, and refer to the map $[a,b] \ni t \mapsto D_\xi f(t) \in \mathbb{R}^{n\times n}$ as the *differential of* $f$ *along* $\xi$.

**Lemma A.5.** *If* $\mathbf{f}$ *is weakly differentiable at* $p_0$ *along a trajectory* $\xi : [a,b] \to \mathbb{R}^n$ *of* $f_{p_0}$*, then* $f_{p_0}$ *is differentiable along* $\xi$ *and the map* $A$ *of (WD3) is such that* $A(t) = D_\xi f_{p_0}(t)$ *for almost all* $t \in [a,b]$.

*PROOF.* We begin by showing that the functions $\psi_\varepsilon$ for which (WD3) holds can be chosen so that $\psi_\varepsilon(t)$ is a monotonically nonincreasing function of $\varepsilon$ for each $t$. To see this, start with some family $\{\psi_\varepsilon\}_{\varepsilon\in]0,\bar\varepsilon]}$ of functions for which (WD3) holds, and define $\hat\psi_\varepsilon = \psi_{2^{-k}\bar\varepsilon}$ whenever $2^{-k-1}\bar\varepsilon < \varepsilon \le 2^{-k}\bar\varepsilon$ and $k$ is a nonnegative integer. Then (WD3) also holds if the $\hat\psi_\varepsilon$ are substituted for the $\psi_\varepsilon$. Next define $\bar\psi_\varepsilon(t) = \min\{\hat\psi_{\varepsilon'}(t) : \varepsilon \le \varepsilon' \le \bar\varepsilon\}$. Then the $\bar\psi_\varepsilon$ are well defined integrable functions of $t$, because for each $\varepsilon$ the set of functions $\{\hat\psi_{\varepsilon'} : \varepsilon \le \varepsilon' \le \bar\varepsilon\}$ is finite. It is clear that (WD3) also holds with the $\bar\psi_\varepsilon$ instead of the $\hat\psi_\varepsilon$, and in addition $\hat\psi_\varepsilon(t)$ is nonincreasing as a function of $\varepsilon$.

Now, if we pick the $\psi_\varepsilon$ to be monotonically nonincreasing with respect to $\varepsilon$, it follows that the limit $\rho(t) = \lim_{\varepsilon\downarrow 0}\psi_\varepsilon(t)$ exists for every $t$. By the dominated convergence theorem, $\int_a^b \rho(t)dt = 0$. Since $\rho(t) \ge 0$ for all $t$, it follows that $\rho(t) = 0$ almost everywhere.

Let $t$ be such that $\rho(t) = 0$. Let $\theta(x) = \psi_{||x-\xi(t)||}(t)$. Then $\lim_{x\to\xi(t)}\theta(x) = 0$. On the other hand, (WD3) implies that

$$||f_{p_0}(x,t) - f_{p_0}(\xi(t),t) - A(t).(x-\xi(t))|| \le \theta(x)||x-\xi(t)||\,. \tag{A.52}$$

Since $\theta(x) \to 0$ as $x \to \xi(t)$, (A.52) says precisely that $A(t)$ is the differential at $\xi(t)$ of the map $x \mapsto f_{p_0}(x,t)$, completing our proof. ∎

If $f \in TVVF(\mathbb{R}^n)$ is differentiable along a trajectory $\xi : [a,b] \to \mathbb{R}^n$ of $f$, and the matrix-valued function $D_\xi f$ is integrable, then the ordinary differential equation

$$\dot\lambda(t) = -\lambda(t).D_\xi f(t)\,, \tag{A.53}$$

for an $\mathbb{R}_n$-valued function $\lambda$ on $[a,b]$, is the *adjoint variational equation* —or, simply, the *adjoint equation*— for $f$ along $\xi$. A solution of the adjoint equation (i.e. an absolutely continuous function $\lambda : [a,b] \mapsto \mathbb{R}_n$ such that (A.53) holds for almost all $t$) is known as an *adjoint vector for $f$ along $\xi$*.

**Theorem A.4.** *Let $P$, $\mathbf{f} = \{f_p\}_{p\in P}$, $p_0 \in P$, $\xi : [a,b] \to \mathbb{R}^n$ be as in Theorem A.3. Let $P \ni p \mapsto \bar x_p \subset \mathbb{R}^n$ be a map which is Fréchet differentiable at $p_0$ and such that $\bar x_{p_0} = \xi(a)$, and let $L_a$ be its differential at $p_0$. Let $\xi_p$ be, for $p$ close enough to $p_0$, maximal trajectories of $f_p$ such that $\xi_p(a) = \bar x_p$, and let $x_p$ be the restriction of $\xi_p$ to $[a,b]$. Then the map $p \mapsto x_p(b)$ is differentiable at $p_0$. Let $L_b$ be its differential at $p_0$. Let $\lambda : [a,b] \to \mathbb{R}^n$ be an adjoint vector for $f_{p_0}$ along $\xi$. Then*

$$\lambda(b)\cdot L_b(v) - \lambda(a)\cdot L_a(v) = \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\int_a^b \Big\langle \lambda(t), f_{p_0+\varepsilon v}(\xi(t),t) - f_{p_0}(\xi(t),t)\Big\rangle dt\,. \tag{A.54}$$

*PROOF.* Assume, without loss of generality, that $p_0 = 0$. According to Theorem A.3 and Lemma A.5, the map $p \mapsto x_p$ is weak* Fréchet differentiable at 0 —as a map from $P$ to $BV([a,b],\mathbb{R}^n)$— and its differential at 0 is the map that sends $v \in P$ to $Y_{A,d(U.v),L_a(v)}$, where $U = D^w\mathcal{W}(0)$, and $\mathcal{W}$ is the map $p \mapsto W_p$, where $W_p(t) = \int_a^t \Big(f_p(\xi(s),s) - f_0(\xi(s),s)\Big)dt$, and $A = D_\xi f$. Then $p \mapsto x_p(b)$ is Fréchet differentiable at 0, and the differential $L_b$ is given by $L_b(v) = Y_{A,d(U.v),L_a(v)}(b)$. Clearly, $L_a(v) = Y_{A,d(U.v),L_a(v)}(a)$. Since $\lambda$ satisfies $\dot\lambda = -\lambda.A$, Lemma A.3 tells us that

$$\lambda(b)\cdot L_b(v) - \lambda(a)\cdot L_a(v) = \int_a^b \lambda.d(U.v)\,.$$

Since $U.v$ is the weak*-limit of $\varepsilon^{-1}W_{\varepsilon v}$ as $\varepsilon \downarrow 0$, our conclusion follows. ∎

We now turn to the differentiation of a Lagrangian cost functional.

If $f \in TVVF(\mathbb{R}^n)$, a *Lagrangian* for $f$ in $\mathbb{R}^n$ is a real-valued function $L$ such that

(L.1) $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(L)$;
(L.2) the function $t \mapsto L(\xi(t), t)$ is locally integrable for every $\xi \in \mathrm{Traj}(f)$.

If $f \in TVVF(\mathbb{R}^n)$ has components $f^1, \ldots, f^n$, and $L$ is a Lagrangian for $f$, then we can consider the time-varying vector field $[L; f]$ in $\mathbb{R}^{n+1}$ whose components are $(f^0, f^1, \ldots, f^n)$, where $f^0 \equiv L$. A trajectory of $[L; f]$ can then be regarded a pair $(\xi^0, \xi)$, where $\xi \in \mathrm{Traj}(f)$ and $\xi^0$ is a real-valued locally absolutely continuous function on $\mathrm{Dom}(\xi)$ such that $\dot{\xi}^0(t) = L(\xi(t), t)$ for a.e. $t$. Any such function $\xi^0$ will be called a *running $L$-cost along $\xi$*. Our definition of a Lagrangian implies that a running $L$-cost along $\xi$ exists for all $\xi \in \mathrm{Traj}(f)$, and it is clear that any two running $L$-costs along $\xi$ must differ by a constant. If $\xi$ is such that $\mathrm{Dom}(\xi)$ is of the form $[a, b]$ or $[a, b[$ or $[a, \infty[$, then there is a canonical choice of running $L$-cost along $\xi$, obtained by letting $\xi^0(t) = \int_a^t L(\xi(s), s)\, ds$. In that case, we will write $\xi^L$ to denote the curve $t \mapsto (\xi^0(t), \xi(t))$, where $\xi^0$ is the canonical running $L$-cost.

Now suppose we are given data $P$, $\mathbf{f}$, $p_0$, $\xi$ as in (D.1,2,3,4), as well as

(D.5) a family $\mathbf{L} = \{L_p\}_{p \in P}$ such that each $L_p$ is a Lagrangian for $f_p$.

Given a trajectory $\xi : [a, b] \mapsto \mathbb{R}^n$ of $f_{p_0}$, we say that the pair $(\mathbf{f}, \mathbf{L})$ is *weakly differentiable at $p_0$ along $\xi$* if the family $\{[L_p; f_p]\}_{p \in P}$ is weakly differentiable along $\xi^{L_{p_0}}$.

It is easy to see that $(\mathbf{f}, \mathbf{L})$ is weakly differentiable at $p_0$ along $\xi$ iff there exists $\bar{\varepsilon} > 0$ such that (WD1,2,3,4) hold, and in addition

(WD5) *There exists $h \in L^1([a, b], \mathbb{R}_n)$ such that*

$$||L_p(x, t) - L_p(\xi(t), t) - h(t).(x - \xi(t))|| \leq \psi_\varepsilon(t)(||x - \xi(t)|| + ||p||) \tag{A.55}$$

*whenever $a \leq t \leq b$, $||x - \xi(t)|| \leq \varepsilon$ and $||p|| \leq \varepsilon$.*

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**COMMENT NO. 94**: It should be $||p - p_0||$, not $||p||$.

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(WD6) *If we let $w_p^0(t) = L_p(\xi(t), t) - L_{p_0}(\xi(t), t)$, $W_p^0(t) = \int_a^t w_p^0(s)\, ds$, then the map $\mathcal{W}^0$ that sends $p$ to $\mathcal{W}^0(p) = W_p^0 \in BV([a, b], \mathbb{R})$ is weak* Fréchet differentiable at $p_0$.*

If $f \in TVVF(\mathbb{R}^n)$, $\xi : [a, b] \mapsto \mathbb{R}^n$ is a trajectory of $f$, and $L$ is a Lagrangian for $f$, we say that *$L$ is differentiable along $\xi$* if the map $x \mapsto L(x, t)$ is differentiable at $\xi(t)$ for almost every $t$. In that case, we write $\nabla_\xi L(t) \stackrel{\text{def}}{=} \nabla_x L(\xi(t), t)$, and refer to the map $[a, b] \ni t \to \nabla_\xi L(t) \in \mathbb{R}^{n \times n}$ as the *differential of $L$ along $\xi$*.

Lemma A.5, applied to the curve $\xi^{L_{p_0}}$, yields

**Lemma A.6.** *If $(\mathbf{f}, \mathbf{L})$ is weakly differentiable at $p_0$ along a trajectory $\xi : [a, b] \to \mathbb{R}^n$ of $f_{p_0}$, then $f_{p_0}$ and $L_{p_0}$ are differentiable along $\xi$ and the maps $A$, $h$ of (WD3,5) are such that $A(t) = D_\xi f(t)$ and $h(t) = \nabla_\xi L_{p_0}(\xi(t), t)$ for almost all $t \in [a, b]$.* ∎

Under the hypothesis of Lemma A.6, we can apply Theorem A.4 to the family $\{[L_p; f_p]\}_{p \in P}$ and the curve $\xi^{L_{p_0}}$. Define

$$J_p(\zeta) = \int_a^b L_p(\zeta(t), t)\, dt\,, \tag{A.56}$$

if $\zeta : [a,b] \to \mathbb{R}^n$ is a curve such that the integral exists. (We will only be using this when $\zeta \in \mathrm{Traj}(f_p)$, in which case $J_p(\zeta)$ is well defined.) An adjoint vector for $[L_{p_0}; f_{p_0}]$ along $\xi^{L_{p_0}}$ is then a pair $(\lambda^0, \lambda)$ such that $\lambda^0 \in \mathbb{R}$, $\lambda : [a,b] \mapsto \mathbb{R}_n$ is absolutely continuous, and

$$\dot{\lambda}(t) = -\lambda(t).A(t) - \lambda^0 h(t) \qquad \text{for} \qquad \text{a.e.} \qquad t \in [a,b]\,. \tag{A.57}$$

Theorem A.4 then gives:

**Theorem A.5.** *Assume that we are given data $P$, $\mathbf{f}$, $p_0$, $\xi$, $\mathbf{L}$ as in (D.1,2,3,4,5). Assume that $(\mathbf{f}, \mathbf{L})$ is weakly differentiable at $p_0$ along $\xi$. Let $P \ni p \mapsto \bar{x}_p \subset \mathbb{R}^n$ be a map which is Fréchet differentiable at $p_0$ and such that $\bar{x}_{p_0} = \xi(a)$, and let $L_a$ be its differential at $p_0$. Let $\xi_p$ be, for $p$ close enough to $p_0$, maximal trajectories of $f_p$ such that $\xi_p(a) = \bar{x}_p$, and let $x_p$ be the restriction of $\xi_p$ to $[a,b]$. Then the maps $p \mapsto x_p(b)$ and $p \mapsto J_p(x_p)$ are differentiable at $p_0$. Let $L_b$, $L_b^0$ be their differentials at $p_0$. Then, if $(\lambda^0, \lambda)$ is an adjoint vector for $[L_{p_0}, f_{p_0}]$ along $\xi$, it follows that*

$$\lambda(b)\cdot L_b(v) + \lambda^0 L_b^0(v) - \lambda(a)\cdot L_a(v) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \int_a^b \Big( H_{p_0+\varepsilon v}(\xi(t), \lambda(t), \lambda_0, t) - H_{p_0}(\xi(t), \lambda(t), \lambda_0, t) \Big)\, dt\,, \tag{A.58}$$

*where*

$$H_p(x, z, z^0, t) = \langle z, f_p(x,t)\rangle + z^0 L_p(x,t) \tag{A.59}$$

*for $x \in \mathbb{R}^n$, $z \in \mathbb{R}_n$, $z^0 \in \mathbb{R}$, $t \in [a,b]$.* ∎

## Appendix B . A lemma .

**Lemma B.1.** *Let $f$ be a real-valued function on a compact interval $[a,b]$. Assume that there exists a finite or countable subset $E$ of $[a,b]$ with the following properties:*

(B.1.a) $\liminf_{h\downarrow 0} \frac{f(x+h)-f(x)}{h} \geq 0$ *for all* $x \in [a,b[\setminus E$,
(B.1.b) $\liminf_{h\downarrow 0} f(x+h) - f(x) \geq 0$ *for all* $x \in [a,b[$,
(B.1.c) $\limsup_{h\downarrow 0} f(x) - f(x-h) \geq 0$ *for all* $x \in ]a,b]$.
*Then* $f(b) \geq f(a)$.

*PROOF.* Let $\varepsilon > 0$ be arbitrary. We will prove that $f(b) \geq f(a) - \varepsilon$. Let $\varepsilon' = \frac{\varepsilon}{1+b-a}$ and $E = \{x_k\}_{k\in N}$, $N \subset I\!N$.

For each $x \in [a,b]$, let $I(x) = \{k : x_k < x\}$. Define $g : [a,b] \mapsto \mathbb{R}$ by

$$g(x) = \sum_{k\in I(x)} 2^{-k}\varepsilon'\,.$$

Then $g$ is left-continuous and monotonically nondecreasing.

Let $S$ be the set of all $x \in [a,b]$ such that $f(y) \geq f(a) - (y-a)\varepsilon' - g(y)$ for all $y \in [a,x]$. It is clear that $a \in S$, and also that $S$ is an interval. So $S$ is of the form $[a,c[$ or $[a,c]$, with $c \in [a,b]$.

Suppose $S = [a,c[$. Then $c > a$. By (B.1.c), given $\delta > 0$ there is a sequence $\{z_j\}$ in $[a,b]$ such that $z_j < c$, $z_j \to c$, and $f(c) - f(z_j) \geq -\delta$. Then $z_j \in S$, so $f(z_j) \geq f(a) - (z_j - a)\varepsilon' - g(z_j) \geq f(a) - (c-a)\varepsilon' - g(c)$. It then follows that $f(c) \geq f(a) - (c-a)\varepsilon' - g(c) - \delta$. Since $\delta$ is arbitrary, we have $f(c) \geq f(a) - (c-a)\varepsilon' - g(c)$, so $c \in S$, which is a contradiction.

Therefore $S = [a,c]$ for some $c \in [a,b]$. We now show that $c = b$. Assume that $c < b$, and consider separately the cases $c \notin E$, $c \in E$.

If $c \notin E$, then (B.1.a) implies that there exists a $c' > c$ such that $f(x) - f(c) \geq -(x-c)\varepsilon'$ for $x \in [c,c']$. But then, since $f(c) \geq f(a) - (c-a)\varepsilon' - g(c)$, we have $f(x) \geq f(a) - (x-c)\varepsilon' - g(c) \geq f(a) - (x-a)\varepsilon' - g(x)$ for $x \in [c,c']$. Therefore $c' \in S$, contradicting the fact that $S = [a,c]$.

Now suppose $c \in E$. Then $c = x_k$ for some $k$, and $x_k \neq b$. By (B.1.b), there exists $h > 0$ such that $c + h \leq b$ and $f(y) - f(c) \geq -2^{-k}\varepsilon'$ for all $y \in [c, c+h]$. If $y \in ]c, c+h]$, then $k \notin I(c)$, and $I(c) \cup \{k\} \subseteq I(y)$. Therefore $g(y) \geq g(c) + 2^{-k}\varepsilon'$. Since $f(c) \geq f(a) - (c-a)\varepsilon' - g(c)$, we have $f(y) \geq f(c) - 2^{-k}\varepsilon' \geq f(a) - (c-a)\varepsilon' - 2^{-k}\varepsilon' - g(c) \geq f(a) - (y-a)\varepsilon' - g(y)$. So $c + h \in S$, which is a contradiction.

We have therefore proved that $c = b$. Then

$$f(b) \geq f(a) - (b-a)\varepsilon' - g(b) \geq f(a) - (b-a)\varepsilon' - \varepsilon' = f(a) - \varepsilon\,.$$

Since $\varepsilon$ is arbitrary, we have shown that $f(b) \geq f(a)$, as desired. ∎

## REFERENCES

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**COMMENT NO. 95**: The puctuation of your bibliographical references is wrong. In English, you put the comma or period before and not after you close the quotation marks. I fixed that all over the place. Also, I corrected other punctuation errors, for example the omission of the blank space before "and" in the Bardi reference.

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**COMMENT NO. 96**: I guess you didn't bother to update the bibliography, did you? The book by Martino and Italo HAS appeared, as I am sure you know.

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**COMMENT NO. 97**: I presume you have been so terribly busy that you didn't even have the time to update the items in the bibliography that were your own publications. THESE TWO PAPERS HAVE APPEARED, HAVEN'T THEY?

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