REGULAR SYNTHESIS AND SUFFICIENCY CONDITIONS FOR OPTIMALITY

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Abstract. The aim of this paper is to analyse the different definitions of synthesis given in the literature and compare them. In particular, we are interested in optimal synthesis. We prove the strongest sufficiency theorem available in the literature (the proof was only sketched in [27]). It is not difficult to check that the synthesis for the famous Fuller problem [12], satisfies the hypotheses of this theorem, but not of other sufficiency theorems [2], [6].

Keywords. Optimal control, regular synthesis, sufficient conditions.

Ref. SISSA 68/96/M

§1. Introduction

The aim of this paper is to analyse the different definitions of synthesis given in the literature and compare them. In particular we are interested in optimal synthesis. Usually the definitions of synthesis are given for two different purposes. The first one is to treat the problem of existence, the second one is to study the problem of sufficiency for optimality. In the first case, see [5], [6], [3], [4], [16], the definition of synthesis should be general enough to prove the existence of syntheses for large class of problems but at the same time a synthesis must be regular enough to obtain optimality. Indeed the general approach is to construct an extremal synthesis satisfying the regularity assumptions of an optimality theorem. The sufficiency theorems ensures the optimality of every extremal synthesis that is sufficiently regular. The use of subanalytic sets has been the most successful approach for proving the existence of optimal syntheses [5], [6], [16], [17], [18], [20], [21], [25], [27]. In this method, the concept of stratification of a manifold plays a key role and the syntheses constructed in this way present a good regularity. In fact, one first proves regularity for extremal trajectories (more precisely one proves that every extremal trajectories can be replaced by a regular trajectory having the same cost) and then, using this, constructs a regular extremal synthesis. It is clear that the strongest is the optimality theorem, the weakest are the assumptions of regularity and the largest is the class of problems for which one can prove the existence of an optimal synthesis. Hence, in general the definition of synthesis given for sufficiency theorems is less restrictive than the definition of synthesis given in existence theorems.

We start by proving the strongest sufficiency theorem available in the literature (the proof was only sketched in [27], indeed we prove a mild generalization of it). It is not difficult to check that the synthesis for the famous Fuller problem [12], satisfies the hypotheses of this theorem, but not of other sufficiency theorems [2], [6]. The main technical problem is the infinite number of switchings and the optimality was originally proved using the symmetry of the problem. This example shows how a synthesis can be *nicer* than the trajectories forming it. In particular, it suggests that there should be an existence theory not strictly related to the regularity properties of trajectories. The subanalytic sets approach requires extremal trajectories with a finite number of switchings in order to obtain a finite dimensional reduction.

There are two essentially different ways of defining a synthesis. In [2], [5], [7], [10], [16], a synthesis is defined as a collection of trajectories generated by a piecewise regular feedback control. Indeed the set of interest (that is the set of points for which there exists an optimal trajectory) is partitioned in a locally finite collection of submanifolds of various dimensions. More precisely the set is stratified and for each stratum there exists a regular feedback control. While in [14], [27], the synthesis is just a collection of trajectories not necessarily arising from a feedback control. The only assignment of the feedback control is not sufficient for describing a synthesis. Indeed a lot of trajectories may be generated by this control and some of them can be not optimal as it is shown in Example 1 of section 5. The same example shows how it is possible to have an optimal synthesis matching the definition given in [27], but not the definitions given in [2], [6]. The only point of the definition given in [6], not satisfied by this synthesis, is the continuity of the value function. But it is clear from the same example that there are nice syntheses presenting a discontinuous value function, hence the existence theory should be enlarged to cover these cases. Example 2 shows another synthesis not matching the definition given in [2].

One interesting open problem is to find the best way of defining a synthesis. A piecewise regular feedback control gives a good description of the known optimal syntheses ([6], [27]), but it presents the problems illustrated by Example 1. Therefore one has to add more structure in the definition restricting the class of systems for which a synthesis exists. A collection of trajectories is a more general object, but it has not enough structure to conceptualize regularity properties shared by many known optimal syntheses and an existence theory based on this weakly regular object seems

impossible.

Another interesting point, that should be further investigated, is the relation between the geometrical approaches of [5], [6], [16], [17], [18], [20], [21], [25], [27], and the theory of viscosity solutions [1], [11]. The latter is based on the study of the value function as a solution of the Hamilton–Jacobi equation.

It is worth it to notice that the optimality theorem we prove is valid for presynthesis (that is a synthesis not necessarily memoryless), but it is not clear if it is possible to construct an existence theory for presynthesis. Moreover, to our knowledge, there are not interesting examples of problems admitting only presyntheses but not syntheses.

Finally, the contents of this paper concerns optimal syntheses, but the discussion about the convenient definitions and the structure of syntheses is interesting also for different problems like asymptotic stabilization problems.

§2. A sufficiency theorem

We start with some definitions that will enable us to state and prove a mild generalization of the theorem whose proof was outlined in [27]. The assumptions made here will be minimal, so as to achieve maximum generality. In subsequent sections we will impose more restrictive conditions and apply our result to other definitions of synthesis given in the literature only for more regular control systems.

We consider a control system:

$$\dot{x} = f(x,u)\,, \qquad x \in \Omega\,, \qquad u \in U\,, \tag{2.1}$$

and the family $\{\mathcal{P}(f,L,\mathcal{T},\tau,x)\}_{x\in\Omega}$ of minimization problems, where $\mathcal{P}(f,L,\mathcal{T},\tau,x)$ is the problem:

$$\text{minimize} \quad \int_a^b L(\gamma(t),\eta(t))dt + \tau(\gamma(b)) \qquad \text{subject to } (2.1)\,,\ \gamma(a)=x\,,\ \text{and } \gamma(b)\in\mathcal{T}\,. \tag{2.2}$$

(A more precise formulation of the problems $\mathcal{P}(f,L,\mathcal{T},\tau,x)$, carefully specifying the space of of curves to be considered, will be given below. Since both f and L have no explicit dependence on t, we can always take $b=0$.)

We assume that

(A1) Ω *is an open subset of* $\mathbb{R}^n$,
(A2) U *is a set,*
(A3) $f:\Omega\times U\to\mathbb{R}^n$ *and* $L:\Omega\times U\to\mathbb{R}$ *are maps,*
(A4) $f(\cdot,u)$ *and* $L(\cdot,u)$ *are maps of class* C^1 *for every fixed* $u\in U$,
(A5) $\mathcal{T}$ *is a nonempty subset of* Ω,
(A6) $\tau:\Omega\to\mathbb{R}$ *is a function of class* C^1.

We use $T^c_x S$ to denote the *contingent tangent* cone to a set S at a point $x\in S$, defined as follows: a vector v is in $T^c_x S$ iff there exists a sequence $\{x_j\}$ of points of S and a sequence $\{h_j\}$ of strictly positive numbers such that $h_j\to 0$ and $x_j = x + h_j v + o(h_j)$ as $j\to\infty$. We write

$$\tilde{T}_x S = T^c_x S \cap (-T^c_x S) = \{v : v\in T^c_x S \quad \text{and} \quad -v\in T^c_x S\}\,. \tag{2.3}$$

As in Sussmann [27], we use $\tilde{f}$ to denote the $\mathbb{R}^{n+1}$ valued map $(x,u)\to(f(x,u),L(x,u))$.

If $\mu: A\to B$ is a map, we use $\mathrm{Dom}(\mu)$ to indicate the domain of μ, i.e. the set A.

A *control* is a U-valued map η such that $\mathrm{Dom}(\eta)$ is a compact subinterval $[a,b]$ of $\mathbb{R}$. A *trajectory* γ for a control η is an absolutely continuous map $\gamma : \mathrm{Dom}(\eta) \to \Omega$ that satisfies $\dot\gamma(t) = f(\gamma(t),\eta(t))$ for almost every $t \in \mathrm{Dom}(\eta)$. If $\mathrm{Dom}(\eta) = [a,b]$ (so that $\mathrm{Dom}(\gamma) = [a,b]$ as well), and $x = \gamma(a)$, $y = \gamma(b)$, we say that γ (or the pair (γ,η)) *goes from x to y*, and that η *steers x to y*, and we write $\gamma^- = x$, $\gamma^+ = y$.

A control $\eta : [a,b] \to U$ is *admissible* if the map $\Omega \times [a,b] \ni (x,t) \to \tilde f_\eta(x,t) = \tilde f(x,\eta(t)) \in \mathbb{R}^{n+1}$ satisfies the following C^1 Carathéodory conditions:

(C.i) *$t \mapsto \tilde f_\eta(x,t)$ is measurable for each $x \in \Omega$,*

(C.ii) *for every compact $K \subseteq \Omega$ there exists an integrable function φ_K such that for every $x \in K$ and $t \in [a,b]$:*

$$\|\tilde f_\eta(x,t)\| + \|D_x \tilde f_\eta(x,t)\| \le \varphi_K(t)\,. \tag{2.4}$$

If η is an admissible control and γ a trajectory corresponding to η, then we say that (γ,η) is an *admissible pair* for $\tilde f$. We use $\mathrm{Adm}(\tilde f)$ to denote the set of all admissible pairs for $\tilde f$, and write $\mathrm{Adm}^0(\tilde f)$, $\mathrm{Adm}^0(\tilde f,\mathcal{T})$, $\mathrm{Adm}^0_x(\tilde f,\mathcal{T})$, to denote, respectively, the set of all $(\gamma,\eta) \in \mathrm{Adm}(\tilde f)$ such that $\mathrm{Dom}(\eta)$ is an interval of the form $[T,0]$ with $T \le 0$, the set of those $(\gamma,\eta) \in \mathrm{Adm}^0(\tilde f)$ such that $\gamma(0) \in \mathcal{T}$, and the set of those $(\gamma,\eta) \in \mathrm{Adm}^0(\tilde f,\mathcal{T})$ such that $\gamma^- = x$. For an admissible pair $(\gamma,\eta) \in \mathrm{Adm}^0(\tilde f,\mathcal{T})$, we use $J(\gamma,\eta)$ to denote the *cost* of (γ,η), so that

$$J(\gamma,\eta) = \int_T^0 L(\gamma(t),\eta(t))\,dt + \tau(\gamma^+)\,.$$

If $\Gamma \subseteq \mathrm{Adm}^0(\tilde f,\mathcal{T})$ is an arbitrary set of admissible pairs, we define a function $V_\Gamma : \Omega \mapsto \mathbb{R} \cup \{+\infty\}$, called the *value function of* Γ, by letting

$$V_\Gamma(x) = \inf\{J(\gamma,\eta) : (\gamma,\eta) \in \Gamma,\ \gamma^- = x\} \tag{2.5}$$

for $x \in \Omega$. (In particular, $V_\Gamma(x) = +\infty$ iff there is no $(\gamma,\eta) \in \Gamma$ starting at x.) If $\Gamma = \mathrm{Adm}^0(\tilde f,\mathcal{T})$, then the function V_Γ is the *value function* of our problem, and in that case we will write $V_{\tilde f,\mathcal{T}}$ rather than $V_{\mathrm{Adm}^0(\tilde f,\mathcal{T})}$.

We can now formulate $\mathcal{P}(f,L,\mathcal{T},\tau,x)$ precisely: $\mathcal{P}(f,L,\mathcal{T},\tau,x)$ is the problem of finding a $(\gamma,\eta) \in \mathrm{Adm}^0_x(\tilde f,\mathcal{T})$ such that $J(\gamma,\eta) = V_{\tilde f,\mathcal{T}}(x)$. Any such pair is said to be a *solution* of $\mathcal{P}(f,L,\mathcal{T},\tau,x)$. A pair $(\gamma,\eta) \in \mathrm{Adm}^0(\tilde f,\mathcal{T})$ is *optimal* if it is a solution of $\mathcal{P}(f,L,\mathcal{T},\tau,\gamma^-)$.

Given $\lambda \in \mathbb{R}_n$ (the space of row n vectors), $\lambda^0 \in \mathbb{R}$, $x \in \Omega$ and $u \in U$, we define:

$$\mathcal{H}(\lambda,\lambda^0,x,u) = \langle \lambda, f(x,u)\rangle + \lambda^0 L(x,u)\,, \tag{2.6}$$

$$H(\lambda,\lambda^0,x) = \inf\{\mathcal{H}(\lambda,\lambda^0,x,u) : u \in U\}\,. \tag{2.7}$$

The functions $\mathcal{H} : \mathbb{R}_n \times \mathbb{R} \times \Omega \times U \to \mathbb{R}$ and $H : \mathbb{R}_n \times \mathbb{R} \times \Omega \to \mathbb{R} \cup \{-\infty\}$ are known as the *Hamiltonian* and the *minimized Hamiltonian* of $\tilde f$.

We say that the pair $(\gamma,\eta) \in \mathrm{Adm}^0(\tilde f,\mathcal{T})$ is *extremal* if there exist an absolutely continuous map $\lambda : \mathrm{Dom}(\gamma) \mapsto \mathbb{R}_n$ and a constant $\lambda^0 \ge 0$, that satisfy:

(EX1) *$(\lambda^0,\lambda(t)) \ne (0,0)$ for some (and hence every) t.*

(EX2) *$\dot\lambda = -\frac{\partial \mathcal{H}}{\partial x}(\lambda(t),\lambda^0,\gamma(t),\eta(t))$ for a.e. $t \in \mathrm{Dom}(\eta)$.*

(EX3) *$H(\gamma(t),\lambda(t),\lambda^0) = \mathcal{H}(\gamma(t),\lambda(t),\lambda^0,\eta(t)) = 0$ for a.e. $t \in \mathrm{Dom}(\eta)$,*

(EX4) *$\langle \lambda(0), v\rangle = \lambda^0 \langle \nabla\tau(\gamma(0)), v\rangle$ for all $v \in \tilde T_{\gamma(0)}\mathcal{T}$.*

Conditions (EX1), (EX2), (EX3), (EX4) are, respectively, the *nontriviality condition*, the *adjoint equation*, the *Hamiltonian minimization condition*, and the *transversality condition*. A pair (λ, λ^0) such that (EX2) holds is called an *adjoint vector for* (γ, η). An adjoint vector for which (EX3) holds is a *minimizing adjoint vector for* (γ, η).

The *maximum principle* says that, under some special conditions on the target set $\mathcal{T}$, *every optimal pair* (γ, η) *is extremal*. (This is true, for example, if $\mathcal{T}$ is a submanifold of Ω of class C^1, or a closed convex subset of Ω, or a set which is locally equivalent to a convex set via a C^1 diffeomorphism. More generally, it is true if the cone $\tilde{T}_{\gamma(0)}\mathcal{T}$ is convex —in which case it is of course a subspace, since $v \in \tilde{T}_{\gamma(0)}\mathcal{T} \Rightarrow -v \in \tilde{T}_{\gamma(0)}\mathcal{T}$— and is an approximating cone to $\mathcal{T}$ at $\gamma(0)$. Recall that a closed convex cone C is an *approximating cone* to a set S at a point $s \in S$ if there exist a neighborhood U of 0, and a continuous map $F : U \cap C \to S$, such that $F(x) = s + x + o(||x||)$ as $x \to 0$ via values in C.)

A *presynthesis* for our family of problems is a subset Γ of $\mathrm{Adm}^0(\tilde{f}, \mathcal{T})$ such that,

(PS) *Whenever* $(\gamma_1, \eta_1) \in \Gamma$, $(\gamma_2, \eta_2) \in \Gamma$, *and* $\gamma_1^- = \gamma_2^-$, *it follows that* $(\gamma_1, \eta_1) = (\gamma_2, \eta_2)$.

The set $\mathrm{Dom}(\Gamma) \stackrel{\mathrm{def}}{=} \{\gamma^- : (\gamma, \eta) \in \Gamma\}$ is called the *domain* of Γ. We say that Γ is a *presynthesis on* S if Γ is a presynthesis and $S = \mathrm{Dom}(\Gamma)$. Clearly, giving a presynthesis on S amounts to choosing, for each $x \in S$, a $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f}, \mathcal{T})$ such that $\gamma^- = x$.

If Γ is a presynthesis such that $\mathrm{Dom}(\Gamma)$ consists of all points that can be steered to the origin in finite time by an admissible pair, then we say that Γ is *total*.

Given a presynthesis Γ and an $x \in \mathrm{Dom}(\Gamma)$ we will always use (γ_x, η_x) to denote the unique pair $(\gamma, \eta) \in \Gamma$ such that $\gamma^- = x$, and will write $T_x = \min \mathrm{Dom}(\gamma_x)$, so $T_x \leq 0$ and $\mathrm{Dom}(\gamma_x) = \mathrm{Dom}(\eta_x) = [T_x, 0]$.

A presynthesis is *memoryless* if whenever $x \in S$ and $t \in \mathrm{Dom}(\eta_x)$ it follows that $y = \gamma_x(t)$ belongs to S and η_y is the restriction of η_x to the interval $[t, 0]$. A *synthesis* is a memoryless presynthesis. If each pair of a presynthesis Γ is optimal (resp. extremal) then we say that Γ is *optimal* (resp. *extremal*). In particular, a presynthesis Γ is a total optimal presynthesis if and only if $V_\Gamma \equiv V_{\tilde{f}}$.

Our goal is to prove that, under suitable regularity assumptions on Γ and V_Γ, a total extremal presynthesis is optimal. We first describe these regularity assumptions in detail.

We say that a function $V : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *weakly upper semicontinuous* (w.u.s.c) at a point $x \in \Omega$ if

$$\liminf_{y \to x} \limsup_{z \to y} V(z) \leq V(x)\,. \tag{2.8}$$

Equivalently, V is w.u.s.c. at x if there exist sequences x_j, ε_j and δ_j such that $x_j \in \Omega$, $x_j \to x$, $\varepsilon_j > 0$, $\delta_j > 0$, $\delta_j \to 0$, having the property that $V(y) \leq V(x) + \delta_j$ whenever $\|y - x_j\| < \varepsilon_j$.

Given a locally Lipschitz vector field X on Ω, we say that V has the *NDJ ("no downward jumps") property along* X if the following holds:

(NDJ) *For every* γ *integral curve of* X, *if* $\mathrm{Dom}(\gamma) = [a, b]$ *then for every* $t \in]a, b]$,

$$\limsup_{h \downarrow 0} V(\gamma(t)) - V(\gamma(t-h)) \geq 0$$

.

Finally we say that V satisfies the *weak continuity conditions* for the control problem (2.1), (2.2) if V is lower semicontinuous and weakly upper semicontinuous, and has the NDJ property along the vector field $x \to f(x, u)$ for every $u \in U$.

We now define the "weak differentiability conditions" for Γ. In this definition it is understood that $\tilde{f}(y, \eta_x(t)) = 0$ for every $t \notin \mathrm{Dom}(\eta_x)$. A presynthesis $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ is (f, L)*–differentiable* at a point $\bar{x} \in \Omega$ if the following condtion is satisfied:

(DC) *there exist a neighborhood N of $\bar{x}$ (in Ω) such that $N \subseteq \mathrm{Dom}(\Gamma)$, and an interval $[T,0]$ such that $\mathrm{Dom}(\eta_x) \subseteq [T,0]$ for all $x \in N$, and an $\bar{\varepsilon} > 0$, with the property that*

(DC1) *there exist, for $\varepsilon \in]0,\bar{\varepsilon}]$, integrable functions $\psi_\varepsilon : [T,0] \to \mathbb{R}$ such that*

$$\lim_{\varepsilon \to 0} \int_T^0 \psi_\varepsilon(t)\, dt = 0\,, \tag{2.9}$$

and the inequality

$$\left\| \tilde{f}(y,\eta_x(t)) - \tilde{f}(\gamma_x(t),\eta_x(t)) - \frac{\partial \tilde{f}}{\partial x}\Big(\gamma_{\bar{x}}(t),\eta_{\bar{x}}(t)\Big)\cdot(y-\gamma_{\bar{x}}(t))\right\| \leq \psi_\varepsilon(t).(||y-\gamma_{\bar{x}}(t)||+||x-\bar{x}||) \tag{2.10}$$

holds for every $y \in \Omega$, $x \in N$, $t \in [T,0]$ such that $\|y-\gamma_{\bar{x}}(t)\| \leq \varepsilon$ and $||x-\bar{x}|| \leq \varepsilon$.

(DC2) *the map $v \to \rho_v$, where ρ_v is the integrable $\mathbb{R}^{n+1}$ valued function on $[T,0]$ given by*

$$\rho_v(t) = \tilde{f}(\gamma_{\bar{x}}(t),\eta_{\bar{x}+v}(t)) - \tilde{f}(\gamma_{\bar{x}}(t),\eta_{\bar{x}}(t))$$

is weak*-differentiable at $v=0$, regarded as a map into the space of $\mathbb{R}^{n+1}$–valued Borel measures. That is: for every continuous function $\alpha : [T,0] \to \mathbb{R}$, the map $\mathbb{R}^n \ni v \to \int_T^0 \rho_v(t)\alpha(t) \in \mathbb{R}^{n+1} dt$ is differentiable at $v=0$.

A set $A \subset \Omega$ is *thin* if there exist A_1, A_2, such that $A = A_1 \cup A_2$, A_1 is a finite or countable union of connected embedded C^1 submanifolds of positive codimension and A_2 has Hausdorff dimension strictly less than $n-1$.

We can now state the main theorem (see [27]):

Theorem 2.1. *let Ω, U, f, L be such that Assumptions (A1) – (A6) hold. Let Γ be a total extremal presynthesis. Assume that*

(2.1.a) the associated value function V_Γ satisfies the weak continuity conditions,

(2.1.b) $V_\Gamma(x) \leq \tau(x)$ for every $x \in \mathcal{T}$,

(2.1.c) Γ is (f,L)–differentiable at all points in the complement of a thin set A.

Then Γ is optimal.

PROOF. We let $S = \mathrm{Dom}(\Gamma)$, so S is exactly the set of all $x \in \Omega$ that can be steered to the target $\mathcal{T}$ in finite time by an admissible control.

We first establish the relation between the differential of the value function V_Γ and the adjoint covector fields along the extremal trajectories of the presynthesis. This result is important in its own right, so we state it as a separate theorem:

Theorem 2.2. *Let Ω, U, f, L be such that Assumptions (A1) – (A6) hold. Let $\Gamma = \{(\gamma_x,\eta_x) : x \in S\}$ be a presynthesis. Let $\bar{x} \in S$ be such that Γ is (f,L)–differentiable at $\bar{x}$. Then*

(2.2.a) the function V_Γ is differentiable at $\bar{x}$;

(2.2.b) if the pair $(\gamma_{\bar{x}},\eta_{\bar{x}})$ is extremal, then every minimizing adjoint vector (λ,λ^0) for $(\gamma_{\bar{x}},\eta_{\bar{x}})$ satisfies the identity $\lambda^0 D_y V_\Gamma(\bar{x}) = \lambda(T_{\bar{x}})$. In particular, $\lambda^0 \neq 0$, and (λ,λ^0) is unique up to multiplication by a positive constant.

PROOF OF THEOREM 2.2. We use Theorem A.5 in Appendix A with $P = \mathbb{R}^n$, letting $f_x(y,t) = f(y,\eta_x(t))$, $L_x(y,t) = L(y,\eta_x(t))$, for $y \in \Omega$, $x \in N$, $t \in [T_x, 0]$, and defining $f_x(y,t) = 0$, $L_x(y,t) = 0$, for $T \leq t < T_x$. Let $\hat{V}_\Gamma(x) = \int_{T_x}^0 L(\gamma_x(t),\eta_x(t))\,dt$, so $\hat{V}_\Gamma$ is the "Lagrangian part" of our cost functional.

The map $p \mapsto \bar{x}_p$ of Theorem A.5 is the identity map, which is certainly differentiable at $\bar{x}$. So Theorem A.5 tells us that the "endpoint map" $x \mapsto \mathcal{E}(x) \stackrel{\text{def}}{=} \gamma_x(0)$ and the function $\hat{V}_\Gamma$ are differentiable at $\bar{x}$, and the identity

$$\lambda(0).D_y\mathcal{E}(\bar{x}).v + \lambda^0 D_y\hat{V}_\Gamma(\bar{x}).v - \lambda(T_{\bar{x}})\cdot v$$

$$= \lim_{\varepsilon\downarrow 0}\ \varepsilon^{-1}\int_T^0 \Big(H(\lambda(t),\lambda^0,\gamma_{\bar{x}}(t),\eta_{\bar{x}+\varepsilon v}(t)) - H(\lambda(t),\lambda^0,\gamma_{\bar{x}}(t),\eta_{\bar{x}}(t))\Big)\,dt. \quad (2.11)$$

holds for every vector v if (λ,λ^0) is an adjoint vector for $(\gamma_{\bar{x}},\eta_{\bar{x}})$. (Here the function λ is extended to $[T,0]$ by letting it equal zero on $[T,T_{\bar{x}}[$. The Hamiltonian $H(\lambda(t),\lambda^0,y,\eta_x(t))$ is also defined for all $t \in [T,0]$, by letting $H((\lambda(t),\lambda^0,y,\eta_x(t)) = \lambda(t).f(y,\eta_x(t)) + \lambda^0 L(y,\eta_x(t))$, and recalling that this expression vanishes for $t \in [T,T_x[$, because $f(y,\eta_x(t))$ and $L(y,\eta_x(t))$ have been defined to be zero for $t \in [T,T_x[$.)

Now, if (λ,λ^0) is minimizing, then the right-hand side of the inequality (2.11) is nonnegative, because $H(\lambda(t),\lambda^0,\gamma_{\bar{x}}(t),\eta_{\bar{x}}(t)) = 0$ and $H(\lambda(t),\lambda^0,\gamma_{\bar{x}}(t),\eta_{\bar{x}+\varepsilon v}(t)) \geq 0$ for almost all $t \in [T,0]$. So

$$\lambda(0).D_y\mathcal{E}(\bar{x}).v + \lambda^0 D_y\hat{V}_\Gamma(\bar{x}).v - \lambda(T_{\bar{x}})\cdot v \geq 0 \qquad \text{for all } v \in \mathbb{R}^n. \quad (2.12)$$

Given v, the curve $r \mapsto \gamma_{\bar{x}+rv}(0)$ is defined for r in a neighborhood of 0 in $\mathbb{R}$, and is differentiable at 0 with derivative $D_y\mathcal{E}(\bar{x}).v$. On the other hand, this curve is contained in $\mathcal{T}$, so the vector $w = D_y\mathcal{E}(\bar{x}).v$ belongs to $\tilde{T}_{\gamma_{\bar{x}}(0)}\mathcal{T}$. The transversality condition (EX4) then says that $\langle\lambda(0),w\rangle = \lambda^0\langle\nabla\tau(\gamma(0)),w\rangle$. Therefore

$$\lambda^0\Big(\nabla\tau(\gamma(0)).D_y\mathcal{E}(\bar{x}).v + D_y\hat{V}_\Gamma(\bar{x}).v\Big) - \lambda(T_{\bar{x}})\cdot v \geq 0 \qquad \text{for all } v \in \mathbb{R}^n. \quad (2.13)$$

Since the left-hand side of this inequality is linear in v, it has to be equal to zero. So

$$\lambda^0\Big(\nabla\tau(\gamma(0)).D_y\mathcal{E}(\bar{x}).v + D_y\hat{V}_\Gamma(\bar{x}).v\Big) = \lambda(T_{\bar{x}})\cdot v \qquad \text{for all } v \in \mathbb{R}^n. \quad (2.14)$$

Clearly, the left-hand side of this equation is $\lambda^0 D_y V_\Gamma(\bar{x}).v$. Since the equality holds for all v, we have $\lambda^0 D_y V_\Gamma(\bar{x}) = \lambda(T_{\bar{x}})$.

In particular, λ^0 cannot vanish, because $\lambda^0 = 0$ would imply $\lambda(T_{\bar{x}}) = 0$, contradicting (EX1). So we may, after multiplication by a positive constant, assume that $\lambda^0 = 1$, and then

$$D_y V_\Gamma(\bar{x}) = \lambda(T_{\bar{x}}). \quad (2.15)$$

This proves, in particular, the uniqueness of $(\lambda(\cdot),\lambda^0)$ up to multiplication by a positive constant, since the Cauchy problem for the adjoint equation has unique solutions. ∎

CONTINUATION OF THE PROOF OF THEOREM 2.1. The minimization condition (EX4) of the Pontryagin Maximum Principle implies that, for a given $x \in S \setminus A$, $u \in U$, the inequality

$$\langle \lambda(t), f(\gamma_x(t), u)\rangle + L(\gamma_x(t), u) \geq 0$$

holds for almost all $t \in [T_x, 0]$, where (λ, λ^0) is a minimizing adpoint vector for (γ_x, η_x), normalized so that $\lambda^0 = 1$. Since $t \mapsto \langle \lambda(t), f(\gamma_x(t), u)\rangle + L(\gamma_x(t), u)$ is continuous on $[T_x, 0]$, the inequality is valid at time T_x. In view of Theorem 2.2, $\lambda(T_x) = dV_\Gamma(x)$, so

$$\langle dV_\Gamma(x), f(x, u)\rangle + L(x, u) \geq 0\,. \tag{2.16}$$

Now consider an arbitrary $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f}, \mathcal{T})$, with domain $[T, 0]$. We want to prove that $J(\gamma, \eta) \geq V_\Gamma(x)$, if $x = \gamma^-$. It follows from Lemma 4.1 of [26]that there exists a sequence of admissible pairs (γ_j, η_j), with domain $[T, 0]$, such that $\gamma_j(0) = \gamma(0)$, η_j is piecewise constant, $\gamma_j \to \gamma$ uniformly on $[T, 0]$, and $J(\gamma_j, \eta_j) \to J(\gamma, \eta)$. Since the function V_Γ is lower semicontinuous, it follows that, if $V_\Gamma(\gamma_j(T)) \leq J(\gamma_j, \eta_j)$, then $V_\Gamma(x) \leq J(\gamma, \eta)$. Hence it suffices to assume that the control η is piecewise constant, and we make this assumption from now on.

We now need a transversality argument to prove that for every open set $W \subset S$ there exists a point $y \in W$ such that the trajectory γ^y, corresponding to the control η and ending at y, intersects the thin set A a finite or countable number of times. We have that $A = A_1 \cup A_2$, and A_1 is union of a finite or countable number of positive codimension manifolds M_j, while $\mathcal{H}^{n-1}(A_2) = 0$ where $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. For every $y \in W$ we have that either the trajectory γ^y is defined on the whole interval $[T, 0]$, or there exists t_y, $T \leq t_y < 0$, such that the trajectory γ^y is defined on the interval $(t_y, 0]$ but can not be prolonged backward in time.

We start considering the case of a constant control $\bar{\eta}$.

Let consider first the set A_1. Each manifold M_j is parametrized by some charts ϕ_k^j defined on open sets $Z_k^j \subset \mathbb{R}^{n_k}$, $n_k \leq n-1$. Here the index k takes value into a subset of the natural numbers. Given $z \in Z_k^j$, let $a_z < 0 < b_z$ (with a_z possibly equal to $-\infty$ and b_z possibly equal to $+\infty$) be such that (a_z, b_z) is the maximal interval of definition for the trajectory $\tilde{\gamma}_z$, corresponding to the control $\bar{\eta}$ and satisfying $\tilde{\gamma}_z(0) = \phi_k^j(z)$. Define $\tilde{Z}_k^j = \{(z, t) : z \in Z_k^j, t \in (a_z, b_z)\}$. and the map $\Psi : \tilde{Z}_k \to \mathbb{R}^n$ by $\Psi(z, t) = \tilde{\gamma}_z(t)$. Since the control $\bar{\eta}$ is constant, the map Ψ is $\mathcal{C}^1$. Moreover

$$D\Psi(z,t) = \left(M(t,0) \cdot (\frac{\partial \phi_k^j}{\partial z_1}(z), \ldots, \frac{\partial \phi_k^j}{\partial z_n}(z)), f(\tilde{\gamma}_z(t), \bar{\eta}) \right) = M(t,0) \left(\frac{\partial \phi_k^j}{\partial z_i}(z), f(\phi_k^j(z), \bar{\eta}) \right),$$

where the M is the fundamental matrix solution for the linear system

$$\dot{v}(t) = D_y f(\tilde{\gamma}_z(t), \bar{\eta}) \cdot v(t).$$

Hence $det(D\Psi(z,t)) = 0$ if and only if the vectors $\frac{\partial \phi_k^j}{\partial z_1}(z), \ldots, \frac{\partial \phi_k^j}{\partial z_{n_k}}(z)$, $f(\phi_k^j(z), \bar{\eta})$ are linearly dependent. Otherwise stated if and only if the trajectory $\tilde{\gamma}_z$ intersects the manifold M_j (more precisely $\phi_k^j(Z_k^j)$) tangentially. By Sard's theorem, the set $N_{jk} \subset \mathbb{R}^n$ of values on which this happens has measure zero. Since $\tilde{\gamma}_z(t) = y$ if and only if $\gamma^y(-t) = z$, this means that for every $y \notin N_{jk}$ the trajectory γ^y intersects the set $\phi_k^j(Z_k^j)$ transversally. Fix $y \notin N_{jk}$. If γ^y is defined on $[T, 0]$ then from the compactness of $[T, 0]$ we obtain that γ^y intersects $\phi_k^j(Z_k^j)$ a finite number of times. Otherwise, for every compact subset $K \subset (t_y, 0]$, it is clear that the trajectory γ^y restricted to K intersects the set $\phi_k^j(Z_k^j)$ a finite number of times. Since there is an increasing sequence of compact sets K_n such that $\cup_n K_n = (t_y, 0]$ we have that γ^y intersects $\phi_k^j(Z_k^j)$ at most a countable

number of times. Let $N = \cup_{jk} N_{jk}$. Since j and k takes values inside countable sets, we have that $meas(N) = 0$ and for every $y \notin N$ the trajectory γ^y intersects A_1 a finite or countable number of times.

Again let Ψ be the flow map associated to the vector field $f(x, \bar{\eta})$, and let $\tilde{A}_2$ be the subset of $A_2 \times [T, 0]$ on which Ψ is defined. We have that $\mathcal{H}^n(A_2 \times [T, 0]) = 0$ and, since Ψ is Lipschitz continuous, $meas(\Psi(\tilde{A}_2)) = \mathcal{H}^n(\Psi(\tilde{A}_2)) = 0$ (see for example [8]). Hence for almost every y the trajectory γ^y does not intersect the set A_2.

Therefore the conclusion holds for the constant control $\bar{\eta}$. Now we can repeat the argument considering separately the time sets on which the control η is constant. We conclude using again the Lipschitz continuity of the flows associated to constant controls.

From the weak upper semicontinuity of V_Γ, there exist x_i, $\varepsilon_i > 0$, $\delta_i > 0$ such that $x_i \to \gamma(0)$, $\delta_i \to 0$ and $V_\Gamma(y) \leq V_\Gamma(\gamma(0)) + \delta_i$ whenever $\|y - x_i\| \leq \varepsilon_i$. By the transversality argument, there exists y_i such that $\|y_i - x_i\| \leq \varepsilon_i$ and the trajectories γ^{y_i} intersect the set A a finite or countable number of times. Since Γ is total and $V_\Gamma(\gamma^{y_j}(0)) < +\infty$, it is clear that, for every j, $\gamma^{y_j}(t) \in S$ for every $t \in \mathrm{Dom}(\gamma^{y_j})$. Hence from (2.16) we have that the function:

$$\varphi(t) = V_\Gamma(\gamma^{y_i}(t)) + \int_T^t L(\gamma^{y_i}(s), \eta(s))ds$$

is differentiable for every t, except a finite or countable number (including the switching times of η), with positive derivative. Thus, φ verifies assumption (B.1.a) of Lemma B.1 (Appendix B). Moreover, the lower semicontinuity of V_Γ and Remark B.1 ensure assumption (B.1.b) of Lemma B.1 for φ, and the (NDJ) condition ensures assumption (B.1.c). Hence we can apply Lemma B.1 to φ. Therefore its value at time T is less or equal its value at time 0, that is

$$V_\Gamma(\gamma^{y_i}(T)) \leq \int_T^0 L(\gamma^{y_i}(t), \eta(t))dt + V_\Gamma(y_i) \leq \int_T^0 L(\gamma^{y_i}(t), \eta(t))dt + V_\Gamma(\gamma(0)) + \delta_i.$$

Since $V_\Gamma(\gamma(0)) \leq \tau(\gamma(0))$ and $\gamma^{y_i} \to \gamma$ uniformly (and in particular $\gamma^{y_i}(T) \to \gamma(T) = x$), using the lower semicontinuity of V_Γ, we conclude $V_\Gamma(x) \leq J(\gamma, \eta)$. This argument can be repeated for every admissible pair (γ, η), thus we conclude that Γ is optimal.

Remark 1. Notice that the weakly upper semicontinuity of the function V_Γ is equivalent to the weakly upper semicontinuity at the points of the target. Indeed consider a point x and its associated admissible pair (γ_x, η_x) (in the case $V_\Gamma(x) = +\infty$, (2.8) is immediately granted), and assume that V_Γ is weakly upper semicontinuous at $\gamma(0)$. Let x_j, ε_j, δ_j be as in the definition of weakly upper semicontinuity. Define γ^y to be the trajectory corresponding to the control η_x and verifying $\gamma^y(0) = y$, and let $y_j = \gamma^{x_j}(T_x)$. Now it is easy to conclude with y_j's playing the role of the x_j's.

§3. The Fuller phenomenon

The aim of this section is to prove that the famous control system, considered by Fuller in [12], satisfies the hypothesis of Theorem 2.1.

Consider the system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \qquad\quad |u| \leq 1 \\ \dot{x}_2 = u \end{cases} \tag{3.1}$$

and the minimization problem:

$$\min \int_0^{+\infty} x_1^2(t)dt. \tag{3.2}$$

In [12] Fuller showed an optimal synthesis for this problem. Let define $\zeta_\pm = \{(x_1, x_2) : x_1 = \pm\xi x_2^2, \pm x_2 \leq 0\}$ where ξ is the unique positive root of the polynomial $x^4 + x^2/12 - 1/18$. These two curves divide the plane in two regions $A^\pm$ (respectively below and above $\zeta_+ \cup \zeta_-$). For every point x there exists a unique optimal trajectory γ_x that corresponds to control $+1$ when $\gamma_x(t) \in A^+$, to control -1 when $\gamma_x(t) \in A^-$ and switches along the curves $\zeta_\pm$ reaching the origin in finite time (and then staying at the origin). See Fig.1.

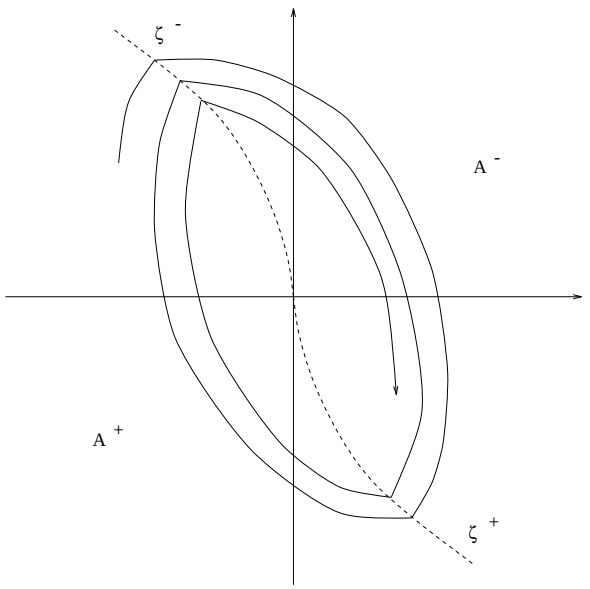

Figure 1

Therefore we can consider it as a problem of reaching the origin in finite time for (3.1) minimizing the cost (3.2). However the sufficiency theorems of [2] and [6] do not apply. Indeed, this synthesis is formed by bang–bang trajectories having an infinite number of switchings and the optimality is proved thanks to the symmetry of the problem. There exists a one parameter family of dilations that send optimal trajectories onto optimal trajectories. Given a solution $(x_1(t), x_2(t))$ of this problem, the trajectory

$$x_\rho(t) = (x_{1,\rho}(t), x_{2,\rho}(t)) \doteq \left(\rho^2 x_1\left(\frac{t}{\rho}\right), \rho x_2\left(\frac{t}{\rho}\right)\right) \tag{3.3}$$

is also an optimal solution corresponding to the control

$$u_\rho(t) = u\left(\frac{t}{\rho}\right). \tag{3.4}$$

Notice that from (3.3), $x_\rho - x = (\rho - 1)(2x, y) + (\rho - 1)^2(x, 0)$.

We want to prove the (f, L)–differentiability of the optimal synthesis at every point in the complement of the switching curves. It is easy to check the property (DC1), then let consider the property (DC2).

Fix a point $\bar{x} \in \mathbb{R}^2$. There exists s_1 such that:

$$\eta_{\bar{x}} = \sum_{k=1}^{+\infty} (\pm 1)^k \chi_{I_k}$$

where:

$$I_k = [s_k, s_{k+1}], \qquad s_k = \theta s_{k-1}, \qquad \theta = \sqrt{\frac{\frac{1}{2} - \xi}{\frac{1}{2} + \xi}},$$

χ is the indicator function and the sign $\pm$ depends on the position of the point in the plane. More precisely it is $+1$ if $\bar{x} \in A^+$ and -1 if $\bar{x} \in A^-$. Moreover from (3.4):

$$\eta_{x_\rho} = \sum_{k=1}^{+\infty} (\pm 1)^k \chi_{I_k^\rho}, \qquad I_k^\rho = [\rho s_k, \rho s_{k+1}]$$

and

$$\tilde{f}(\gamma_{\bar{x}}(t), \eta_{x_\rho}(t)) - \tilde{f}(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) = \begin{pmatrix} 0 \\ \eta_{x_\rho}(t) - \eta_{\bar{x}}(t) \\ 0 \end{pmatrix}. \tag{3.5}$$

Let $[T, 0]$ be a set containing $\mathrm{Dom}(\eta_x)$ for x near $\bar{x}$, fix a continuous function $\alpha : [T, 0] \to \mathbb{R}$ and let w be its module of continuity. The only nonzero component of (3.5) is the second one. Hence we have to estimate only the integral:

$$\int_T^0 (\eta_{x_\rho}(t) - \eta_{\bar{x}}(t))\alpha(t)dt = \int_T^0 \sum_{k=1}^{+\infty} (\pm 1)^k (\chi_{I_k^\rho}(t) - \chi_{I_k}(t))\alpha(t)dt =$$

$$= \int_T^0 \mp \chi_{[s_1, \rho s_1]}(t)\alpha(t)dt + \int_T^0 \sum_{k=2}^{+\infty} ((\pm 1)^{k-1} - (\pm 1)^k)\chi_{[s_k, \rho s_k]}(t)\alpha(t)dt =$$

$$= \pm(1-\rho)s_1\alpha(s_1) \pm 2\sum_{k=2}^{+\infty} (-1)^k(\rho - 1)s_k\alpha(s_k) +$$

$$\pm \int_{s_1}^{\rho s_1} (\alpha(s) - \alpha(s_1))ds \pm 2\sum_{k=2}^{+\infty} (-1)^k(\rho-1)s_k \int_{s_k}^{\rho s_k} (\alpha(s) - \alpha(s_k))ds \leq$$

$$\leq \mp(\rho - 1)s_1\alpha(s_1) + (\pm 2)\sum_{k=2}^{+\infty} (-1)^k(\rho-1)s_k\alpha(s_k)$$

$$\pm(1-\rho)s_1 w((1-\rho)s_1) \pm 2\sum_{k=2}^{+\infty} (-1)^k(\rho-1)s_k w((\rho-1)s_k).$$

Therefore

$$\lim_{\rho \to 1} \frac{1}{\rho - 1} \int_T^0 (\eta_{x_\rho}(t) - \eta_{\bar{x}}(t))\alpha(t)dt = \mp s_1\alpha(s_1) + (\pm 2)\sum_{k=2}^{+\infty} (-1)^k\theta^k s_1\alpha(s_k) \tag{3.6}$$

being the sum convergent since α is continuous on a compact set, hence bounded, and $\theta < 1$.

This establishes the existence of the derivative in the direction of the vector $(2x, y)$. It is easy to verify the existence of the derivative in the direction $f(\gamma_{\bar{x}}(T_{\bar{x}}), \eta_{\bar{x}}(T_{\bar{x}}))$. Let $x_\sigma = \gamma_{\bar{x}}(\sigma T_{\bar{x}}))$. In this case we have:

$$\lim_{\sigma \to 1} \frac{1}{\sigma - 1} \int_T^0 (\eta_{x_\sigma}(t) - \eta_{\bar{x}}(t))\alpha(t)dt = T_{\bar{x}}\eta_{\bar{x}}(T_{\bar{x}})\alpha(T_{\bar{x}}). \tag{3.7}$$

Now the two vectors $(2x, y)$, $f(\gamma_x(T_x), \eta_x(T_x))$ form a basis for every x in a neighborhood of $\bar{x}$ and are smooth functions of x. Moreover, from (3.6),(3.7) the derivatives in these two directions are continuous in x, hence we obtain the conclusion.

The other assumptions are easily checked and then Theorem 2.1 applies ensuring the optimality of the synthesis described above.

§4. The Boltianskii and Brunovsky definitions of synthesis

We start by reviewing the definition of synthesis proposed by Brunowsky [6] and then we recall some further restrictions existing in the definition given by Boltianskii [2].

We consider the system (2.1), (2.2), with the restriction that f is $\mathcal{C}^1$ as a function of (x, u).

Given a manifold M, a stratification of M is a locally finite collection $\mathcal{P}$ of disjoint regular submanifolds such that the following holds. If $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap clos(P_2) \neq \emptyset$ then $P_1 \subset clos(P_2)$ and $dim(P_1) < dim(P_2)$ (here *clos* denotes the topological closure and *dim* the topological dimension of a manifold). A set for which there exists a stratification is said to be *stratified*.

Let $S \subset \Omega$ be an open domain such that the origin belongs to S. A *regular synthesis* Γ in S for the control problem (1.1),(1.2) with target the origin is a 6–tuple $(\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \Pi, \Sigma, v)$ where

1) $\mathcal{P}$ is disjoint union of the origin and a locally finite (in S) partition of $S \setminus \{0\}$ into $\mathcal{C}^1$ connected submanifolds (called cells),

2) $\mathcal{P}$ is a disjoint union of $\mathcal{P}_1$ (cells of type I) and $\mathcal{P}_2$ (cells of type II), $\Pi : \mathcal{P}_1 \to \mathcal{P}$, $\Sigma : \mathcal{P}_2 \to \mathcal{P}_1$ and $v : S \to U$ (the closed loop control) are maps

such that the following properties are satisfied:

A. The set $S' = \bigcup\{P \in \mathcal{P} : \dim P < n\} \setminus \{0\}$ is a stratified subset of S of dimension $< n$

B. The function v is $\mathcal{C}^1$ on each cell. If $P_1 \in \mathcal{P}$ and $x \in P_1$, then $f(x, v(x)) \in T_x P_1$ (the tangent space to P_1 at x) and there exists a time t_x such that the trajectory ξ_x of the equation

$$\dot{x} = f(x, v(x))$$

with $\xi_x(0) = x$ satisfies $\xi_x(t) \in P_1$ for every $t \in [0, t_x[$ and $\lim_{t \to t_x} \xi_x(t) \in \Pi(P_1)$. The control v is continuous on $P_2 \cup \Sigma(P_2)$ for every cell $P_2 \in \mathcal{P}_2$. For each $x \in P_2$ there exists a unique trajectory ξ_x such that $\xi_x(0) = x$ and $\xi_x(t) \in \Sigma(P_2)$ for $t > 0$ small. In this case t_x is the time in which ξ_x reach $\Pi(\Sigma(P_2))$. On every cell P, t_x, ξ_x, $u_x(t) = v(\xi_x(t))$ are continuously differentiable functions of t, x for every $x \in P$, $t \in [0, t_x]$ (that is can be prolonged to $\mathcal{C}^1$ functions on $[0, t_x + \varepsilon_x)$ for some $\varepsilon_x > 0$)

C. The trajectory ξ_x (obtained piecing together the trajectories on every single cell) is an admissible extremal trajectory ending at the origin and passing through a finite number of cells

D. The value function V_Γ is continuous.

The definition given by Boltianskii [2] has the following restrictions:

i) Every trajectory ξ_x leaving a first type cell of dimension k must enter a cell of dimension $k - 1$. Every trajectory ξ_x leaving a second type cell of dimension k must enter a first type cell of dimension $k + 1$.

ii) The only zero dimensional cell must be the point target. The only cell of dimension one must be a trajectory ending at the target.

iii) Every trajectory ξ_x must enter a new cell in a transversal way.

Remark 2. Another definition was given in [16] using a feedback control. This definition is more general but strictly related to the definition given by Brunovsky. Also in this case the definition was

given to prove an existence theorem for stabilizing synthesis. While the definition in [27] is more general since it is not based on the existence of any feedback control generating the trajectories.

The examples of the following section show optimal syntheses that do not match the definitions above. Let Γ be the synthesis of Example 1. We have that V_Γ is discontinuous, hence Γ does not satisfy the point D of the Brunowsky definition.

The same example show another important feature. It shows the necessity of the definition of the map Σ. Indeed, in this case if we do not define the map Σ then some not optimal trajectories would be admissible solutions of the discontinuous feedback. Hence the feedback alone is not sufficient to describe the synthesis. The hypothesis of the uniqueness of the ξ_x entering $\Sigma(P_2)$ can be relaxed after adding a specification of ξ_x on an initial interval of time. This was done in [16], [25], where for every $x \in P_2 \in \mathcal{P}_2$ we have an assigned trajectory ξ_x, defined for an interval of time of length ε_x, that enters $\Sigma(P_2)$.

In any case it is important to specify the cell the trajectory ξ_x is supposed to enter. If we allow all the trajectories of the discontinuous feedback then we obtain some not optimal trajectories. For example, starting from the point $(-1,0)$ we can follow the control $+1$ for time 2π, reaching again the same point, and then go to the origin using the control -1. This trajectory γ_1 is admissible indeed at the point $(-1,0)$ the two controls ± 1 determine the same velocity vector. However, γ_1 is clearly not time optimal. We can also define the trajectory γ_2, with domain $[0,+\infty[$, corresponding to the constant control $+1$ and starting from $(-1,0)$. Also this trajectory is admissible and it neither reaches the origin in finite time nor approaches it as time goes to infinity. Notice that the trajectories γ_1 and γ_2 satisfy the equation $\dot\gamma(t) = f(\gamma(t), v(\gamma(t)))$ for every t in their domain. Hence they satisfy the equation corresponding to the discontinuous feedback in any reasonable *weak* sense. For definition of solutions to a discontinuous ODE refer to [9]. This phenomenon happens for a single point because the example is in two dimensions.

The Fuller example itself does not match the definitions above but it presents a quite regular synthesis. Since every trajectory has an infinite number of switchings the synthesis does not satisfy point C.

More dramatic is the situation for the definition given by Boltianskii. The synthesis of Example 1 does not match the restriction ii). There are many cells of dimension one and zero. This restriction is not satisfied even by very simple examples in which V_Γ is continuous. In [14], [15], [3], [4], [19], [20], [23], [24], [25], it was studied the synthesis problem and the optimal trajectories properties for control affine single–input two dimensional systems. There are some singular trajectories called turnpike. In general these turnpikes are one dimensional cells of the time optimal synthesis with target the origin and they are not necessarily ending at the origin. For some explicit examples see [15].

The synthesis of Example 2 does not match the restriction i) as it is easily shown, since some trajectories leave a three dimensional cell entering a one dimensional cell.

Finally, in [6] it was shown an example not satisfying condition iii).

§5. Examples

Example 1.
We study the time-optimal synthesis for the planar system:

$$\dot q = F(q) + u\,G(q)\,, \qquad |u| \le 1\,, \tag{5.1}$$

where

$$q \doteq \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2\,, \qquad F(q) \doteq \begin{pmatrix} 1 - \frac{y}{2} \\ \\ \frac{x+1}{2} \end{pmatrix}\,, \qquad G(q) \doteq \begin{pmatrix} -\frac{y}{2} \\ \\ \frac{x+1}{2} \end{pmatrix}\,, \tag{5.2}$$

and the target is the origin of $\mathbb{R}^2$.

The trajectories of (5.1) corresponding to the constant control $u \equiv -1$ are straight horizontal lines going from left to right, while those corresponding to $u \equiv +1$ are circles centered at the point (-1,1), running counterclockwise. The optimal synthesis, to be determined below, is shown in Fig.2.

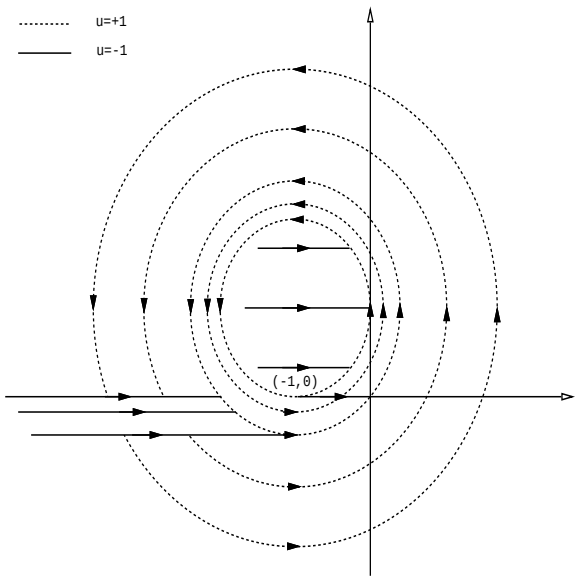

Figure 2

An arc $\gamma : [a,b] \to \mathbb{R}^2$ is *simple* if γ is an injective map. It is clear that an optimal trajectory is simple.

Since $G(q) = 0$ only for $q = (-1,0)$, it is clear that a trajectory $\gamma : [a,b] \to \mathbb{R}^2$ uniquely determines the corresponding control $u(\cdot)$, unless the set $\{t : \gamma(t) = (-1,0)\}$ is of positive measure. In particular, for simple trajectories γ —the only ones that will be of interest to us— the control is uniquely determined by γ.

We use $\mathbb{R}^k_*$ to denote the space of real row vectors with k entries, while $\mathbb{R}^k$ denotes the space of column vectors. If Ω is an open subset of $\mathbb{R}^k$ and $X : \Omega \to \mathbb{R}^k$ is a vector field of class C^1, then ∇X will denote the Jacobian matrix of X, i.e. the $k \times k$-matrix-valued function on Ω whose rows are the gradients of the components of X. If X, Y are vector fields of class C^1, then $[X,Y]$ will denote the Lie bracket of X and Y, so $[X,Y] = (\nabla Y).X - (\nabla X).Y$.

If X, Y are $\mathbb{R}^2$-valued functions on a subset of $\mathbb{R}^2$, then (X,Y) will denote the 2×2 matrix with columns X, Y. In particular, we can define the functions, see [23]:

$$\Delta_A = det\,(F,G) \qquad \Delta_B = det\,(G,[F,G]) \tag{5.3}$$

where "*det*" stands for determinant. From the results of [23], we know that every extremal trajectory of (5.1) is a finite concatenation of "bang" and "singular" arcs. A bang arc is a trajectory for a constant control with values in the set $\{-1,1\}$ of endpoints of the control interval. Singular arcs lie in the set of zeros of the function Δ_B. In our case,

$$\Delta_A = \frac{x+1}{2} \quad \text{and} \quad \Delta_B = -\frac{y}{4} \tag{5.4}$$

so the only singular arcs are those entirely contained in the x axis, and these correspond to the control -1, so they actually are bang-bang. Therefore *all the extremals of our problem are bang-bang.*

If $\Delta_A(q) \neq 0$ then the vectors $F(q)$ and $G(q)$ are linearly independent, and we can define the numbers $f(q)$ and $g(q)$ as the coefficients of the linear combination:

$$[F, G](q) = f(q)F(q) + g(q)G(q). \tag{5.5}$$

Let Ω_A, Ω_B be, respectively, the sets $\{q : \Delta_A(q) \neq 0\}$, $\{q : \Delta_B(q) \neq 0\}$, and define $\Omega_{AB} = \Omega_A \cap \Omega_B$. As was proved in [23], if γ is an extremal trajectory and $\gamma(t) \in \Omega_{AB}$ for every t, then γ is bang-bang with at most one switching, and this switching can only be from the control value -1 to the value $+1$ if $f > 0$, and from $+1$ to -1 if $f < 0$. In our case, it follows from (5.4) that

$$\Omega_A = \{(x, y) : x \neq -1\}\,, \quad \Omega_B = \{(x, y) : y \neq 0\}\,, \tag{5.6}$$

and

$$[F, G] = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} = \frac{y}{2(x+1)}F + \frac{2-y}{2(x+1)}G\,, \tag{5.7}$$

so Ω_{AB} is the union of four open quadrants $\Omega_{AB,i}$, $i = 1, 2, 3, 4$, defined by

$$\begin{aligned} \Omega_{AB,1} &= \{(x, y) : x > -1\,,\ y > 0\}\,, \\ \Omega_{AB,2} &= \{(x, y) : x < -1\,,\ y > 0\}\,, \\ \Omega_{AB,3} &= \{(x, y) : (x, -y) \in \Omega_{AB,2}\}\,, \\ \Omega_{AB,4} &= \{(x, y) : (x, -y) \in \Omega_{AB,1}\}\,. \end{aligned} \tag{5.8}$$

Clearly, $f > 0$ on $\Omega_{AB,1} \cup \Omega_{AB,3}$ and $f < 0$ on $\Omega_{AB,2} \cup \Omega_{AB,4}$. So the control switchings are from -1 to 1 on $\Omega_{AB,1} \cup \Omega_{AB,3}$ and from 1 to -1 on $\Omega_{AB,2} \cup \Omega_{AB,4}$.

To construct an optimal synthesis, we will actually construct a collection $\mathcal{F}$ of simple trajectories such that every optimal trajectory ending at the target is in $\mathcal{F}$, and for every initial point q we will find a $\gamma_q \in \mathcal{F}$ such that if $\delta \in \mathcal{F}$ steers Q to 0 and $\delta \neq \gamma_q$ then δ is not optimal. Since the existence of optimal trajectories from any initial point follows by elementary arguments, we will be able to conclude that γ_q is optimal.

Suppose that $\gamma : [a, b] \to \mathbb{R}^2$ is time-optimal, $\gamma(b) = (0, 0)$, $a < b$, and γ arises from an open-loop control $t \to u(t)$. Then $\gamma$is simple, and is a concatenation of finitely many maximal bang arcs. Since γ is an extremal, we can fix a nontrivial minimizing adjoint vector λ along γ and a constant $\lambda_0 \geq 0$ for which (EX4) holds. Define the *switching function*

$$\varphi(t) = \lambda(t) \cdot G(\gamma(t))\,.$$

The minimization condition implies that

$$\varphi(t) \neq 0 \implies u(t) = -sgn(\varphi(t))\,. \tag{5.9}$$

Let $\delta : [c, d] \to \mathbb{R}^2$ be a maximal $u = -1$ piece of γ, so $c < d$ and δ is of the form:

$$\delta(t) = (x_0 + t, y_0)\,. \tag{5.10}$$

For a.e. $t \in [c, d]$, we have $\dot{\lambda} = -\lambda \cdot \nabla(F - G) = 0$. So $\lambda = (\lambda_1, \lambda_2)$ is constant and:

$$\varphi(t) = -\lambda_1 \frac{y_0}{2} + \lambda_2 \frac{x_0 + t + 1}{2} \tag{5.11}$$

is a linear function of t. Since a linear function cannot have more than one zero unless it vanishes identically, we conclude that $a < c < d < b$ can only be true if $y_0 = 0$. If $d = b$, then of course $y_0 = 0$, since $\gamma(b) = (0, 0)$. So one of the following two possibilities occurs:

(I) δ is contained in the x axis,
(II) $a = c$.

This shows that all the maximal $u = -1$ pieces of γ are entirely contained in the x axis, with the only possible exception that γ may contain an initial horizontal segment $[a, d]$ for which $y \neq 0$.

Let us now analyze the adjoint equation for a maximal $u = +1$ piece $\delta : [c, d] \to \mathbb{R}^2$ of γ, such that $c < d$. Clearly, δ is of the form:

$$x(t) = -1 + r\cos(t + \theta)\,, \qquad y(t) = 1 + r\sin(t + \theta) \qquad c \leq t \leq d\,, \tag{5.12}$$

where $r > 0$ is a constant and $\theta \in [0, 2\pi]$. (The possibility that $r = 0$ is obviously excluded, because γ is time-optimal and $c < d$.) The components λ_1, λ_2 of λ satisfy $\dot\lambda_1 = -\lambda_2$ and $\dot\lambda_2 = -\lambda_1$, so there exist constants $A > 0$, $\theta_1 \in [0, 2\pi]$, such that

$$\lambda_1(t) = A\cos(t + \theta_1)\,, \qquad \lambda_2(t) = A\sin(t + \theta_1) \qquad c \leq t \leq d\,. \tag{5.13}$$

Therefore, up to multiplication by a positive constant,

$$\varphi(t) = -\cos(t + \theta_1) + r\sin(\theta_1 - \theta)\,. \tag{5.14}$$

Now suppose that one of the two endpoints c, d of the domain of δ is a switching time and the corresponding point in $\mathbb{R}^2$ lies in the x axis. That is, we assume that $h \in \{c, d\}$ is such that $\varphi(h) = 0$ and $y(h) = 0$. Then

$$r\sin(h + \theta) = -1\,, \tag{5.15}$$

so $\varphi(t) = -\cos(t + \theta_1) + r\Big(\sin(h + \theta_1)\cos(h + \theta) - \cos(h + \theta_1)\sin(h + \theta)\Big)$, and then

$$\varphi(t) = -\cos(t + \theta_1) + \cos(h + \theta_1) + r\sin(h + \theta_1)\cos(h + \theta)\,. \tag{5.16}$$

In particular,

$$0 = \varphi(h) = r\sin(h + \theta_1)\cos(h + \theta)\,, \tag{5.17}$$

so $\varphi(t) = -\cos(t + \theta_1) + \cos(h + \theta)$. If $r \neq 1$ then (5.15) implies $0 \geq \sin(h + \theta) \neq -1$, so $\cos(h + \theta) \neq 0$. But then (5.17) implies $\sin(h + \theta_1) = 0$, so $\cos(h + \theta_1) = \pm 1$. The possibility that $\cos(h + \theta_1) = 1$ is excluded, for in that case φ would be > 0 throughout (c, d), contradicting the minimization condition. So $\varphi(t) = -1 - \cos(t + \theta_1)$. This shows that $\varphi(t) < 0$ for $t \in [c, d] - \{h\}$, unless $d - c = 2\pi$. But if $d - c = 2\pi$ then γ would not be simple. So $d - c < 2\pi$, and we have shown that the other endpoint h' of $[c, d]$ cannot be a switching point, and therefore $h' \in \{a, b\}$.

It then follows that one of the following possibilities must occur:

(i) $d = b$;
(ii) $a = c < d < b$ and $\gamma(d)$ lies in the x axis;
(iii) $a < c < d < b$ and $\gamma(d) = (-1, 0)$.

Indeed, if $d < b$ but $\gamma(d)$ is not in the x axis, then we could let δ' be the maximal $u = -1$ piece of γ that starts at time d, and our analysis of the $u = -1$ pieces of γ, applied to δ', would show that δ' cannot in fact occur, since δ' is not an initial segment of γ and is not contained in the x axis. So, if $d < b$ then $\gamma(d)$ lies in the x axis, and d is a switching time of γ. Our previous argument shows that $c = a$, unless $r = 1$. But if $r = 1$, and $\gamma(d)$ is in the x axis, then $\gamma(d) = (-1, 0)$. So $d < b$ implies that either (ii) or (iii) must hold.

We now combine the results about the two types of pieces and provide a complete description of the optimal trajectories ending at the origin. Let us use L, L_0, C to denote, respectively, the

trajectory types "$u = -1$ and not contained in the x axis," "$u = -1$ and contained in the x axis," and "$u = 1$." (The symbols L and C stand for "line" and "circle.") Let us use $T_1T_2 \ldots T_m$ to denote the trajectory type "T_1 followed by T_2 ... followed by T_m".

With this notation, we have established that a nontrivial optimal trajectory γ ending at the origin is of one of the following seven types: L_0, C, L_0C, LC, CL_0, LCL_0, CL_0C. (Concatenations of six or more types are ruled out because any such concatenation would have to contain two different C pieces for which (iii) above holds, so γ would go at least twice through the point $(-1,0)$, and would not be simple. excluded for the same reason. The type L_0CL_0 is excluded because the last switching would have to happen at $(-1,0)$, so C would have to be the circle with center $(-1,1)$ and radius 1, which intersects the x axis only at $(-1,0)$. So the other switching would also happen at $(-1,0)$, and γ would not be simple. The exclusion of L_0CL_0 implies in turn that the types CL_0CL_0, LCL_0CL_0 and $L_0CL_0CL_0$ are ruled out as well. The types L_0CL_0C and LCL_0C are excluded because the switching from C to L_0 would have to happen at $(-1,0)$, so the L_0 piece following the switching has to be a segment going from $(-1,0)$ to a point $(\alpha,0)$ and α must satisfy $-1 < \alpha < 0$, for if α was ≥ 0 then the L_0 piece would already reach the origin before switching to the last C. But if $-1 < \alpha < 0$ then the C arc starting at $(\alpha,0)$ does not go through the origin. Finally, having excluded L_0CL_0C we are also forced to exclude CL_0CL_0C.) The LCL_0 trajectories are further restricted by the condition that the switching from C to L_0 must happen at the point $(-1,0)$. Let us call those LCL_0 trajectories that satisfy this extra condition "good."

It turns out that the optimal synthesis involves all seven types listed above, but not all trajectories of one of these types are optimal, so we need a finer analysis. We let $\mathcal{F}$ be the set of all simple trajectories that end at the origin and are either of one of the six types L_0, C, L_0C, LC, CL_0, CL_0C, or of type LCL_0 and good.

We now observe that $\dot{x} = 1 - y$ along a $u = 1$ arc, so $\dot{x} > 1$ as long as $y < 0$, $u = -1$. Therefore, if $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$ satisfy $y_1 = y_2 \leq 0$, $x_1 < x_2$, $x_1 + x_2 = -2$, so that q_1 and q_2 can be joined both by a C arc and an L (or L_0) arc, then the C arc is contained in the lower half-plane, and is faster than the L or L_0 arc. In particular, if we let $L_0(\beta)$ denote, for $\beta > 0$, the L_0 arc going from $(-\beta, 0)$ to $(0,0)$, we see that $L_0(\beta)$ *is not optimal if* $\beta > 1$, because the piece of L_0 going from $(-\beta, 0)$ to $(-2+\beta, 0)$ can be replaced by a faster type C arc.

Next we show that $L_0(1)$ *is optimal.* It suffices to notice that there is no trajectory from $(-1,0)$ to $(0,0)$ of any of the types C, LC, LCL_0, and exactly one trajectory of each of the types L_0, L_0C, CL_0, and CL_0C, but the last three are not simple, so $L_0(1)$ is left as the only possible candidate.

We now study the optimal trajectories that are of one of the types C, L_0C, LC, and CL_0C. Let $\gamma : [a,b] \to \mathbb{R}^2$ be such a trajectory, and take $b = 0$, so $\gamma(0) = (0,0)$.

If γ is a C arc, then $a > -2\pi$, for otherwise γ would contain a full loop. For $0 < a < 2\pi$, we let $C(a)$ be the C arc $\gamma : [-a, 0] \to \mathbb{R}^2$ such that $\gamma(0) = (0,0)$. It will be shown below that the arcs $C(a)$ are all optimal.

Next suppose that γ is LC. Since the C piece satisfies (5.12), we must have $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$. Let c be the time when the switching from L to C occurs. On $[c, 0]$, the switching function φ is given, up to multiplication by a positive constant, by:

$$\varphi(t) = \sqrt{2}\sin\Big(\theta_1 - \frac{\pi}{4}\Big) - \cos(t + \theta_1) = \sin\theta_1 - \cos\theta_1 - \cos(t + \theta_1).$$

So

$$\varphi(t) = (1 + \sin t)\sin\theta_1 - (1 + \cos t).\cos\theta_1$$

In addition, c cannot be $-\pi$, because $c = -\pi$ would correspond to $\gamma(c) = (-2, 2) \in Q_{AB,2}$, and we know that a switching from $u = -1$ to $u = 1$ is not permitted at points in $Q_{AB,2}$.

If $\sin\theta_1 = 0$, then $\cos\theta_1 = \pm 1$, so $\cos\theta_1 = 1$, because $\varphi(0) \leq 0$. So $\varphi(t) = 1 + \cos t$, and $\varphi(c) = 0$ would imply $c = -\pi$. Since this possibility has been ruled out, we conclude that $\sin\theta_1 \neq 0$. We can then define

$$\psi(t) = \frac{1+\sin t}{1+\cos t}\,, \quad \nu = \frac{\cos\theta_1}{\sin\theta_1}\,.$$

Then ψ is well defined and smooth except at values of t such that $t = \pi + 2k\pi$, k an integer. In our case, ψ is defined on the whole interval $[c, 0]$, except at $t = -\pi$ (if $-\pi \in [c, 0]$). Since $c \neq -\pi$, $\psi(c)$ is defined, and the switching condition $\varphi(c) = 0$ says that $\psi(c) = \nu$. If $\sin\theta_1 > 0$, then $\varphi(t) < 0$ iff $\psi(t) < \nu$ and $\varphi(t) = 0$ iff $\psi(t) = \nu$. Then $\psi(0) \leq \nu$ and, since $\lim_{t\to-\pi}\psi(t) = +\infty$, we see that c must satisfy $-\pi < c < 0$. If $\sin\theta_1 < 0$, then $\varphi(t) < 0$ iff $\psi(t) > \nu$. So $\frac{1}{2} = \psi(0) \geq \nu$. A straightforward computation shows that $\frac{d\psi}{dt}(t) > 0$ for $-2\pi \leq t < -\pi$. Since $\psi(-2\pi) = \psi(0) \geq \nu$, we conclude that $\psi(t) > \nu$ for $-2\pi \leq t < -\pi$. So, once again, the switching cannot occur at a point in $[-2\pi, -\pi)$, and then $c \in (-\pi, 0)$.

So we have shown in all possible cases that $-\pi < c < 0$. If $-\pi < c < -\frac{\pi}{2}$, then $\gamma(c) \in Q_{AB,2}$, so a switching from L to C is not permitted. If $c = -\frac{\pi}{2}$, then $\gamma(c)$ lies in the x axis, so the previous piece is L_0 rather than L. So $-\frac{\pi}{2} < c < 0$. If $\frac{\pi}{4} < c$, then $\gamma(c) \in Q_{AB,4}$, and the switching is not permitted either. So the only possibility left to us is $-\frac{\pi}{2} < c \leq -\frac{\pi}{4}$.

For $0 < \sigma < 2\pi$, $s > 0$, let $LC(s,\sigma)$ denote the trajectory that consists of a $u = -1$ arc during time s, followed by a $u = 1$ arc during time σ, and ending at the origin. We have shown that $LC(s,\sigma)$ *is not optimal unless* $\frac{\pi}{4} \leq \sigma < \frac{\pi}{2}$.

We now show that *when* $\frac{\pi}{4} \leq \sigma < \frac{\pi}{2}$ *then* $LC(s,\sigma)$ *is optimal.* To see this, let q be the starting point of $LC(s,\sigma)$, for $s > 0$, $\frac{\pi}{4} \leq \sigma < \frac{\pi}{2}$. It is clear that $LC(s,\sigma)$ is the only LC trajectory from q to $(0,0)$. Also, it is easy to see that there are no trajectories from q to $(0,0)$ of the types L_0, C, L_0C, or good LCL_0. There is exactly one CL_0 trajectory, but it is easy to see that the time $\hat{\tau}$ along this trajectory is larger than the time τ along $LC(s,\sigma)$. (Actually, $\tau < |x| < \hat{\tau} - \frac{3\pi}{2}$.) Finally, there is exactly one CL_0C trajectory γ, which can also be ruled out. (If $\bar{x} = -1 - \sqrt{2-(y+1)^2}$, so $\bar{q} = (\bar{x}, y)$ is the point where $LC(s,\sigma)$switches from L to C, then both $LC(s,\sigma)$ and γ go through $\bar{q}$ and coincide from that point on. So it suffices to compare the parts of $LC(s,\sigma)$ and γ up to $\bar{q}$. The time along $LC(s,\sigma)$ is $\bar{x} - x$, and that along γ is $> \sqrt{(1+x)^2+(1-y)^2} - 1 + \frac{3\pi}{2}$, so it suffices to notice that $\sqrt{(1+x)^2+(1-y)^2} - 1 + \frac{3\pi}{2} + x - \sqrt{2-(y+1)^2} > 0$.)

For $s > 0$, we let $L_0C(s)$ denote the trajectory that consists of a $u = -1$ arc during time s followed by a $u = 1$ arc during time $\frac{\pi}{2}$, and ending at the origin. Then is is easily shown that $L_0C(s)$ *is optimal for every* $s > 0$. (Indeed, if $q = (x, 0)$ is the starting point of $L_0C(s)$, for $s > 0$, then $L_0C(s)$ is the only L_0C trajectory from q to $(0,0)$, and there are no simple trajectories from q to $(0,0)$ of the types C, LC, CL_0, LCL_0 or CL_0C. There is an L_0 arc, namely, $L_0(|x|)$, but we know that this arc is not optimal, because $|x| > 1$.)

Now define $R_1 = \{(0,0)\}$, $R_2 = (-1,0) \times \{0\}$, $R_3 = \{(-1,0)\}$. For $q = (0,0)$, let Γ_q be the trajectory γ with domain $\{0\}$ such that $\gamma(0) = (0,0)$. For $q = (x,0) \in R_2 \cup R_3$, let $\Gamma_q = L_0(|x|)$. Then the Γ_q, $q \in R_1 \cup R_3 \cup R_3$ are optimal.

For $r > 0$, let $D(r)$ be the open disc with center $(-1,1)$ and radius r, and let $\partial D(r)$ denote the boundary of $D(r)$. Let $R_4 = D(1)$, $R_5 = \partial D(1) \backslash \{(-1,0)\}$. Then a point $q \in \mathbb{R}^2$ can be joined to $(0,0)$ by a good LCL_0 trajectory if and only if $q \in R_4$, and in that case the good trajectory is unique, and we will use $LCL_0(q)$ to denote it. Then *the arcs* $LCL_0(q)$, *for* $q \in R_4$, *are optimal.* To see this, observe that if $q \in R_4$ then the only trajectories in $\mathcal{F}$ other than $LCL_0(q)$ that go from q to the origin are either LCL_0 but not good, or LC with a switching from L to C at a time c that violates the requirement that $-\frac{\pi}{2} < c$. So all the alternative candidates are excluded, and $LCL_0(q)$ must be the optimal trajectory.

A similar argument shows that if $q \in R_5$ then the unique CL_0 trajectory going from q to $(0,0)$

—henceforth labeled $CL_0(q)$— is optimal.

Next, let $R_6 = D(\sqrt{2})\backslash(R_2 \cup D(1))$. If $q \in R_6$, then the obvious candidate for optimality is the arc Γ_q of type CL_0 obtained by following the C arc through q until R_2 is reached, and then continuing along R_2 towards the origin. We now prove that *the arcs Γ_q, $q \in R_6$, are optimal.* To see this, observe that if $q = (x, y) \in R_6$ and $y \neq 0$ then there are no simple arcs from q to $(0, 0)$ of the types L_0, C, L_0C, CL_0C. There is an LC arc, but (i) if $y < 0$ then this arc switches from L to C in $Q_{AB,4}$, so it is not optimal; (ii) if $y > 0$, then the arc is $LC(s, \sigma)$ for some $\sigma \in \pi, 2\pi)$, so it is not optimal either. If $y < 0$ or $y > 2$ or $x > -1$, then there is no good LCL_0 arc. If $0 < y < 2$ and $x < -1$ then there is a good LCL_0 arc, but it switches from L to C at a point in $Q_{AB,2}$, so this arc is not optimal. Finally, if $y < 0$ then Γ_q is the only CL_0 arc, and if $y > 0$ then there are two such arcs, but the other one is not optimal, because it contains $L_0(\beta)$ for some $\beta > 1$. This proves the optimality of the arcs Γ_q for $q \in R_6$, $y \neq 0$. A similar analysis works for $y = 0$. (Alternatively, if $q = (x, 0)$ then Γ_q is a cofinal piece of $\Gamma_{q'}$ for some $q' = (x', y') \in R_6$ such that $y' \neq 0$, so Γ_q is optimal.)

Next, we let $R_7 = \partial D(\sqrt{2})\backslash\{(0, 0)\}$. For $q \in R_7$ there is a unique trajectory Γ_q of type C going from q to $(0, 0)$. The optimality of Γ_q for $q \in R_7$ is proved by considerations similar to those used for R_6.

We let $R_8 = (-\infty, -2) \times \{0\}$, $R_9 = \{(x, y) : 1 - \sqrt{2} < y < 0\ ,\ x < -1\ ,\ (x + 1)^2 + (1 - y)^2 > 2\}$, $R_{10} = (-\infty, -1) \times \{1 - \sqrt{2}\}$. Then $R_9 \cup R_{10}$ is exactly the set of starting points q of the trajectories $LC(s, \sigma)$, for $s > 0$, $\frac{\pi}{4} \leq \sigma < \frac{\pi}{2}$, and we know that for each $q \in R_9 \cup R_{10}$ the $LC(s, \sigma)$ trajectory is unique and optimal. We define Γ_q to be this trajectory. A similar fact is true for R_8, with $L_0C(s)$ rather than $LC(s, \sigma)$, and Γ_q is then defined to be the optimal trajectory.

Finally, we let $R_{11} = \mathbb{R}^2\backslash(R_1 \cup \ldots \cup R_{10})$. For $q \in R_{11}$, we let Γ_q be the unique simple CL_0C trajectory from q to $(0, 0)$. We show that Γ_q *is optimal for all* $q \in R_{11}$. Let $\gamma \in \mathcal{F}$ be an extremal trajectory starting at q and verifying $\gamma(0) = 0$. There is no such trajectory of type L_0,C or L_0C. The case LC is impossible, otherwise γ would have a switching at time $c \in [-5\pi/4, -\pi/2[$. If γ is of type CL_0 then it contains a not optimal trajectory $L_0(\beta)$, $\beta > 1$. Finally, if γ is of type LCL_0 good then $\gamma(t) = p \in R_6$ for some t, but Γ_p is not the restriction of γ to $[t, 0]$. Hence, if γ is optimal then it is of type CL_0C.

Since we have considered all the extremal trajectories this concludes the construction.

This ends the analysis and proves that the synthesis of Fig.1 is the optimal one.

Let define the feedback control $u(x, y)$ to be equal to 0 on R_1, to -1 on the set $R_2 \cup R_3 \cup R_4 \cup R_8 \cup R_9 \cup R_{10}$, and to $+1$ on $R_5 \cup R_6 \cup R_7 \cup R_{11}$. Then the $+1$ trajectory starting from the point $(-1, 0)$ is an admissible trajectory for this feedback control but it is not optimal.

Example 2.

The following example provides a nice synthesis that is optimal but does not match the restriction i) of the definition of Boltianskii. Consider the system:

$$\begin{cases} \dot{x} = u\frac{\sqrt{x^2+y^2}}{\sqrt{1+x^2+y^2}} \\ \dot{y} = v\frac{\sqrt{x^2+y^2}}{\sqrt{1+x^2+y^2}} \\ \dot{z} = w\frac{1}{\sqrt{1+x^2+y^2}} \end{cases} \qquad u^2 + v^2 + w^2 \leq 1 \ \cdot$$

and the problem of stabilization to the origin in minimum time. Consider the following regions:

$$R_0 = \{(0, 0, 0)\}, \qquad R_1 = \{(x, y, z) : x = y = 0, z > 0\}, \qquad R_2 = \{(x, y, z) : x = y = 0, z < 0\},$$

$$R_3 = \{(x, y, z) : x^2 + y^2 > 0\}.$$

We assign a feedback control $(u_i(x,y,z), v_i(x,y,z), w_i(x,y,z))$ for every region R_i in the following way:

$$u_0 = v_0 = w_0 = 0, \qquad u_1 = v_1 = 0,\ w_1 = -1 \qquad u_2 = v_2 = 0,\ w_2 = 1,$$
$$u_3 = -\frac{x}{\sqrt{x^2+y^2}},\ v_3 = -\frac{y}{\sqrt{x^2+y^2}},\ w_3 = 0.$$

Hence starting from the point (x,y,z) we first reach the z axis using control (u_3, v_3, w_3), and then we reach the origin running on the z axis. In this way we have constructed a synthesis Γ. It is possible to prove that this synthesis Γ is optimal applying Theorem 2.1. However, in this case it is immediate to verify that the associated value function

$$V_\Gamma = |z| + \sqrt{x^2+y^2}$$

satisfies the Hamilton–Jacobi equation on the set $\{(x,y,z) : x^2+y^2 > 0, z \neq 0\}$. Indeed:

$$\inf_{u^2+v^2+z^2\le 1}\left(\frac{\partial V}{\partial x}\,u\,\frac{\sqrt{x^2+y^2}}{\sqrt{1+x^2+y^2}} + \frac{\partial V}{\partial y}\,v\,\frac{\sqrt{x^2+y^2}}{\sqrt{1+x^2+y^2}} + \frac{\partial V}{\partial z}\,w\,\frac{1}{\sqrt{1+x^2+y^2}}\right) =$$
$$= -\left\|\left(\frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}}\right)\right\| = -1.$$

Notice that every trajectory starting from a point of the three dimensional cell R_3 either reaches a one dimensional cell, i.e. R_1, R_2, or reaches the zero dimensional cell R_0. Hence Γ does not match the restriction i) of section 4.

Appendix A . Differentiation of trajectories with respect to a parameter .

The purpose of this appendix is to prove a general theorem on differentiation with respect to a parameter p, taking values in a normed space P, of a family of trajectories $x_p : [a,b] \mapsto \mathbb{R}^n$ of vector fields f_p. We will state and prove the theorem under minimal hypotheses, much weaker than what is actually needed for the main results of this paper, because the theorem has other applications, such as the theory of envelopes, where the more general statement is useful.

If $\dot{x}_p(t) = f_p(x_p(t), t)$, then formal differentiation with respect to p at $p = p_0$ in some direction v yields the variational equation

$$\dot{y}_{p_0,v}(t) = \frac{\partial f_{p_0}}{\partial x}\Big(x^{p_0}(t), t\Big).y_{p_0,v}(t) + \lim_{\varepsilon\downarrow 0}\frac{w_{p_0+\varepsilon v}(t) - w_{p_0}(t)}{\varepsilon}, \tag{A.1}$$

where $y_{p_0,v}(t)$ is the directional derivative of $x_p(t)$ at p_0 in the v-direction, and $w_p(t) = f_p(x_{p_0}(t), t)$. The technical problem that will concern us here is to make (A.1) rigorous even when the limit in the right-hand side only exists in a weak sense. For example, suppose that $n = 1$ and we are looking at the control system $\dot{x} = u$, and a family of "bang-bang" controls u_s depending on $s \in [0,\infty[$, such that $u_s(t) = 1$ for $t < s$, and $u_s(t) = -1$ for $t \ge s$. Let $x_s : [0,\infty[\mapsto \mathbb{R}$ be the solution of $\dot{x} = u_s$, $x(0) = 0$. Then $x_s(t) = t$ for $t \le s$, and $x_s(t) = 2s - t$ for $t \ge s$. Given any $T > 0$, the limit $\lim_{\varepsilon\to 0}\frac{u^T_{s+\varepsilon} - u^T_s}{\varepsilon}$ —where u^T_s is the restriction of u_s to $[0,T]$— does not exist pointwise, but exists in the weak* sense —regarding the u_s as members of the dual of the space of continuous functions on $[0,T]$— and equals $2\delta_s$, where δ_s is the Delta function at s. If we let $y_s(t)$ be the derivative of $x_s(t)$ with respect to s —so $y_s(t) = y_{s,1}(t)$— then y_s satisfies the differential equation $\dot{y}_s(t) = 2\delta_s(t)$, which makes better sense in integrated form: $y_s(t) = 2\chi_{[s,\infty[}(t)$. This says, in particular, that $y_s(T) = 2$ for $s < T$, and $y_s(T) = 0$ for $s > T$.

As this example shows, the variational equation that arises when the limit of (A.1) exists in a weak sense is a linear equation $\dot{y}(t) = A(t)y(t) + w(t)$ where the "input" w is the formal derivative of a function W which is not necessarily absolutely continuous or even continuous. So our first task will be to study the solutions of linear systems with "generalized inputs" w.

Let $-\infty < a < b < \infty$, and let n be a positive integer. If $A \in L^1([a,b];\mathbb{R}^{n\times n})$, and $\bar{Y} \in \mathbb{R}^n$, then it is well known that the initial value problem

$$\dot{Y}(t) = A(t)\cdot Y(t) + w(t)\,, \quad Y(a) = \bar{Y} \tag{A.2}$$

has a unique solution for every initial condition and every map $w \in L^1([a,b],\mathbb{R}^n)$. Because (A.2) can also be written formally as

$$dY(t) = A(t)\cdot Y(t)\,dt + w(t)\,dt\,, \quad Y(a) = \bar{Y} \tag{A.3}$$

we will use $Y_{A,w\,dt,\bar{Y}}$ to denote the unique solution of (A.2).

As explained before, it will be important for us to be able to solve (A.2) for "inputs" w more general than integrable functions. The appropriate class of inputs turns out to be that of *formal derivatives of bounded measurable functions*. Precisely, we let $BM([a,b],\mathbb{R}^n)$ denote the space of all bounded Lebesgue measurable functions $W : [a,b] \to \mathbb{R}^n$, and use $BM_0([a,b],\mathbb{R}^n)$ to denote the set of those $W \in BM([a,b],\mathbb{R}^n)$ such that $W(a) = 0$. (In $BM([a,b],\mathbb{R}^n)$ we do *not* identify two functions that are equal almost everywhere.) We endow $BM([a,b],\mathbb{R}^n)$ with the norm

$$||W||_{BM} = \sup\{||W(t)|| : t \in [a,b]\}\,. \tag{A.4}$$

Then $BM([a,b],\mathbb{R}^n)$ is a Banach space, and $BM_0([a,b],\mathbb{R}^n)$ is a closed subspace of $BM([a,b],\mathbb{R}^n)$. The space $C^0([a,b],\mathbb{R}^n)$ of continuous maps $W : [a,b] \mapsto \mathbb{R}^n$ is of course a closed subspace of $BM([a,b],\mathbb{R}^n)$, but $L^\infty([a,b],\mathbb{R}^n)$ is *not* a subspace of $BM([a,b],\mathbb{R}^n)$, because in $L^\infty([a,b],\mathbb{R}^n)$ we identify functions that are equal a.e., but in $BM([a,b],\mathbb{R}^n)$ we do not.

If $W \in BM_0([a,b],\mathbb{R}^n)$ is absolutely continuous, and $w = \dot{W}$, so that, formally, $dW = w\,dt$, then we can rewrite (A.2) formally as

$$dY(t) = A(t)\cdot Y(t)\,dt + dW(t) \qquad Y(a) = \bar{Y}\,. \tag{A.5}$$

A solution of (A.5) is then a continuous map $t \to Y(t)$ such that

$$Y(t) = \bar{Y} + \int_a^t A(s)\cdot Y(s)\,ds + W(t) \qquad \text{for} \quad \text{all} \quad t \in [a,b]\,. \tag{A.6}$$

Now (A.6) makes sense for arbitrary $Y \in BM([a,b],\mathbb{R}^n)$, $W \in BM_0([a,b],\mathbb{R}^n)$, even if W is not absolutely continuous. So we turn it into a definition: for a $W \in BM_0([a,b],\mathbb{R}^n)$, a *solution* of (A.5) is a map $Y \in BM([a,b],R^n)$ such that (A.6) holds. It then follows that

(EU) *for every $W \in BM_0([a,b],\mathbb{R}^n)$ and every $\bar{Y} \in \mathbb{R}^n$ there exists a unique solution Y of (A.5).*

Indeed, uniqueness is trivial, since the difference $Y^1 - Y^2$ of two solutions is a solution of the homogeneous problem (i.e. of (A.2) with $w \equiv 0$ and $\bar{Y} = 0$), so $Y^1 - Y^2 \equiv 0$. To prove existence, we observe that (A.6) is equivalent to

$$Y(t) - W(t) = \bar{Y} + \int_a^t A(s)\cdot(Y(s) - W(s))\,ds + \int_a^t A(s)\cdot W(s)\,ds \qquad \text{for} \quad \text{all} \quad t \in [a,b]\,, \tag{A.7}$$

i.e. to $dZ = AZ\,dt + AW\,dt$, $Z(a) = \bar{Y}$, with $Z = Y - W$. Since $AW \in L^1$, because $A \in L^1$ and W is bounded and measurable, the equation $\dot{Z}(t) = A(t)\cdot Z(t) + A(t)W(t)$ has a unique solution Z^* for which $Z^*(a) = \bar{Y}$. If we let $Y(t) = Z^*(t) + W(t)$, and recall that $W(a) = 0$, we see that Y is the desired solution of (A.5).

We will use $Y_{A,dW,\bar{Y}}$ to denote the solution of (A.5), whose existence and uniqueness has just been established. In addition to proving (EU), the preceding argument also gives an explicit formula for $Y_{A,dW,\bar{Y}}$, namely:

$$Y_{A,dW,\bar{Y}} = W + Y_{A,AW\,dt,\bar{Y}}\,. \tag{A.8}$$

Clearly, the function $Y_{A,AW\,dt,\bar{Y}}$ is absolutely continuous, so the regularity properties of $Y_{A,dW,\bar{Y}}$ —modulo absolutely continuous functions— are exactly the same as those of W. (For example: Y is continuous, right-continuous, or left-continuous at a point $t \in [a,b]$ if and only if W is; Y is of bounded variation iff W is; Y is absolutely continuous iff W is.)

Gronwall's inequality, applied to (A.7), tells us that

$$||Y_{A,dW,\bar{Y}}(t) - W(t)|| \le e^{||A||_{L^1}}\Big(||\bar{Y}|| + ||A||_{L^1}||W||_{BM}\Big)\,. \tag{A.9}$$

Therefore

$$||Y_{A,dW,\bar{Y}}(t)||_{BM} \le ||W||_{BM} + e^{||A||_{L^1}}\Big(||\bar{Y}|| + ||A||_{L^1}||W||_{BM}\Big)\,. \tag{A.10}$$

On the other hand, (A.6) also implies that

$$||W||_{BM} \le ||\bar{Y}|| + ||Y_{A,dW,\bar{Y}}||_{BM}\Big(1 + ||A||_{L^1}\Big)\,. \tag{A.11}$$

Theorem A.1. *Let $\{W_j\}$, $\{\bar{Y}_j\}$, $\{A_j\}$ be sequences in $BM_0([a,b],\mathbb{R}^n)$, $\mathbb{R}^n$, $L^1([a,b],\mathbb{R}^{n\times n})$, respectively. Assume that $\{A_j\}$ is bounded in $L^1([a,b],\mathbb{R}^{n\times n})$. Let $Y_j = Y_{A_j,dW_j,\bar{Y}_j}$. Then*

(A.1.1) *$\{Y_j\}$ is bounded in $BM([a,b],\mathbb{R}^n)$ if and only if $\{W_j\}$ is bounded in $BM([a,b],\mathbb{R}^n)$ and $\{\bar{Y}_j\}$ is bounded in $\mathbb{R}^n$.*

(A.1.2) *if $\{A_j\}$ converges in $L^1([a,b],\mathbb{R}^n)$ to a limit A, $\bar{Y} \in \mathbb{R}^n$, $\bar{Y}_j \to \bar{Y}$, $\{W_j\}$ is bounded in $BM([a,b],\mathbb{R}^n)$, $W \in BM_0([a,b],\mathbb{R}^n)$, and $Y = Y_{A,dW,\bar{Y}}$, then $Y_j \to Y$ almost everywhere iff $W_j \to W$ almost everywhere. In that case, $Y_j - W_j$ converges to $Y - W$ uniformly on $[a,b]$. In particular, for every $t \in [a,b]$, $Y_j(t) \to Y(t)$ iff $W_j(t) \to W(t)$.*

PROOF. Statement (A.1.1) follows from the bounds (A.10) and (A.11). To prove (A.1.2), we first observe that under the hypotheses of (A.1.2) we can apply (A.1.1) and conclude that $\{Y_j\}$ is bounded in $BM([a,b],\mathbb{R}^n)$.

Let $\tilde{Y}_j = Y_j - W_j = Y_{A_j,A_jW_j\,dt,\bar{Y}_j}$, $\tilde{Y} = Y - W = Y_{A,AW\,dt,\bar{Y}_j}$. Then

$$\tilde{Y}_j(t) = \bar{Y}_j + \int_a^t A_j(s)Y_j(s)\,ds = \bar{Y}_j + \int_a^t A_j(s)\tilde{Y}_j(s)\,ds + \int_a^t A_j(s)W_j(s)\,ds\,, \tag{A.12}$$

so

$$\tilde{Y}_j(t) - \tilde{Y}(t) = \int_a^t A(s)\Big(\tilde{Y}_j(s) - \tilde{Y}(s)\Big)\,ds + R_j(t)\,, \tag{A.13}$$

where

$$R_j(t) = \bar{Y}_j - \bar{Y} + \int_a^t \Big(A_j(s) - A(s)\Big)\tilde{Y}_j(s)\,ds - \int_a^t A(s)\Big(W_j(s) - W(s)\Big)\,ds\,, \tag{A.14}$$

and also

$$\tilde{Y}_j(t) - \tilde{Y}(t) = \bar{Y}_j - \bar{Y} + \int_a^t \Big(A_j(s) - A(s)\Big)Y_j(s)\,ds + \int_a^t A(s)\Big(Y_j(s) - Y(s)\Big)\,ds\,. \tag{A.15}$$

If $W_j - W \to 0$ a.e., then, since the sequences $\{\tilde{Y}_j\}$, $\{Y_j\}$ and $\{W_j\}$ are uniformly bounded, and $A_j \to A$ in L^1, it is clear that the first two terms of the sum defining $R_j(t)$ go to zero uniformly as $j \to \infty$. Also,

$$\left\| \int_a^t A(s)\Big(W_j(s) - W(s)\Big)\, ds \right\| \leq \int_a^b ||A(s)|| \cdot ||W_j(s) - W(s)||\, ds\,, \tag{A.16}$$

which goes to zero because $s \to ||A(s)||$ is integrable and the sequence $\{W_j - W\}$ is uniformly bounded and goes to zero almost everywhere. So, if we let $K_j = \sup\{||R_j(t)|| : a \leq t \leq b\}$, we see that $K_j \to 0$ as $j \to \infty$. Then Gronwall's inequality, applied to (A.13), gives

$$||\tilde{Y}_j(t) - \tilde{Y}(t)|| \leq K_j e^{||A||_{L^1}}\,. \tag{A.17}$$

So $\tilde{Y}_j - \tilde{Y} \to 0$ uniformly.

If $Y_j - Y \to 0$ a.e., then the first two terms of the right-hand side of (A.15) go to zero uniformly, and the third one is bounded by $\int_a^b ||A(s)|| \cdot ||Y_j(s) - Y(s)||\, ds$, which also goes to zero. So $\tilde{Y}_j - \tilde{Y} \to 0$ uniformly.

We have thus shown that $\tilde{Y}_j - \tilde{Y} \to 0$ uniformly if either $W_j \to W$ a.e. or $Y_j \to Y$ a.e. This clearly implies all the conclusions of (A.1.2). ∎

A very important subspace of $BM([a,b], \mathbb{R}^n)$ is $BV([a,b], \mathbb{R}^n)$, the space of all functions $W : [a,b] \mapsto \mathbb{R}^n$ such that (a) W is of bounded variation, (b) W is right-continuous at every point of $]a,b]$, and (c) $W(a) = 0$. There is a canonical correspondence between $BV([a,b], \mathbb{R}^n)$ and the dual space $C^0([a,b], \mathbb{R}_n)^*$ of the Banach space $C^0([a,b], \mathbb{R}_n)$ of continuous functions $[a,b] \mapsto \mathbb{R}_n$. (We write R_n rather than R^n because we want to think of the members of $C^0([a,b], \mathbb{R}_n)$ as *row-vector-valued functions*, cf. below.)

The members of $C^0([a,b], \mathbb{R}_n)^*$ are the $\mathbb{R}^n$*-valued Borel measures on* $[a,b]$. When $n = 1$, they are the *finite signed Borel measures on* $[a,b]$. Clearly, $C^0([a,b], \mathbb{R}_n)^*$ is canonically identified with $\Big(C^0([a,b], \mathbb{R})^*\Big)^n$, so the members of $C^0([a,b], \mathbb{R}_n)^*$ can be thought of as n-tuples of finite signed Borel measures. The identification map from $BV([a,b], \mathbb{R}^n)$ to $C^0([a,b], \mathbb{R}_n)^*$ is the one that assigns to a function $W \in BV([a,b], \mathbb{R}^n)$ the unique Borel measure μ_W such that $\mu_W([a,t]) = W(t)$ for $a < t \leq b$. (The measure of a single-point set $\{t\}$ is then $\mu_W(t+) - \mu_W(t-)$, which is equal to $\mu_W(t) - \mu_W(t-)$ if $t > a$ and to $\mu_W(a+)$ if $t = a$. In particular, $\mu_W(\{t\}) = 0$ iff μ_W is continuous at t.)

From now on we identify a function $W \in BV([a,b], \mathbb{R}^n)$ with its corresponding measure μ_W, and write $\int \alpha \cdot dW$ for $\int \alpha \cdot d\mu_W$. As explained above, it is convenient to think of the members of $C^0([a,b], \mathbb{R}_n)$ as row-vector-valued functions $\alpha = (\alpha_1, \ldots, \alpha_n)$, and of those of $BV([a,b], \mathbb{R}^n)$ as column-vector-valued functions $W = (W^1, \ldots, W^n)$, so the expression $\alpha \cdot dW$ stands for the inner product $\sum_k \alpha_k\, dW^k$. Clearly, $\int \alpha \cdot dW$ is defined, more generally, for an arbitrary bounded Borel measurable $\mathbb{R}_n$-valued map α on $[a,b]$.

Naturally, the integral $\int \alpha\, dW$ of a scalar bounded Borel measurable function $\alpha : [a,b] \mapsto \mathbb{R}$ is also well defined, and is a vector in $\mathbb{R}^n$. We emphasize that, if W is not continuous, then for an integral such as $\int_s^t dW$ to be well defined we have to be careful to specify whether the integral is to be interpeted as $\int \chi_{[s,t]}\, dW$ or $\int \chi_{]s,t]}\, dW$ or $\int \chi_{[s,t[}\, dW$ or $\int \chi_{]s,t[}\, dW$, so we will never write $\int_s^t dW$ unless W is continuous at s and t.

On $BV([a,b], R^n)$ there are at least three important topologies , namely, (a) the one induced by the norm $||\ldots||_{BM}$, (b) the strong topology of $BV([a,b], \mathbb{R}^n)$ as the dual of the Banach space $C^0([a,b], \mathbb{R}_n)$, (c) the weak* topology, also corresponding to the duality with $C^0([a,b], \mathbb{R}_n)$. The

first one will not be of direct interest to us, except for the fact that, in order to apply Theorem A.1, we will often need to know that a set is bounded in the norm $||\ldots||_{BM}$. The second one is induced by the *total variation norm*:

$$||W||_{TV} = \sup\left\{ \left| \int \alpha \cdot dW \right| : \alpha \in C^0([a,b], \mathbb{R}_n)\,,\ ||\alpha||_{sup} \le 1 \right\}. \tag{A.18}$$

It is well known that $||W||_{TV}$ is the supremum of the sums $S_\pi(W)$, ranging over all partitions $\pi \in \Pi([a,b]) = \cup_m \Pi_m([a,b])$, where $\Pi_m([a,b])$ is the set of all $m+1$-tuples $(t_0, t_1, \ldots, t_m)$ of members of $[a,b]$ such that $a = t_0 < t_1 < \ldots < t_m = b$ and, for $\pi = (t_0, t_1, \ldots, t_m) \in \Pi([a,b])$, $S_\pi(W) \stackrel{\text{def}}{=} \sum_{i=1}^m ||W(t_i) - W(t_{i-1})||$. In particular, it is clear that $||W||_{BM} \le ||W||_{TV}$ for all W in $BV([a,b], \mathbb{R}^n)$. Therefore, *a subset of* $BV([a,b], \mathbb{R}^n)$ *which is bounded in TV norm is also bounded in BM norm.*

We will need the following

Lemma A.1. *If a sequence* $\{W_j\}$ *in* $BV([a,b], \mathbb{R}^n)$ *weak*-converges to a* $W \in BV([a,b], \mathbb{R}^n)$*, then (a)* $\{W_j\}$ *is bounded in TV norm, (b)* $W_j(a) \to W(a)$ *and* $W_j(b) \to W(b)$*, (c) every subsequence of* $\{W_j\}$ *has a subsequence that converges to* W *at all but at most countably many points of* $[a,b]$*, (d)* $\int_a^b \beta(s)||W_j(t) - W(t)||\, dt \to 0$ *for every function* $\beta \in L^1([a,b], \mathbb{R})$*.*

PROOF. Statement (a) follows from the uniform boundedness theorem, and Statement (b) is trivial, since $W_j(a) = 0 = W(a)$, $W_j(b) = \int 1\, dW_j$, and $W(b) = \int 1\, dW$. Statement (d) clearly follows from (c), using the dominated convergence theorem together with the fact that boundedness in TV norm implies boundedness in BM norm.

So all we need to do is prove (c). If Z^i is the i-th component of Z for $Z \in BM([a,b], \mathbb{R}^n)$, $i = 1, \ldots, n$, then $\{W_j\}$ weak*-converges to W iff $\{W_j^i\}$ weak*-converges to W^i for each i. From this it clearly follows that it suffices to prove (c) for $n = 1$.

If $n = 1$, then every finite signed Borel measure μ on $[a,b]$ admits a unique decomposition $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are nonnegative Borel measures that are mutually singular (that is: there exist two μ-measurable sets E and F, such that $[a,b] = E \cup F$, $E \cap F = \emptyset$, $\mu^+(E) = \mu^-(F) = 0$). This gives, using the identification of $BV([a,b], R)$ with $C^0([a,b], R)^*$, a decomposition $Z = Z^+ - Z^-$ of every $Z \in BV([a,b], \mathbb{R})$ into a difference of two members Z^+, Z^- of $\mathcal{M}(a,b)$, the space of all monotonically nondecreasing functions $W : [a,b] \mapsto \mathbb{R}$ such that $W(a) = 0$ and W is right-continuous at every $t \in]a,b]$. It is well known that $||Z||_{TV} = Z^+(b) + Z^-(b)$.

Now let $\{W_{j(k)}\}_{k=1}^\infty$ be an arbitrary subsequence of $\{W_j\}$, and write $Z_k = W_{j(k)}$. Letting $Z_k = Z_k^+ - Z_k^-$, we have $||Z_k^+||_{TV} = Z_k^+(b) \le ||Z_k||_{TV}$, so the sequence $\{Z_k^+\}$ is bounded in $BV([a,b], \mathbb{R})$. Similarly, $\{Z_k^-\}$ is bounded in $BV([a,b], \mathbb{R})$. Since the closed balls of $BV([a,b], R)$ are weak*-compact and metrizable, there exists a subsequence $\{Z_{k(\ell)}\}$ of $\{Z_k\}$ such that $Z_{k(\ell)}^+$ and $Z_{k(\ell)}^-$ weak*-converge to limits U, V. Then of course $Z_{k(\ell)} = Z_{k(\ell)}^+ - Z_{k(\ell)}^-$ weak*-converges to $U - V$, and also to W, so $U - V = W$. (But this does *not* mean that $U = W^+$ and $V = W^-$.)

Since the $Z_{k(\ell)}^+$ are in $\mathcal{M}(a,b)$, it is well known (and very easy to prove directly) that $\{Z_{k(\ell)}^+(t)\}$ converges to $U(t)$ at every point of continuity of U. Since U is monotonically nondecreasing, $Z_{k(\ell)}^+(t) \to U(t)$ at all but countably many t's in $[a,b]$. A similar conclusion holds for $Z_{k(\ell)}^-$ and V. So $Z_{k(\ell)}(t) \to W(t)$ for all but countably many t's, concluding our proof. ∎

As a corollary of Lemma A.1, we get the integration by parts formula:

Lemma A.2. *If* $\lambda : [a,b] \mapsto \mathbb{R}_n$ *is absolutely continuous, and* $W \in BV([a,b], \mathbb{R}^n)$*, then*

$$\lambda(b) \cdot W(b) = \int \lambda \cdot dW + \int_a^b \dot{\lambda}(t) \cdot W(t)\, dt\,. \tag{A.19}$$

PROOF. The result is obviously true, from the standard product rule for derivatives, if W is also absolutely continuous. In the general case, we use the fact that an arbitrary $W \in BV([a,b],\mathbb{R}^n)$ is weak*-limit of a sequence $\{W_j\}$ of absolutely continuous functions. Then (i) $\lambda(b)\cdot W_j(b) \to \lambda(b)\cdot W(b)$, because $W_j(b) \to W(b)$, (ii) $\int \lambda\cdot dW_j \to \int \lambda\cdot dW$, because $\{W_j\}$ weak*-converges to W, and λ is continuous, and (iii) $\int_a^b \dot\lambda(t)\cdot W_j(t)\,dt \to \int_a^b \dot\lambda(t)\cdot W(t)\,dt$, because Part (d) of Lemma A.1 implies that $\int_a^b ||\dot\lambda(t)||.||W_j(t)-W(t)||\,dt \to 0$, since $\dot\lambda \in L^1$. ∎

An important consequence of Lemma A.2 is the following

Lemma A.3. *Let $W \in BV([a,b],\mathbb{R}^n)$, $A \in L^1([a,b],R^{n\times n})$, $\bar Y \in \mathbb{R}^n$, and write $Y = Y_{A,dW,\bar Y}$. Suppose $\lambda : [a,b] \to \mathbb{R}_n$ is absolutely continuous and satisfies $\dot\lambda(t) = -\lambda(t)A(t)$ for almost all t. Then*

$$\lambda(b)\cdot Y(b) - \lambda(a)\cdot Y(a) = \int \lambda\cdot dW\,. \tag{A.20}$$

PROOF. Write $Y(t) = V(t) + W(t)$, where $V(t) = Y(a) + \int_a^t A(s).Y(s)\,ds$. Then

$$\lambda(b)\cdot Y(b) - \lambda(a)\cdot Y(a) = \lambda(b)\cdot V(b) - \lambda(a)\cdot V(a) + \lambda(b)\cdot W(b)\,. \tag{A.21}$$

The function $t \to \lambda(t)\cdot V(t)$ is absolutely continuous and its derivative is $\dot\lambda(t)\cdot V(t) + \lambda(t)\cdot \dot V(t)$, which equals $-\lambda(t)A(t)Y(t) + \lambda(t)A(t)Y(t) - \dot\lambda(t)\cdot W(t)$, i.e. $-\dot\lambda(t)\cdot W(t)$. So

$$\lambda(b)\cdot Y(b) - \lambda(a)\cdot Y(a) = \lambda(b)\cdot W(b) - \int_a^b \dot\lambda(t)\cdot W(t)\,dt\,. \tag{A.22}$$

Equation (A.20) follows from this and the integration by parts formula (A.19). ∎

It is clear that, if $A \in L^1([a,b],\mathbb{R}^{n\times n})$, $\bar Y \in \mathbb{R}^n$, and $W \in BM_0([a,b],\mathbb{R}^n)$, then $Y_{A,dW,\bar Y} - W$ is absolutely continuous, so $W \in BV([a,b],\mathbb{R}^n)$ iff $Y_{A,dW,\bar Y}$ is of bounded variation, which happens iff $Y_{A,dW,\bar Y} - \bar Y \in BV([a,b],\mathbb{R}^n)$.

Theorem A.2. *Let $\{W_j\}$, $\{\bar Y_j\}$, $\{A_j\}$ be sequences in $BV([a,b],\mathbb{R}^n)$, $\mathbb{R}^n$, $L^1([a,b],\mathbb{R}^{n\times n})$, respectively. Assume that $\{A_j\}$ is bounded in $L^1([a,b],\mathbb{R}^{n\times n})$ and $\{\bar Y_j\}$ is bounded in $\mathbb{R}^n$. Let $Y_j = Y_{A_j,dW_j,\bar Y_j}$. Then*

(A.2.1) *$\{Y_j - \bar Y_j\}$ is bounded in $BV([a,b],\mathbb{R}^n)$ if and only if $\{W_j\}$ is bounded in $BV([a,b],\mathbb{R}^n)$.*

(A.2.2) *If $\{A_j\}$ converges in $L^1([a,b],\mathbb{R}^n)$ to a limit A, $\bar Y_j \to \bar Y$, $W \in BV([a,b],\mathbb{R}^n)$, and we write $Y = Y_{A,dW,\bar Y}$, then $\{W_j\}$ weak*-converges to W iff $\{Y_j - \bar Y_j\}$ weak*-converges to $Y - \bar Y$. In that case, $Y_j - W_j - \bar Y_j$ converges to $Y - W - \bar Y$ strongly in $BV([a,b],\mathbb{R}^n)$. In particular, (a) $W_j(b) \to W(b)$, (b) $Y_j(b) \to Y(b)$, (c) for every $t \in [a,b]$, $Y_j(t) \to Y(t)$ iff $W_j(t) \to W(t)$.*

PROOF. We know from theorem A.1 that $\{Y_j - \bar Y_j\}$ is bounded in BM norm iff $\{W_j\}$ is. Since boundedness in BV norm implies boundedness in BM norm, we see that boundedness of one of the two sequences in BV norm implies that there is a constant C such that $||Y_j(t)|| \le C$ for all j,t, and then the integral $I_j = \int_a^b ||A_j(t)Y_j(t)||\,dt$ is bounded by a fixed constant K. But $I_j = ||Y_j - \bar Y_j - W_j||_{BV}$. So boundedness in BV of one of the sequences $\{W_j\}$, $\{Y_j\}$ implies boundedness in BV of the other one.

If either W_j weak*-converges to W, or $Y_j - \bar Y_j$ weak*-converges to $Y - \bar Y$, then the uniform boundedness theorem implies that either $\{W_j\}$ or $\{Y_j - \bar Y_j\}$ is bounded in BV norm, so both

sequences are bounded in BV norm —and hence also in BM— norm by a constant C. Writing $Z_j = Y_j - \bar{Y}_j - W_j$, $Z = Y - \bar{Y} - W$, we have then $Z_j - Z = R_j^1 + R_j^2$, where

$$R_j^1(t) = \int_a^t \Big(A_j(s) - A(s)\Big) Y_j(s)\, ds\,, \qquad R_j^2(t) = \int_a^t A(s)\Big(Y_j(s) - Y(s)\Big)\, ds\,. \tag{A.23}$$

Then $||R_j^1||_{BV} \le C||A_j - A||_{L^1}$, so $||R_j^1||_{BV} \to 0$. Also,

$$||R_j^2||_{BV} \le ||A||_{L^1}.||\bar{Y}_j - \bar{Y}|| + \int_a^b ||A(t)||.||Y_j(t) - \bar{Y}_j - (Y(t) - \bar{Y})||\, dt\,. \tag{A.24}$$

If $\{Y_j - \bar{Y}_j\}$ weak*-converges to $Y - \bar{Y}$, then Lemma A.1 implies that $||R_j^2||_{BV} \to 0$, so $Z_j \to Z$ in $BV([a,b], \mathbb{R}^n)$.

If $\{W_j\}$ weak*-converges to W, then we rewrite R_j^2 as

$$R_j^2(t) = \int_a^t A(s)\Big(Z_j(s) - Z(s)\Big)\, ds + R_j^3(t) + R_j^4(t)\,, \tag{A.25}$$

where $R_j^3(t) = \int_a^t A(s)\Big(\bar{Y}_j - \bar{Y}\Big)\, ds$ and $R_j^4(t) = \int_a^t A(s)\Big(W_j(s) - W(s)\Big)\, ds$. We then have $||R_j^3||_{BV} \le ||A||_{L^1}.||\bar{Y}_j - \bar{Y}||$ and $||R_j^4||_{BV} \le \int_a^b ||A(s)||.||W_j(s) - W(s)||\, ds$. Lemma A.1 then imples that $||R_j^4||_{BV} \to 0$. So $Z_j(t) - Z(t) = \int_a^t A(s).(Z_j(s) - Z(s))\, ds + R_j(t)$, where $R_j = R_j^1 + R_j^3 + R_j^4$, so $||R_j||_{BV} \to 0$. Then Gronwall's inequality implies that $||Z_j - Z||_{sup} \le e^{||A||_{L^1}} ||R_j||_{BV}$. If $Q_j(t) = \int_a^t A(s).(Z_j(s) - Z(s))\, ds$, then $||Q_j||_{BV} \le ||A||_{L^1}||Z_j - Z||_{sup}$, so $||Q_j||_{BV} \to 0$. Since $Z_j - Z = Q_j + R_j$, we see that $||Z_j - Z||_{BV} \to 0$.

So we have shown that if one of the two sequences $\{W_j - W\}$, $\{Y_j - \bar{Y}_j - (Y - \bar{Y})\}$ weak*-converges to 0, then $Y_j - \bar{Y}_j - W_j \to Y - \bar{Y} - W$ strongly in $BV([a,b], \mathbb{R}^n)$. This implies all the conclusions of (A.2.2). ∎

Recall that, if P, Q are normed spaces, and $F : \Omega \to Q$ is a map defined on a neighborhood Ω of a point p of P, then F is said to be *Fréchet differentiable* at p if there exists a bounded linear map L —called the *Fréchet differential* of F at p— such that $F(p') = F(p) + L(p' - p) + o(||p' - p||)$ as $p' \to p$. If X is a Banach space and $Q = X^*$, then we call F *weak* Fréchet differentiable* at p if for every $x \in X$ the map $\Omega \ni p' \mapsto \langle F(p'), x\rangle \in \mathbb{R}$ is Fréchet differentiable at p. (We use $\langle x^*, x\rangle$, for $x \in X$, $x^* \in X^*$, as an alternative notation for $x^*(x)$.)

It is easy to see that in general weak* Fréchet differentiability does not imply Fréchet differentiability. (For example, if $X = C^0([0,1], \mathbb{R})$, $P = \mathbb{R}$, $\Omega = [0,1]$, and $F(t) = \chi_{[0,t]}$ for $t \in \Omega$, then for every $\alpha \in X$ the map $t \mapsto \langle F(t), \alpha\rangle$ is just the map $t \mapsto \int_a^t \alpha(s)\, ds$, which is differentiable at every $t \in]0,1[$. The weak*-limit $\lim_{h\to 0} h^{-1}(F(t+h) - F(t))$ is δ_t, the delta function at t (i.e. the functional $\alpha \to \alpha(t)$), but $||h^{-1}(F(t+h) - F(t)) - \delta_t||_{BV} = 2$ if $h \neq 0$ and $t, t+h \in [0,1]$.)

Suppose P is a normed space, X is a Banach space, $p \in P$, $F : \Omega \to X^*$ is a map defined on a neighborhood Ω of p, and F is weak* Fréchet differentiable at p. Then for each $x \in X$ there is a bounded linear functional θ_x on P such that $\langle F(p+v), x\rangle = \langle F(p), x\rangle + \theta_x(v) + o(||v||)$ as $v \to 0$ in P. Clearly, θ_x is unique, and is given by the formula

$$\theta_x(v) = \lim_{h\to 0} h^{-1}\langle F(p + hv) - F(p), x\rangle\,. \tag{A.26}$$

For any given $v \in P$, use θ^v to denote the map $x \to \theta_x(v)$. Then θ^v is a limit of linear maps, so it is linear. If $\{h_j\}$ is any sequence going to 0, then the sequence of linear functionals ν_j on X

given by $\nu_j(x) = h_j^{-1}\langle F(p+h_jv) - F(p), x\rangle$ converges pointwise to θ^v. Since each ν_j is a bounded linear functional on X, it follows for the uniform boundedness theorem that θ^v is bounded. Since $\theta^v(x) = \theta_x(v)$, which is linear with respect to v, we can conclude that $v \mapsto \theta_v$ is a linear map from P to X^*. We use $D^wF(p)$ to denote this map, and $D^wF(p).v$ to denote its value for a particular v, so $D^wF(p).v = \theta^v$, and

$$\langle D^wF(p).v, x\rangle = \lim_{h\to 0} h^{-1}\langle F(p+hv) - F(p), x\rangle\,. \tag{A.27}$$

Lemma A.4. *If P is a normed space, X is a Banach space, $p \in P$, $F : \Omega \to X^*$ is a map defined on a neighborhood Ω of p, and F is weak* Fréchet differentiable at p. Then:*

(A.4.i) *there is a $C > 0$ such that*

$$||F(p+v) - F(p)|| \le C||v|| \qquad \textit{for all sufficiently small} \qquad v \in P\,, \tag{A.28}$$

(A.4.ii) *the linear map $D^wF(p) : P \mapsto X^*$ defined by (A.27) is bounded.*

PROOF. Suppose (A.4.i) is not true. Then there exists a sequence $\{v_j\}$ in P, converging to 0 and such that $||F(p+v_j) - F(p)|| > j||v_j||$. Define, for each j, a member σ_j of X^*, by

$$\sigma_j = \frac{F(p+v_j) - F(p) - \theta^{v_j}}{||v_j||}\,. \tag{A.29}$$

Then $\sigma_j(x) \to 0$ for each x, so the sequence $\{\sigma_j\}$ is pointwise bounded and hence uniformly bounded.

On the other hand, for each $x \in X$, the numbers $\frac{\theta^{v_j}(x)}{||v_j||}$ are bounded, because $\theta^{v_j}(x) = \theta_x(v_j)$, and θ_x is a bounded linear functional on P, so $|\theta^{v_j}(x)| \le ||\theta_x||.||v_j||$. So, using the uniform boundedness theorem once again, we conclude that the sequence $\left\{\frac{\theta^{v_j}}{||v_j||}\right\}$ is bounded in X^*. Since $\{\sigma_j\}$ is bounded, we conclude that the sequence $\left\{\frac{F(p+v_j)-F(p)}{||v_j||}\right\}$ is bounded, contradicting the fact that $||F(p+v_j) - F(p)|| > j||v_j||$. So (A.28) holds for some $C > 0$, and we have proved (A.4.i).

It follows from (A.4.i) that $|\langle F(p+v) - F(p), x\rangle| \le C||x||.||v||$ for all $x \in X$, if $v \in P$ is small enough. Applying this with hv in the role of v, and letting $h \downarrow 0$, we find that

$$|\langle D^wF(p).v, x\rangle| \le C||x||.||v|| \qquad \text{for all} \qquad x \in X\,,\ \ v \in P\,. \tag{A.30}$$

Therefore $||D^wF(p).v|| \le C||v||$, so $D^wF(p)$ is a bounded linear operator from P to X^*. ∎

We are now ready to study the differentiability properties with respect to a parameter p, belonging to a normed space P, of a family of trajectories $x_p : [a,b] \to \mathbb{R}^n$ of time-varying vector fields f_p. We will prove, under minimal technical hypotheses, that if the initial condition map $p \mapsto x_p(a)$ is differentiable at $p = 0$, and the map $p \mapsto f_p$ satisfies a "weak differentiability condition" at the trajectory x_0, then the map $p \mapsto x_p$, is weak*- differentiable at $p = 0$, as a map into $BV([a,b], \mathbb{R}^n)$, and in particular the endpoint map $p \mapsto x_p(b)$ is differentiable at $p = 0$.

A *curve in* $\mathbb{R}^n$ is a continuous map $\xi : I \mapsto \mathbb{R}^n$, whose domain $I = \mathrm{Dom}(\xi)$ is a nonempty interval. A curve ξ is an *arc* if $\mathrm{Dom}(\xi)$ is compact. The *graph* $G(\xi)$ of a curve ξ is the set $G(\xi) = \{(\xi(t), t) : t \in \mathrm{Dom}(\xi)\}$. If $\xi : [a,b] \mapsto \mathbb{R}^n$ is an arc in $\mathbb{R}^n$, and $\varepsilon > 0$, then the *ε-tube about* ξ is the set $\mathcal{T}(\xi, \varepsilon) = \{(x,t) : x \in \mathbb{R}^n, a \le t \le b, ||x - \xi(t)|| \le \varepsilon\}$.

A *time-varying vector field* on $\mathbb{R}^n$ is an $\mathbb{R}^n$-valued map f whose domain $\mathrm{Dom}(f)$ is a —possibly empty— subset S of the product $\mathbb{R}^n \times \mathbb{R}$. We will use $TVVF(\mathbb{R}^n)$ to denote the set of all time-varying vector fields on $\mathbb{R}^n$. A *trajectory*, or *integral curve*, of an $f \in TVVF(\mathbb{R}^n)$ is a locally absolutely continuous curve ξ in $\mathbb{R}^n$ such that $G(\xi) \subseteq \mathrm{Dom}(f)$ and $\dot{\xi}(t) = f(\xi(t), t)$ for almost all $t \in \mathrm{Dom}(\xi)$. We use $\mathrm{Traj}(f)$ to denote the set of all trajectories of f, and $\mathrm{Traj}_c(f)$ to denote the set of all $\xi \in \mathrm{Traj}(f)$ such that $\mathrm{Dom}(\xi)$ is compact.

If $f \in TVVF(\mathbb{R}^n)$ and $\xi \in \mathrm{Traj}(f)$, we say that ξ is a *maximal* trajectory of f if it cannot be extended to a trajectory $\tilde{\xi} : \tilde{I} \mapsto \mathbb{R}^n$ of f defined on an interval $\tilde{I}$ such that $I \subseteq \tilde{I}$ but $I \neq \tilde{I}$. We use $\mathrm{MTraj}(f)$ to denote the set of all maximal trajectories of f. Given $\bar{x} \in \mathbb{R}^n$, $\bar{t} \in \mathbb{R}$, we use $\mathrm{Traj}(f, \bar{x}, \bar{t})$, $\mathrm{MTraj}(f, \bar{x}, \bar{t})$ to denote, respectively, the set of all $\xi \in \mathrm{Traj}(f)$ such that $\xi(\bar{t}) = \bar{x}$, and the set $\mathrm{Traj}(f, \bar{x}, \bar{t}) \cap \mathrm{MTraj}(f)$. Zorn's lemma implies that every $\xi \in \mathrm{Traj}(f, \bar{x}, \bar{t})$ can be extended to a $\tilde{\xi} \in \mathrm{MTraj}(f, \bar{x}, \bar{t})$, so $\mathrm{Traj}(f, \bar{x}, \bar{t}) \neq \emptyset$ iff $\mathrm{MTraj}(f, \bar{x}, \bar{t}) \neq \emptyset$. Clearly, $\mathrm{Traj}(f, \bar{x}, \bar{t}) \neq \emptyset$ iff $(\bar{x}, \bar{t}) \in \mathrm{Dom}(f)$. (Indeed, if $(\bar{x}, \bar{t}) \in \mathrm{Dom}(f)$, then the map ξ with domain $\{\bar{t}\}$ such that $\xi(\bar{t}) = \bar{x}$ is in $\mathrm{Traj}(f)$.)

We say that an $f \in TVVF(\mathbb{R}^n)$ has the *forward existence property* at a point $(\bar{x}, \bar{t})$ if for some $\varepsilon > 0$ there exists a $\xi \in \mathrm{Traj}(f, \bar{x}, \bar{t})$ with domain $[\bar{t}, \bar{t} + \varepsilon]$. We say that f has the *forward limit property* on a subset S of $\mathrm{Dom}(f)$ if, whenever $\xi : [a, b[\mapsto \mathbb{R}^n$ is a trajectory of f which is contained in a compact subset of S, it follows that $\lim_{t \uparrow b} \xi(t)$ exists.

Suppose we are given the following data:

(D.1) a normed space P,
(D.2) a family $\mathbf{f} = \{f_p\}_{p \in P}$ of time-varying vector fields on $\mathbb{R}^n$, depending on a parameter $p \in P$,
(D.3) a $p_0 \in P$,
(D.4) a trajectory $\xi : [a, b] \mapsto \mathbb{R}^n$ of f_{p_0}.

We then define functions $w_p : [a, b] \mapsto \mathbb{R}^n$ by

$$w_p(t) = f_p(\xi(t), t) - f_{p_0}(\xi(t), t) \quad \text{for} \quad t \in [a, b]\,, \tag{A.31}$$

so w_p is defined on $[a, b]$ whenever $G(\xi) \subseteq \mathrm{Dom}(f_p)$. If w_p is integrable, we let

$$W_p(t) = \int_a^t w_p(s)\, ds \qquad \text{for} \qquad t \in [a, b]\,. \tag{A.32}$$

We then let $\mathcal{W}$ be the map that sends p to $\mathcal{W}(p) = W_p$, with domain

$$\mathrm{Dom}(\mathcal{W}) = \{p \in P : G(\xi) \subseteq \mathrm{Dom}(f_p),\ w_p \in L^1([a, b], \mathbb{R}^n)\}\,. \tag{A.33}$$

We regard $\mathcal{W}$ as a map from $\mathrm{Dom}(\mathcal{W})$ to $BV([a, b], \mathbb{R}^n)$.

We then say that $\mathbf{f}$ is *weakly differentiable at p_0 along ξ* if there exists $\bar{\varepsilon} > 0$ such that

(WD1) *$\mathcal{T}(\xi, \bar{\varepsilon}) \subseteq \mathrm{Dom}(f_p)$ for every p such that $||p - p_0|| \leq \bar{\varepsilon}$.*
(WD2) *For every p such that $||p - p_0|| \leq \bar{\varepsilon}$, f_p has the forward limit property on $\mathcal{T}(\xi, \bar{\varepsilon})$ and the forward existence property at every (x, t) such that $a \leq t < b$ and $||x - \xi(t)|| < \bar{\varepsilon}$.*
(WD3) *There exist $A \in L^1([a, b], \mathbb{R}^{n \times n})$, and functions $\psi_\varepsilon \in L^1([a, b], \mathbb{R})$, for $\varepsilon \in]0, \bar{\varepsilon}]$, such that*

$$||f_p(x, t) - f_p(\xi(t), t) - A(t).(x - \xi(t))|| \leq \psi_\varepsilon(t)(||x - \xi(t)|| + ||p - p_0||) \tag{A.34}$$

whenever $a \leq t \leq b$, $||x - \xi(t)|| \leq \varepsilon$, $||p - p_0|| \leq \varepsilon$, and

$$\lim_{\varepsilon \downarrow 0} \int_a^b \psi_\varepsilon(t)\, dt = 0\,. \tag{A.35}$$

(WD4) *The function w_p defined by (A.31) is integrable for every p such that $||p - p_0|| \leq \bar{\varepsilon}$ and, if $\mathcal{W}$ is defined as above, then $\mathcal{W}$ is weak* Fréchet differentiable at p_0.*

The second part of condition (WD4) says that for each $\alpha \in C^0([a,b],\mathbb{R}_n)$ the map $p \to \int \alpha \cdot dW_p$ is Fréchet differentiable at p_0. Lemma A.4 then implies —since $W_{p_0} = 0$— that there exists a bounded linear map $D^w\mathcal{W}(p_0) : P \mapsto BV([a,b],\mathbb{R}^n)$ having the property that

$$\int \alpha \cdot dW_p = \langle \mathcal{W}(p) - \mathcal{W}(p_0), \alpha \rangle = \int \alpha \cdot \Big(D^w\mathcal{W}(p_0).(p-p_0) \Big) + o(||p-p_0||) \qquad \text{as} \qquad p \to p_0 \quad \text{(A.36)}$$

for each $\alpha \in C^0([a,b],\mathbb{R}_n)$, and also that there is a constant C such that

$$||W_p||_{TV} \leq C||p - p_0|| \qquad \text{for all } p \in P \text{ such that } \|p - p_0\| \text{ is sufficiently small.} \qquad \text{(A.37)}$$

Theorem A.3. *Let P, $\mathbf{f}$, p_0, ξ be data as in (D.1,2,3,4). Assume that $\mathbf{f}$ is weakly differentiable at p_0 along ξ. Let $\bar{\varepsilon} > 0$ be such that (WD1) to (WD4) hold. Then*

(A.3.a) *For every $\varepsilon \in]0,\bar{\varepsilon}]$ there exists an $\delta \in]0,\varepsilon[$ such that, whenever $||p-p_0|| \leq \delta$, $||\bar{x}-\xi(a)|| \leq \delta$, and $\zeta \in \mathrm{MTraj}(f_p,\bar{x},a)$, it follows that $[a,b] \subseteq \mathrm{Dom}(\zeta)$ and $||\zeta(t)-\xi(t)|| \leq \varepsilon$ for all $t \in [a,b]$.*

(A.3.b) *If $\{\bar{x}_p\}_{||p-p_0|| \leq \bar{\varepsilon}}$ is a family of points of $\mathbb{R}^n$ such that the map $p \mapsto \bar{x}_p$ is Fréchet differentiable at p_0 and $\bar{x}_{p_0} = \xi(a)$, then, if $\{\xi_p\}_{||p-p_0|| \leq \bar{\varepsilon}}$ is a family of curves such that $\xi_p \in \mathrm{MTraj}(f_p,\bar{x}_p,a)$, and x_p denotes the restriction of ξ_p to $[a,b]$ —which is well defined for small enough $||p-p_0||$ by (A.3.a)— then the map $\mathcal{X}$ that sends each p to the function x_p is weak* Fréchet differentiable at p_0 as a map from P to $BV([a,b],\mathbb{R}^n)$. Moreover, $D^w\mathcal{X}(p_0)$ can be computed as follows. For $v \in P$,*

$$D^w\mathcal{X}(p_0).v = Y_{A,\, d(D^w\mathcal{W}(0).v),\, L.(p-p_0)}\,, \qquad \text{(A.38)}$$

where $L : P \to \mathbb{R}^n$ is the differential at p_0 of the map $p \to \bar{x}_p$.

PROOF. We will assume, as we clearly may without loss of generality, that $p_0 = 0$. Also, we know that there is a $C > 0$ such that the bound (A.37) holds. So we can asssume, by making $\bar{\varepsilon}$ smaller, if necessary, that $||W_p||_{TV} \leq C||p||$ whenever $||p|| \leq \bar{\varepsilon}$. By making C larger and $\bar{\varepsilon}$ smaller, if necessary, we may also assume that $C \geq 1$, and

$$8e^{||A||_{L^1}}||\psi_\varepsilon||_{L^1} < 1 \qquad \text{for} \qquad \varepsilon \in]0,\bar{\varepsilon}]\,. \qquad \text{(A.39)}$$

Pick $\varepsilon \in]0,\bar{\varepsilon}[$, and choose $\delta > 0$ such that

$$C\delta < \varepsilon\,. \qquad \text{(A.40)}$$

Notice that $\delta < \varepsilon$, since $C \geq 1$.

Let p, $\bar{x}$ be such that $||p|| \leq \delta$ and $||\bar{x} - \xi(a)|| \leq \delta$. Let $\zeta \in \mathrm{MTraj}(f_p,\bar{x},a)$, and let $I = \mathrm{Dom}(\zeta)$. Then $a \in I$. Let c be the supremum of those t such that $a \leq t \leq b$ and $||\zeta(s) - \xi(s)|| \leq \varepsilon$ for $a \leq s \leq t$. Then $a < c \leq b$, $c \in I$, and either $c = b$ or $||\zeta(c) - \xi(c)|| = \varepsilon$. (The fact that $a < c$ follows from the local existence property, since $||\bar{x} - \xi(a)|| \leq \delta < \varepsilon < \bar{\varepsilon}$, and ζ is continuous. The fact that $c \in I$ follows from the forward limit property, since the graph of the restriction of ζ to $[a,c[$ is contained in $\mathcal{T}(\xi,\varepsilon)$. If $c < b$ and $||\zeta(c) - \xi(c)|| < \varepsilon$, then the forward existence property implies that ζ is not maximal.)

We will estimate $||\zeta(t) - \xi(t)||$ for $t \in [a,c]$, and show that —if ε is small enough, and δ is suitably chosen— $||\zeta(t) - \xi(t)|| < \varepsilon$ for all such t, including $t = c$. This will imply that $c = b$,

From now on, we work on the interval $[a,c]$. We have

$$\zeta(t)-\xi(t)=\bar{x}-\xi(a)+\int_a^t\Big(f_p(\zeta(s),s)-f_0(\xi(s),s)\Big)\,ds\,, \tag{A.41}$$

so

$$\zeta(t)-\xi(t)=\bar{x}-\xi(a)+W_p(t)+\int_a^t\Big(f_p(\zeta(s),s)-f_p(\xi(s),s)\Big)\,ds\,. \tag{A.42}$$

Let $r_p(t)=f_p(\zeta(t),t)-f_p(\xi(t),t)-A(t).(\zeta(t)-\xi(t))$. Then

$$\zeta(t)-\xi(t)=\bar{x}-\xi(a)+W_p(t)+\int_a^t A(s).(\zeta(s)-\xi(s))\,ds+\int_a^t r_p(s)\,ds\,, \tag{A.43}$$

so

$$||\zeta(t)-\xi(t)||\le||\bar{x}-\xi(a)||+||W_p||_{sup}+\int_a^t||A(s)||.||\zeta(s)-\xi(s)||\,ds+\int_a^t r_p(s)\,ds\,, \tag{A.44}$$

and then

$$||\zeta(t)-\xi(t)||\le||\bar{x}-\xi(a)||+||W_p||_{sup}+2\varepsilon||\psi_\varepsilon||_{L^1}+\int_a^t||A(s)||.||\zeta(s)-\xi(s)||\,ds\,, \tag{A.45}$$

because $||r_p(s)||\le 2\varepsilon\psi_\varepsilon(s)$ for $s\in[a,c]$, since $||\zeta(s)-\xi(s)||\le\varepsilon$ and $||p||\le\varepsilon$.

Since $||\bar{x}-\xi(a)||\le\delta$, and $||W_p||_{sup}\le||W||_{TV}\le C||p||\le C\delta$, Gronwall's inequality implies

$$||\zeta(t)-\xi(t)||\le e^{||A||_{L^1}}\Big((C+1)\delta+2\varepsilon||\psi_\varepsilon||_{L^1}\Big)\,. \tag{A.46}$$

In view of (A.39),we have

$$2\varepsilon e^{||A||_{L^1}}||\psi_\varepsilon||_{L^1}<\frac{\varepsilon}{4}\,. \tag{A.47}$$

Now suppose that

$$\delta\le\kappa\varepsilon\,,\qquad\text{where}\qquad\kappa=(4C+4)^{-1}e^{-||A||_{L^1}}\,. \tag{A.48}$$

Then

$$||\zeta(t)-\xi(t)||\le\frac{\varepsilon}{2}\qquad\text{for}\qquad\text{all}\qquad t\in[a,c]\,. \tag{A.49}$$

This is our desired estimate, from which it follows, as explained above, that $c=b$, and then $||\zeta(t)-\xi(t)||\le\varepsilon$ for all $t\in[a,b]$. This completes the proof of (A.3.a).

Besides proving (A.3.a), we have also shown that, if κ is the constant given by (A.48), then the conclusion of (A.3.a) holds for every δ such that $0<\delta\le\kappa\varepsilon$. (Notice that if $\delta\le\kappa\varepsilon$ then (A.40) holds.)

Now suppose we are given a map $P\ni p\mapsto\bar{x}_p\in\mathbb{R}^n$ which is defined for $||p||\le\bar{\delta}$ for some $\bar{\delta}>0$. Assume that this map is Fréchet differentiable at 0, with differential L, and is such that $\bar{x}_0=\xi(a)$. Write $z_p=Y_{A,dV_p,L.p}$, where $V_p=D^w\mathcal{W}(0).p$.

It then follows that there exist a constant $C'\ge1$ and a $\bar{\delta}'\in]0,\bar{\delta}]$ such that (i) $C'\bar{\delta}'\le\kappa\bar{\varepsilon}$, and (ii) $||\bar{x}_p-\xi(a)||\le C'||p||$ for $||p||\le\bar{\delta}'$.

Pick in an arbitrary fashion $\xi_p\in\mathrm{MTraj}(f_p,\bar{x}_p,a)$ for $||p||\le\bar{\delta}'$. Then, if $||p||\le\bar{\delta}'$, it follows that (i) ξ_p is defined, (ii) $\xi_p\in\mathrm{MTraj}(f_p,\bar{x},a)$ for an $\bar{x}$ —namely, $\bar{x}_p$— such that $||\bar{x}-\xi(a)||\le C'\bar{\delta}'\le\kappa\bar{\varepsilon}$. Since $C'\ge1$, we also have $||p||\le\kappa\bar{\varepsilon}$. Therefore we can conclude —using $\varepsilon=\bar{\varepsilon}$ and $\delta=\kappa\bar{\varepsilon}$, that $[a,b]\subseteq\mathrm{Dom}(\xi_p)$ and $(\xi_p(t),t)\in\mathcal{T}(\xi,\bar{\varepsilon})$ for $t\in[a,b]$.

Let x_p be the restriction of ξ_p to $[a,b]$. For $||p|| \leq \bar{\delta}'$, we have

$$x_p(t) - \xi(t) - z_p(t) = \bar{x}_p - \xi(a) - L.p + \int_a^t A(s).(x_p(s) - \xi(s) - z_p(s))\, ds + W_p(t) - V_p(t) + R_p(t)\,, \quad \text{(A.50)}$$

where $R_p(t) = \int_a^t r_p(s)\, ds$. If $p \neq 0$, let

$$Y_p = ||p||^{-1}(x_p - x_0 - z_p)\,, \qquad U_p = ||p||^{-1}(W_p - V_p + R_p)\,, \qquad \bar{Y}_p = ||p||^{-1}(\bar{x}_p - x_0(a) - L.p)\,. \quad \text{(A.51)}$$

Then $Y_p = Y_{A, dU_p, \bar{Y}_p}$.

We have to prove that Y_p weak* converges to 0 as $p \to 0$. Since $\bar{Y}_p \to 0$, our conclusion will follow if we show that U_p weak* converges to 0. Since $V_p = D^w\mathcal{W}(0).p$, it is clear that $||p||^{-1}(W_p - V_p)$ weak*-converges to 0. So all we need is to prove that $||p||^{-1}R_p$ weak*-converges to 0. If $||p||$ is sufficiently small, then we can apply the previous estimates using $\delta = ||p||$, $\varepsilon = \frac{\delta}{\kappa}$. We conclude that $||r_p(s)|| \leq 2\varepsilon\psi_\varepsilon(s)$ for all $s \in [a,b]$. This implies that

$$||r_p||_{L^1} \leq \frac{2||p||}{K} ||\psi_{\kappa^{-1}||p||}||_{L^1}\,. \quad \text{(A.52)}$$

This shows that $||R_p||_{TV} = ||r_p||_{L^1} = o(||p||)$ as $p \to 0$. So $||p||^{-1}R_p$ converges to 0 strongly —and *a fortiori* weakly*— in $BV([a,b], \mathbb{R}^n)$. This completes our proof. ∎

If $f \in TVVF(\mathbb{R}^n)$ and $\xi : [a,b] \mapsto \mathbb{R}^n$ is a trajectory of f, we say that f *is differentiable along* ξ if the map $x \mapsto f(x,t)$ is differentiable at $\xi(t)$ for almost every t. In that case, we write $D_\xi f(t) \stackrel{\text{def}}{=} \frac{\partial f}{\partial x}(\xi(t), t)$, and refer to the map $[a,b] \ni t \mapsto D_\xi f(t) \in \mathbb{R}^{n\times n}$ as the *differential of* f *along* ξ.

Lemma A.5. *If* $\mathbf{f}$ *is weakly differentiable at* p_0 *along a trajectory* $\xi : [a,b] \to \mathbb{R}^n$ *of* f_{p_0}, *then* f_{p_0} *is differentiable along* ξ *and the map* A *of (WD3) is such that* $A(t) = D_\xi f_{p_0}(t)$ *for almost all* $t \in [a,b]$.

PROOF. We begin by showing that the functions ψ_ε for which (WD3) holds can be chosen so that $\psi_\varepsilon(t)$ is a monotonically nonincreasing function of ε for each t. To see this, start with some family $\{\psi_\varepsilon\}_{\varepsilon \in]0,\bar{\varepsilon}]}$ of functions for which (WD3) holds, and define $\hat{\psi}_\varepsilon = \psi_{2^{-k}\bar{\varepsilon}}$ whenever $2^{-k-1}\bar{\varepsilon} < \varepsilon \leq 2^{-k}\bar{\varepsilon}$ and k is a nonnegative integer. Then (WD3) also holds if the $\hat{\psi}_\varepsilon$ are substituted for the ψ_ε. Next define $\bar{\psi}_\varepsilon(t) = \min\{\hat{\psi}_{\varepsilon'}(t) : \varepsilon \leq \varepsilon' \leq \bar{\varepsilon}\}$. Then the $\bar{\psi}_\varepsilon$ are well defined integrable functions of t, because for each ε the set of functions $\{\hat{\psi}_{\varepsilon'} : \varepsilon \leq \varepsilon' \leq \bar{\varepsilon}\}$ is finite. It is clear that (WD3) also holds with the $\bar{\psi}_\varepsilon$ istead of the $\hat{\psi}_\varepsilon$, and in addition $\hat{\psi}_\varepsilon(t)$ is nonincreasing as a function of ε.

Now, if we pick the ψ_ε to be monotonically nonincreasing with respect to ε, it follows that the limit $\rho(t) = \lim_{\varepsilon \downarrow 0} \psi_\varepsilon(t)$ exists for every t. By the dominated convergence theorem, $\int_a^b \rho(t)dt = 0$. Since $\rho(t) \geq 0$ for all t, it follows that $\rho(t) = 0$ almost everywhere.

Let t be such that $\rho(t) = 0$. Let $\theta(x) = \psi_{||x-\xi(t)||}(t)$. Then $\lim_{x\to\xi(t)} \theta(x) = 0$. On the other hand, (WD3) implies that

$$||f_{p_0}(x,t) - f_{p_0}(\xi(t),t) - A(t).(x - \xi(t))|| \leq \theta(x)||x - \xi(t)||\,. \quad \text{(A.53)}$$

Since $\theta(x) \to 0$ as $x \to \xi(t)$, (A.53) says precisely that $A(t)$ is the differential at $\xi(t)$ of the map $x \mapsto f_{p_0}(x,t)$, completing our proof. ∎

If $f \in TVVF(\mathbb{R}^n)$ is differentiable along a trajectory $\xi : [a,b] \to \mathbb{R}^n$ of f, and the matrix-valued function $D_\xi f$ is integrable, then the ordinary differential equation

$$\dot{\lambda}(t) = -\lambda(t).D_\xi f(t)\,, \tag{A.54}$$

for an $\mathbb{R}_n$-valued function λ on $[a,b]$, is the *adjoint variational equation* —or, simply, the *adjoint equation*— for f along ξ. A solution of the adjoint equation (i.e. an absolutely continuous function $\lambda : [a,b] \mapsto \mathbb{R}_n$ such that (A.54) holds for almost all t) is known as an *adjoint vector for f along ξ*.

Theorem A.4. *Let P, $\mathbf{f} = \{f_p\}_{p\in P}$, $p_0 \in P$, $\xi : [a,b] \to \mathbb{R}^n$ be as in Theorem A.3. Let $P \ni p \mapsto \bar{x}_p \subset \mathbb{R}^n$ be a map which is Fréchet differentiable at p_0 and such that $\bar{x}_{p_0} = \xi(a)$, and let L_a be its differential at p_0. Let ξ_p be, for p close enough to p_0, maximal trajectories of f_p such that $\xi_p(a) = \bar{x}_p$, and let x_p be the restriction of ξ_p to $[a,b]$. Then the map $p \mapsto x_p(b)$ is differentiable at p_0. Let L_b be its differential at p_0. Let $\lambda : [a,b] \to \mathbb{R}^n$ be an adjoint vector for f_{p_0} along ξ. Then*

$$\lambda(b)\cdot L_b(v) - \lambda(a)\cdot L_a(v) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\int_a^b \Big\langle \lambda(t), f_{p_0+\varepsilon v}(\xi(t),t) - f_{p_0}(\xi(t),t)\Big\rangle dt\,. \tag{A.55}$$

PROOF. Assume, without loss of generality, that $p_0 = 0$. According to Theorem A.3 and Lemma A.5, the map $p \mapsto x_p$ is weak* Fréchet differentiable at 0 —as a map from P to $BV([a,b],\mathbb{R}^n)$— and its differential at 0 is the map that sends $v \in P$ to $Y_{A,d(U.v),L_a(v)}$, where $U = D^w\mathcal{W}(0)$, and $\mathcal{W}$ is the map $p \mapsto W_p$, where $W_p(t) = \int_a^t \Big(f_p(\xi(s),s) - f_0(\xi(s),s)\Big)dt$, and $A = D_\xi f$. Then $p \mapsto x_p(b)$ is Fréchet differentiable at 0, and the differential L_b is given by $L_b(v) = Y_{A,d(U.v),L_a(v)}(b)$. Clearly, $L_a(v) = Y_{A,d(U.v),L_a(v)}(a)$. Since λ satisfies $\dot{\lambda} = -\lambda.A$, Lemma A.3 tells us that

$$\lambda(b)\cdot L_b(v) - \lambda(a)\cdot L_a(v) = \int_a^b \lambda.d(U.v)\,.$$

Since $U.v$ is the weak*-limit of $\varepsilon^{-1}W_{\varepsilon v}$ as $\varepsilon \downarrow 0$, our conclusion follows. ∎

We now turn to the differentiation of a Lagrangian cost functional.

If $f \in TVVF(\mathbb{R}^n)$, a *Lagrangian* for f in $\mathbb{R}^n$ is a real-valued function L such that

(L.1) $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(L)$;
(L.2) the function $t \mapsto L(\xi(t),t)$ is locally integrable for every $\xi \in \mathrm{Traj}(f)$.

If $f \in TVVF(\mathbb{R}^n)$ has components $f^1,\ldots,f^n$, and L is a Lagrangian for f, then we can consider the time-varying vector field $[L;f]$ in $\mathbb{R}^{n+1}$ whose components are $(f^0,f^1,\ldots,f^n)$, where $f^0 \equiv L$. A trajectory of $[L;f]$ can then be regarded a pair (ξ^0,ξ), where $\xi \in \mathrm{Traj}(f)$ and ξ^0 is a real-valued locally absolutely continuous function on $\mathrm{Dom}(\xi)$ such that $\dot{\xi}^0(t) = L(\xi(t),t)$ for a.e. t. Any such function ξ^0 will be called a *running L-cost along ξ*. Our definition of a Lagrangian implies that a running L-cost along ξ exists for all $\xi \in \mathrm{Traj}(f)$, and it is clear that any two running L-costs along ξ must differ by a constant. If ξ is such that $\mathrm{Dom}(\xi)$ is of the form $[a,b]$ or $[a,b[$ or $[a,\infty[$, then there is a canonical choice of running L-cost along ξ, obtained by letting $\xi^0(t) = \int_a^t L(\xi(s),s)\,ds$. In that case, we will write ξ^L to denote the curve $t \mapsto (\xi^0(t),\xi(t))$, where ξ^0 is the canonical running L-cost.

Now suppose we are given data P, $\mathbf{f}$, p_0, ξ as in (D.1,2,3,4), as well as

(D.5) a family $\mathbf{L} = \{L_p\}_{p\in P}$ such that each L_p is a Lagrangian for f_p.

Given a trajectory $\xi : [a,b] \mapsto \mathbb{R}^n$ of f_{p_0}, we say that the pair $(\mathbf{f}, \mathbf{L})$ is *weakly differentiable at* p_0 *along* ξ if the family $\{[L_p; f_p]\}_{p \in P}$ is weakly differentiable along $\xi^{L_{p_0}}$.

It is easy to see that $(\mathbf{f}, \mathbf{L})$ is weakly differentiable at p_0 along ξ iff there exists $\bar{\varepsilon} > 0$ such that (WD1,2,3,4) hold, and in addition

(WD5) *There exists* $h \in L^1([a,b], \mathbb{R}_n)$ *such that*

$$||L_p(x,t) - L_p(\xi(t),t) - h(t).(x - \xi(t))|| \leq \psi_\varepsilon(t)(||x - \xi(t)|| + ||p||) \tag{A.56}$$

whenever $a \leq t \leq b$, $||x - \xi(t)|| \leq \varepsilon$ *and* $||p|| \leq \varepsilon$.

(WD6) *If we let* $w_p^0(t) = L_p(\xi(t),t) - L_{p_0}(\xi(t),t)$, $W_p^0(t) = \int_a^t w_p^0(s)\, ds$, *then the map* $\mathcal{W}^0$ *that sends* p *to* $\mathcal{W}^0(p) = W_p^0 \in BV([a,b], \mathbb{R})$ *is weak* Fréchet differentiable at* p_0.

If $f \in TVVF(\mathbb{R}^n)$, $\xi : [a,b] \mapsto \mathbb{R}^n$ is a trajectory of f, and L is a Lagrangian for f, we say that L *is differentiable along* ξ if the map $x \mapsto L(x,t)$ is differentiable at $\xi(t)$ for almost every t. In that case, we write $\nabla_\xi L(t) \stackrel{\text{def}}{=} \nabla_x L(\xi(t),t)$, and refer to the map $[a,b] \ni t \to \nabla_\xi L(t) \in \mathbb{R}^{n \times n}$ as the *differential of* L *along* ξ.

Lemma A.5, applied to the curve $\xi^{L_{p_0}}$, yields

Lemma A.6. *If* $(\mathbf{f}, \mathbf{L})$ *is weakly differentiable at* p_0 *along a trajectory* $\xi : [a,b] \to \mathbb{R}^n$ *of* f_{p_0}, *then* f_{p_0} *and* L_{p_0} *are differentiable along* ξ *and the maps* A, h *of (WD3,5) are such that* $A(t) = D_\xi f(t)$ *and* $h(t) = \nabla_\xi L_{p_0}(\xi(t),t)$ *for almost all* $t \in [a,b]$. ∎

Under the hypothesis of Lemma A.6, we can apply Theorem A.4 to the family $\{[L_p; f_p]\}_{p \in P}$ and the curve $\xi^{L_{p_0}}$. Define

$$J_p(\zeta) = \int_a^b L_p(\zeta(t),t)\, dt\,, \tag{A.57}$$

if $\zeta : [a,b] \to \mathbb{R}^n$ is a curve such that the integral exists. (We will only be using this when $\zeta \in \mathrm{Traj}(f_p)$, in which case $J_p(\zeta)$ is well defined.) An adjoint vector for $[L_{p_0}; f_{p_0}]$ along $\xi^{L_{p_0}}$ is then a pair (λ^0, λ) such that $\lambda^0 \in \mathbb{R}$, $\lambda : [a,b] \mapsto \mathbb{R}_n$ is absolutely continuous, and

$$\dot{\lambda}(t) = -\lambda(t).A(t) - \lambda^0 h(t) \qquad \text{for} \qquad \text{a.e.} \qquad t \in [a,b]\,. \tag{A.58}$$

Theorem A.4 then gives:

Theorem A.5. *Assume that we are given data* P, $\mathbf{f}$, p_0, ξ, $\mathbf{L}$ *as in (D.1,2,3,4,5). Assume that* $(\mathbf{f}, \mathbf{L})$ *is weakly differentiable at* p_0 *along* ξ. *Let* $P \ni p \mapsto \bar{x}_p \subset \mathbb{R}^n$ *be a map which is Fréchet differentiable at* p_0 *and such that* $\bar{x}_{p_0} = \xi(a)$, *and let* L_a *be its differential at* p_0. *Let* ξ_p *be, for* p *close enough to* p_0, *maximal trajectories of* f_p *such that* $\xi_p(a) = \bar{x}_p$, *and let* x_p *be the restriction of* ξ_p *to* $[a,b]$. *Then the maps* $p \mapsto x_p(b)$ *and* $p \mapsto J_p(x_p)$ *are differentiable at* p_0. *Let* L_b, L_b^0 *be their differentials at* p_0. *Then, if* (λ^0, λ) *is an adjoint vector for* $[L_{p_0}, f_{p_0}]$ *along* ξ, *it follows that*

$$\lambda(b) \cdot L_b(v) + \lambda^0 L_b^0(v) - \lambda(a) \cdot L_a(v) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_a^b \Big(H_{p_0 + \varepsilon v}(\xi(t), \lambda(t), \lambda_0, t) - H_{p_0}(\xi(t), \lambda(t), \lambda_0, t) \Big)\, dt\,, \tag{A.59}$$

where

$$H_p(x, z, z^0, t) = \langle z, f_p(x,t) \rangle + z^0 L_p(x,t) \tag{A.60}$$

for $x \in \mathbb{R}^n$, $z \in \mathbb{R}_n$, $z^0 \in \mathbb{R}$, $t \in [a,b]$. ∎

Appendix B . A lemma .

Lemma B.1. *Let f be a real-valued function on a compact interval $[a,b]$. Assume that there exists a finite or countable subset E of $[a,b]$ with the following properties:*

(B.1.a) $\liminf_{h\downarrow 0} \frac{f(x+h)-f(x)}{h} \geq 0$ *for all* $x \in [a,b[\backslash E$,
(B.1.b) $\liminf_{h\downarrow 0} f(x+h) - f(x) \geq 0$ *for all* $x \in [a,b[$,
(B.1.c) $\limsup_{h\downarrow 0} f(x) - f(x-h) \geq 0$ *for all* $x \in]a,b]$.
Then $f(b) \geq f(a)$.

PROOF. Let $\varepsilon > 0$ be arbitrary. We will prove that $f(b) \geq f(a) - \varepsilon$. Let $\varepsilon' = \frac{\varepsilon}{1+b-a}$ and $E = \{x_k\}_{k\in N}$, $N \subset \mathbb{N}$.

For each $x \in [a,b]$, let $I(x) = \{k : x_k < x\}$. Define $g : [a,b] \mapsto \mathbb{R}$ by

$$g(x) = \sum_{k\in I(x)} 2^{-k}\varepsilon' .$$

Then g is left-continuous and monotonically nondecreasing.

Let S be the set of all $x \in [a,b]$ such that $f(y) \geq f(a) - (y-a)\varepsilon' - g(y)$ for all $y \in [a,x]$. It is clear that $a \in S$, and also that S is an interval. So S is of the form $[a,c[$ or $[a,c]$, with $c \in [a,b]$.

Suppose $S = [a,c[$. Then $c > a$. By (B.1.c), given $\delta > 0$ there is a sequence $\{z_j\}$ in $[a,b]$ such that $z_j < c$, $z_j \to c$, and $f(c) - f(z_j) \geq -\delta$. Then $z_j \in S$, so $f(z_j) \geq f(a) - (z_j - a)\varepsilon' - g(z_j) \geq f(a) - (c-a)\varepsilon' - g(c)$. It then follows that $f(c) \geq f(a) - (c-a)\varepsilon' - g(c) - \delta$. Since δ is arbitrary, we have $f(c) \geq f(a) - (c-a)\varepsilon' - g(c)$, so $c \in S$, which is a contradiction.

Therefore $S = [a,c]$ for some $c \in [a,b]$. We now show that $c = b$. Assume that $c < b$, and consider separately the cases $c \notin E$, $c \in E$.

If $c \notin E$, then (B.1.a) implies that there exists a $c' > c$ such that $f(x) - f(c) \geq -(x-c)\varepsilon'$ for $x \in [c,c']$. But then, since $f(c) \geq f(a) - (c-a)\varepsilon' - g(c)$, we have $f(x) \geq f(a) - (x-c)\varepsilon' - g(c) \geq f(a) - (x-a)\varepsilon' - g(x)$ for $x \in [c,c']$. Therefore $c' \in S$, contradicting the fact that $S = [a,c]$.

Now suppose $c \in E$. Then $c = x_k$ for some k, and $x_k \neq b$. By (B.1.b), there exists $h > 0$ such that $c + h \leq b$ and $f(y) - f(c) \geq -2^{-k}\varepsilon'$ for all $y \in [c, c+h]$. If $y \in]c, c+h]$, then $k \notin I(c)$, and $I(c) \cup \{k\} \subseteq I(y)$. Therefore $g(y) \geq g(c) + 2^{-k}\varepsilon'$. Since $f(c) \geq f(a) - (c-a)\varepsilon' - g(c)$, we have $f(y) \geq f(c) - 2^{-k}\varepsilon' \geq f(a) - (c-a)\varepsilon' - 2^{-k}\varepsilon' - g(c) \geq f(a) - (y-a)\varepsilon' - g(y)$. So $c + h \in S$, which is a contradiction.

We have therefore proved that $c = b$. Then

$$f(b) \geq f(a) - (b-a)\varepsilon' - g(b) \geq f(a) - (b-a)\varepsilon' - \varepsilon' = f(a) - \varepsilon .$$

Since ε is arbitrary, we have shown that $f(b) \geq f(a)$, as desired. ∎

Remark B.1. If f is lower semicontinuous then (B.1.b) automatically holds. ∎

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