HIGHER ORDER VARIATIONS: HOW CAN THEY BE DEFINED IN ORDER TO HAVE GOOD PROPERTIES?

GIANNA STEFANI*

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Abstract

Good properties of variations in connection with their use are discussed. A definition for variations at a point of a trajectory is proposed with properties sufficiently good to state a higher order maximum principle for the minimum time problem and for an optimal control problem with constraints on the end-point. The definition allows to define variations on the base of the relations at a point in the Lie Algebra associated to the system.

1 Introduction

Consider a control system on an n-dimensional manifold M

$$\dot{x}(t) = f(x(t), u(t)) \ , \ x(0) = \xi_0, \tag{Σ}$$

where $f : M \times R^m \to TM$ is a smooth function and the set $\mathcal{U}$ of admissible controls is the class of those integrable maps u from the compact interval $J = [0, T]$ into a subset Ω of R^m for which the time dependent vector field $(t, \xi) \mapsto f(\xi, u(t))$ and its first derivative with respect to the state is locally L^1-bounded. Let $t \to x(t, \xi_0, u)$ denote the solution of (Σ) relative to the control u and let $R(\xi_0, t)$ be the reachable set at time t.

Consider a reference control $\hat{u}$ and suppose that the corresponding reference trajectory $t \to \hat{x}(t) \equiv x(t, \xi_0, \hat{u})$ is defined on J. There are classical problems in optimal control theory linked to the study of the reachable sets "near" a point $\hat{x}(t)$ of the reference trajectory.

*University of Naples. email: stefani@facec.cce.unifi.it

A first problem, related to minimum time, is to decide whether the point $\hat{x}(t)$ belongs to the interior of the reachable set at time t. In fact if it does, then the trajectory cannot be of minimum time after t.

A second problem is to decide whether $\hat{x}$ minimizes a cost of the final point, say $\beta_0(x(T))$, possibly in presence of constraints $\beta_1(x(T)) = \cdots = \beta_r(x(T)) = 0$. This problem can be transformed in the following way. If we define $\xi_1 = \hat{x}(T)$ and $\beta \equiv (\beta_0, \ldots, \beta_r) : M \to R^{r+1}$, then the reference trajectory $\hat{x}$, solves the original problem if and only if it satisfies the constraints and $\beta(R(\xi_0, T))$ does not intersect the half-line $\{\beta(\xi_1) - \lambda(1, 0, \ldots, 0) \ : \ \lambda > 0\}$. In particular $\beta(\xi_1)$ cannot be interior to $\beta(R(\xi_0, T))$.

If ξ_0 is an equilibrium point and $\hat{x}(t) \equiv \xi_0$, then the first problem reduces to decide whether $\xi_0 \in int\, R(\xi_0, T)$ and, if we allow T to vary, then we deal with the so called small time local controllability (STLC) property, i.e. ξ_0 belongs to the interior of the reachable set at each time $t > 0$.

A way to attempt the above problems is to study the *tangent vectors* to a subset A of a manifold N at a point y. The simplest way to define tangent vectors is to define them as tangent directions to curves starting at y and lying in A.

Definition 1.1 *We say that $v \in T_yN$ is tangent to A at y and we write $v \in T_yA$, if and only if there exists $\gamma : [0, \bar{\epsilon}] \to A$, such that $\gamma(\epsilon) = y + \epsilon v + o(\epsilon)$.*

With this definition T_yA is a cone, possibly nonconvex. In general T_yA has not the following "good property"

$$T_yA = T_yN \Rightarrow y \in int\, A, \tag{1}$$

so that we have to select particular subsets of T_yA to obtain it. The choice of such sets may depend on the problem.

In our cases we have to study the sets $R(\xi_0, T)$ and $\beta(R(\xi_0, T))$. It is clear that if $v \in T_{\xi_1}R(\xi_0, T)$ then $T_{\xi_1}\beta(v) \in T_{\beta(\xi_1)}\beta(R(\xi_0, T))$, therefore we are mainly interested in studying subsets of tangent vectors to $R(\xi_0, T)$.

A natural way to obtain tangent vectors in the reachable sets is to consider control variations

$$\epsilon \to u_\epsilon \ , \ u_0 = \hat{u}$$

of the reference control $\hat{u}$ and the curve

$$\epsilon \mapsto \gamma(\epsilon) = x(T, \xi_0, u_\epsilon) \in R(\xi_0, T).$$

If $\gamma(\epsilon) \equiv \xi_1 + \epsilon v + o(\epsilon)$, then $v \in T_{\xi_1}R(\xi_0, T)$. Such vectors v with "good properties" are sometimes called *variations*.

In particular consider the so-called "Pontryagin variations" obtained in the following way. Take the controls $u_{\epsilon t\omega} : [0,T] \to \Omega$ obtained replacing the reference control $\hat{u}$ by the constant value ω in the interval $[t, t+\epsilon]$. The final point of the relative trajectory gives rise to a curve

$$\gamma : \epsilon \mapsto x(T, \xi_0, u_{\epsilon t\omega})$$

in the reachable set $R(\xi_0, T)$. If t is a Lebesgue point of $t \mapsto f(\hat{x}(t), \hat{u}(t))$, see [10], then the tangent vector to this curve can be thought as the transport along the reference trajectory of a "trajectory's variation" v_t produced at time t, given by

$$v_t = f(\hat{x}(t), \omega) - f(\hat{x}(t), \hat{u}(t)).$$

Namely, let $\xi \mapsto \hat{\Phi}_{t,T}$ be the flow from time t to time T of the reference time-dependent vector field $(t, \xi) \mapsto f(\xi, \hat{u}(t))$. If t is a Lebesgue point of $\hat{u}$, then

$$T_{\hat{x}(t)}\hat{\Phi}_{t,T}\Big(f(\hat{x}(t), \omega) - f(\hat{x}(t), \hat{u}(t))\Big)$$

belongs to $T_{\xi_1}R(\xi_0, T)$ for all $\omega \in \Omega$.

The Pontryagin variations are a subclass of the so-called *needle variations*, for which the support (i.e. the set in which the control differs from the reference one) goes to zero with the variational parameter ϵ.

Definition 1.2 *We call $v_t \in T_{\hat{x}(t)}$ a needle trajectory variation of order k at time t if there exists a control variation $\epsilon \mapsto u_{\epsilon t}$ such that the relative trajectory gives rise to a continuous curve*

$$\gamma : \epsilon \mapsto \hat{\Phi}_{t+\epsilon^{1/k},t}(x(\epsilon^{1/k}, \hat{x}(t), u_{\epsilon t})) \equiv \hat{x}(t) + \epsilon v_t + o(\epsilon).$$

We easily obtain as before that $T_{\hat{x}(t)}\hat{\Phi}_{t,T}(v_t)$ is a tangent vector to the reachable set $R(\xi_0, T)$.

Let us review some results which apply in the above considered cases.

1.1 STLC Problem

In [6] a sufficient condition of STLC for a multivalued inclusion is given using a suitable definition of variations. If we apply that definition to system (Σ) we obtain the following:

Definition 1.3 *We say that $v \in T_{\xi_0}M$ is a STLC variation of order k if and only if for all ϵ sufficiently small*

$$\xi_0 + \epsilon v + o(\epsilon) \in R(\xi_0, \epsilon^{1/k}).$$

The result in [6], applied to our case is the following:

Theorem 1.4 *Let Ω be bounded. If the convex cone generated by the STLC variations of any order defined above is the whole tangent space, then*

$$\xi_0 \in int\, R(\xi_0, [0,t]) \ , \ \forall t > 0.$$

Notice that no continuity far from zero is required in the definition of STLC variations. Moreover the above variations are tangent vectors to the sets $R(\xi_0, [0,t])$, for all $t > 0$, but only particular tangent vectors are variations.

A suitable generalization to unbounded control of the above definition and result has been obtained recently by R.M. Bianchini [3].

1.2 Minimum cost without constraints

It is clear that if (-1) is tangent to $\beta_0(R(\xi_0, T))$ at $\beta_0(\xi_1)$, then $\hat{x}$ does not minimize $\beta_0(x(T))$ if there are no constraints on the final point. In this case all the tangent vectors to $\beta_0(R(\xi_0, T))$ have to be considered and we obtain:

Theorem 1.5 *If $\hat{x}$ minimizes $\beta_0(x(T))$ over all the solutions of the system (Σ), then*

$$-T_{\xi_1}\beta_0(v) \le 0 \ , \ \forall v \in T_{\xi_1} R(\xi_0, T).$$

In particular if we apply the above result to the Pontryagin variations, then we obtain

$$-T_{\xi_1}\beta_0 \circ T_{\hat{x}(t)}\hat{\Phi}_{t,T}\Big(f(\hat{x}(t),\omega) - f(\hat{x}(t),\hat{u}(t))\Big) \le 0 \ , \ \forall \omega \in \Omega \ , \ a.e.\ t \in [0,T]. \tag{2}$$

The covector $\hat{p}(t) \equiv -T_{\xi_1}\beta_0 \circ T_{\hat{x}(t)}\hat{\Phi}_{t,T}$ is the solution of the adjoint equation, given in a chart by

$$\dot{p}(t) = -p(t)\, \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t)) \ , \tag{3}$$

with boundary condition

$$p(T) = -T_{\xi_1}\beta_0. \tag{4}$$

Therefore the inequality (2) leads to the Pontryagin Maximum Principle, i. e.

If $\hat{x}$ minimizes $\beta_0(x(T))$ over all the solutions of the system (Σ), then the solution $\hat{p} : [0,T] \to (R^n)^*$ of the adjoint equation (3) with boundary condition (4) is such that , for all $\omega \in \Omega$ and almost all $t \in [0,T]$,

$$\hat{p}(t) f(\hat{x}(t), \hat{u}(t)) \ge \hat{p}(t) f(\hat{x}(t), \omega).$$

Every further tangent vector to $R(\xi_0, T)$ at ξ_1 gives a strengthening of the Pontryagin Maximum Principle. In particular needle variations give conditions along the trajectory, see also [5].

In this case where no constraints are imposed on the end point one can obtain tangent vectors to $\beta_0(R(\xi_0, T))$ besides the ones coming from $R(\xi_0, T)$ (see for example [7], [9]).

1.3 Minimum time and minimum cost with constraints

The case of minimum time and the one of minimum cost with constraints are very similar because in both cases we want to select a subset $\mathcal{K}$ of tangent vectors to A at y with the property (1). We shall deal with a property little stronger than (1), but which is basically equivalent to Maximum Principle when applied to needle variations. Consider the following property which says that $y \in int\, A$ if the convex cone generate by $\mathcal{K}$ is the whole tangent space $T_y N$

$$y \in \partial A \;\Rightarrow\; \exists p^* \in T^*_y N \; s.t. \; p^* \neq 0 \; and \; p^* v \leq 0 \;,\; \forall v \in \mathcal{K}. \qquad (5)$$

For sake of simplicity, in what follows we consider the case when $N = M \;,\; y = \xi_1 \;,\; A = R(\xi_0, T)$.

A nice example in [4] shows that tangent vectors generated by general control variations do not have the property (5). Following [4], we introduce the summability property which is easily seen to imply (5).

Definition 1.6 *A subset $\mathcal{K}$ of $T_{\xi_1} R(\xi_0, T)$ has the summability property if for all finite subset $\{v_1, \cdots, v_p\}$ of $\mathcal{K}$ there exists a constant $\bar{\epsilon} > 0$ and a continuous map*

$$\varphi : [0, \bar{\epsilon}] \times \Big\{(c_1, \ldots, c_p) : c_i \geq 0 \;,\; \sum c_i = 1\Big\} \to R(\xi_0, T)$$

such that

$$\varphi(\epsilon, c_1, \ldots, c_p) = \xi_1 + \epsilon(c_1 v_1 + \cdots + c_p v_p) + o(\epsilon).$$

In [4] systems linear with respect to the control are considered and an order for tangent vectors is defined in connection with the L^1-norm of the control variation. It is shown that the subsets consisting of tangent vectors of order one and at most one tangent vector of higher order have the summability property. Notice that the control variations considered in [4] have no restriction on the support.

In general we can obtain a Maximum Principle each time that we can select a set $\mathcal{V}_t$, $t \in [0,T)$, of needle variations in such a way that the set

$$\mathcal{K} = \left\{ T_{\hat{x}(t)} \hat{\Phi}_{t,T}(\mathcal{V}_t) : t \in [0,T) \right\}$$

has the summability property. Namely, if this is the case we have that if $\xi_1 \in \partial R(\xi_0, T)$ then there exists a nonzero solution $\hat{p} : [0,T] \to (R^n)^*$ of the adjoint equation (3) such that

$$\hat{p}(t) v_t \leq 0 \ , \ \forall v_t \in \mathcal{V}_t \ , \ t \in [0,T).$$

It is possible to prove that the needle variations of order one (like the Pontryagin variations) have the sommability property, by concatenating the controls. They give rise to the Pontryagin Maximum Principle for this case.

For nonlinear systems it is possible that Pontryagin Maximum Principle is not sufficient to single out a unique candidate and additional conditions may be required. To obtain such additional conditions one can add some new variations.

The main difficulties one meets in order to prove the “summability property” arise when two variations are obtained at the same time. In fact if v_t and v_{t_1} are variations at different times, then it is possible to sum them simply concatenating the two control variations. If v and v_1 are higher order variations at the same time, concatenating their control variations, a new principal part may appear.

To our knowledge there is no example which proves that there are needle variations which do not have the summability property (the example in [4] concerns variations with finite support). Nevertheless this property has not been proved for general higher order needle variations.

The higher order maximum principles developed in [8] and [7] are based on needle variations that are required to be obtained continuously on a small but finite arc of trajectory, so that each variation can be thought as produced at different times. Basically if $v \in T_{\hat{x}(t)}M$ is a variation at t as defined either in [8] or in [7], then there is a positive constant h and a continuous map $w : [t-h, t+h] \to TM$ such that $w(t) = v$ and $w(\tau)$ is a variation at τ, for all $\tau \in [t-h, t+h]$. In particular if the convex cone generated by the variations at a time is the whole tangent space, then the same holds for neighboring times.

In the next section a definition of variations developed in [2] is presented (see also [12]). Such variations verify the summability property and may occur at an isolated time. This aim is obtained by requiring that a variation at time t can be obtained with control variations of time length ϵ and starting times $t + \gamma\epsilon$, depending on a new parameter γ.

The original motivation to this definition was to extend the known results of controllability at a point to the case of nonstationary trajectories.

In particular our aim was to obtain higher order variations at time t using the relations at $\hat{x}(t)$ in the Lie Algebra of the vector fields associated to the system, see [13] and [1]. This kind of "thin conditions" which depend only on a finite number of derivatives at $\hat{x}(t)$, cannot give variations on an arc of trajectory (see Example 3.3 below). Of course if the relations in the Lie Algebra are verified on an arc of trajectory, then the variations are produced in the same arc and some additional properties can be developed.

The variations defined in the next section are more general than the ones defined in [8]. The comparison with the ones defined in [7] is more difficult, nevertheless the variation which appear in Example 3.3 is not a variation as defined in [7].

2 The variational cone

In this section we give a definition of variation which is a slight modification of the one given in [2], where more general time-dependent control systems and admissible control functions are considered.

Definition 2.1 *A vector $v \in T_{\hat{x}(t)}M$ is a* right variation *of order k of $(\hat{u}, \hat{x})$ at time $t \in [0, T)$ if there are positive numbers $\bar{\gamma}, \bar{c}, \bar{\epsilon}$ and a three-parameters control variation*

$$\eta : [0, \bar{\gamma}] \times [0, \bar{c}] \times [0, \bar{\epsilon}] \to \mathcal{U}$$

such that the maps

$$(\xi, \gamma, c, \epsilon) \mapsto \varphi_{\gamma,c,\epsilon}(\xi) \equiv x(\epsilon^{1/k}, \xi, \eta(\gamma, c, \epsilon))$$

$$(\xi, \gamma, c, \epsilon) \mapsto T_\xi \varphi_{\gamma,c,\epsilon}$$

are continuous for ξ belonging to a neighborhood of $\hat{x}(t)$ and

$$\hat{\Phi}_{t+\epsilon^{1/k}(1+\gamma),t}(\varphi_{\gamma,c,\epsilon}(\hat{x}(t + \gamma\epsilon^{1/k}))) = \hat{x}(t) + \epsilon\, c\, v + o(\epsilon)$$

where

$$\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$$

uniformly with respect to (γ, c).

Remark 2.2 *Here only right variations are considered, i.e. relative to control variations which take place after t. In an analogous way left variations could be considered, see [2].*

Remark 2.3 *Notice that with this definition a topology on $\mathcal{U}$ is not needed. In [2] the continuity of the map φ is obtained through the continuity of η with respect to a suitable topology introduced in the set of admissible control $\mathcal{U}$. This topology may depend on the map f. For example, if f is not linear with respect to the control, then in general the L^1-topology cannot be chosen.*

If we denote by $\mathcal{V}_t$ the set of variations at time t, it is easy to see that $\mathcal{V}_t$ is a positive cone which may be not convex. For the properties of $\mathcal{V}_t$ we refer to [2] where the proofs of the results described in this section are also given. It is easily seen that the slight different definition of variation does not change the proofs.

To explain the meaning of the parameters γ and c let us remark that the role of γ is to allow "place" to concatenate the control variations at the same time. Basically we require the same trajectory variation in an interval of time which goes to zero as the time-length of the control variation. The role of c is more technical and it is the same as the one of the c_i's in the summability property.

The following one is the main result on the variations proved in [2] in a little different form.

Theorem 2.4 *The subset $\mathcal{K}$ of $T_{\xi_1}R(\xi_0,T)$ defined by*

$$\mathcal{K} = \textit{convex hull}\left\{T_{\hat{x}(t)}\hat{\Phi}_{t,T}\left(\mathcal{V}_t\right) : t \in [0,T)\right\}, \tag{6}$$

has the summability property. Therefore if

$$\mathcal{K} = T_{\xi_1}M$$

then

$$\xi_1 \in intR(\xi_0,T).$$

Remark 2.5 *Pontryagin variations are of this type, namely*

$$f(\hat{x}(t),\omega) - f(\hat{x}(t),\hat{u}(t))$$

is a right (left) variation of order one at each time t that is a right (left) Lebesgue point for $\hat{u}$. Therefore this kind of variations gives a strengthening of the Pontryagin Maximum Principle.

3 Higher order variations induced by the relations in the Lie Algebra

In what follows we give a result proved in [2] which states that suitable relations in the Lie Algebra associated to (Σ) at a point $\hat{x}(t)$ generate variations at t. The result concerns affine systems and a trajectory relative to a control taking values in the interior of Ω.

Let (Σ) be of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x),$$

and let Ω be the hypercube

$$H_\rho = \Big\{(\omega_1, .., \omega_m) \in R^m : |\omega_i| \le \rho, i = 1, .., m\Big\},$$

possibly $\rho = +\infty$, i.e. $H_\infty = R^m$. We choose $\hat{u} \equiv 0$ so that $\hat{x}(t) = exp\, t f_0(\xi_0)$.

The same arguments can be used to analyze trajectories relative to C^∞ controls, by modifying the system in a standard way.

In order to state the result we need the notations introduced in [1], [2]. Let $LieX$ be the free Lie Algebra on R generated by the noncommutative indeterminate $X = \Big\{X_0, \dots, X_m\Big\}$. $\mathcal{S}$ will denote the ideal of $LieX$ generated by $\{X_1, \dots, X_m\}$. Replacing X_i by f_i in an element $\chi \in LieX$, we obtain a vector field which will be denoted by χ_f. For any subset A of $LieX$, $A_f = \Big\{\chi_f : \chi \in A\Big\}$ will denote a subset of the Lie Algebra $Lief$ generated by $\Big\{f_0, \dots, f_m\Big\}$.

By means of a set $l = \{l_0, \dots, l_m\}$ of nonnegative integers we define a weight on $LieX$ which will induce a weight on $Lief$. Let Λ be a bracket in $\mathcal{S}$. We denote the "length" of Λ with respect to X_i (i.e. the number of times that X_i appears in Λ) by $|\Lambda|_i$. The weight of Λ is defined by

$$\|\Lambda\|_l = \sum_{i=0}^{m} l_i |\Lambda|_i,$$

and the subspace of $\mathcal{S}_f$ consisting of the elements of weight not greater than i is given by

$$V_i^l = span\Big\{\Lambda_f : \Lambda \in \mathcal{S}\ ,\ \|\Lambda\|_l \le i\Big\}.$$

An element $\chi \in \mathcal{S}$ is called $l-homogeneous$ if it is a linear combination of brackets with the same weight, which will be called the weight of χ.

Following [13] we say that an $l-$ *homogeneous* element $\chi \in \mathcal{S}$ is $l-$ *neutralized* for (Σ) at ξ if χ_f is a linear combination at ξ of brackets with less weight. In other words

Definition 3.1 *χ is $l-$ neutralized at ξ if there is an $i < \|\chi\|_l$ such that $\chi_f(\xi) \in V_i^l(\xi)$.*

For more details and examples see [1], [2].

To characterize the elements of $Lief$ which give rise to variations, we introduce the set of obstructions relative to a weight l, see [1], [2]. Let

$$\mathcal{B} = span\Big\{\Lambda \in \mathcal{S} : |\Lambda|_0 \text{ } is \text{ } odd \text{ } , \text{ } |\Lambda|_i \text{ } is \text{ } even \text{ } , \text{ } i = 1, \ldots, m\Big\}$$

$$\mathcal{B}_S^l = \Big\{\chi \in \mathcal{B} : \chi \text{ } is \text{ } symmetric \text{ } w.r.t. \text{ } the \text{ } X_i's \text{ } with \text{ } the \text{ } same \text{ } weight\Big\}.$$

The set of obstructions relative to the weight l is the set:

$$\mathcal{B}_l^* = Lie\{X_0, \mathcal{B}_S^l)\} \cap \mathcal{S}.$$

The following result states that an element of $\mathcal{S}_f$ determines a variation at time t if every obstruction with less or equal weight is neutralized at $\hat{x}(t)$.

Theorem 3.2 *[2]. Let $l = \Big\{l_0, \ldots, l_m\Big\}$ be a set of nonnegative weights. If $\rho < +\infty$, then we also suppose $l_0 \leq l_i$, $i = 1, \ldots, m$. If Φ is a bracket in $\mathcal{S}$ such that each $\Lambda \in \mathcal{B}_l^*$ with $\|\Lambda\|_l \leq \|\Phi\|_l$ is $l-$ neutralized at $\hat{x}(t)$, then $\Phi_f(\hat{x}(t))$ is a variation at t.*

Applying the above theorem it is possible to prove that particular brackets induce variations at any time, see [1], [2]. As an example we have that for all $t \in [0, T)$

$$\mathcal{C}_t \equiv \Big\{ad_{f_0}^k f_i(\hat{x}(t)) : k \geq 0 \text{ } , \text{ } i = 1 \ldots m\Big\} \subset \mathcal{V}_t$$

and

$$[f_i, f_j]\,(\hat{x}(t)) \subset \mathcal{V}_t \text{ } , \text{ } i, j = 1 \ldots m.$$

Such vectors are variations also in the sense of [8] and [7].

We are going now to provide an example of a vector field which defines a variation v at $t = 0$, but the set of variations at positive times is separated from v. In particular, otherwise than what happens in [8] and [7], the convex cone generated by the variations is the whole tangent space at an isolated point.

Example 3.3 *Let us consider the following system on* R^3

$$\begin{aligned} \dot{x} &= 1+u \\ \dot{y} &= ux \\ \dot{z} &= u(x^3y+y^2) \end{aligned}$$

with $\xi_0 = (0,0,0)$ *and* $|u| \leq 1$. *Let the reference control be* $\hat{u} \equiv 0$ *so that the reference trajectory is given by* $\hat{x}(t) = (t,0,0)$. *We have that the convex cone generated by* $\mathcal{C}_0$ *is given by*

$$span\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big\}.$$

Consider the bracket

$$\chi_f = [ad_{f_0}f_1, ad^2_{f_1}f_0] = (2-6x)\frac{\partial}{\partial z}.$$

$\pm\chi_f(\xi_0)$ *define variations at* 0. *In fact they satisfy the hypotheses of Theorem 3.2 with* $l = \{2,3\}$, *for details see [1],[2]. Therefore*

$$convex\,cone\,\mathcal{V}_0 = R^3.$$

To show that the convex cone generated by $\mathcal{V}_t$, t *"near 0", cannot be* R^3, *consider* $\bar{\xi} = \hat{x}(\bar{t})$, $0 < \bar{t} < 1/3$ *and define*

$$\hat{y} : t \mapsto exp\,t f_0(\bar{\xi}).$$

The convex cone generated by the variations at $\bar{t}$ *for* $(\hat{u},\hat{x})$ *is the same as the one generated by the variations at* 0 *for* $(\hat{u},\hat{y})$. *If this cone would be* R^3, *then* $\hat{y}(t)$ *would be interior to* $R(\bar{\xi},t)$ *for all* $t > 0$. *But*

$$ad^2_{f_1}f_0 = -(2x^3+6xy-2y)\frac{\partial}{\partial z},$$

moreover also for the couple $(\hat{u},\hat{y})$ *the convex cone generated by* $\mathcal{C}_0$ *is equal to span* $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ *for all* $t > 0$. *Therefore*

$$ad^2_{f_1}f_0(\bar{\xi}) = (6-2\bar{t}^3)\frac{\partial}{\partial z} \notin span\,\mathcal{C}_0.$$

The results in [11] imply that for each $t > 0$ *sufficiently small* $\hat{y}(t) = exp\,t\,f_0(\bar{\xi})$ *belongs to the boundary of* $R(\bar{\xi},t)$. *A contradiction.*

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