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MINIMUM TIME OPTIMALITY OF A BANG-BANG TRAJECTORY

LAURA POGGIOLINI AND GIANNA STEFANI

Abstract. We announce a sufficient condition for a bang-bang extremal $\widehat{\xi}$ to be a *strong* local optimizer for the minimum time problem with fixed endpoints. Strong local optimizer means that $\widehat{\xi}$ joins the two given points in time less than the time needed by any other trajectory belonging to a state-neighborhood of $\widehat{\xi}$.

1. Introduction

We state sufficient conditions for a bang-bang extremal $\widehat{\xi}$ to be a strong local optimizer for the minimum time problem with fixed endpoints. We underline that the conditions lead that the optimum is local with respect to the state and not to the time. To be more precise we consider the minimum time problem for an affine control system with controls in a polyhedron Δ, namely we consider the problem

$$\textit{minimize} \quad T$$

under the conditions

$$\begin{aligned}
&\dot{\xi}(t) = X_0(\xi(t)) + \sum_{k=1}^{m} u_k(t) X_k(\xi(t)) \qquad \text{a.e. } t \in [0, T] \\
&\xi(0) = x_0\,; \quad \xi(T) = y_0 \\
&u \equiv (u_1, \ldots, u_m) \in L^\infty([0, T], \Delta)
\end{aligned}$$

The state space is $\mathbb{R}^n$, x_0, y_0 are given points of $\mathbb{R}^n$ and $X_0, \ldots, X_m : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields.

We consider an admissible bang-bang trajectory $\widehat{\xi} : [0, \widehat{T}] \to \mathbb{R}^n$ and we want to give conditions for $\widehat{\xi}$ to be a local optimizer in the sense of the following definition

Definition 1.1. *An admissible trajectory $\widehat{\xi} : [0, \widehat{T}] \to \mathbb{R}^n$ is a strong local optimizer for the above problem if there is an open neighborhood $\mathcal{U}$ of $\{\widehat{\xi}(t) : t \in [0, \widehat{T}]\}$ in $\mathbb{R}^n$ such that if $\xi : [0, T] \to \mathbb{R}^n$ is any admissible trajectory and $\{\xi(t) : t \in [0, T]\}$ is contained in $\mathcal{U}$, then $\widehat{T} \leq T$.*

The result is given in the spirit of the paper [2], see also [1] for partial results, where sufficient conditions for a Mayer problem in a fixed interval of time are given.

As a matter of fact, applying a suitable time reparametrization our minimum time problem fits with the results in [2] and we obtain a sufficient condition for a local optimum in the sense of the *graphs* of the trajectories, in particular only with respect to the trajectories defined on a time interval $[0, T]$ with T *near* $\widehat{T}$.

On the contrary, applying directly the Hamiltonian methods we obtain that the reference trajectory is optimal with respect to the trajectories defined on any interval $[0, T]$.

The sufficient optimality conditions include obviously that $\widehat{\xi}$ satisfies the Pontryagin Maximum Principle (PMP), contain conditions on the regularity of the maximized Hamiltonian (see Assumptions 2.2, 2.3 and 2.4) and require that a suitable second variation is definite positive.

The second variation is the one relative to the finite dimensional subproblem obtained by moving the switching times of the reference trajectory.

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All the conditions are invariant under change of coordinates and they can be stated on any smooth manifold. Moreover the conditions concern the adjoint covector introduced by the PMP and the vector fields defining the reference trajectory, so that they result to be feedback invariant.

Here we give only the result, complete proofs and an extension to a more general Bolza problem will appear elsewhere.

The literature on second order sufficient conditions for the optimality of a bang-bang trajectory is scarce, we refer to [5] and the reference therein for results based on the existence of a regular synthesis. In [4] and [3] conditions are given for time optimality local with respect to both state and time.

2. Basic definitions and assumptions

Let $\widehat{u}\colon [0, \widehat{T}] \to \Delta$ be the bang-bang control associated to the reference trajectory $\widehat{\xi}$ and let

$$0 \le \widehat{\tau}_0 < \cdots < \widehat{\tau}_{r+1} = \widehat{T}$$

be the switching times, i.e. $\widehat{u}$ takes a constant value $\widehat{u}^i$, which is a vertex of Δ, in each interval $(\widehat{\tau}_i, \widehat{\tau}_{i+1})$, $i = 1, \ldots, r$. We call h_i the vector fields that define the reference trajectory $\widehat{\xi}$ in the interval $[\widehat{\tau}_i, \widehat{\tau}_{i+1}]$ i.e.

$$h_i = X_0 + \sum_{k=1}^{m} \widehat{u}_k^i X_k \,,$$

therefore the reference trajectory satisfies the differential equations

$$\dot{\xi}(t) = h_i(\xi(t)) \qquad t \in [\widehat{\tau}_{i-1}, \widehat{\tau}_i].$$

Notice that the vector fields h_i need not to be different each other, for example if u is scalar and $\Delta = [-1, 1]$ then h_i is either $X_0 + X_1$ or $X_0 - X_1$.

To each h_i we associate an Hamiltonian $H_i\colon\ (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$ given by

$$(p, x) \to \langle p, h_i(x)\rangle = \langle p, X_0(x)\rangle + \sum_{k=1}^{m} \widehat{u}_k^i \langle p, X_k(x)\rangle.$$

In the scalar case we have only two different Hamiltonians

$$H^{\pm}\colon (p, x) \to \langle p, h_i(x)\rangle = \langle p, X_0(x)\rangle \pm \langle p, X_1(x)\rangle.$$

and the H_i's are equal either to H^+ or to H^-.

With the notation stated above, we can make the assumptions concerning the maximum principle and the regularity of the **maximized Hamiltonian**

$$H\colon (p, x) \to \langle p, X_0(x)\rangle + \max_{u \in \Delta} \sum_{k=1}^{m} u_k \langle p, X_k(x)\rangle$$

Notice that in our case H is a Lipschitz function.

Assumption 2.1. *The couple $(\widehat{\xi}, \widehat{u})$ is a normal Pontryagin extremal, i.e. it satisfies the PMP in the normal form.*

In our case the PMP can be expressed in the following form: there exists a Lipschitz solution $\widehat{\lambda}\colon [0, \widehat{T}] \to (\mathbb{R}^n)^*$ of the adjoint equation

$$\dot{\lambda}(t) = -\langle \lambda(t), Dh_i(\widehat{\xi}(t))\rangle, \quad t \in [\widehat{\tau}_{i-1}, \widehat{\tau}_i], \quad i = 1, \ldots, r+1$$

such that

$$H_i(\widehat{\lambda}(t), \widehat{\xi}(t)) = H(\widehat{\lambda}(t), \widehat{\xi}(t)) = 1, \quad t \in [\widehat{\tau}_{i-1}, \widehat{\tau}_i], \qquad i = 1, \ldots, r+1$$

Assumption 2.2. *The couple $(\widehat{\xi}, \widehat{u})$ is a regular bang-bang extremal i.e.*

$$\langle \widehat{\lambda}(t), X_0(\widehat{\xi}(t))\rangle + \sum_{k=1}^{m} u_k \langle \widehat{\lambda}(t), X_k(\widehat{\xi}(t))\rangle < 1$$

for all $t \neq \widehat{\tau}_i$ and all $u \in \Delta$ such that $u \neq \widehat{u}(t)$

Assumption 2.3. *The couple $(\widehat{\xi}, \widehat{u})$ has simple switching points, i.e. at any switching time $\widehat{\tau}_i$*

$$\langle \widehat{\lambda}(\widehat{\tau}_i), X_0(\widehat{\xi}(\widehat{\tau}_i))\rangle + \sum_{k=1}^{m} u_k \langle \widehat{\lambda}(\widehat{\tau}_i), X_k(\widehat{\xi}(\widehat{\tau}_i))\rangle < 1$$

for all $u \in \Delta$ different from $\widehat{u}^i$ and $\widehat{u}^{i+1}$.

From the PMP we can easily deduce for $i = 1, \ldots, r$

$$\langle \widehat{\lambda}(\widehat{\tau}_i), (h_i - h_{i+1})(\widehat{\xi}(\widehat{\tau}_i))\rangle = 0$$
$$\langle \widehat{\lambda}(\widehat{\tau}_i), [h_i, h_{i+1}](\widehat{\xi}(\widehat{\tau}_i))\rangle \geq 0$$

where $[h_i, h_{i+1}]$ denotes the Lie-brackets between the vector fields h_i and h_{i+1}. We require that the above inequality is strict, i.e. we make the following assumption

Assumption 2.4. *The reference couple satisfies the strict bang-bang Legendre condition*

$$\langle \widehat{\lambda}(\widehat{\tau}_i), [h_i, h_{i+1}](\widehat{\xi}(\widehat{\tau}_i))\rangle > 0 \quad i = 1, \ldots, r$$

Remark 2.5. Assumptions 2.2, 2.3, 2.4, imply that the Hamiltonian system associated to the maximized Hamiltonian H is piece-wise smooth in a neighborhood of $\{(\widehat{\lambda}(t), \widehat{\xi}(t)) \colon t \in [0, \widehat{T}]\}$ in $(\mathbb{R}^n)^* \times \mathbb{R}^n$ and it has the existence and unicity property for the solutions, see [2].

3. The result

In order to give the main result, we consider the finite dimensional subproblem of the given one obtained by moving the switching times, namely we define $\Theta = \{\theta = (\tau_1, \ldots, \tau_{r+1}) \in \mathbb{R}^{r+1} \colon 0 < \tau_1 < \cdots < \tau_{r+1}\}$ and we take into account only the piecewise constant controls u such that $u(t) = \widehat{u}^i$ for $t \in (\tau_{i-1}, \tau_i)$.

For each θ in a neighborhood of $\widehat{\theta} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_{r+1})$ in Θ, we denote by $S(\theta)$ the solution at time τ_{r+1} of the system

$$\dot{\xi}(t) = h_i(\xi(t)) \qquad t \in [\tau_{i-1}, \tau_i], \quad i = 1, \ldots, r \tag{3.1}$$

$$\xi(0) = x_0 \tag{3.2}$$

The finite dimensional problem becomes the following problem on Θ

$$\textit{minimize} \qquad \gamma(\theta) = \tau_{r+1} \qquad \textit{subject to} \qquad S(\theta) = y_0,$$

with reference point $\widehat{\theta}$.

The PMP implies that the finite dimensional subproblem satisfies the Lagrange multipliers rule, namely

$$D\gamma(\widehat{\theta}) + \widehat{\lambda}(\widehat{T}) DS(\widehat{\theta}) = 0$$

We define the **second variation at the switching points** as the quadratic form of the second order condition for the finite dimensional subproblem associated to the Lagrange multiplier $\widehat{\lambda}(\widehat{T})$ i.e. the quadratic form

$$\widehat{\lambda}(\widehat{T}) D^2 S(\widehat{\theta})\Big|_{\ker DS(\widehat{\theta})}$$

Theorem 3.1. *Assume that the given bang-bang trajectory is a normal Pontryagin extremal (Assumption 2.1), it is regular (Assumption 2.2), it has simple switching points (Assumption 2.3) and that the strict bang-bang Legendre condition is satisfied (Assumption 2.4). If either $\ker DS(\widehat{\theta}) = \{0\}$ or the second variation at the switching points is positive definite then $\widehat{\xi}$ is a strong local minimizer in the sense of Definition 1.1.*

Remark 3.2. Notice that $\ker DS(\widehat{\theta}) = \{0\}$ means that $\theta \to S(\theta)$ defines locally around $\widehat{\theta}$ an $(r+1)$-dimensional sub-manifold of $\mathbb{R}^n$ containing y_0.

Remark 3.3. Here we consider the normal case for the sake of simplicity, indeed analogous conditions for the abnormal case lead to the conclusion that there is a neighborhood $\mathcal{U}$ of $\mathbb{R}^n$ such that $\widehat{\xi}$ is the only admissible trajectory whose values are contained in $\mathcal{U}$.

4. The second variation

Denoting by $S_{ts}(\theta, x)$ the solution at time t of (3.1) with initial condition $\xi(s) = x$ we have

$$S(\theta) = S_{\tau_{r+1},0}(\theta, x_0) \qquad \text{and} \qquad \widehat{\xi}(t) = S_{t,0}(\widehat{\theta}, x_0)\,,$$

moreover it is not difficult to prove that

$$\begin{aligned} &\partial_{\tau_{r+1}} S(\widehat{\theta}) = h_{r+1}(S(\widehat{\theta})) \\ &\partial_{\tau_i} S(\theta) = \partial_x S_{\widehat{\tau}_{r+1}\widehat{\tau}_i}(\widehat{\theta}, x_0)(h_i - h_{i+1})(S_{\widehat{\tau}_i 0}(\widehat{\theta}, x_0)) \qquad i = 1, \dots, r\,. \end{aligned}$$

If we introduce the vector fields

$$x \to g_i(x) = [\partial_x S_{\widehat{\tau}_i 0}(\widehat{\theta}, x)]^{-1} h_i(S_{\widehat{\tau}_i 0}(\widehat{\theta}, x))$$

we can write, in a more compact form,

$$\begin{aligned} &\partial_{\tau_{r+1}} S(\widehat{\theta}) = DS(\widehat{\theta}) g_{r+1}(x_0) \\ &\partial_{\tau_i} S(\widehat{\theta}) = DS(\widehat{\theta})[g_i(x_0) - g_{i+1}(x_0)], \qquad i = 1, \dots, r \end{aligned}$$

Remark 4.1. $DS(\widehat{\theta})$ is injective if and only if the vector fields $g_i(x_0)$, $i = 1, \dots, r+1$, are linearly independent.

Using a suitable reparametrization of the time we can prove that the positivity of the second variation at the switching points is equivalent to the positivity of the quadratic form

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_{r+1}) \to \sum_{i=2}^{r+1} \sum_{j=1}^{i-1} \varepsilon_i \varepsilon_j \langle \widehat{\lambda}(0), [g_i, g_j](x_0) \rangle$$

over $\varepsilon \in \mathbb{R}^{r+1}$ such that

$$\sum_{i=1}^{r+1} \varepsilon_i g_i(x_0) = 0\,.$$

References

[1] Andrei A. Agrachev, Gianna Stefani, and PierLuigi Zezza. Symplectic methods for strong local optimality in the bang-bang case. In J.P. Gauthier F. Monroy-Perez A. Anzaldo-Meneses, B. Bonnard, editor, *Contemporary Trends in Non-linear Geometric Control Theory and its Applications*, London, 2002. World Scientific.

[2] Andrei A. Agrachev, Gianna Stefani, and PierLuigi Zezza. Strong optimality for a bang-bang trajectory. *SIAM J. Control Optim.*, to appear.

[3] Helmut Maurer and Nikolai P. Osmolovskii. Second order sufficient conditions for time optimal bang-bang control problems. *Preprint*, 2002.

[4] Andrei V. Sarychev. First and second order sufficient optimality conditions for bang-bang controls. *SIAM J. Control Optimization*, 1(35):315–340, 1997.

[5] Hector J. Sussmann and Benedetto Piccoli. Regular synthesis and sufficient conditions for optimality. *SIAM J. Control and Optimization*, 39(2):359–410, 2000.

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