

A TOWER OF GENUS TWO CURVES RELATED TO THE KOWALEWSKI TOP

FRANCK LEPREVOST AND DIMITRI MARKUSHEVICH

ABSTRACT. Several curves of genus 2 are known, such that the equations of motion of the Kowalewski top are linearized on their Jacobians. One can expect from transcendental approaches via solutions of equations of motion in theta-functions, that their Jacobians are isogeneous. The paper focuses on two such curves: Kowalewski's and that of Bobenko–Reyman–Semenov-Tian-Shansky, the latter arising from the solution of the problem by the method of spectral curves. An isogeny is established between the Jacobians of these curves by purely algebraic means, using Richelot's transformation of a genus 2 curve. It is shown that this isogeny respects the Hamiltonian flows. The two curves are completed into an infinite tower of genus 2 curves with isogeneous Jacobians.

INTRODUCTION

Several authors writing on the Kowalewski top remarked that there are a few apparently different curves of genus 2 arising in the problem of integrating the equations of motion of the top. The one classically known is Kowalewski's curve [8]; see also a modern exposition of her approach in [1] or [2]. The remarkable property of this curve C_1 is that the flow of solutions of the equations of motion is linearized on its Jacobian J_1 , and so, the solutions can be expressed in terms of theta-functions of two variables. Bobenko–Reyman–Semenov-Tian-Shansky [3] constructed another curve of genus 2 C_2 with the same property, but arising in a different way, namely, from the Lax representation for the equations of motion of the top. Their construction leads to a genus 2 curve only in the case when the angular momentum l of the top is orthogonal to the gravity vector g . So, in this case, there are two different genus 2 curves associated to the Kowalewski top. It is interesting to study more closely the relation between the two curves.

The authors of [3] claim that the Jacobians of the two curves are isogeneous. They do not give an explicit proof, but write out the solutions of the equations of motion in terms of theta-functions on the Jacobian J_2 of their curve C_2 . The formulas for the solutions

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define, in fact, a map from J_2 onto the corresponding Liouville torus T . The knowledge of the smallest periods of the solutions would give an information on the nature of this map. The authors claim that this map is an isogeny. It is known also that solutions of the equations of motion on the Jacobian of Kowalewski's curve J_1 yield an isogeny from J_1 onto T . Thus, J_1 and J_2 are isogeneous to the same abelian surface T , hence isogeneous to one another. It is a natural problem to search for an algebraic expression for such an isogeny, avoiding cumbersome formulas with theta-functions. An elegant and purely algebraic solution to this problem is given in the present paper.

To our knowledge, so far only two more curves of genus 2, related to the Kowalewski top and different from the curve of Kowalewski have been mentioned in the literature. They are introduced in [6]. The authors establish the existence of an isogeny between the Jacobians of these curves and of Kowalewski's. Their analytic approach is completely different from the one purely algebraic applied in the present paper, and they do not address the question on the relation of their curves to that of [3].

In the present paper, we show that the curve C_1 of Kowalewski is obtained from the curve C_2 of Bobenko–Reyman–Semenov–Tian–Shansky by Richelot's transformation ([10],[11]) inducing an isogeny of degree 4 between their Jacobians. Furthermore, we show, that in iterating Richelot's construction in a convenient way, one can obtain a tower of countably many curves of genus 2, whose Jacobians are all isogeneous to that of the curve of Kowalewski. Thus, this approach gives an infinity of Jacobians, on which the Hamiltonian flow of the Kowalewski top is linearized.

In Section 1, we describe briefly the equations of motion of the Kowalewski top and the procedures leading to C_1 and C_2 . We explain, why the flow of solutions of the equations is linearized on the Jacobians of the two curves. This certainly provides a linear map between the universal covers of the Jacobians, but still does not explain, why they are isogeneous.

In Section 2, we describe Richelot's construction, and apply it to obtain a (2,2)-correspondence between the curves of Bobenko–Reyman–Semenov–Tian–Shansky and of Kowalewski. We show that this correspondence induces an isogeny of the Jacobians with kernel $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and that the isogeny transforms the solutions of the Lax equations on J_2 into Kowalewski's solutions of the equations of motion of the top on J_1 .

In Section 3, we describe briefly Richelot's algorithm which leads to a tower of isogeneous abelian surfaces, and apply it to our situation; we obtain a tower whose ending segment is the Jacobian of the curve of Bobenko et al. followed by that of the curve of Kowalewski.

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1. KOWALEWSKI'S TOP

We will follow the notations of [3]. In fact, the integrable system introduced there is Kowalewski's top in constant electric and gravitational fields, called Kowalewski's gyrostat. As soon as we are interested in the classical situation, we will specialize all the formulas to the case when the electric field is zero. The motion of the top can be described by the following system:

$$\begin{aligned}\frac{dl}{dt} &= [l, \omega] + [c, g], \\ \frac{dg}{dt} &= [g, \omega] \\ \omega &= \mathcal{J}l\end{aligned}\tag{1}$$

Here $[\cdot, \cdot]$ is the vector product in \mathbb{R}^3 , l is the angular momentum, g the gravity vector, c the vector of the center of mass, ω the angular velocity, and $\mathcal{J} = I^{-1}$ the inverse of the inertia tensor $I = (I_{ij})_{1 \leq i, j \leq 3}$, everything in a moving frame (e_1, e_2, e_3) , attached to the solid. This system is Hamiltonian, with Hamiltonian

$$H = \frac{1}{2}(\mathcal{J}l, l) - (g, c).$$

In Kowalewski's integrable case, the inertia tensor is $I = \text{diag}(1, 1, 1/2)$, and c lies in the plane spanned by e_1, e_2 . If we choose the moving frame so that the center of mass is the endpoint of e_1 , then

$$H = \frac{1}{2}(l_1^2 + l_2^2 + 2l_3^2) - g_1,$$

and there are two additional integrals of motion

$$I_1 = (l, g)^2, \quad I_2 = (l_1^2 - l_2^2 + 2g_1)^2 + 4(l_1 l_2 + g_2)^2.$$

The equations of motion admit a Lax representation

$$\frac{dL}{dt} = [L, M]\tag{2}$$

with Lax matrix

$$L(\lambda) = \begin{bmatrix} \frac{g_1}{\lambda} & \frac{g_2}{\lambda} & -l_2 + \frac{g_3}{\lambda} & -l_1 \\ \frac{g_2}{\lambda} & -\frac{g_1}{\lambda} & l_1 & -l_2 - \frac{g_3}{\lambda} \\ l_2 + \frac{g_3}{\lambda} & -l_1 & -2\lambda - \frac{g_1}{\lambda} & -2l_3 + \frac{g_2}{\lambda} \\ l_1 & l_2 - \frac{g_3}{\lambda} & 2l_3 + \frac{g_2}{\lambda} & 2\lambda + \frac{g_1}{\lambda} \end{bmatrix},$$

where $[L, M] = LM - ML$ for some matrix M , which we will not explicitize here. One can verify, that the invariants H, I_1 and I_2 belong to the algebra generated by the coefficients of λ^{-2} and λ^0 in the Laurent expansions of $\text{Tr}(L(\lambda)^2)$ and $\text{Tr}(L(\lambda)^4)$. Since

these coefficients are invariant under the flow of (2), the spectral curve $P(\lambda, \mu) = 0$ is also invariant, where

$$P(\lambda, \mu) = \det (L(\lambda) - \mu).$$

Let Γ be the non-singular compactification of the spectral curve, and $L(t)$ a solution of (2). Then we have the line bundle E_t of eigenvectors of $L(t)$ on Γ (it is defined a priori on a Zariski open subset of Γ , but it is uniquely extended to all of Γ as a line subbundle of a fixed vector bundle, namely, of the trivial one $\mathbb{C}^4 \times \Gamma$). It is proved in [9] that the evolution of the class of E_t on the Jacobian of Γ is linear, and the velocity $V = d[E_t]/dt$ is given by

$$\omega(V) = \sum_{p: \lambda(p)=\infty} \text{res}_p \left(\frac{1}{2} \mu \omega \right) \quad \forall \quad \omega \in H^0(\Gamma, \Omega_\Gamma^1). \quad (3)$$

Moreover, the flow is confined to the Jacobian of the curve $C_2 = \Gamma / \langle \tau_1 \rangle$ and parallel to the Prym variety $P(C_2/E)$, where $E = \Gamma / \langle \tau_1, \tau_2 \rangle$, and $\tau_1 : (\lambda, \mu) \mapsto (-\lambda, \mu)$, $\tau_2 : (\lambda, \mu) \mapsto (\lambda, -\mu)$.

From the physical point of view, it is natural to think of $|g|^2$ and I_1 as of trivial constants of motion. The slices $|g|^2 = \gamma$, $I_1 = \kappa$ represent 4-dimensional symplectic manifolds $M_{\gamma\kappa}$ (see (1.3) of [3] for corresponding Poisson brackets), and the remaining first integrals (H, I_2) yield the complete integrability of the Hamiltonian system on $M_{\gamma\kappa}$ in the sense of Liouville. They define the moment map $\mu : M_{\gamma\kappa} \rightarrow \mathbb{C}^2$, whose (compactified) fibers are disjoint unions of Liouville tori, and the Hamiltonian flow linearizes on their universal cover. It turns out, that the Liouville tori can be identified with the Prym variety $P(C_2/E)$, if $\kappa \neq 0$; in this case, C_2 is of genus 3, E elliptic, and $\dim P(C_2/E) = 2$. If $\kappa = 0$, the genus of C_2 (resp. E) goes down to 2 (resp. 0), and $P(C_2/E)$ becomes simply the Jacobian of C_2 .

The curve C_2 is that of [3] mentioned in the introduction, and our aim is to compare it to the curve of Kowalewski. So, we will suppose from now on that $I_1 = \kappa = 0$, and C_2 is of genus 2. We can also normalize the constants so that $|g|^2 = \gamma = 1$. We have for Γ the equation

$$\mu^4 - 2d_1(\lambda^2)\mu^2 + d_2(\lambda^2) = 0,$$

where

$$d_1(z) = z^{-1} - 2H + 2z, \quad d_2(z) = z^{-2} - 4Hz^{-1} + I_2,$$

and the equations of C_2, E are obtained by substituting $\lambda^2 = z$, resp. $\mu^2 = y$. One can check that the 1-forms

$$\omega_0 = \frac{dz}{\mu z(\mu^2 - d_1(z))}, \quad \omega_1 = \frac{1}{2} \left(\mu^2 - \frac{1}{z} \right) \omega_0 \quad (4)$$

yield a basis of $H^0(C_2, \Omega_{C_2}^1)$. Applying (3), we obtain the following statement:

Proposition 1. *The coordinates of the velocity vector V in the basis ω_0, ω_1 are $(0, -1)$. Hence, the equation $d[E_t]/dt = V$ induces on $\text{Sym}^2(C_2)$ the following system:*

$$\begin{aligned} \sum_{i=1,2} \frac{dz_i/dt}{\mu_i z_i (\mu_i^2 - d_1(z_i))} &= 0, \\ \sum_{i=1,2} \frac{1}{2} \left(\mu_i^2 - \frac{1}{z_i} \right) \frac{dz_i/dt}{\mu_i z_i (\mu_i^2 - d_1(z_i))} &= -1. \end{aligned} \quad (5)$$

Following [3], where the analogous system is written out for the case when C_2 is of genus 3, we will call (5) the Dubrovin form of the equations of the motion of the top.

The change of variables $x = \frac{1}{2}(\mu^2 - z^{-1})$, $u = \frac{\mu}{\sqrt{2}}(x^2 + 2Hx - 1 + \frac{1}{4}I_2)$ brings the equation of C_2 to the canonical form:

$$u^2 = x(x^2 + 2Hx + \frac{1}{4}I_2)(x^2 + 2Hx - 1 + \frac{1}{4}I_2),$$

and the basis (4) of $H^0(C_2, \Omega_{C_2}^1)$ becomes

$$\omega_0 = \frac{dx}{\sqrt{2}u}, \quad \omega_1 = \frac{x dx}{\sqrt{2}u}.$$

So, the Dubrovin equations can be rewritten as follows:

$$\frac{dx_1/dt}{u_1} + \frac{dx_2/dt}{u_2} = 0, \quad \frac{x_1 dx_1/dt}{u_1} + \frac{x_2 dx_2/dt}{u_2} = -\sqrt{2}. \quad (6)$$

The linearized equations of Kowalewski have the same form, but on another curve of genus 2. We will write out her solution in omitting details of calculations. We are using formulas from Audin [1]. Some differences in coefficients are explained by the choice of different dimensionless parameters: $I_{33} = 1/2$, $|c| = |g| = 1$ here, and $I_{33} = |c| = |g| = 1$ in [1]. When comparing solutions, we should keep in mind that the corresponding times are related by the equation $\tilde{t} = \sqrt{2}t$, where t is the time of Bobenko et al. ([3]) and \tilde{t} Audin's ([1]).

Let $x = l_1 + il_2$, $y = l_1 - il_2$. Considering them as independent complex variables, define the new variables ξ_1, ξ_2 by

$$\xi_1 = H + \frac{R(xy) - \sqrt{R(x^2)R(y^2)}}{(x-y)^2}, \quad \xi_2 = H + \frac{R(xy) + \sqrt{R(x^2)R(y^2)}}{(x-y)^2},$$

where

$$R(x) = -x^2 + 2Hx + 1 - \frac{1}{4}I_2.$$

Then the equations (1) are reduced to the following system

$$\frac{d\xi_1/d\tilde{t}}{\eta_1} + \frac{d\xi_2/d\tilde{t}}{\eta_2} = 0, \quad \frac{\xi_1 d\xi_1/d\tilde{t}}{\eta_1} + \frac{\xi_2 d\xi_2/d\tilde{t}}{\eta_2} = i \quad (7)$$

on $\text{Sym}^2(C_1)$, where C_1 is the genus 2 curve defined by the equation

$$\eta^2 = 2\xi((\xi - H)^2 + 1 - \frac{1}{4}I_2)((\xi - H)^2 - \frac{1}{4}I_2).$$

Like (6), these equations describe a linearized flow on the Jacobian of the hyperelliptic curve. Its velocity vector with respect to the time t is $V_1 = (0, \sqrt{-2})$. It is a natural question to ask whether the two flows can be transformed into each other by a holomorphic (and hence algebraic) map between the Jacobians. Considering the differentials of the first kind as coordinate functions on the universal covering of the Jacobian of a curve, we can represent such a map in the form $\nu_0 = a\omega_0, \nu_1 = b\omega_0 + \sqrt{-2}\omega_1$, where $\nu_0 = d\xi/\eta, \nu_1 = \xi d\xi/\eta$, and $a, b \in \mathbb{C}$. The question is whether it can be realized for some a, b by an algebraic correspondence between C_1 et C_2 . The answer can be obtained by expressing both solutions in terms of theta functions, but there is also a beautiful purely algebraic construction of such a correspondence, using only the equations of the curves. It is described in the next section.

2. RICHELLOT ISOGENY

In this section, we apply Richelot's construction ([10], [11]). We follow the approaches of [5], p. 89 and of [4]. Let C be a genus 2 curve defined over the ground field K by an equation

$$u^2 = f(x) = G_1(x)G_2(x)G_3(x),$$

where

$$G_j(x) = g_{j2}x^2 + g_{j1}x + g_{j0} \in K[x].$$

Let \widehat{C} be the genus 2 curve defined by the following equation

$$\Delta Y^2 = F(X) = L_1(X)L_2(X)L_3(X),$$

where

$$L_1(X) = [G_2, G_3] = G_2'(X)G_3(X) - G_2(X)G_3'(X)$$

and so on, cyclically, and $\Delta = \det(g_{ij})$. A $(2, 2)$ -correspondence between C and \widehat{C} is defined by the curve Z given over $C \times \widehat{C}$ by the equations

$$\begin{cases} G_1(x)L_1(X) + G_2(x)L_2(X) &= 0, \\ G_1(x)L_1(X)(x - X) &= yY. \end{cases}$$

The correspondence Z induces the isogeny $\varphi : J \longrightarrow \widehat{J}$ between J and \widehat{J} , the Jacobians of C and \widehat{C} respectively, given by the formula $\varphi([\sum n_i P_i]) = [\sum n_i p_2 p_1^{-1} P_i]$ for all divisor $\sum n_i P_i$ of degree zero, where p_1 (resp. p_2) is the restriction to Z of the projection of $C \times \widehat{C}$ to C (resp. to \widehat{C}). The kernel of φ is an abelian group of type $(2, 2)$, whose non-zero elements are explicitly given in terms of the roots of the G_i 's (see [4], p. 52). In other words, φ is a $(2, 2)$ -isogeny of abelian surfaces, the so-called Richelot isogeny, and it factors the multiplication by 2 on \widehat{J} .

Now let C_1 be Kowalewski's curve, $C = C_2$ the curve obtained by Bobenko et al., and denote by J_1 and J_2 their Jacobians respectively. These curves are defined over $K = \mathbb{Q}(H, I_2)$ and, with the notations of the current section, the equation of C_2 is given by

$$u^2 = G_1(x)G_2(x)G_3(x)$$

where

$$\begin{aligned} G_1(x) &= x, \\ G_2(x) &= x^2 + 2Hx + \frac{1}{4}I_2, \\ G_3(x) &= x^2 + 2Hx + \frac{1}{4}I_2 - 1. \end{aligned}$$

It is worthwhile to compute Δ and the L_i 's. It follows that $\widehat{C_2}$ is given by the equation

$$W^2 = -2(X + H)[X^2 + 1 - \frac{1}{4}I_2][X^2 - \frac{1}{4}I_2].$$

By the translation $\tilde{X} = X + H$, it is transformed into

$$W^2 = -2\tilde{X}[(\tilde{X} - H)^2 + 1 - \frac{1}{4}I_2][(\tilde{X} - H)^2 - \frac{1}{4}I_2].$$

We see that $\widehat{C_2}$ is isomorphic to C_1 via the map $\nu : (\tilde{X}, W) \mapsto (\xi, \eta) = (\tilde{X}, iW)$. So, the Jacobians J_2 and J_1 are isogenous via the composition $\psi = \nu_* \circ \varphi$, where φ is Richelot's isogeny, defined above.

There are several ways to prove that J_1 and J_2 are generically non-isomorphic. One of them is to compute their Igusa invariants [7] and to check that they are different. We used this procedure to complete the proof of the following result.

Theorem 1. *J_1 and J_2 are isogenous over $\mathbb{Q}(H, I_2)(i)$ via the isogeny ψ , and are generically non-isomorphic.*

Corollary 1. *The curves C_1 and C_2 are not isomorphic. Moreover there are no non-constant morphisms between C_1 and C_2 .*

Although it directly follows from the previous theorem, the first part of the above corollary could be proved directly. The second part is obvious, for a morphism between curves having the same genus ≥ 2 should be an isomorphism.

Corollary 2. *The isogeny ψ transforms the flow of solutions of Dubrovin equations (6) on J_2 into that of Kowalewski's equations (7) on J_1 .*

Proof follows from the calculation of the differential of Richelot's isogeny. Its adjoint $\delta = (d_0\varphi)^*$ can be understood as a linear map $\delta : H^0(\widehat{C_2}, \Omega_{\widehat{C_2}}^1) \rightarrow H^0(C_2, \Omega_{C_2}^1)$, and it follows from definitions that $\delta = p_{1*}p_2^*$ (p_{1*} being the trace map for the double covering

p_1). This map δ is computed in [4]: $\delta\left(S(X)\frac{dX}{W}\right) = S(x)\frac{dx}{u}$ for S a polynomial of degree ≤ 1 . As $d_0\nu_*$ is the multiplication by $-i$, we obtain:

$$(d_0\psi)^* : \frac{d\xi}{\eta} \mapsto -i\frac{dx}{u} \quad , \quad (d_0\psi)^* : \frac{\xi d\xi}{\eta} \mapsto -i\frac{xdx}{u} - iH\frac{dx}{u} \quad .$$

This implies that $d_0\psi$ transforms the generating vector $(0, -\sqrt{2})$ of Dubrovin's flow (with respect to the basis $\frac{dx}{u}, \frac{xdx}{u}$) into $(0, \sqrt{-2})$. This ends the proof.

3. A TOWER OF ABELIAN SURFACES

As explained in [4], Richelot's method allows to construct a tower of $(2, 2)$ -isogenies of abelian surfaces:

$$\cdots \longrightarrow \mathcal{J}_{n+1} \xrightarrow{\varphi_n} \mathcal{J}_n \longrightarrow \cdots \longrightarrow \mathcal{J}_2 \xrightarrow{\varphi_1} \mathcal{J}_1,$$

where \mathcal{J}_n is the Jacobian of a genus 2 curve \mathcal{C}_n defined by an equation $y^2 = F_n(x)$. The algorithm takes as input a suitable factorisation of $F_{n-1}(x) = P_{n-1}(x)Q_{n-1}(x)R_{n-1}(x)$ in real polynomials of degree 2, applies Richelot's construction on it, and outputs the polynomial $F_n(x)$ as a product $P_n(x)Q_n(x)R_n(x)$ of real polynomials of degree 2: see [4] for a more complete description and for some applications.

By applying the above method with $\mathcal{J}_1 = J_1$ the Jacobian of Kowalewski's curve C_1 , one obtains the following tower of isogenous Jacobians of computable curves of genus 2

$$\cdots \longrightarrow J_{n+1} \xrightarrow{\psi_n} J_n \longrightarrow \cdots \longrightarrow J_2 \xrightarrow{\psi_1=\psi} J_1,$$

whose ending segment is the Jacobian of the curve of Bobenko et al. followed by the Jacobian of the curve of Kowalewski.

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F.L.: TECHNISCHE UNIVERSITÄT BERLIN, FACHBEREICH MATHEMATIK MA 8-1, STRASSE DES 17. JUNI 136, D-10623 BERLIN GERMANY

E-mail address: `leprevot@math.tu-berlin.de`

D. M.: MATHÉMATIQUES - BÂT. M2, UNIVERSITÉ LILLE 1, F-59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

E-mail address: `markushe@gat.univ-lille1.fr`