Geometric Completeness Of Distribution Spaces

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Abstract

This paper introduces a notion of *geometric completeness* for spaces of distributions, modelled after the notion of a complete variety in algebraic geometry. It is related to the following *elimination problem* for systems of PDE: consider the set of homogeneous solutions of a system of PDE in some space of distributions. When is the projection of this set onto some of its co-ordinates also the set of homogeneous solutions of a system of PDE?

1 Introduction

Let $\mathcal{A} = \mathbb{C}[X_1, \ldots, X_n]$ be the polynomial ring in n indeterminates. Classical (i.e. pre-Grothendieck) algebraic geometry studies the category of affine varieties (i.e. the set of common zeros in $\mathbb{C}^n$ of ideals of $\mathcal{A}$) or of projective varieties (i.e. the set of common zeros in $\mathbb{P}^{n-1}$ of homogeneous ideals of $\mathcal{A}$). A basic question here is that of classical elimination theory - given a variety, when is its projection onto some of its co-ordinates also a variety? An affirmative answer to this question is the notion of a complete variety (for instance Mumford [6]). In this paper I consider the analogous question for the set of homogeneous solutions of a system of PDE in some space of distributions. This question is of fundamental importance in the behavioural formulation of control theory due to J.C.Willems. The affirmative answer here is the notion of geometric completeness.

Thus suppose now that $\mathcal{A} = \mathbb{C}[\partial_1, \ldots, \partial_n]$, the ring of constant coefficient partial differential operators on $\mathbb{R}^n$ (which being isomorphic to $\mathbb{C}[X_1, \ldots, X_n]$, I have denoted it also by $\mathcal{A}$). Let $\mathcal{D}'$ be the space of distributions on $\mathbb{R}^n$; then $\mathcal{A}$ acts on it by differentiation and gives it the structure of an $\mathcal{A}$-module.

$$p(\partial) : \mathcal{D}' \longrightarrow \mathcal{D}'$$

The *zeros* now of $p(\partial)$ is $Ker_{\mathcal{D}'}(p(\partial))$, the kernel of the above morphism, and is the object analogous to the algebraic subset of $\mathbb{C}^n$ defined by the corresponding polynomial $p(X)$ in $\mathbb{C}[X_1, \ldots, X_n]$. More generally, if $\mathcal{F}$ is any $\mathcal{A}$-submodule of $\mathcal{D}'$, then the kernel of the $\mathcal{A}$-module morphism

$$p(\partial) : \mathcal{F} \longrightarrow \mathcal{F}$$

denoted $Ker_{\mathcal{F}}(p(\partial))$ (and equal to $\mathcal{F} \cap Ker_{\mathcal{D}'}(p(\partial))$) is the set of zeros of $p(\partial)$ in $\mathcal{F}$. The zeros in $\mathcal{F}$ of an ideal $\mathfrak{i}$ of $\mathcal{A}$ is then $\bigcap_{p(\partial)\in\mathfrak{i}} Ker_{\mathcal{F}}(p(\partial))$, the set of common zeros of all the elements in $\mathfrak{i}$, and is denoted $Ker_{\mathcal{F}}(\mathfrak{i})$.

Even more generally, let $p(\partial) = (p_1(\partial), \ldots, p_k(\partial))$ be an element in $\mathcal{A}^k$. It defines an $\mathcal{A}$-module morphism

$$\begin{array}{rccc} p(\partial) : & \mathcal{F}^k & \longrightarrow & \mathcal{F} \\ & (f_1, \ldots, f_k) & \mapsto & \sum p_i(\partial) f_i \end{array}$$

Its zeros, i.e. its kernel, is denoted $Ker_{\mathcal{F}}(p(\partial))$. If $\mathcal{P}$ is a submodule of $\mathcal{A}^k$, then $Ker_{\mathcal{F}}(\mathcal{P}) = \bigcap_{p(\partial)\in\mathcal{P}} Ker_{\mathcal{F}}(p(\partial))$ is the set of common zeros of elements in $\mathcal{P}$. As $\mathcal{A}$ is Noetherian, $\mathcal{P}$ is finitely generated, say by $p_1, \ldots, p_l$, $p_i = (p_{i1}(\partial), \ldots, p_{ik}(\partial))$. Let $P(\partial)$ be the $l \times k$ matrix whose (i,j)th element is $p_{ij}(\partial)$. Then $P(\partial)$ defines an $\mathcal{A}$-module morphism

$$P(\partial) : \mathcal{F}^k \longrightarrow \mathcal{F}^l \tag{1}$$

Clearly the kernel of this morphism is $Ker_{\mathcal{F}}(\mathcal{P})$. Thus the objects of study here, called henceforth *differential kernels*, are the homogeneous solutions in an $\mathcal{A}$-submodule $\mathcal{F}$ of $\mathcal{D}'$, of matrices of partial differential operators on $\mathbb{R}^n$.

While these differential kernels have been studied for long, ubiquitous as they are in mathematics and physics, their study using homological algebraic methods (which are the methods employed here) was initiated in the 1950s by the work of Ehrenpreis, Malgrange, Palamodov and others. It stems from the observation that $Ker_{\mathcal{F}}(\mathcal{P})$ is isomorphic to $Hom_{\mathcal{A}}(\mathcal{A}^k/\mathcal{P}, \mathcal{F})$, the isomorphism given by

$$\phi \in Hom_{\mathcal{A}}(\mathcal{A}^k/\mathcal{P}, \mathcal{F}) \mapsto (\phi(\overline{e_1}), \ldots, \phi(\overline{e_k})) \in Ker_{\mathcal{F}}(\mathcal{P})$$

where $\overline{e_1}, \ldots, \overline{e_k}$ are the images in $\mathcal{A}^k/\mathcal{P}$ of the standard basis of $\mathcal{A}^k$. Recently these kernels have also come to be used in modelling engineering phenomena where they are called *behaviours* (see Willems [13] and the references therein). It has been realised in the light of these developments the many formal similarities between differential kernels and algebraic varieties. In this paper, for instance, I study the PDE notion analogous to the notion of a complete variety in algebraic geometry. More precisely I calculate the obstruction to the following - when is the projection of a differential kernel onto some of its co-ordinates also a differential kernel?

To calculate this obstruction I need to first consider a restricted instance of this question - given a *differential system* such as (1), when is the image of $P(\partial) : \mathcal{F}^k \to \mathcal{F}^l$ equal to the kernel of the differential system defined by the module $\mathcal{P}_1$ of relations between the l rows of $P(\partial)$? (To ask if this image is the kernel of *some* differential system, not necessarily of this module of relations, is the more general question posed in the paragraph above and is analogous to asking when the image of a morphism $f : X \to Y$ of varieties is (Zariski) closed.) I also need to consider the dual question - when is the kernel of (1) equal to the image of the differential system defined by a matrix whose columns generate the relations between the columns of $P(\partial)$?

An affirmative answer to the first question above would amount to an effective solvability test (with respect to the space $\mathcal{F}$) for the system $P(\partial)$. Indeed let $\mathcal{P}_1$ be the submodule of $\mathcal{A}^l$ defined above. Suppose that $\mathcal{P}_1$ is generated by some l_1 elements. Writing out these l_1 elements in rows yields an $l_1 \times l$ matrix $P_1(\partial)$ such that $P_1(\partial)P(\partial) = 0$. Then given some element g in $\mathcal{F}^l$, a necessary condition for the existence of an f in $\mathcal{F}^k$ such that $P(\partial)f = g$, is that $P_1(\partial)g = 0$. Now if the image of $P(\partial)$ were equal to the kernel of the morphism

$$P_1(\partial) : \quad \mathcal{F}^l \quad \longrightarrow \quad \mathcal{F}^{l_1}$$

determined by the matrix $P_1(\partial)$, then the above necessary condition for solvability would also be sufficient.

Answers to such questions clearly depend not only on the system $P(\partial)$, but also on the $\mathcal{A}$-module structure of $\mathcal{F}$. For instance, suppose that for some $\mathcal{A}$-submodule $\mathcal{F}$ of $\mathcal{D}'$, the image of every differential system is the kernel of the morphism determined by the module of relations as above - then $\mathcal{F}$ is said to admit the *Fundamental Principle* (the terminology is due to Ehrenpreis). This is equivalent to $\mathcal{F}$ being an injective $\mathcal{A}$-module, and it is a celebrated theorem of Ehrenpreis, Malgrange and Palamodov that $\mathcal{D}'$ and the space $\mathcal{C}^\infty$ of all smooth functions on $\mathbb{R}^n$ are injective $\mathcal{A}$-modules (see Hörmander [3] for a detailed exposition). The space $\mathcal{S}'$ of temperate distributions is also an injective $\mathcal{A}$-module - a theorem of Malgrange [5] - so that the Fundamental Principle is again valid here. Earlier Hörmander [2] and Łojasiewicz [4] had proved that $\mathcal{S}'$ was a divisible $\mathcal{A}$-module, or in other words that any nonzero element $p(\partial)$ of $\mathcal{A}$ defines a surjective $\mathcal{A}$-module morphism on $\mathcal{S}'$. This was an important result as it implied the existence of a fundamental solution in $\mathcal{S}'$ for any nonzero $p(\partial)$, and simpler proofs by Bernstein and Gelfand, and by Atiyah (using Hironaka's resolution of singularities) had soon followed.

On the other hand, the Schwartz space $\mathcal{S}$ of rapidly decreasing functions, the spaces $\mathcal{E}'$ and $\mathcal{D}$ of compactly supported distributions and smooth functions are not injective, not even divisible $\mathcal{A}$-modules as can be easily seen by Fourier transformation. Instead they are flat $\mathcal{A}$-modules (the canonical injection $i : \mathcal{A} \to \mathcal{E}'$ is a faithfully flat ring morphism). The usual characterization of flatness in terms of relations implies that for these submodules of $\mathcal{D}'$, every kernel is the image of a morphism (determined by the relations between columns as above) - in other words an affirmative answer to the dual question for every system.

This question also arises in physics and engineering in the following context. The laws governing many phenomena can be expressed in terms of systems of differential equations, the admissible phenomena being the homogeneous solutions of the system. In other words, the admissible phenomena appears as the kernel of a differential system $P(\partial) : \mathcal{F}^k \to \mathcal{F}^l$. A fundamental question here is whether this kernel is exact, i.e. whether there is a system

$$P_{-1}(\partial) : \mathcal{F}^{k_1} \longrightarrow \mathcal{F}^k$$

such that the image of $P_{-1}(\partial)$ equals the kernel of $P(\partial)$. In physics such a $P_{-1}(\partial)$ is called a *potential function*, in engineering an *image representation* (for the relationship between image representations and the fundamental concept of controllability, see [9, 13]). Notice here that the domain of the potential function $P_{-1}(\partial)$ is (a cartesian product of) the same space $\mathcal{F}$ as the domain of $P(\partial)$ (this is more restrictive than the usual notion of a potential, and is dictated by applications in physics and engineering). Thus it is demanded, for instance, that the compactly supported kernel of $P(\partial)$ should be the image of compactly supported elements. This means a more careful choice of a potential function from the, *a priori*, many that might be available. For example the results of this paper show that exactness of

$$(\mathcal{C}^\infty)^3 \xrightarrow{curl} (\mathcal{C}^\infty)^3 \xrightarrow{div} \mathcal{C}^\infty$$

is preserved under restriction to the space $\mathcal{D}$. Thus

$$\mathcal{D}^3 \xrightarrow{curl} \mathcal{D}^3 \xrightarrow{div} \mathcal{D}$$

is also exact. On the other hand if cr is the $\mathcal{A}$-module morphism defined by some two columns of $curl$, say

$$cr = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix}$$

then

$$(\mathcal{C}^\infty)^2 \xrightarrow{cr} (\mathcal{C}^\infty)^3 \xrightarrow{div} \mathcal{C}^\infty$$

is still exact (this is because the three columns of $curl$ are $\mathcal{A}$-dependent; therefore as $\mathcal{C}^\infty$ is a divisible, in fact an injective $\mathcal{A}$-module, it follows that the image of cr equals the image of $curl$ - see the proof of Theorem 2.4 below). But now the restriction

$$\mathcal{D}^2 \xrightarrow{cr} \mathcal{D}^3 \xrightarrow{div} \mathcal{D}$$

is *no longer* exact. This explains, one supposes, why the de-Rham complex is the preferred choice of resolution of *divergence* instead of the above complex with cr, or why *curl* is the preferred choice for the magnetic potential. One could further demand that certain *conserved* quantities in the kernel of $P(\partial)$ be the image of similarly conserved quantities, and this paper considers one such situation.

Clearly the two questions discussed above, viz. the existence of a Fundamental Principle on the one hand, and the existence of a potential function on the other, are 'dual' to one another, in the sense that while the first demands exactness of

$$\mathcal{F}^k \stackrel{P(\partial)}{\longrightarrow} \mathcal{F}^l \stackrel{P_1(\partial)}{\longrightarrow} \mathcal{F}^{l_1}$$

the second requires exactness of

$$\mathcal{F}^{k_1} \stackrel{P_{-1}(\partial)}{\longrightarrow} \mathcal{F}^k \stackrel{P(\partial)}{\longrightarrow} \mathcal{F}^l$$

One of the results of this paper, viz. Theorem 3.1, states that when $\mathcal{F}$ is either $\mathcal{D}'$, $\mathcal{C}^\infty$, $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$, i.e. one of the *classical spaces*, then the answers to these dual questions for spaces dual to one another, are dual to each other as well. More generally, one can study the problem of embedding the system in a two sided complex, i.e. to construct a complex

$$\ldots \to \mathcal{F}^{k_i} \stackrel{P_{-i}(\partial)}{\longrightarrow} \ldots \stackrel{P_{-2}(\partial)}{\longrightarrow} \mathcal{F}^{k_1} \stackrel{P_{-1}(\partial)}{\longrightarrow} \mathcal{F}^k \stackrel{P(\partial)}{\longrightarrow} \mathcal{F}^l \stackrel{P_1(\partial)}{\longrightarrow} \mathcal{F}^{l_1} \stackrel{P_2(\partial)}{\longrightarrow} \ldots \stackrel{P_i(\partial)}{\longrightarrow} \mathcal{F}^{l_i} \to \ldots$$

with 'minimal' homology. The study of this problem was initiated in [11] in the context of a Nullstellensatz for partial differential operators, and is continued here (see also [10, 14]). This paper shows that if $\mathcal{F}$ is $\mathcal{D}'$, $\mathcal{C}^\infty$ or $\mathcal{S}'$, then any differential system $P(\partial) : \mathcal{F}^k \to \mathcal{F}^l$ can be extended to the *right* by a finite exact complex, whereas if $\mathcal{F}$ is $\mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$, then it can be extended similarly to the *left.*

The paper is organized as follows. As remarked above, the results presented here are consequences of the $\mathcal{A}$-module structure of the classical spaces. The description of this structure has an illustrious history going back to classical work of Ehrenpreis, Malgrange and Palamadov (see [8]). Recently, Oberst [7] has proved a powerful theorem regarding the $\mathcal{A}$-module structure of $\mathcal{D}'$ and $\mathcal{C}^\infty$, and which allows simpler and more transparent proofs of many classical results such as the theorem of Malgrange [5]. The next section contains an exposition of this, after which I prove the duality result about kernels and images of differential systems described above. The last section introduces the notion of geometric completeness, and calculates the obstruction to the elimination problem for systems of PDE.

2 Oberst's theorem and the structure of the classical spaces

Let $P(\partial)$ be an $l \times k$ matrix with entries $p_{ij}(\partial)$ from $\mathcal{A}$. Let $\mathcal{P}$ be the submodule of $\mathcal{A}^k$ spanned by the rows of $P(\partial)$. Recall from the introduction that the kernel of $P(\partial) : \mathcal{F}^k \to \mathcal{F}^l$ is $Ker_{\mathcal{F}}(\mathcal{P})$, and is isomorphic to $Hom_{\mathcal{A}}(\mathcal{A}^k/\mathcal{P}, \mathcal{F})$.

Given a family $\{\mathcal{P}_\alpha\}$ of submodules of $\mathcal{A}^k$, it is immediate that $Ker_{\mathcal{F}}(\sum \mathcal{P}_\alpha) = \bigcap Ker_{\mathcal{F}}(\mathcal{P}_\alpha)$. It follows that for a submodule $\mathcal{P}$ of $\mathcal{A}^k$ there is a largest submodule of $\mathcal{A}^k$ (with respect to inclusion), denoted $\overline{\mathcal{P}}_{\mathcal{F}}$, or simply $\overline{\mathcal{P}}$ if the $\mathcal{F}$ under consideration is clear, such that $Ker_{\mathcal{F}}(\overline{\mathcal{P}}) = Ker_{\mathcal{F}}(\mathcal{P})$. This $\overline{\mathcal{P}}$, which is the PDE analogue of a radical ideal, is called the $\mathcal{F}$-*closure* of $\mathcal{P}$ (also called the Willems closure in engineering literature). If $\mathcal{P} = \overline{\mathcal{P}}$, then $\mathcal{P}$ is said to be closed with respect to $\mathcal{F}$, or $\mathcal{F}$-closed. The calculation of this closure with respect to the classical spaces is the PDE analogue of the Hilbert Nullstellensatz [11, 12], and is a consequence of the following fundamental results.

Theorem 2.1 *(Ehrenpreis-Malgrange-Palamodov): The spaces $\mathcal{D}'$ and $\mathcal{C}^\infty$ are injective $\mathcal{A}$-modules [3, 8].*

While this theorem is quite classical, Oberst has recently proved the following deep theorem which allows elementary proofs of the $\mathcal{A}$-module structure of $\mathcal{S}$ and of $\mathcal{S}'$.

Theorem 2.2 *(Oberst): $\mathcal{D}'$ and $\mathcal{C}^\infty$ are also large cogenerators [7].*

As observed above, injectivity is equivalent to the existence of the Fundamental Principle, viz.

Corollary 2.1 *(Lemma 31 of [7]): The Fundamental Principle holds over $\mathcal{D}'$ and $\mathcal{C}^\infty$.*

The results that I need from [11, 12] - all consequences of the above - are the following:

Theorem 2.3 *(The Nullstellensatz for systems of PDE): (i) Every submodule $\mathcal{P}$ of $\mathcal{A}^k$ is closed with respect to $\mathcal{D}'$ or $\mathcal{C}^\infty$.*
(ii) Let $\mathcal{P} = \bigcap_{i=1}^t \mathcal{Q}_i$ be an irredundant primary decomposition of $\mathcal{P}$ in $\mathcal{A}^k$, where $\mathcal{Q}_i$ is $\mathfrak{p}_i$-primary. Suppose that the affine varieties in $\mathbb{C}^n$ of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ contain purely imaginary points (i.e. intersect $\imath\mathbb{R}^n$) and that of $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_t$ do not. Then the closure of $\mathcal{P}$ with respect to $\mathcal{S}'$ is $\bigcap_{i=1}^r \mathcal{Q}_i$, so that $\mathcal{P}$ is closed with respect to $\mathcal{S}'$ if and only if the variety of every associated prime of $\mathcal{A}^k/\mathcal{P}$ contains purely imaginary points.
(iii) Let $\pi : \mathcal{A}^k \to \mathcal{A}^k/\mathcal{P}$ be the canonical projection. Then the closure of $\mathcal{P}$ with respect to $\mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$ is $\pi^{-1}(T(\mathcal{A}^k/\mathcal{P}))$, where $T(\mathcal{A}^k/\mathcal{P})$ is the submodule of torsion elements of $\mathcal{A}^k/\mathcal{P}$. Thus $\mathcal{P}$ is closed with respect to $\mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$ if and only if $\mathcal{A}^k/\mathcal{P}$ is torsion free (or equivalently if and only if $\mathcal{P}$ is 0-primary). □

Corollary 2.2 *(Corollaries 2.2 and 2.3 of [11]): For every submodule $\mathcal{P}$ of $\mathcal{A}^k$, $Ker_{\mathcal{D}}(\mathcal{P})$ is dense in $Ker_{\mathcal{S}}(\mathcal{P})$. Furthermore, each of $Ker_{\mathcal{D}}(\mathcal{P})$, $Ker_{\mathcal{E}'}(\mathcal{P})$, and $Ker_{\mathcal{S}}(\mathcal{P})$ is dense in $Ker_{\mathcal{D}'}(\mathcal{P})$ if and only if $\mathcal{P}$ is closed with respect to $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{S}$ (or equivalently if and only if $\mathcal{P}$ is 0-primary).* □

Finally observe that if $\mathcal{P}$ is equal to its closure with respect to $\mathcal{D}$, then it is also equal to its closure with respect to any of the other classical spaces, and in particular, with respect to $\mathcal{S}'$.

I first describe the $\mathcal{A}$-module structure of $\mathcal{D}$, $\mathcal{E}'$ and $\mathcal{S}$. This, together with the theorem of Hörmander and Lojasiewicz (that $\mathcal{S}'$ is a divisible $\mathcal{A}$-module) easily implies the Fundamental Principle over $\mathcal{S}'$ (viz. that $\mathcal{S}'$ is an injective $\mathcal{A}$-module).

Proposition 2.1 *: $\mathcal{D}$ is a faithfully flat $\mathcal{A}$-module, and $\mathcal{E}'$, a faithfully flat $\mathcal{A}$-algebra.*

Proof: Let $P(\partial)$ be any $l \times k$ matrix with entries $p_{ij}(\partial)$ from $\mathcal{A}$. Consider the $\mathcal{A}$-module morphism

$$P^t(\partial) : (\mathcal{C}^\infty)^l \longrightarrow (\mathcal{C}^\infty)^k$$

determined by the transpose of $P(\partial)$ (i.e. by the $k \times l$ matrix whose (i,j)-th entry is $p_{ji}(-\partial)$). By the Fundamental Principle of Ehrenpreis in the $\mathcal{C}^\infty$-category, the image $ImP^t(\partial)$ of the above morphism equals the kernel, $KerQ(\partial)$, of the differential system $Q(\partial) : (\mathcal{C}^\infty)^k \to (\mathcal{C}^\infty)^m$ determined by $Q(\partial)$ whose m rows generate all the relations between the rows of $P^t(\partial)$). Thus the sequence

$$(\mathcal{C}^\infty)^l \stackrel{P^t(\partial)}{\longrightarrow} (\mathcal{C}^\infty)^k \stackrel{Q(\partial)}{\longrightarrow} (\mathcal{C}^\infty)^m$$

is exact. Consider next the dual sequence

$$(\mathcal{E}')^m \xrightarrow{Q^t(\partial)} (\mathcal{E}')^k \xrightarrow{P(\partial)} (\mathcal{E}')^l$$

Now $KerP(\partial) = (ImP^t(\partial))^{\perp} = (KerQ(\partial))^{\perp} = (ImQ^t(\partial))^{\perp\perp}$ - the superscript $\perp$ denotes the orthogonal - which is the closure of $ImQ^t(\partial)$ in $(\mathcal{E}')^k$. (Being a subspace in a locally convex space, there is no difference between the weak and strong closures of $ImQ^t(\partial)$.) On the other hand, the image $Q(\partial)(\mathcal{C}^\infty)^k$ of $Q(\partial)$ is also closed in $(\mathcal{C}^\infty)^m$, it being the kernel of some differential system (invoking again the Fundamental Principle of Ehrenpreis). This implies, as $\mathcal{C}^\infty$ is Frechet, that the image $Q^t(\partial)(\mathcal{E}')^m$ is closed as well.

Thus it follows that $KerP(\partial) = ImQ^t(\partial)$. As $P(\partial)$ was arbitrary, this implies flatness of the $\mathcal{A}$-algebra $\mathcal{E}'$ (by the usual characterization of flatness in terms of relations).

Now let $\mathfrak{m}$ be any maximal ideal of $\mathcal{A}$. The Fourier transforms of the functions in $\mathfrak{m}\mathcal{E}'$ (which extend to functions holomorphic on $\mathbb{C}^n$) all vanish at the zero of $\mathfrak{m}$. Thus $\mathfrak{m}\mathcal{E}' \neq \mathcal{E}'$, so that $\mathcal{E}'$ is faithfully flat.

As the image of the morphism $Q(\partial) : (\mathcal{C}^\infty)^k \to (\mathcal{C}^\infty)^m$ is closed, it follows that it is a strict morphism with respect to the weak topology, i.e. the $\mathcal{E}'$-topology (see for instance Bourbaki [1], Chapter IV, §4, Theorem 1). Now consider $\mathcal{C}^\infty$ as a subspace of $\mathcal{D}'$, i.e. with the (even weaker) $\mathcal{D}$-topology. The morphism $Q(\partial)$ remains strict with respect to this topology (for instance its image remains closed, being equal to the same kernel as before). This implies by density of the subspace $\mathcal{C}^\infty$ in $\mathcal{D}'$ that $Q(\partial) : (\mathcal{D}')^k \to (\mathcal{D}')^m$ is also a strict morphism (exercise 4 of Exercises on §4, Chapter IV of [1]). This implies that $Q^t(\partial) : \mathcal{D}^m \to \mathcal{D}^k$ has closed image (Proposition 2 of Chapter IV, §4 of [1]). Now the above proof for $\mathcal{E}'$ (with $\mathcal{D}'$ replacing $\mathcal{C}^\infty$ and $\mathcal{D}$ replacing $\mathcal{E}'$) carries over to $\mathcal{D}$, so that $\mathcal{D}$ is a faithfully flat $\mathcal{A}$-module. □

Remark: As observed in the proof above, every morphism $P(\partial) : (\mathcal{C}^\infty)^k \to (\mathcal{C}^\infty)^l$ is strict. This implies (as $\mathcal{C}^\infty$ is Frechet) that every $P(\partial) : (\mathcal{E}')^k \to (\mathcal{E}')^l$ is also strict.

Proposition 2.2 *: $\mathcal{S}$ is a flat $\mathcal{A}$-module.*

Proof: Let $\mathfrak{i}$ be any ideal of $\mathcal{A}$. I show that the canonical morphism injects $\mathfrak{i} \otimes_{\mathcal{A}} \mathcal{S}$ into $\mathcal{S}$.

First consider $\mathcal{A}$ as the $\mathbb{C}$-subspace of $\mathcal{E}'$ consisting of those distributions supported at 0 and endowed with the subspace topology (which is locally convex). Let $\mathfrak{i}$ inherit this topology, while $\mathcal{S}$ with its usual Frechet topology is a nuclear space. Endow $\mathfrak{i} \otimes_{\mathbb{C}} \mathcal{S}$ with the projective tensor product topology, then the canonial projection $\pi : \mathfrak{i} \times \mathcal{S} \to \mathfrak{i} \otimes_{\mathbb{C}} \mathcal{S}$ is continuous. Consider $\mathfrak{i} \otimes_{\mathcal{A}} \mathcal{S}$ as a quotient of $\mathfrak{i} \otimes_{\mathbb{C}} \mathcal{S}$ with the quotient topology.

Now let $\sum_{j=1}^k a_j(\partial) \otimes_{\mathcal{A}} f_j$ map to $\sum_{j=1}^k a_j(\partial) f_j = 0$. In other words, let $f = (f_1, \ldots, f_k)$ in $\mathcal{S}^k$ be in the kernel of the morphism defined by the $1 \times k$ matrix $a(\partial) = (a_1(\partial), \ldots, a_k(\partial))$

$$a(\partial) : \mathcal{S}^k \longrightarrow \mathcal{S}$$

By Corollary 2.2 of [11] quoted as Corollary 2.2 in the beginning of this section, there is a sequence $\{g_m = (g_{m_1}, \ldots, g_{m_k})\}$ in the kernel of the morphism $a(\partial)$, each g_m compactly supported, and converging to f in $\mathcal{S}^k$. But $\sum a_j(\partial) g_{m_j} = 0$ implies that $\sum a_j(\partial) \otimes_{\mathcal{A}} g_{m_j} = 0$ in $\mathfrak{i} \otimes_{\mathcal{A}} \mathcal{D}$ for each m (each g_{m_j} is in $\mathcal{D}$, which is by the above proposition, $\mathcal{A}$-flat). Hence $\sum a_j(\partial) \otimes_{\mathcal{A}} g_{m_j} = 0$ in $\mathfrak{i} \otimes_{\mathcal{A}} \mathcal{S}$ as well. But $\sum a_j(\partial) \otimes_{\mathbb{C}} g_{m_j}$ converges to $\sum a_j(\partial) \otimes_{\mathbb{C}} f_j$ which implies that $\sum a_j(\partial) \otimes_{\mathcal{A}} g_{m_j}$ also converges to $\sum a_j(\partial) \otimes_{\mathcal{A}} f_j$. Thus $\sum_{j=1}^k a_j(\partial) \otimes_{\mathcal{A}} f_j$ also equals 0. This shows that $\mathfrak{i} \otimes_{\mathcal{A}} \mathcal{S}$ injects into $\mathcal{S}$, so that $\mathcal{S}$ is $\mathcal{A}$-flat. □

Remark: $\mathcal{S}$ is not however a faithfully flat $\mathcal{A}$-module. For let $p(x)$ be any polynomial such that $p(x)^{-1}f$ is in $\mathcal{S}$ for every f in $\mathcal{S}$ (for instance the polynomial $1 + x_1^2 + \ldots + x_n^2$). Then Fourier transforming shows that the morphism $p(\partial) : \mathcal{S} \to \mathcal{S}$ is surjective, so that $\mathfrak{m}\mathcal{S} = \mathcal{S}$ for any maximal ideal $\mathfrak{m}$ of $\mathcal{A}$ that contains $p(\partial)$.

Theorem 2.4 *(Malgrange): $\mathcal{S}'$ is an injective $\mathcal{A}$-module; equivalently the Fundamental Principle holds over $\mathcal{S}'$.*

Proof: Let $P(\partial)$ be any $l \times k$ matrix, it defines the morphism $P(\partial) : (\mathcal{S}')^k \to (\mathcal{S}')^l$. I show that its image equals the kernel of the differential system determined by the module of relations between the rows of $P(\partial)$.

Let $\mathcal{P}_c$ be the submodule of $\mathcal{A}^l$ generated by the *columns* of $P(\partial)$. Without loss of generality, it can be assumed that $\mathcal{A}^l/\mathcal{P}_c$ is torsion free. For otherwise, suppose that $c(\partial)$ is some element of $\mathcal{A}^l \setminus \mathcal{P}_c$ such that $a(\partial)c(\partial) = \sum_{j=i}^{k} a_j(\partial)c_j(\partial)$, where $c_1(\partial), \ldots c_k(\partial)$ are the columns of $P(\partial)$, and $a(\partial), a_1(\partial), \ldots a_k(\partial)$ are elements of $\mathcal{A}$, with $a(\partial)$ nonzero. I claim that the image of $c(\partial) : \mathcal{S}' \to (\mathcal{S}')^l$ is contained in the image of $P(\partial) : (\mathcal{S}')^k \to (\mathcal{S}')^l$. For let g be any element of $\mathcal{S}'$. Then as $\mathcal{S}'$ is a divisible $\mathcal{A}$-module (the theorem of Hörmander-Lojasiewicz), there is a f in $\mathcal{S}'$ such that $a(\partial)f = g$. Hence $c(\partial)g = c(\partial)a(\partial)f = a(\partial)c(\partial)f = \sum a_j(\partial)c_j(\partial)f = \sum c_j(\partial)a_j(\partial)f$, which is in the image of $P(\partial)$. Thus augmenting the columns of $P(\partial)$ by this $c(\partial)$ will not change its image. Similarly, one can augment the columns of $P(\partial)$ by the representatives of any set of generators of the torsion module of $\mathcal{A}^l/\mathcal{P}_c$ without changing its image. In other words one can assume that $\mathcal{A}^l/\mathcal{P}_c$ is torsion free as claimed.

Let $\mathcal{P}_1$ be the submodule of $\mathcal{A}^l$ consisting of all the relations between the l rows of $P(\partial)$. Suppose that it is generated by some l_1 elements, say. Writing out these l_1 elements in rows yields an $l_1 \times l$ matrix $P_1(\partial)$, which defines a morphism

$$P_1(\partial) : (\mathcal{E}')^l \longrightarrow (\mathcal{E}')^{l_1}$$

By Proposition 2.1, $\mathcal{E}'$ is a flat $\mathcal{A}$-algebra, hence the kernel, $Ker_{\mathcal{E}'}P_1(\partial)$, of the above morphism is also an image, in fact the image of a morphism determined by a matrix whose columns generate all the relations between the columns of $P_1(\partial)$. This module of all the relations between the columns of $P_1(\partial)$ certainly contains $\mathcal{P}_c$; in fact it *equals* $\mathcal{P}_c$, since $\mathcal{A}^l/\mathcal{P}_c$ has been assumed to be torsion free. In other words, the following sequence is exact

$$(\mathcal{E}')^k \xrightarrow{P(\partial)} Ker_{\mathcal{E}'}P_1(\partial) \to 0$$

Now consider the morphism $P^t(\partial) : \mathcal{S}^l \to \mathcal{S}^k$ given by the transpose of $P(\partial)$. As $\mathcal{S}$ is also $\mathcal{A}$-flat (Proposition 2.2), the kernel $Ker_{\mathcal{S}}P^t(\partial)$ of this morphism is equal to an image, indeed equal to the image $Im_{\mathcal{S}}P_1^t(\partial)$ of $P_1^t(\partial) : \mathcal{S}^{l_1} \to \mathcal{S}^l$, where $P_1^t(\partial)$ is the transpose of $P_1(\partial)$ defined above. Thus it follows that

$$Im_{\mathcal{S}'}P(\partial) \subset (Im_{\mathcal{S}'}P(\partial))^{\perp\perp} = (Ker_{\mathcal{S}}P^t(\partial))^{\perp} = (Im_{\mathcal{S}}P_1^t(\partial))^{\perp} = Ker_{\mathcal{S}'}P_1(\partial)$$

where $Im_{\mathcal{S}'}P(\partial)$ denotes the image of $P(\partial) : (\mathcal{S}')^k \to (\mathcal{S}')^l$, and $Ker_{\mathcal{S}'}P_1(\partial)$ the kernel of $P_1(\partial) : (\mathcal{S}')^l \to (\mathcal{S}')^{l_1}$. This implies that $Im_{\mathcal{S}'}P(\partial)$ is *dense* in $Ker_{\mathcal{S}'}P_1(\partial)$. Thus to prove the theorem it suffices to show that $Im_{\mathcal{S}'}P(\partial)$ is closed.

This folows, just as in the second half of Proposition 2.1, from a strictness argument. In more detail, consider the submodule of compactly supported elements of $Im_{\mathcal{S}'}P(\partial)$; by exactness of the sequence above, it follows that this submodule equals $P(\partial)(\mathcal{E}')^k$. Thus if a sequence $\{g_i\}$ in

$Im_{\mathcal{S}'}P(\partial)$, each g_i compactly supported, converges to a compactly supported g, it follows that there is a sequence $\{f_i\}$, each f_i also compactly supported and such that $P(\partial)f_i = g_i$, converging to a compactly supported f with $P(\partial)f = g$ (by the remark following Proposition 2.1). This implies, as $P(\partial)(\mathcal{E}')^k$ is densely contained, that $Im_{\mathcal{S}'}P(\partial)$ is closed. $\square$

Remark: As $Hom_{\mathcal{A}}(\mathcal{A}/(p), \mathcal{S}') = 0$ for p any polynomial whose variety does not contain purely imaginary points, it follows that $\mathcal{S}'$ is not a cogenerator.

Corollary 2.3 : *Let $P(\partial) : \mathcal{F}^k \to \mathcal{F}^l$, where $\mathcal{F}$ is a classical space. Then the image $P(\partial)\mathcal{F}^k$ is always closed in $\mathcal{F}^l$.*

Proof: For $\mathcal{D}'$, $\mathcal{C}^\infty$ and $\mathcal{S}'$, the corollary follows from the Fundamental Principle. It has already been observed above for $\mathcal{E}'$ and $\mathcal{D}$, and it follows for $\mathcal{S}$ from the corresponding fact for $\mathcal{S}'$ (as $\mathcal{S}$ is Frechet). $\square$

Corollary 2.4 : *Let $P(\partial) : \mathcal{F}^k \to \mathcal{F}^l$ be a differential system. If $\mathcal{F}$ is $\mathcal{D}'$, $\mathcal{C}^\infty$ or $\mathcal{S}'$, then $P(\partial)$ can be extended to the right by an exact sequence*

$$\mathcal{F}^k \xrightarrow{P(\partial)} \mathcal{F}^l \xrightarrow{P_1(\partial)} \mathcal{F}^{l_1} \xrightarrow{P_2(\partial)} \ldots \longrightarrow \mathcal{F}^{l_{n-1}} \xrightarrow{P_n(\partial)} \mathcal{F}^{l_n} \to 0$$

On the other hand if $\mathcal{F}$ is $\mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$, then $P(\partial)$ can be extended to the left by an exact sequence

$$0 \to \mathcal{F}^{k_n} \xrightarrow{P_{-n}(\partial)} \mathcal{F}^{k_{n-1}} \longrightarrow \ldots \xrightarrow{P_{-2}(\partial)} \mathcal{F}^{k_1} \xrightarrow{P_{-1}(\partial)} \mathcal{F}^k \xrightarrow{P(\partial)} \mathcal{F}^l$$

Proof: The proof of the first part of the corollary, when $\mathcal{F}$ is $\mathcal{D}'$, is in [11]. It relies only on the Fundamental Principle and thus holds for $\mathcal{C}^\infty$ and $\mathcal{S}'$ as well (that the length of the sequence is no more than n follows from the fact that the global dimension of $\mathcal{A}$ is n). This same proof holds for the second part of the corollary, with flatness of $\mathcal{F}$ replacing injectivity. $\square$

3 Duality

In this section I prove a duality theorem for systems of PDE.

Theorem 3.1 : *Let $\mathcal{P}$ be a submodule of $\mathcal{A}^k$. Let $g = \{p_1(\partial), \ldots, p_l(\partial)\}$ be a set of generators for $\mathcal{P}$, and let $P_g(\partial) : \mathcal{F}^k \to \mathcal{F}^l$ be the morphism defined by the matrix P_g whose rows are the $p_i(\partial)$. Let $\mathcal{P}_g^t$ be the submodule of $\mathcal{A}^l$ generated by the rows of the transpose $P_g^t(\partial)$.*
(i) Let $\mathcal{F}$ be $\mathcal{D}'$, $\mathcal{C}^\infty$ or $\mathcal{S}'$. Then the image of $P_g(\partial)$ is always a kernel. Dually, if $\mathcal{F}$ is $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{S}$, then the kernel of $P_g(\partial)$ is always an image.
(ii) Let $\mathcal{F}$ be $\mathcal{D}'$ or $\mathcal{C}^\infty$. Then the kernel of $P_g(\partial)$ is an image if and only if $\mathcal{P}$ is 0-primary. Dually, if $\mathcal{F}$ is $\mathcal{D}$ or $\mathcal{E}'$, then the image of $P_g(\partial)$ is a kernel if and only if $\mathcal{P}_g^t$ is 0-primary.
(iii) Let $\mathcal{F}$ be $\mathcal{S}'$. Then the kernel of $P_g(\partial)$ is an image if and only if the varieties of the nonzero associated primes of $\mathcal{A}^k/\mathcal{P}$ do not contain purely imaginary points. Dually, if $\mathcal{F}$ is $\mathcal{S}$, then the image of $P_g(\partial)$ is a kernel if and only if the varieties of the nonzero associated primes of $\mathcal{A}^l/\mathcal{P}_g^t$ do not contain purely imaginary points.

Proof: (i) If $\mathcal{F}$ is $\mathcal{D}'$, $\mathcal{C}^\infty$ or $\mathcal{S}'$, then the Fundamental Principle in these categories assert that an image is always a kernel. On the other hand, if $\mathcal{F}$ is $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{S}$, then by flatness of these $\mathcal{A}$-modules, a kernel is always an image.
(ii) Let $\mathcal{F}$ be either $\mathcal{D}'$ or $\mathcal{C}^\infty$. Then the kernel of $P_g(\partial) : \mathcal{F}^k \to \mathcal{F}^l$ is an image if and only if $\mathcal{A}^k/\mathcal{P}$ is torsion free, or equivalently if and only if $\mathcal{P}$ is 0-primary ($\mathcal{P}$ is the submodule of $\mathcal{A}^k$ generated by the rows of $P_g(\partial)$) - see Theorem 2 in [9] or Proposition 3.1 in [11].

Suppose that $\mathcal{F}$ is either $\mathcal{D}$ or $\mathcal{E}'$. As the image $ImP_g(\partial)$ of $P_g(\partial)$ is closed (Corollary 2.3), it follows that

$$ImP_g(\partial) = (ImP_g(\partial))^{\perp\perp} = (KerP_g^t(\partial))^\perp$$

where $P_g^t(\partial) : (\mathcal{F}')^l \to (\mathcal{F}')^k$ is the morphism defined by the transpose of $P_g(\partial)$. But $KerP_g^t(\partial)$ is an image, say of $Q(\partial) : (\mathcal{F}')^{l_1} \to (\mathcal{F}')^l$, if and only if $\mathcal{P}_g^t$ is 0-primary, where $\mathcal{P}_g^t$ is the submodule of $\mathcal{A}^l$ generated by the rows of $P_g^t(\partial)$ (by the result quoted just above). Under this assumption, and only then, is

$$(KerP_g^t(\partial))^\perp = (ImQ(\partial))^\perp = KerQ^t(\partial)$$

Thus it follows that when $\mathcal{F}$ is either $\mathcal{D}$ or $\mathcal{E}'$, the image of $P_g(\partial) : \mathcal{F}^k \to \mathcal{F}^l$ is a kernel if and only if $\mathcal{P}_g^t$ is 0-primary.
(iii) Now consider $P_g(\partial) : (\mathcal{S}')^k \to (\mathcal{S}')^l$. Let $\mathcal{P}_{-1}$ be the submodule of $\mathcal{A}^k$ consisting of all the relations between the columns of $P_g(\partial)$. Generators of this module, say k_1 in number, determine a morphism $P_{-1}(\partial) : (\mathcal{S}')^{k_1} \to (\mathcal{S}')^k$. Clearly the image of this morphism is contained in the kernel of $P_g(\partial) : (\mathcal{S}')^k \to (\mathcal{S}')^l$, indeed it is the largest such image contained in this kernel. On the other hand, the image of $P_{-1}(\partial) : (\mathcal{S}')^{k_1} \to (\mathcal{S}')^k$ is by Theorem 2.4, equal to the kernel of $P_0(\partial) : (\mathcal{S}')^k \to (\mathcal{S}')^{l'}$, where the l' rows of $P_0(\partial)$ generate the submodule $\mathcal{P}_0$ of $\mathcal{A}^k$ of all the relations between the rows of $P_{-1}(\partial)$. Thus the kernel of $P_g(\partial)$ would be an image if and only if it were equal to the kernel of $P_0(\partial)$. But observe (as in the proof of Theorem 2.4) that $\mathcal{P}_0/\mathcal{P}$ is the torsion module of $\mathcal{A}^k/\mathcal{P}$, so that $\mathcal{P}_0$ is 0-primary. This then implies, by Theorem 2.3 quoted in the beginning of Section 2, that $\mathcal{P}_0$ is equal to its closure $\overline{\mathcal{P}_0}$ with respect to $\mathcal{D}$, and hence also with respect to $\mathcal{S}'$. Thus it finally follows that the kernel of $P_g(\partial)$ is an image if and only if the closure $\overline{\mathcal{P}}$ of $\mathcal{P}$ with respect to $\mathcal{S}'$ is equal to the submodule $\mathcal{P}_0$ described above, in other words if and only if this closure is 0-primary. By Theorem 2.3 again, this means that the varieties of the nonzero associated primes of $\mathcal{A}^k/\mathcal{P}$ should not contain purely imaginary points.

Consider finally the remaining question: when is the image of $P_g(\partial) : \mathcal{S}^k \to \mathcal{S}^l$ equal to a kernel? Dualizing as in (ii) yields the following dual criterion: Let $\mathcal{P}_0'/\mathcal{P}_g^t$ be the torsion module of $\mathcal{A}^l/\mathcal{P}_g^t$, so that $\mathcal{P}_0'$ is 0-primary. Then the image of $P_g(\partial)$ equals a kernel if and only if the closure of $\mathcal{P}_g^t$ with respect to $\mathcal{S}'$ equals $\mathcal{P}_0'$, which is if and only if the varieties of the nonzero associated primes of $\mathcal{A}^l/\mathcal{P}_g^t$ do not contain purely imaginary points. □

Observe in the above theorem that the submodule $\mathcal{P}_g^t$ of $\mathcal{A}^l$ depends upon the choice of generators g of the submodule $\mathcal{P}$ of $\mathcal{A}^k$. Thus while the conditions in the above theorem which specify when a kernel is an image is in terms of the submodule $\mathcal{P}$ (viz. the given data), the conditions which specify when an image is a kernel seem to however depend upon the choice of generators g. I now show that this is not the case, i.e. these latter conditions are independent of the choice of g as well.

So let $g = \{p_1(\partial), \ldots, p_l(\partial)\}$ and $g' = \{p_{l+1}(\partial), \ldots, p_{l+l'}(\partial)\}$ be two sets of generators for $\mathcal{P}$. Writing the elements of these two sets as rows of matrices, say $P_g(\partial)$ and $P_{g'}(\partial)$, yields two $\mathcal{A}$-module morphisms

$$P_g(\partial) : \mathcal{F}^k \longrightarrow \mathcal{F}^l$$

and

$$P_{g'}(\partial) : \mathcal{F}^k \longrightarrow \mathcal{F}^{l'}$$

Proposition 3.1 *: Let $\mathcal{F}$ be a classical space. Then the image of the morphism $P_g(\partial)$ above is a kernel if and only if the image of the morphism $P_{g'}(\partial)$ is also a kernel.*

Proof: This is immediate if $\mathcal{F}$ is $\mathcal{D}'$, $\mathcal{C}^\infty$ or $\mathcal{S}'$, for then by the injectivity of these spaces, every image is a kernel. Thus assume now that $\mathcal{F}$ is $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{S}$. Let $P_1(\partial)$ be the $(l+1) \times k$ matrix obtained by adjoining p_{l+1} (i.e. as another row) to $P_g(\partial)$. This matrix determines a morphism $P_1(\partial) : \mathcal{F}^k \to \mathcal{F}^{l+1}$. I prove that the image of $P_g(\partial)$ is a kernel if and only if this is true of the morphism $P_1(\partial)$. Clearly this suffices to prove the proposition.

Let $\mathcal{P}_g^t$ and $\mathcal{P}_1^t$ be submodules of $\mathcal{A}^l$ and $\mathcal{A}^{l+1}$ generated by the rows of the transposes $P_g^t(\partial)$ and $P_1^t(\partial)$ respectively. I claim that the associated primes of $\mathcal{A}^l/\mathcal{P}_g^t$ and $\mathcal{A}^{l+1}/\mathcal{P}_1^t$ differ by at most the zero prime. More precisely the following holds

$$Either \quad Ass(\mathcal{A}^l/\mathcal{P}_g^t) = Ass(\mathcal{A}^{l+1}/\mathcal{P}_1^t) \quad or \quad Ass(\mathcal{A}^l/\mathcal{P}_g^t) \cup \{0\} = Ass(\mathcal{A}^{l+1}/\mathcal{P}_1^t)$$

This is a straight forward calculation. As $p_{l+1}(\partial)$ is in $\mathcal{P}$, it is an $\mathcal{A}$-linear combination of $p_1(\partial), \ldots, p_l(\partial)$, say $p_{l+1} = a_1 p_1 + \cdots + a_l p_l$. Therefore if $z = (x_1, \ldots, x_l)$ in $\mathcal{A}^l \setminus \mathcal{P}_g^t$ is such that az is in $\mathcal{P}_g^t$ for some a in $\mathcal{A}$, then for $z_1 = (x_1, \ldots, x_l, a_1 x_1 + \cdots + a_l x_l)$ in $\mathcal{A}^{l+1}$, it is true that az_1 belongs to $\mathcal{P}_1^t$. Furthermore z_1 is not in $\mathcal{P}_1^t$. This shows that $Ass(\mathcal{A}^l/\mathcal{P}_g^t) \subset Ass(\mathcal{A}^{l+1}/\mathcal{P}_1^t)$.

Conversely, suppose $z_1 = (x_1, \ldots, x_l, x_{l+1})$ in $\mathcal{A}^{l+1} \setminus \mathcal{P}_1^t$ is such that az_1 is in $\mathcal{P}_1^t$ for some a in $\mathcal{A}$. If $z = (x_1, \ldots, x_l)$ is not in $\mathcal{P}_g^t$, then the annihilators of the residue classes of z_1 in $\mathcal{A}^{l+1}/\mathcal{P}_1^t$ and of z in $\mathcal{A}^l/\mathcal{P}_g^t$ coincide. On the other hand if z is in $\mathcal{P}_g^t$, an elementary calculation shows that the annihilator of the residue class of z_1 is now the 0 ideal. Then 0 is also an associated prime of $\mathcal{A}^{l+1}/\mathcal{P}_1^t$, and this is therefore the only prime that is perhaps not in $Ass(\mathcal{A}^l/\mathcal{P}_g^t)$. This establishes the claim.

From the above theorem it now follows that the image of the morphism $P_g(\partial)$ is a kernel precisely when this is true of the morphism $P_1(\partial)$. This proves the proposition. □

Remark: It follows from the above proof that the set of *nonzero* associated primes of $\mathcal{A}^l/\mathcal{P}_g^t$ depends only on $\mathcal{P}$ and not on the choice of g. They will be denoted by $NAss(\mathcal{A}^*/\mathcal{P}^t)$.

The statements about images in parts (ii) and (iii) of Theorem 3.1 now become

Corollary 3.1 *: If $\mathcal{F}$ is $\mathcal{D}$ or $\mathcal{E}'$, then the image of the morphism defined by any matrix whose rows generate $\mathcal{P}$, is a kernel if and only if $NAss(\mathcal{A}^*/\mathcal{P}^t)$ is empty. When $\mathcal{F}$ is $\mathcal{S}$, then this image is a kernel if and only if the varieties of the primes in $NAss(\mathcal{A}^*/\mathcal{P}^t)$ do not contain purely imaginary points.* □

4 Geometric completeness and the elimination problem for PDE

I now consider the problem of elimination for systems of PDE. Motivated by the definition of a complete variety, I make the following

Definition 4.1 *: Let $\mathcal{F}$ be an $\mathcal{A}$-submodule of $\mathcal{D}'$. Suppose that all the projections of all the differential kernels (i.e. all the $Ker_{\mathcal{F}}(\mathcal{P})$ as $\mathcal{P}$ varies over all submodules of $\mathcal{A}^k$, $k = 1, 2, \ldots$) onto various subsets of its co-ordinates are also differential kernels. Then $\mathcal{F}$ is said to be geometrically complete.*

Example 1: Let $\mathcal{A} = \mathbb{C}[\frac{d}{dt}]$, and let $\pi_2 : \mathcal{D}^2 \to \mathcal{D}$ be the projection onto the second factor. Let $\mathcal{P}$ be the cyclic submodule of $\mathcal{A}^2$ generated by $(\frac{d}{dt}, -1)$. Then $Ker_{\mathcal{D}}(\mathcal{P}) = \{(f, \frac{df}{dt}) \mid f \in \mathcal{D}\}$, but $\pi_2(Ker_{\mathcal{D}}(\mathcal{P})) = \{\frac{df}{dt} \mid f \in \mathcal{D}\}$ is not a differential kernel in $\mathcal{D}$. Thus $\mathcal{D}$ is *not* geometrically complete. Similarly, $\mathcal{E}'$ and $\mathcal{S}$ are also not geometrically complete. □

I first show that an $\mathcal{A}$-submodule $\mathcal{F}$ of $\mathcal{D}'$ is geometrically complete if it is an injective $\mathcal{A}$-module. Thus $\mathcal{D}'$, $\mathcal{C}^\infty$ and $\mathcal{S}'$ are geometrically complete. I then calculate the obstruction to a projection of a differential kernel being a differential kernel in the case of the other classical spaces, i.e. when $\mathcal{F}$ is $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{S}$.

Towards this end, consider the split exact sequence

$$0 \to \mathcal{A}^p \underset{\overset{\longleftarrow}{\pi_1}}{\overset{i_1}{\longrightarrow}} \mathcal{A}^{p+q} \underset{\overset{\longleftarrow}{i_2}}{\overset{\pi_2}{\longrightarrow}} \mathcal{A}^q \to 0$$

Applying the functor $Hom_{\mathcal{A}}(-, \mathcal{F})$ yields the split exact sequence

$$0 \to \mathcal{F}^p \underset{\overset{\longrightarrow}{i_1}}{\overset{\pi_1}{\longleftarrow}} \mathcal{F}^{p+q} \underset{\overset{\longrightarrow}{\pi_2}}{\overset{i_2}{\longleftarrow}} \mathcal{F}^q \to 0$$

Let $\mathcal{P}$ be any submodule of $\mathcal{A}^{p+q}$, so that $Ker_{\mathcal{F}}(\mathcal{P})$ is a differential kernel in $\mathcal{F}^{p+q}$. In this notation, the following proposition relates the $\mathcal{A}$-submodules $i_2^{-1}(Ker_{\mathcal{F}}(\mathcal{P}))$, $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$, $Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$ and $Ker_{\mathcal{F}}(\pi_2(\mathcal{P}))$ of $\mathcal{F}^q$.

Proposition 4.1 *: Let $\mathcal{F}$ be an $\mathcal{A}$-submodule of $\mathcal{D}'$. Then*

$$i_2^{-1}(Ker_{\mathcal{F}}(\mathcal{P})) = Ker_{\mathcal{F}}(\pi_2(\mathcal{P})) \subset \pi_2(Ker_{\mathcal{F}}(\mathcal{P})) \subset Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$$

If $\mathcal{F}$ is an injective $\mathcal{A}$-module, then also

$$\pi_2(Ker_{\mathcal{F}}(\mathcal{P})) = Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$$

so that $\mathcal{F}$ is then geometrically complete.

Proof: Suppose first that $p_2(\partial)$ (in $\mathcal{A}^q$) is in $i_2^{-1}(\mathcal{P})$. Then $(0, p_2(\partial))$ is in $\mathcal{P}$, so that $(0, p_2(\partial))(f) = 0$ for all f in $Ker_{\mathcal{F}}(\mathcal{P})$. It follows that $p_2(\partial)(\pi_2(f)) = 0$, and hence that $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$ is contained in $Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$.

Suppose next that f (in $\mathcal{F}^q$) is in $i_2^{-1}(Ker_{\mathcal{F}}(\mathcal{P}))$; then $i_2(f) = (0, f)$ is in $Ker_{\mathcal{F}}(\mathcal{P})$. Thus for all $p(\partial) = (p_1(\partial), p_2(\partial))$ in $\mathcal{P}$, $(p_1(\partial), p_2(\partial))(0, f) = p_2(\partial)(f) = 0$, which is to say that f is in $Ker_{\mathcal{F}}(\pi_2(\mathcal{P}))$.

Finally suppose that f is in $Ker_{\mathcal{F}}(\pi_2(\mathcal{P}))$, i.e. suppose that $p_2(\partial)(f) = 0$ for all $p_2(\partial)$ in $\pi_2(\mathcal{P})$. Then for all $p_1(\partial)$ in $\mathcal{A}^p$, $(p_1(\partial), p_2(\partial))(0, f) = 0$. In particular $(0, f)$ is in $Ker_{\mathcal{F}}(\mathcal{P})$, so that $Ker_{\mathcal{F}}(\pi_2(\mathcal{P}))$ is contained in both $i_2^{-1}(Ker_{\mathcal{F}}(\mathcal{P}))$ and $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$. This proves the first part of the proposition.

The first split exact sequence above implies that the following sequence

$$0 \to \mathcal{A}^q / i_2^{-1}(\mathcal{P}) \longrightarrow \mathcal{A}^{p+q}/\mathcal{P} \longrightarrow \mathcal{A}^p/\pi_1(\mathcal{P}) \to 0 \qquad (2)$$

is exact. Thus

$$0 \to Hom_{\mathcal{A}}(\mathcal{A}^p/\pi_1(\mathcal{P}), \mathcal{F}) \xrightarrow{i_1} Hom_{\mathcal{A}}(\mathcal{A}^{p+q}/\mathcal{P}, \mathcal{F}) \xrightarrow{\pi_2} Hom_{\mathcal{A}}(\mathcal{A}^q/i_2^{-1}(\mathcal{P}), \mathcal{F})$$

$$\xrightarrow{\delta} Ext^1_{\mathcal{A}}(\mathcal{A}^p/\pi_1(\mathcal{P}), \mathcal{F}) \xrightarrow{i_1} Ext^1_{\mathcal{A}}(\mathcal{A}^{p+q}/\mathcal{P}, \mathcal{F}) \longrightarrow \cdots$$

is also exact, where the morphism π_2 above is just the restriction of the π_2 in the second split exact sequence above to the $\mathcal{A}$-submodule $Hom_{\mathcal{A}}(\mathcal{A}^{p+q}/\mathcal{P}, \mathcal{F}) \simeq Ker_{\mathcal{F}}(\mathcal{P})$ of $\mathcal{F}^{p+q}$.

Now if $\mathcal{F}$ is injective, then the Ext terms in the above sequence are zero, and π_2 surjects onto $Hom_{\mathcal{A}}(\mathcal{A}^q/i_2^{-1}(\mathcal{P}), \mathcal{F}) \simeq Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$. As this is true of every $\mathcal{P}$ and of every projection of $Ker_{\mathcal{F}}(\mathcal{P})$ to various subsets of its co-ordinates, it follows that $\mathcal{F}$ is geometrically complete. This proves the second statement of the proposition. □

Corollary 4.1 : *The spaces $\mathcal{D}'$, $\mathcal{C}^\infty$ and $\mathcal{S}'$ are geometrically complete.* □

By Example 1 above, $\mathcal{D}$, $\mathcal{E}'$ and $\mathcal{S}$ are not geometrically complete. Thus, when $\mathcal{F}$ is one of these three spaces, there is a $Ker_{\mathcal{F}}(\mathcal{P})$ (in the notation of the above proposition) such that $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$ is not a differential kernel. I now calculate the obstruction that needs to be overcome for this projection to be a kernel.

Proposition 4.2 : *Let $\mathcal{F}$ be $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{S}$. Then $Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$ is the smallest kernel containing $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$. Thus $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$ is a differential kernel if and only if the morphism π_2 (in the long exact sequence above) is surjective.*

Proof: Suppose that $Ker_{\mathcal{F}}(\mathcal{N})$ contains $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$. I show that it must then also contain $Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$.

As before, let an overline denote the closure of a submodule with respect to any of the three $\mathcal{F}$s being considered here (by Theorem 2.3 (iii) these closures are all equal). Thus $Ker_{\mathcal{F}}(\mathcal{P}) = Ker_{\mathcal{F}}(\overline{\mathcal{P}})$, so that also $\pi_2(Ker_{\mathcal{F}}(\overline{\mathcal{P}})) \subset Ker_{\mathcal{F}}(\mathcal{N})$. Now $\overline{\mathcal{P}}$ is $\mathcal{F}$-closed, hence by Corollary 2.2 quoted earlier, $Ker_{\mathcal{F}}(\overline{\mathcal{P}})$ is dense in $Ker_{\mathcal{D}'}(\overline{\mathcal{P}})$. It follows from this (as π_2 is continuous) that $\pi_2(Ker_{\mathcal{D}'}(\overline{\mathcal{P}})) \subset Ker_{\mathcal{D}'}(\mathcal{N})$. But by the previous proposition, $\pi_2(Ker_{\mathcal{D}'}(\overline{\mathcal{P}}))$ equals $Ker_{\mathcal{D}'}(i_2^{-1}(\overline{\mathcal{P}}))$. Thus $Ker_{\mathcal{D}'}(i_2^{-1}(\overline{\mathcal{P}})) \subset Ker_{\mathcal{D}'}(\mathcal{N})$, and therefore also $Ker_{\mathcal{F}}(i_2^{-1}(\overline{\mathcal{P}})) \subset Ker_{\mathcal{F}}(\mathcal{N})$. To prove the proposition, it now suffices to show that $Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$ is equal to $Ker_{\mathcal{F}}(i_2^{-1}(\overline{\mathcal{P}}))$, and for this it is enough to show that $i_2^{-1}(\overline{\mathcal{P}}) = \overline{i_2^{-1}(\mathcal{P})}$. This follows easily from Theorem 2.3 (iii) above, since $p(\partial)$ is in $i_2^{-1}(\overline{\mathcal{P}})$, i.e. $(0, p(\partial))$ is in $\overline{\mathcal{P}}$, if and only if there is a nonzero $a(\partial)$ in $\mathcal{A}$ with $a(\partial)(0, p(\partial)) = (0, a(\partial)p(\partial))$ in $\mathcal{P}$. But this is so if and only if $a(\partial)p(\partial)$ is in $i_2^{-1}(\mathcal{P})$, which is to say that $p(\partial)$ is in $\overline{i_2^{-1}(\mathcal{P})}$.

Therefore $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$ is a differential kernel if and only if π_2 is surjective. □

I translate this proposition in terms of the given data, viz. the submodule $\mathcal{P}$. First, I determine a necessary and sufficient condition for the connecting morphism δ above to be the zero morphism - this is exactly when π_2 is surjective. This condition involves the module of relations between the columns of a matrix whose rows generate $i_2^{-1}(\mathcal{P})$. Next, I specialize to the case when the image of the morphism determined by a matrix whose rows generate $\mathcal{P}$, equals a kernel. By Proposition 3.3, this is independent of the choice of generators of $\mathcal{P}$, and thus of the corresponding matrix. This case corresponds to the vanishing of the Ext terms in the long exact sequence above.

Thus consider the following simultaneous free resolution of the sequence (2)

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{A}^{n_2} & \longrightarrow & \mathcal{A}^{q} & \longrightarrow & \mathcal{A}^{q}/i_2^{-1}(\mathcal{P}) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{A}^{n_1+n_2} & \longrightarrow & \mathcal{A}^{p+q} & \longrightarrow & \mathcal{A}^{p+q}/\mathcal{P} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{A}^{n_1} & \longrightarrow & \mathcal{A}^{p} & \longrightarrow & \mathcal{A}^{p}/\pi_1(\mathcal{P}) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

where, by assumption, the submodule $\mathcal{P}$ of $\mathcal{A}^{p+q}$ is generated by n_1 elements, and therefore the images of these generators under the projections π_1 (and π_2) also generate $\pi_1(\mathcal{P})$ (and $\pi_2(\mathcal{P})$). This yields the bottom row of the above diagram. Assume further that the submodule $i_2^{-1}(\mathcal{P})$ of $\mathcal{A}^q$ is generated by n_2 elements; this gives the top row, and thus also the simultaneous resolution. Applying the functor $Hom(-,\mathcal{F})$ yields

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \uparrow & & \uparrow & \\
0 \to & \mathcal{F}^{q} & \stackrel{i_2^{-1}(P)(\partial)}{\longrightarrow} & \mathcal{F}^{n_2} & \to \cdots \\
 & \pi_2 \uparrow & & \uparrow & \\
0 \to & \mathcal{F}^{p+q} & \stackrel{P(\partial)}{\longrightarrow} & \mathcal{F}^{n_1+n_2} & \to \cdots \\
 & \uparrow & & i_1 \uparrow & \\
0 \to & \mathcal{F}^{p} & \stackrel{\pi_1(P)(\partial)}{\longrightarrow} & \mathcal{F}^{n_1} & \to \cdots \\
 & \uparrow & & \uparrow & \\
 & 0 & & 0 &
\end{array}$$

In this diagram $i_2^{-1}(P)(\partial)$ and $\pi_1(P)(\partial)$ are matrices whose rows are the generators of the submodules $i_2^{-1}(\mathcal{P})$ and $\pi_1(\mathcal{P})$ chosen above in the construction of the simultaneous resolution (see [11] for details). This construction also implies the following description of the matrix $P(\partial)$

$$P(\partial) = \begin{pmatrix} \pi_1(P)(\partial) & \pi_2(P)(\partial) \\ 0 & i_2^{-1}(P)(\partial) \end{pmatrix}$$

(here the first n_1 rows of $P(\partial)$ are the generators of the submodule $\mathcal{P}$ chosen above). Now let f_2 be in the kernel of the morphism $i_2^{-1}(P)(\partial)$. Then $\delta(f_2)$ equals the class of $i_1^{-1}(P(\partial)((0,f_2))) = \pi_2(P)(\partial)(f_2)$ in $Ext^1_{\mathcal{A}}(\mathcal{A}^p/\pi_1(\mathcal{P}),\mathcal{F})$. Thus the connecting morphism δ of the long exact sequence is the zero morphism precisely when the submodule $\{\pi_2(P)(\partial)(f_2) \mid f_2 \in Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))\}$ is contained in $Im(\pi_1(P)(\partial))$.

To determine when this is so, recollect that the $\mathcal{F}$s being considered here, viz. $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{S}$, are all flat $\mathcal{A}$-modules (Propositions 2.1 and 2.2). Therefore $Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))$ equals an image (Theorem 3.1 (i)), in fact the image of the morphism determined by a matrix, say $R(\partial)$, whose columns generate all the relations between the columns of $i_2^{-1}P(\partial)$. Clearly then, the submodule $\{\pi_2(P)(\partial)(f_2) \mid f_2 \in Ker_{\mathcal{F}}(i_2^{-1}(\mathcal{P}))\}$ is equal to the image of the morphism given by the composition $\pi_2(P)(\partial) \circ R(\partial)$, so that δ is the zero morphism if and only if this image is contained in the image of $\pi_1(P)(\partial)$. But these images are closed subspaces of the topological vector space $\mathcal{F}^{n_1}$ (Corollary 2.3), hence

$$Im(\pi_1(P)(\partial)) = Im(\pi_1(P)(\partial))^{\perp\perp} = Ker_{\mathcal{F}'}(\pi_1(P)^t(\partial))^{\perp}$$

(where $\pi_1(P)^t(\partial) : (\mathcal{F}')^{n_1} \to (\mathcal{F}')^p$ is the morphism on the dual determined by the transpose). Similarly $Im(\pi_2(P)(\partial) \circ R(\partial)) = Ker_{\mathcal{F}'}(R^t(\partial) \circ \pi_2(P)^t(\partial))^{\perp}$. Thus $Im(\pi_2(P)(\partial) \circ R(\partial)) \subset Im(\pi_1(P)(\partial))$ if and only if $Ker_{\mathcal{F}'}(\pi_1(P)^t(\partial)) \subset Ker_{\mathcal{F}'}(R^t(\partial) \circ \pi_2(P)^t(\partial))$. Let $\pi_1(\mathcal{P})^t$ and $\mathcal{R}^t\pi_2(\mathcal{P})^t$ be submodules of $\mathcal{A}^{n_1}$ generated by the rows of the matrices $\pi_1(P)^t(\partial)$ and $R^t(\partial) \circ \pi_2(P)^t(\partial)$ respectively. It then follows that δ is the zero morphism if and only if $Ker_{\mathcal{F}'}(\pi_1(\mathcal{P})^t) \subset Ker_{\mathcal{F}'}(\mathcal{R}^t\pi_2(\mathcal{P})^t)$. But this inclusion holds if and only if the $\mathcal{F}'$-closure of $\mathcal{R}^t\pi_2(\mathcal{P})^t$ is contained in the $\mathcal{F}'$-closure of $\pi_1(\mathcal{P})^t$. This leads, in the above notation, to the following theorem.

Theorem 4.1 *: (i) Let $\mathcal{F}$ be $\mathcal{D}$ or $\mathcal{E}'$. Then $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$ is a differential kernel if and only if the submodule $\mathcal{R}^t\pi_2(\mathcal{P})^t$ is contained in $\pi_1(\mathcal{P})^t$.*
(ii) Let $\pi_1(\mathcal{P})^t = \bigcap_{i=1}^{t} \mathcal{Q}_i$ and $\mathcal{R}^t\pi_2(\mathcal{P})^t = \bigcap_{j=1}^{t'} \mathcal{Q}'_j$ be irredundant primary decompositions in $\mathcal{A}^{n_1}$, where $\mathcal{Q}_i$ and $\mathcal{Q}'_j$ are $\mathfrak{p}_i$ and $\mathfrak{p}'_j$ primary respectively. Suppose that the affine varieties in $\mathbb{C}^n$ of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_{r'}$ contain purely imaginary points and that of $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_t, \mathfrak{p}'_{r'+1}, \ldots, \mathfrak{p}'_{t'}$ do not. Then $\pi_2(Ker_{\mathcal{S}}(\mathcal{P}))$ is a differential kernel if and only if $\bigcap_{j=1}^{r'} \mathcal{Q}'_j$ is contained in $\bigcap_{i=1}^{r} \mathcal{Q}_i$. □

I specialize the above to the case when $Ext^1_{\mathcal{A}}(\mathcal{A}^{p+q}/\mathcal{P}, \mathcal{F})$ vanishes, that is when the image of $P(\partial)$ is a kernel. Then for δ to be the zero morphism it is necessary that also $Ext^1_{\mathcal{A}}(\mathcal{A}^p/\pi_1(\mathcal{P}), \mathcal{F})$ vanish, viz. that the image of $\pi_1(P)(\partial)$ also be a kernel. Conditions for this are given by Corollary 3.1, so that this special case of the vanishing of the Ext terms can be formulated in terms of associated primes.

Theorem 4.2 *: (i) Let $\mathcal{F}$ be $\mathcal{D}$ or $\mathcal{E}'$. Suppose that $NAss(\mathcal{A}^*/\mathcal{P}^t)$ is empty. Then the projection $\pi_2(Ker_{\mathcal{F}}(\mathcal{P}))$ is a kernel if and only if $NAss(\mathcal{A}^*/\pi_1(\mathcal{P})^t)$ is also empty.*
(ii) Suppose that the varieties of the primes in $NAss(\mathcal{A}^*/\mathcal{P}^t)$ do not contain purely imaginary points. Then $\pi_2(Ker_{\mathcal{S}}(\mathcal{P}))$ is a kernel if and only if the varieties of the primes in $NAss(\mathcal{A}^*/\pi_1(\mathcal{P})^t)$ also do not contain purely imaginary points. □

Finally I point out that geometric completeness is a notion weaker than injectivity, viz. I exhibit an example of an $\mathcal{A}$-module which is geometrically complete but which is not an injective $\mathcal{A}$-module.

Example 2: Let $\mathcal{A} = \mathbb{C}[\frac{d}{dt}]$. Consider $\mathbb{C}$ as an $\mathcal{A}$-submodule of $\mathcal{D}'$. Let $P(\frac{d}{dt})$ be any matrix (say of size $l \times k$) determining an $\mathcal{A}$-module morphism $P(\frac{d}{dt}) : \mathbb{C}^k \to \mathbb{C}^l$. This morphism is identical to the $\mathbb{C}$-morphism determined by the matrix $P(c)$ whose entries are the constant terms of the corresponding entries of $P(\frac{d}{dt})$. Its kernel is a $\mathbb{C}$-subspace of $\mathbb{C}^k$. All its projections to various subsets of its co-ordinates are then $\mathbb{C}$-subspaces, and these are all differential kernels (indeed kernels of morphisms given by matrices whose entries can be taken to be from $\mathbb{C} \subset \mathbb{C}[\frac{d}{dt}]$). Thus $\mathbb{C}$ (as an $\mathcal{A}$-submodule of $\mathcal{D}'$) is geometrically complete, but is clearly not injective. □

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