The Evolution Of The Concept Of Controllability

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Abstract

This paper traces the evolution of the central concept of controllability of state space systems introduced by Kalman to its behavioural generalisation by Willems, and beyond to distributed behaviours. It turns out that controllability in this new formalism is identical to the notion of a potential in physics or of exactness in mathematics.

1 Introduction

It is perhaps not an exaggeration to say that the central concept of the body of results that constitute the subject of Systems and Control is the one of controllability. Introduced by Kalman [4] it was quickly realised to be of fundamental importance, and the whole edifice of the subject was erected essentially on this notion. The important problems of stabilizability, trajectory tracking, disturbance rejection etc. were understood and solved in terms of controllability, and it became the defining feature of the post war *paradigm* of control theory.

In this language of Kuhn [5] then followed a period of *crisis*. The paradigm of the Kalman theory had been immensely successful in the setting of linear systems described by ordinary differential operators of the first order, i.e. in the state space description of lumped systems. Here the theory was even effective in the sense of finite computable procedures. For instance controllability could be decided by calculating the rank of some matrix. However, the theory was tied to a state space description. Such a description rarely arose naturally when modelling a system, the natural model had to be reduced to it. In this process the variables in the description lost their natural meanings. More serious, a state space model was tied to a particular input-output structure of the system. Often the need arose to consider other input-output structures, sometimes all possible input-output structures. Even more serious was the shortcomings in the generalisation of the theory to distributed systems, i.e. to systems described by partial differential operators. The Kalman paradigm was applicable only to small class of distributed systems, the state linear ones defined by infinitesimal generators of continuous semigroups on Hilbert spaces, and even here the concept of controllability did not carry over perfectly but spawned the many notions of exact, approximate, finite time etc. controllability. Finally the Kalman paradigm in the distributed case was far from being computationally effective, dealing as it did with operators on infinite dimensional spaces. For instance there was no finite procedure to determine when a distributed system was controllable.

The *paradigm shift* comes with the concept of a behaviour introduced by J.C.Willems in a series of fundamental papers summarised in [16, 17] (see also the text [8]). In this new paradigm all the problems with the state space theory described above are overcome in a most spectacular manner. The behavioural theory, first introduced for lumped systems, is not tied to a state space description but contains it as a special case. Thus the orders of the differential operators that occur in the behavioural description of a system can be arbitrary. The original variables that occur in the modelling process can therefore be retained along with their meanings. The modelling process then is completely natural and consequently enormously simplified. Quite amazing is the fact that the behavioural description is not tied to any input-output structure, indeed it is free of such structures altogether. Finally the theory extends in toto to distributed systems and remains computationally effective even here! For instance the lumped behavioural notion of controllability not only generalises perfectly to distributed systems, it can also be decided by a finite procedure [7].

In this article I argue that the most impressive achievement of the behavioural theory is the aesthetic by focussing on the notion of controllability. It turns out that this notion coincides with two great ideas - that of a potential in physics and of exactness (or vanishing cohomology) in mathematics. It is always exciting for the practitioners of a discipline to realise that a central notion of their subject is really an instance of a grand older tradition.

2 State space versus behaviours

The Kalman paradigm studies non-autonomous systems whose state x, driven by an input u, evolves in time according to a first order differential equation

$$\begin{array}{ll} \frac{dx}{dt} = Ax + Bu, & \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ x(0) = x_o & \end{array} \tag{1}$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R}^m \to \mathbb{R}^n$ are linear maps. (If higher order derivatives appear in the modelling process, they are reduced to first order derivatives by enlarging state space in the usual way.) In this setting, a point x in $\mathbb{R}^n$ is reachable from x_o if there is an input u such that the solution x of the initial value problem (1) satisfies $x(T) = x$ for some $T > 0$.

Definition 2.1 *: The system (1) is (state) controllable if every point in $\mathbb{R}^n$ is reachable from any arbitrary point x_0.*

Theorem 2.1 *: The system (1) is (state) controllable if and only if the matrix $[B, AB, \ldots, A^{n-1}B]$ has rank n.*

The remarkable fact about the above result is that the determination of controllability is reduced to a finite calculation. It was this effectiveness of the theory that led to its widespread applicability.

I have already indicated some of the problems of the state space theory, even in this lumped setting. More serious were the shortcomings in the extension of the theory to distributed systems, where $\mathbb{R}^n$ is replaced by an infinite dimensional Hilbert space and the map A by the infinitesimal generator of a continuous semigroup. This model itself did not admit an effective representation, and the crucial feature of the Kalman paradigm in the lumped setting was lost in its distributed analogue.

The change in paradigm introduced by Willems is a radical departure from the state space theory. In his weltanschauung, the root cause of the problems with the state space theory is in its focus on a particular model, instead of on the system itself or its *behaviour* that the model attempts to describe. The problems with the state space paradigm can therefore be overcome if one turns ones attention back to the original system. The system, or its behaviour, is the set of all possible trajectories, as originally and naturally concieved, considered all together as an object of study. (This is analogous to Poincare's conception of the collection of all the phase curves of a vector field - the phase portrait - as an object of study.) The system is this set of trajectories, it defines this set and is in turn defined by it. These trajectories could be the evolution of the state as in the Kalman paradigm, so that the state space theory is merely a special instance of a behaviour. As we do not wish to list every possible trajectory in a behaviour, often because we cannot, we ask whether there is a more efficient and concise description of it. This description constitutes the laws of the behaviour and they allow us to decide when a trajectory belongs to it. An expression of these laws is a model of the behaviour. Different expressions are simply different models of the same behaviour. Thus to say that a property is true of a behaviour and not just of *a* model of it is also to say that it is true of *every* model of it.

As an example, consider the behaviour given by all possible smooth (i.e. $\mathcal{C}^\infty$) static magnetic fields. A priori, perhaps any smooth function of space could occur, but observations indicate the operation of a law, viz. that there are no magnetic monopoles. An expression of this law is that a trajectory is in this behaviour if and only if its divergence equals 0. Observe here that this behaviour occurs as the kernel of a differential operator

$$\begin{array}{rccc} div: & (\mathcal{C}^\infty(\mathbb{R}^3))^3 & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}^3) \\ & f & \mapsto & div(f) \end{array}$$

While this notion of a behaviour is completely general and universal I restrict now to those given by 'local laws', i.e. those whose trajectories are functions that satisfy differential equations (like the example above). Almost all of the laws of physics are local, as are the models of engineering systems. Amongst these behaviours the linear time invariant ones, i.e. those whose trajectories satisfy constant coefficient differential equations, occupy a special place both because of their simplicity as well as because of their ability to approximate more general behaviours. Consider henceforth those trajectories in such a behaviour that are smooth (complex valued) functions, in short a $\mathcal{C}^\infty$-behaviour. This is only for the sake of the exposition here, behaviours in other distribution spaces are important and are considered in [13, 14, 18] for example.

Willems starts his theory with the observation that the general nature of laws that determine the trajectories in the behaviour $\mathcal{B}$ of a lumped (linear, time invariant) system is of the form

$$\mathcal{B} = \{f \in (\mathcal{C}^\infty(\mathbb{R}))^k \mid M(\frac{d}{dt})f \in ImN(\frac{d}{dt})\} \tag{2}$$

where $M(\frac{d}{dt})$ and $N(\frac{d}{dt})$ are $g \times k$ and $g \times m$ matrices whose entries are from the ring $\mathcal{A} = \mathbb{C}[\frac{d}{dt}]$ of constant coefficient ordinary differential operators (see especially [8]). Here the matrix $M(\frac{d}{dt})$ defines a map

$$M(\frac{d}{dt}) : (\mathcal{C}^\infty)^k \longrightarrow (\mathcal{C}^\infty)^g$$

and similarly for $N(\frac{d}{dt})$. The space $\mathcal{C}^\infty$ is an $\mathcal{A}$-module, and the maps $M(\frac{d}{dt})$ and $N(\frac{d}{dt})$ are $\mathcal{A}$-module maps. The first step in this study consists in rewriting $\mathcal{B}$ as the kernel of another map

$$P(\frac{d}{dt}) : (\mathcal{C}^\infty)^k \longrightarrow (\mathcal{C}^\infty)^l$$

given by some matrix $P(\frac{d}{dt})$. Thus $\mathcal{B} = \{f \in (\mathcal{C}^\infty)^k \mid P(\frac{d}{dt})f = 0\}$ is a *kernel representation* (in the view point adopted here, this kernel, or for that matter (2) are just different models of the behaviour $\mathcal{B}$). This is the process of *elimination*, and it is a theorem that this can always be done for a $\mathcal{C}^\infty$-behaviour of the general form (2) - this is however not valid in some spaces such as the space $\mathcal{D}$ of compactly supported smooth functions [15, 18].

The crucial insight of Willems is the following

Definition 2.2 : *A behaviour $\mathcal{B}$ is controllable if for all f_1, f_2 in $\mathcal{B}$, there is a $\tau \geq 0$ and an f in $\mathcal{B}$ such that $f(t) = f_1(t)$ for $t < 0$ and $f(t) = f_2(t)$ for $t \geq \tau$.*

The geometric meaning of this definition is clear - in a controllable behaviour one can switch from one trajectory to another while remaining within the behaviour provided that one allows a delay. Observe also that this definition is a statement about the primary object itself, i.e. the trajectories of a behaviour, instead of about an artificially introduced 'state'.

I have pointed out above that the set $\{(x,u)\}$ satisfying (1) is a special instance of a behaviour, indeed it is the kernel of

$$[diag(\frac{d}{dt}) - A, \quad -B] : (\mathcal{C}^\infty)^n \times (\mathcal{C}^\infty)^m \longrightarrow (\mathcal{C}^\infty)^n$$

For such a state space behaviour there are a priori two notions of controllability. Quite amazing then is the following theorem of Willems -

Theorem 2.2 *: A state space behaviour is state controllable if and only if it is controllable in the sense of a behaviour.*

Thus the behavioural theory of lumped systems generalises perfectly the central concept of controllability of the state space theory. The proof of the above equivalence is via the following characterisation.

Theorem 2.3 *: Let $\mathcal{B}$ be a $\mathcal{C}^\infty$-behaviour, and let it be given as the kernel of $P(\frac{d}{dt}) : (\mathcal{C}^\infty)^k \to (\mathcal{C}^\infty)^g$ (in other words let $Ker(P(\frac{d}{dt}))$ be any model of $\mathcal{B}$ in a kernel representation). Then the following are equivalent -*
(i) $\mathcal{B}$ is controllable.
(ii) $P(\lambda)$ as a function of the complex variable λ has constant rank.
(iii) $\mathcal{B}$ is an image, i.e. there is a map $P_1(\frac{d}{dt}) : (\mathcal{C}^\infty)^{k_1} \to (\mathcal{C}^\infty)^k$ whose image is $\mathcal{B}$.
(iv) The compactly supported trajectories of $\mathcal{B}$ are dense in it.

These results follow quite easily from the Smith form of a matrix. Instead of explaining them here, or their computational effectiveness, I first describe the generalisation of this theory to distributed systems. Not only do the basic concepts of the lumped theory carry over to the distributed setting, but so also do these results. Note however that there is no Smith form available now as the ring of partial differential operators is not a principal ideal domain. We have instead to take recourse to deep results from several complex variables and the theory of partial differential equations as developed by Ehrenpreis, Hormander, Palamodov, Malgrange and others. In turn, the behavioural theory of distributed systems has raised important mathematical issues - while the classical theory of PDE was mostly concerned with over-determined systems, the behavioural theory must contend with under-determined ones, for they are the ones that occur in modelling engineering systems. This interplay between mathematics and system theory is the content of recent work, an outstanding example being the theorem of Oberst [6] that I explain below. It is the foundational result on which the behavioural theory of distributed systems rests.

3 Distributed behaviours

Consider now the extension of the behavioural theory of lumped systems to the case of constant coefficient partial differential operators, i.e. when the ring $\mathcal{A}$ is $\mathbb{C}[\partial_1, \ldots, \partial_n]$. This is just the polynomial ring in n-indeterminates, and crucial to the computational effectiveness of the theory is that $\mathcal{A}$ is a Noetherian ring. The solution space, assumed again for simplicity to be the space $\mathcal{C}^\infty$ of smooth functions on $\mathbb{R}^n$, is an infinite dimensional topological vector space, *not* finitely generated as an $\mathcal{A}$-module. The effort of the theory lies therefore in describing the properties of the behaviour, for instance controllability, in terms of the equations describing the behaviour. This entire set of equations, which corresponds to all possible models, *is* a finitely generated $\mathcal{A}$-module, and recent computational techniques such as Gröbner basis etc, ensure the effectiveness of the theory, even in this distributed setting (for a discussion of these computational issues see [1]).

Thus assume now, as in the lumped case, that a linear shift invariant distributed behaviour $\mathcal{B}$ is given by

$$\mathcal{B} = \{f \in (\mathcal{C}^\infty(\mathbb{R}^n)^k \mid M(\partial)f \in ImN(\partial)\}$$

where $M(\partial)$ and $N(\partial)$ are $g \times k$ and $g \times m$ matrices whose entries now are from the ring $\mathcal{A} = \mathbb{C}[\partial_1, \ldots, \partial_n]$. Here, as before, $M(\partial)$ and $N(\partial)$ define $\mathcal{A}$-module maps on direct sums of the space $\mathcal{C}^\infty$ of smooth functions on $\mathbb{R}^n$ ($\mathcal{C}^\infty$ is an $\mathcal{A}$-module, this structure given by differentiation). Again as in the lumped case elimination is possible [6, 13]. Thus there is a

$$P(\partial) : (\mathcal{C}^\infty)^k \longrightarrow (\mathcal{C}^\infty)^l \tag{3}$$

such that $\mathcal{B} = \{f \in (\mathcal{C}^\infty)^k \mid P(\partial)f = 0\}$. That such a kernel representation for a smooth behaviour $\mathcal{B}$ exists is a consequence of the Fundamental Principle of Ehenpreis-Malgrange-Palamodov [3] which is a deep and difficult result (its analogue in the lumped case from which elimination there is a consequence, is far simpler and follows easily from the fact that $\mathbb{C}[\frac{d}{dt}]$ is a principal ideal domain).

The Fundamental Principle answers the most basic question in the theory of PDE, that of solvability - given a system as in (3), and a g in $(\mathcal{C}^\infty)^l$, is there an f in $(\mathcal{C}^\infty)^k$ such that $P(\partial)f = g$? To answer this consider the submodule $\mathcal{P}_1$ of $\mathcal{A}^l$ consisting of all the relations between the rows of $P(\partial)$. As $\mathcal{A}$ is Noetherian $\mathcal{P}_1$ is finitely generated, say by some l_1 elements. Writing out these l_1 elements in rows yields an $l_1 \times l$ matrix $P_1(\partial)$ such that $P_1(\partial)P(\partial) = 0$ ($P_1(\partial)$ is the 'largest' such matrix in the sense that if $Q(\partial)P(\partial) = 0$ for some matrix $Q(\partial)$, then the submodule of $\mathcal{A}^l$ spanned by its rows is contained in $\mathcal{P}_1$). Clearly a necessary condition for the existance of an f such that $P(\partial)f = g$ is that $P_1(\partial)g = 0$. In other words (by definition) one can solve for all g in the image of (3) and this image is always contained in the kernel of the map $P_1(\partial) : (\mathcal{C}^\infty)^l \to (\mathcal{C}^\infty)^{l_1}$ defined by the matrix $P_1(\partial)$. The Fundamental Principal states that the above necessary condition is also sufficient. In other words the image of $P(\partial)$ *equals* the kernel of $P_1(\partial)$. As there exist algorithms to calculate $P_1(\partial)$, this solvability criterion is an effective one.

From the view point of our subject, it is the question dual to the Fundamental Principal that is of greater interest - viz. when is the kernel of the map (3) (which is a kernel representation of a behaviour $\mathcal{B}$) equal to an image? In the lumped case such images are precisely the set of controllable behaviours (Theorem 2.3 above). It turns out that this is the case with distributed behaviours as well. Thus controllability corresponds to *exactness* of a kernel representation. In physics such an image is a (vector) *potential* for the dynamics given by the kernel (see the discussion on magnetic potentials and the Aharanov-Bohm experiment in [2]). But before I explain this I need to weave in another strand from mathematics, one which also has a rich history.

Let $\mathfrak{i}$ be an ideal of the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$ in n indeterminates. Let $\mathcal{V}(\mathfrak{i})$ be the set of common zeros in $\mathbb{C}^n$ of all the polynomials in $\mathfrak{i}$. Consider the ideal of all polynomials that vanish on $\mathcal{V}(\mathfrak{i})$. The famous theorem of Hilbert (the Nullstellensatz) states that this ideal equals $\sqrt{\mathfrak{i}}$, the radical ideal of $\mathfrak{i}$. It is a foundational result in algebraic geometry which connects geometry to algebra.

Consider now the analogous question in PDE. The ring here is $\mathcal{A} = \mathbb{C}[\partial_1, \ldots, \partial_n]$, which is again the polynomial ring, the indeterminate written as ∂_i instead of X_i. Given a $p(\partial)$ in $\mathcal{A}$, its zeros are now not points in $\mathbb{C}^n$ but functions f in some space $\mathcal{F}$ such that $p(\partial)f = 0$. More generally a zero of $p(\partial) = (p_1(\partial), \dots p_k(\partial))$ in $\mathcal{F}^k$ is a k-tuple $(f_1, \dots, f_k)$ such that $\sum p_i(\partial) f_i = 0$. Given a submodule $\mathcal{P}$ of $\mathcal{A}^k$, let $\mathcal{B}_{\mathcal{F}}(\mathcal{P})$ denote the set of common zeros in $\mathcal{F}^k$ of all the elements in $\mathcal{P}$. The question of Hilbert in the PDE setting is then to characterise the submodule of all elements of $\mathcal{A}^k$ which vanish on $\mathcal{B}_{\mathcal{F}}(\mathcal{P})$. This submodule (which obviously contains $\mathcal{P}$) is the analogue of a radical ideal and is called the Willems closure of $\mathcal{P}$ with respect to $\mathcal{F}$, or in short the $\mathcal{F}$-closure of $\mathcal{P}$. Call $\mathcal{P}$ $\mathcal{F}$-closed (or simply closed if the space $\mathcal{F}$ is clear from context) if its closure equals $\mathcal{P}$ itself.

Observe here that if $\mathcal{P}$ is generated by some l elements, and if $P(\partial)$ is the $l \times k$ matrix formed by writing these elements as rows, then $\mathcal{B}_{\mathcal{F}}(\mathcal{P})$ is nothing but the $\mathcal{F}$-behaviour of $P(\partial)$, i.e. the kernel of the map $P(\partial) : \mathcal{F}^k \to \mathcal{F}^l$. Thus a choice of generators results in a model for a behaviour, a different choice is a different model. All possible models are however "contained" in the $\mathcal{F}$-closure of $\mathcal{P}$. To make a statement about a behaviour, and not merely of a model of it, is to make a statement about all possible models and therefore about this closure. This, as has been already noted, is the behavioural way.

Of interest here (in the context of controllability) is the closure of $\mathcal{P}$ with respect to $\mathcal{C}^\infty$ and the space $\mathcal{D}$ of smooth functions with compact support. This characterisation [7], as well as those with respect to the other classical spaces [13, 14], is based on the following fundamental result of Oberst [6].

Theorem 3.1 : *Let $\mathcal{P}$ be a submodule of $\mathcal{A}^k$. Then $\mathcal{B}_{\mathcal{C}^\infty}(\mathcal{P})$ consists of just the 0 function if and only if $\mathcal{P} = \mathcal{A}^k$.*

Thus if $P(\partial)$ is a matrix whose rows do not generate all of $\mathcal{A}^k$, then the kernel of the map (3) contains nonzero functions. Together with the Fundamental Principle, Oberst's Theorem further implies the following - let $P(\partial)$ and $P'(\partial)$ be two matrices of sizes $l \times k$ and $l' \times k$ respectively. Then the ($\mathcal{C}^\infty$) kernels of the maps defined by them are equal if and only if the submodules (of $\mathcal{A}^k$) generated by their rows are equal.

The PDE analogue of the Hilbert Nullstellensatz for the spaces $\mathcal{C}^\infty$ and $\mathcal{D}$ is

Theorem 3.2 : *(i) Every submodule $\mathcal{P}$ of $\mathcal{A}^k$ is closed with respect to $\mathcal{C}^\infty$.*
(ii) A submodule $\mathcal{P}$ of $\mathcal{A}^k$ is closed with respect to D if and only if the quotient $\mathcal{A}^k/\mathcal{P}$ is torsion free.

(For the Nullstellensatz with respect to other spaces see [11, 13, 14, 18].)

The relationship between potentials (or exactness) and the above Nullstellensatz is the behavioural notion of controllability ([7]).

Definition 3.1 : *A behaviour $\mathcal{B}$ is said to be controllable if for f_1 and f_2, any two elements in $\mathcal{B}$, and for U_1 and U_2 any two open subsets of $\mathbb{R}^n$ such that their closures are disjoint, there exists an element f in $\mathcal{B}$ which equals f_1 on U_1 and f_2 on U_2.*

Thus controllability solves a patching problem. Such a patching allows one to move from one trajectory to another. In this sense controllable behaviours are the other end of the spectrum from *autonomous* ones where one is constrained to remain on a single trajectory.

The characterisation of controllable behaviours is [7]

Theorem 3.3 : *Let $\mathcal{P}$ be a submodule of $\mathcal{A}^k$ and let $\mathcal{B}$ be the smooth behaviour defined by it. Then the following are equivalent -*
(i) $\mathcal{B}$ is controllable.
(ii) $\mathcal{B}$ admits a potential (or an image representation). Therefore compactly supported trajectories

are dense in $\mathcal{B}$.
(iii) $\mathcal{P}$ is closed with respect to D (which happens exactly when the quotient $\mathcal{A}^k/\mathcal{P}$ is torsion free).

Thus controllability asserts the existence of a potential, and conversely. Especially important is the fact that this can be determined by a finite procedure, viz. (iii) above.

We have come a long way, from state controllability to distributed controllable behaviours, ensuring at each stage that our generalisations included the earlier notion. It is possible in this setting to define an a priori weaker notion of asymptotic (or approximate) controllability but it turns out that we get nothing new, even for distributed systems ([13]). Thus controllability in the behavioural sense seems to be the most faithful extension of Kalman's definition.

For further variations on this theme, see [9, 10, 12, 19, 20] and the references therein.

References

[1] T.Cotroneo, *Algorithms in Behavioural Systems Theory*, Doctoral Thesis, University of Groningen, 2001.

[2] R.P.Feynman, R.Leighton and M.Sands, *The Feynman Lectures in Physics, vol. 2*, Addison-Wesley, 1964.

[3] L.Hörmander, *An Introduction to Complex Analysis in Several Variables*, North Holland, 3rd ed., 1990.

[4] R.E.Kalman, On the general theory of control systems, In *Proceedings of the 1st World Congress of the International Federation of Automatic Control*, 481-493, Moscow, 1960.

[5] T.Kuhn, *The Structure of Scientific Revolutions*, University of Chicago Press, 1962,

[6] U.Oberst, Multidimensional constant linear systems, *Acta Applicandae Mathematicae*, 20:1-175, 1990.

[7] H.Pillai and S.Shankar, A behavioural approach to control of distributed systems, *SIAM jl. Control Optim.*, 37:388-408, 1998.

[8] J.W.Polderman and J.C.Willems, *Introduction to Mathematical Systems Theory*, Springer Verlag, 1998.

[9] J.F.Pommaret and A.Quadrat, Algebraic analysis of linear multidimensional control systems, *IMA J. Mathematical Control Information*, 16:275-297, 1999.

[10] P.Rocha and J.Wood, Trajectory control and interconnection of 1D and nD systems, *SIAM jl. Control Optim.*, 40:107-134, 2001.

[11] A.Sasane, On the Willems closure with respect to $\mathcal{W}_s$, *preprint of the University of Twente*, 2002.

[12] A.Sasane, E.G.F.Thomas and J.C.Willems, Time-autonomy versus time-controllability, *Systems and Control Letters*, 45:145-153, 2002.

[13] S.Shankar, The Nullstellensatz for systems of PDE, *Advances in Applied Mathematics*, 23:360-374, 1999.

[14] S.Shankar, The lattice structure of behaviours, *SIAM jl. Control Optim.*, 39:1817-1832, 2001.

[15] S.Shankar, Geometric completeness of distribution spaces, *preprint of the Chennai Mathematical Institute*, 2001.

[16] J.C.Willems, Paradigms and puzzles in the theory of dynamical systems, *IEEE Trans Automatic Control*, 36:259-294, 1991.

[17] J.C.Willems, Open dynamical systems and their control, *Proceedings of the International Congress of Mathematicians, Berlin*, Vol III: Invited lectures, 697-706, 1998.

[18] J.Wood, Key problems in the extension of module-behaviour duality, *Linear Algebra and its Applications*, 351-352:761-798, 2002.

[19] J.Wood and E.Zerz, Notes on the definition of behavioural controllability *Systems and Control Letters*, 37:31-37, 1999.

[20] E.Zerz, Extension modules in behavioural linear systems theory, *Multidimensional Systems and Signal Processing*, 12:309-327, 2001.