Applications of Lefschetz numbers in control theory

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Abstract. The goal of this paper is to develop some applications of the Lefschetz fixed point theory techniques, already available in dynamics, in control theory. A dynamical system on a manifold M is a map $f : M \to M$. The current state, $x \in M$, of the system determines the next state, $f(x)$, and its equilibria are the fixed points of f, $f(x) = x$. More generally, one deals with coincidences of a pair of maps $f, g : N \to M$, $f(x) = g(x)$, between manifolds of the same dimension. The main tool is the Lefschetz number λ_{fg} defined in terms of the homology of M, N, f: if $\lambda_{fg} \neq 0$ then there is at least one coincidence. Moreover, since all properties established through homology are "robust by nature", any pair f', g' homotopic to f, g has a coincidence as well. In the control situation, the next state $f(x, u)$ depends on the current one, $x \in M$, as well as the input, $u \in U$. It is described by a map $f : N = U \times M \to M$ and its equilibria are the coincidences of f and the projection $g : N = U \times M \to M$. Since in this case the dimensions of N and M are not equal, the Lefschetz number has to be replaced with the so-called Lefschetz homomorphism. In this paper the Lefschetz homomorphism is applied to detection of equilibria and controllability. The secondary objective is to study robustness of these properties; for example, we find out when small perturbations of the systems can lead to the loss of equilibria.

1. Introduction.

The goal of this paper is to develop some applications of the Lefschetz fixed point theory techniques, already available in dynamics, in control theory. A (discrete) dynamical system on a manifold M is simply a map $f : M \to M$. Then the next state $f(x)$ of the system depends only on the current one, $x \in M$. The equilibrium set $C = \{x \in M : f(x) = x\}$ of the system is the set of fixed points of f. It is treated via the so-called Coincidence Problem [**2**], [**30**]: "Given two maps $f, g : N \to M$ between two n-dimensional manifolds, what can be said about the coincidence set C of all x such that $f(x) = g(x)$?" One of the main tools is the Lefschetz number λ_{fg}. The famous Lefschetz coincidence theorem states that if $\lambda_{fg} \neq 0$ then there is at least one coincidence, i.e., $C \neq \varnothing$. Using this and similar invariants one can find out whether a dynamical system has an equilibrium or a periodic point.

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In case of a *controlled* dynamical system, the next state $f(x,u)$ depends not only on the current one, $x \in M$, but also on the *input*, $u \in U$. Suppose we have a fiber bundle given by the bundle projection $U \to N \xrightarrow{g} M$ and a map $f : N \to M$. The this is a discrete time control system with U the space of inputs, M the space of states, and N the space of pairs of states and inputs of the system. Just as above, the equilibrium set of the system $C = \{x \in M : f(x,u) = x\}$ is the coincidence set of the pair (f,g). However now we have to deal with the fact that the dimensions of N and M are *not equal* anymore! As a result, the Lefschetz *number* does not do as good a job detecting coincidences and has to be replaced with the Lefschetz *homomorphism* $\Lambda_{fg} : H_k(N,A) \to H_{k-n}(M)$, $k = 0,1,...$ [**26**].

The state space M as a nontrivial manifold appears naturally in numerous applications. For example, $M = \mathbf{T}^n = (\mathbf{S}^1)^n$, where $\mathbf{T}^n$ is the n dimensional torus, is the configuration space of a robotic arm with n revolving joints [**22**, p. 1]; or $M = R^3 \times SO(3)$ is the configuration space of a rigid body in space [**20**, Chapter 2]. Typically, we have $N = M \times U$. However nontrivial bundles are also common. For example, if $M = T\mathbf{S}^2$, the tangent bundle of the 2-sphere $\mathbf{S}^2$, is the state space of a spherical pendulum with a gas jet control which is always directed in the tangent space, then N is an $\mathbf{R}^2$-bundle over M not homeomorphic to $\mathbf{R}^2 \times M$ [**22**, p. 17].

A model of a "plant" is a control system as a triple (M,N,f) of the spaces M,N and the map f as described above. Since our knowledge of the model is inevitably imprecise, we have to deal with perturbations of the system. The space of inputs U is the set of known, or "uncertain", parameters. On the other hand, perturbations may be understood as variations of the unknown, or "certain", parameters of the system. Therefore, if the system depends continuously on these parameters, the change of M,N, and f is also continuous. This means that we have to consider spaces homeomorphic to M,N and maps homotopic to f. An appropriate instrument is homology. Indeed, the homology of M,N,f remains constant under homeomorphisms and homotopies and can be rigorously and effectively computed, see Mischaikow [**21**].

Normally the homotopies of f are assumed to be "small" (in particular, this is the basis of the notion of structural stability). However unless actual estimates are available, we don't know how "small" these perturbations are in real life. For example, the cascading failures of the electric grid may be explained by the lack of "robustness" of the system and the underestimation of the magnitude of possible disturbances. Therefore in order to take into account the "worst possible scenario" we should consider *arbitrary homotopies* of f. An example of such a control system is the human organism as it preserves stability inside under extraordinary amount of uncertainty on the outside. In this paper Lefschetz theory is applied to study existence of equilibria and controllability for (discrete and continuous time) systems determined by maps homotopic to f (Theorems 5.1, 6.2, 7.1).

The secondary objective of this paper is to study robustness of some of these properties under "small" perturbations because sometimes they may produce dramatic changes in the behavior of the system. We consider situations when this change is the loss of equilibria of the system (Theorem 5.3).

The paper is organized as follows. In Section 2 we review the classical theory of Lefschetz numbers and show its inadequacy for control theory. In Section 3 we consider the necessary generalization, the Lefschetz homomorphism, of the Lefschetz number and state several relevant results about existence of coincidences.

In Section 4 we state some results about removability of coincidences. In Section 5 we provide sufficient conditions of existence of equilibria and their robustness. In Section 6 we provide sufficient conditions of controllability. In Section 7 we discuss how these results can be applied to existence of equilibria and controllability of continuous time control systems. In Section 8 we consider applications of the coincidence results to robust stability. Notions of control theory are defined as needed, for details see [**22**], [**24**], [**29**]. The necessary background from algebraic topology is outlined in the Appendix.

2. Review of Lefschetz theory.

Consider the *Fixed Point Problem*: "If M is a manifold and $f : M \to M$ is a map, what can be said about the set of points $x \in M$ such that $f(x) = x$?" Applications of fixed point theorems (Kakutani, Banach, etc.) to control problems are abundant, [**1**], [**6**], [**7**], [**15**], [**18**], [**23**]. However the methods we suggest in this paper go far beyond those.

One may associate to f an integer λ_f called the Lefschetz number [**3**]. It detects fixed points and is computable by a simple formula:

$$\lambda_f = \sum_n (-1)^n Trace(f_{*n}),$$

where $f_{*n} : H_n(M) \to H_n(M)$ is induced by f. The Lefschetz fixed point theorem states that if $\lambda_f \neq 0$, then f has a fixed point.

The *Coincidence Problem* is concerned with a similar question about two maps $f, g : N \to M$ and their *coincidences* $x \in N : f(x) = g(x)$. One of the main tools is the *Lefschetz coincidence number* λ_{fg} defined similarly to λ_f as the alternating sum of traces of a certain endomorphism on the homology group of M. A Lefschetz type coincidence theorem states that if $\lambda_{fg} \neq 0$ then f, g (and any pair homotopic to them) have a coincidence.

Until recently such theorems have been mostly considered in the following two settings. Case 1: [**2**, VI.14], [**30**, Chapter 7] $f : N \to M$ is a map between two n-manifolds. Case 2: [**14**] $f : N \to M$ is a map from an arbitrary topological space to an open subset of $\mathbf{R}^n$ and all fibers $f^{-1}(y)$ are acyclic, i.e., $H_k(f^{-1}(y)) = 0$ for $k = 1, 2, \ldots$.

Neither case is broad enough for the purposes of control theory. Case 1 deals with equilibria of dynamical systems but fails to include a simplest control system because the dimensions of N and M have to be equal. In Case 2 the dimensions are also equal in the sense that $H_*(N) = H_*(M)$ (Vietoris Theorem). Therefore if $N = M \times U$, then the input space U must be acyclic.

As an example from dynamics, one can consider the problem of existence of closed orbits of a flow given by a map $f : [a, b] \times M \to M$. Closed orbits correspond to coincidences of f and the projection $p : [a, b] \times M \to M$. More generally one considers $f : X \times M \to M$, where X is a topological space. This situation was studied in [**19**], [**11**], [**12**], [**10**] under the name "parametrized fixed point theory". These results can be applied to detection of equilibria (Section 5), but the setting is not general enough to study controllability (Section 6). The author [**25**], [**26**] extended some of the results of [**12**] to the general case of two arbitrary maps $f, g : N \to M$ to an orientable compact manifold. The content of these papers is briefly outlined in the next section.

3. Detecting coincidences.

Consider the classical definition of the Lefschetz number of a pair of maps $f:(N,\partial N)\to(M,\partial M)$, $g:N\to M$, where both M and N are orientable compact connected manifolds with boundaries $\partial M,\partial N$, and $\dim M=\dim N=n$. It relies on the following: if $h:E\to E$ is a degree 0 endomorphism of a finitely dimensional graded module $E=\{E_k\}, h_k:E_k\to E_k$, then its *Lefschetz number* is $L(h)=\sum_k(-1)^k Trace(h_k)$. To apply this formula we let $E=H_*(M)$, then the Lefschetz number is defined as $\lambda_{fg}=L(g_*D_Nf^*D_M^{-1})$, where $D_M:H^*(M,\partial M)\to H_{n-*}(M)$, $D_N:H^*(N,\partial N)\to H_{n-*}(N)$ are the Poincaré duality isomorphisms.

Now suppose N *is an arbitrary topological space,* $A\subset N$, M *is an orientable compact connected manifold,* $\dim M=n$, *and* $f:(N,A)\to(M,\partial M)$, $g:N\to M$ *are maps.* The generalization is based on the fact that since $E=H_*(M)$ is equipped with the cap product $\frown: E^*\otimes E\to E$, one can define the "Lefschetz class" $L(h)\in E$ of an endomorphism h given by $h_k:E_k\to E_{k+m}$ of any degree m. There is an explicit formula for it.

Theorem 3.1. [**26**, Proposition 2.2] *If* $h:H_*(M)\to H_{*+m}(M)$ *is a homomorphism of degree* m *then*

$$L(h)=\sum_k(-1)^{k(k+m)}\sum_j x_j^k\frown h(a_j^k),$$

where $\{a_1^k,...,a_{m_k}^k\}$ *is a basis for* $H_k(M)$ *and* $\{x_1^k,...,x_{m_k}^k\}$ *the corresponding dual basis for* $H^k(M)$.

For a given $z\in H_*(N,A)$, suppose h_{fg}^z is defined as the composition

$$H_*(M)\xrightarrow{D_M^{-1}}H_{n-*}(M,\partial M)\xrightarrow{f^*}H_{n-*}(N,A)\xrightarrow{\frown z}H_{k-n+*}(N)\xrightarrow{g_*}H_{k-n+*}(M).$$

i.e., $h_{fg}^z=g_*((f^*D_M^{-1})\frown z))$. Its degree is $m=|z|-n$.

Definition 3.2. The *Lefschetz homomorphism* $\Lambda_{fg}:H_k(N,A)\to H_{k-n}(M)$, $k=0,1,...$, is defined by

$$\Lambda_{fg}(z)=L(h_{fg}^z).$$

Proposition 3.3. *If the degree* m *of* h *is zero,* $L(h)=\sum_k(-1)^k Trace(h_k)$.

In particular, the degree of h_{fg}^z is zero for all $z\in H_n(N,A)$. If, moreover, N is a orientable compact connected manifold of dimension n, we have $H_n(N,\partial N)=\mathbf{Q}$. It is generated by the fundamental class $O_N\in H_n(N,\partial N)$ of N. Since $D_N(x)=x\frown O_N$, we recover the Lefschetz number, $\lambda_{fg}=\Lambda_{fg}(O_N)$.

Theorem 3.4. [**26**, Theorem 6.1] ***(Existence of coincidences)*** *If* $\Lambda_{fg}\neq 0$ *then any pair of maps* f',g' *homotopic to* f,g *has a coincidence.*

Especially important for control theory are the following corollaries.

Corollary 3.5. ***(Existence of fixed points)*** *Let* $g:M\times U\to M$ *be a map. Given* $v\in H_*(U)$, *suppose the homomorphism (Knill trace)* $g_v:H_*(M)\to H_{*+k}(M)$ *of degree* $|v|$ *is defined by*

$$g_v(x)=(-1)^{(n-|x|)|v|}g_*(x\otimes v),$$

$x\in H_*(M)$. *Then, if*

$$L(g_v)\neq 0 \textit{ for some } v\in H_*(U)$$

then any map $g' : M \times U \to M$ homotopic to g has a fixed point x, $g'(x,u) = x$ for some u.

Proof. Suppose $(N, A) = (M, \partial M) \times U$ and apply the above theorem to the pair p, g, where p is the projection $p : (M, \partial M) \times U \to (M, \partial M)$. Also according to Corollary 5.7 in [**26**], $\Lambda_{pg}(O_M \otimes v) = L(g_v)$. □

Corollary 3.6. ***(Sufficient condition of surjectivity)*** *If*

$$f_* : H_n(N, A) \to H_n(M, \partial M) = \mathbf{Q} \text{ is nonzero}$$

then any map $f' : (N, A) \to (M, \partial M)$ homotopic to f is onto.

Proof. Apply the theorem to the pair f, c, where c is any constant map (as in Section 5 in [**25**] and Proposition 6.8 in [**26**]). □

The condition of this corollary is equivalent to nonvanishing of the degree of f in case of manifolds of equal dimensions.

4. Removing coincidences.

Suppose M *and* N *are manifolds,* $\dim M = n$, $f : (N, A) \to (M, \partial M), g : N \to M$ *are maps,* A *is a closed subset of* N.

When $\dim N = \dim M = n > 2$, the vanishing of the Lefschetz number λ_{fg} implies that the coincidence set can be removed by homotopies of f, g [**5**]. If $\dim N = n + m, m > 0$, this is no longer true even if λ_{fg} is replaced with Λ_{fg}. Some progress has been made for $m = 1$. In this case the secondary obstruction to the removability of a coincidence set was considered in [**9**], [**8**], [**16**]. These results can be used to study removability of equilibria when the dimension of the input space is 1 (Section 5). Necessary conditions of the global removability for arbitrary m were considered in [**13**, Section 5] with N a torus and M a nilmanifold. For some $m > 1$, a partial converse of the Theorem 3.4 is provided by the author [**27**]. A version of this theorem is given below.

Let F be a closed submanifold of N. We say that F satisfies condition (A) if one of the following three conditions holds [**27**, Section 4]:

- *(a1) M is a surface, i.e., $n = 2$; or*
- *(a2) F is acyclic, i.e., $H_k(F) = 0$ for $k = 1, 2, \ldots$; or*
- *(a3) every component of F is a homology m-sphere, i.e., $H_k(F) = 0$ for $k \neq 0, m$, for the following values of m and n:*

  (1) $m = 4$ and $n \geq 6$;
  (2) $m = 5$ and $n \geq 7$;
  (3) $m = 12$ and $n = 7, 8, 9, 14, 15, 16, \ldots$.

Theorem 4.1. ***(Removability of coincidences)*** *Suppose condition (A) is satisfied for $F = C_{fg}$, the coincidence set of f, g, $f(C_{fg}) = g(C_{fg}) = \{x\}$, $x \in M \backslash \partial M$, and $C_{fg} \cap A = \varnothing$. Then, if*

$$\Lambda_{fg}(z) = \sum_k (-1)^k Trace(h^z_{fg}) \in H_0(M) = \mathbf{Q} \text{ is zero for all } z \in H_n(N, A)$$

then there is a homotopy of f (or g) to a map f' (or g') such that the new pair has no coincidences. The homotopy can be chosen arbitrarily small and constant on the compliment of a neighborhood of F.

PROOF. Let $I_{fg}^N(\tau) \in H^n(N,A)$ be the cohomology coincidence index [**27**, Section 2]. Here $I_{fg}^N = (f,g)^* : H^n(M^\times) \to H^n(N,A)$, τ is the generator of $H^n(M^\times) = \mathbf{Q}$, $M^\times = (M \times M, M \times M \backslash d(M))$, and $d(M)$ is the diagonal. Let I_{fg} be the homology coincidence homomorphism defined by $I_{fg} = (f,g)_* : H_*(N,A) \to H_*(M^\times)$. Since $C_{fg} \cap A = \varnothing$, both are well defined. By Theorem 6.1 in [**26**], $\Lambda_{fg}(z) = \pi_*(\tau \frown I_{fg}(z))$, where $\pi : M \times M \to M$ is the projection on the first factor. Then, for any $z \in H_n(N,A)$

$$\begin{aligned}\Lambda_{fg}(z) &= \pi_*(\tau \frown (f,g)_*(z)) = \pi_*(f,g)_*((f,g)^*(\tau) \frown z) \\ &= < (f,g)^*(\tau), z > = < I_{fg}^N(\tau), z > .\end{aligned}$$

Therefore $\Lambda_{fg}(z) = 0$ for all $z \in H_n(N,A)$ if and only if $I_{fg}^N(\tau) = 0$. The trace formula for Λ_{fg} comes from Proposition 3.3.

Let G be an open neighborhood of x. Choose tubular neighborhoods W and V of C_{fg} such that $V \subset \overline{V} \subset W \subset N \backslash A$, $f(W) \subset G$, and $W \cap C_{fg} = C_{fg}$. Condition (A) ensures that a more general condition is satisfied [**27**, Section 4]:

$$\text{(A)} \quad H^{k+1}(W, W \backslash V; \pi_k(\mathbf{S}^{n-1})) = 0 \text{ for } k \geq n+1.$$

Now we proceed as in the proof of Theorem 2 in [**27**] to show that C_{fg} can be removed by a homotopy of f or g relative to $N \backslash W$ provided the local cohomology index $I_{fg}^W(\tau)$ vanishes. This index is defined as the one above: $I_{fg}^W = (f,g)^* : H^n(M^\times) \to H^n(W, W \backslash V)$. Now if $i : W \to N$ is the inclusion, then $I_{fg}^W(\tau) = i^* I_{fg}^N(\tau) = 0$. □

Condition (A) ensures that only the primary obstruction to removability may be nonzero. Further investigation of necessary conditions of removability of coincidences (and its applications in control theory, see Sections 5 - 8) will require consideration of higher order obstructions. The case when $f(C_{fg})$ is not a single point is best addressed in the context of Nielsen theory via Wecken type theorems [**28**]. In general, the homotopy of f is not local.

Especially important for control theory are the following corollaries.

COROLLARY 4.2. ***(Removability of fixed points)*** *Given a parametrized map $g : M \times U \to M$ with only one fixed point, $g(a,u) = a \in M \backslash \partial M$, suppose condition (A) is satisfied for the fixed point set $F = \{u \in U : g(a,u) = a\}$ of g. Then, if*

$$L(g_1) = \sum_k (-1)^k Trace(\overline{g}_{*k}) = 0,$$

where $\overline{g}(\cdot) = g(\cdot, u_0)$, then there is a homotopy of g to a map g' such that g' has no fixed points. The homotopy can be chosen arbitrarily small and constant on the compliment of a neighborhood of $\{a\} \times F$.

PROOF. Suppose $(N,A) = (M, \partial M) \times U$ and apply the theorem to the pair p, g, where p is the projection $p : (M, \partial M) \times U \to (M, \partial M)$. Next we use the formula $\Lambda_{pg}(O_M \otimes v) = L(g_v)$ from Corollary 5.7 in [**26**]. Finally, we observe also that $C_{pg} = \{a\} \times F$. □

COROLLARY 4.3. ***(Removability of images)*** *Suppose that there is a fiber $F = f^{-1}(x_0)$ of f satisfying condition (A). Then, if*

$$f_* : H_n(N,A) \to H_n(M, \partial M) = \mathbf{Q} \text{ is zero}$$

then there is a homotopy f to a map f' which is not onto; specifically, $x_0 \notin f(N)$. The homotopy can be chosen arbitrarily small and constant on the compliment of a neighborhood of F.

PROOF. Suppose $c(x,u) = x_0$ is the constant map. Next, $f_* = 0$ if and only if $\Lambda_{fc}(z) = 0$ for all $z \in H_n(N,A)$ (see Section 5 in [**25**]). Now apply the theorem to the pair f, c (cf. Theorem 3 in [**27**]). Observe also that $C_{fc} = f^{-1}(x_0)$. □

Applications of these results in control theory are considered below.

5. Existence of equilibria.

Suppose that $(M, \partial M)$ is a compact orientable manifold with $\dim M = n$. A map $g : U \times M \to M$ determines a *discrete time control system* D_g, with U the space of inputs, M the space of states of the system.

As before suppose $\{a_1^k, ..., a_{m_k}^k\}$ is a basis for $H_k(M)$ and $\{x_1^k, ..., x_{m_k}^k\}$ the corresponding dual basis for $H^k(M)$.

THEOREM 5.1. ***(Existence of equilibria)*** *If*

$$L(g_v) = (-1)^{n|v|} \sum_k (-1)^k \sum_j x_j^k \frown g_*(a_j^k \otimes v) \neq 0 \text{ for some } v \in H_*(U)$$

then every perturbation of the discrete time system D_g has an equilibrium.

PROOF. In light of Corollary 3.5 we only need to show that the above formula for $L(g_v)$ is true. From Theorem 3.1 we have the following:

$$L(g_v) = \sum_k (-1)^s \sum_j x_j^k \frown (-1)^t g_*(a_j^k \otimes v),$$

where $s = k(k + |v|)$ and $t = (n - |a_j^k|)|v| = (n-k)|v|$. Now $s + t = k^2 + n|v|$ and the formula follows. □

COROLLARY 5.2. *Suppose $M = \mathbf{S}^n$, and suppose one of the following conditions is satisfied:*
(1) $g_*(1 \otimes v) \neq 0$ for some $v \in H_n(U)$; or
(2) $g_*(O_M \otimes 1) \neq (-1)^{n+1} O_M$;
Then the discrete time system D_f has an equilibrium.

PROOF. Since $\dim M = n$, only $v \in H_0(U), ..., H_n(U)$ can appear in the formula for $L(g_v)$. In fact, $k + |v| \leq n$. Now, for $M = \mathbf{S}^n$ we have

$$L(g_v) = (-1)^{n|v|}(1 \frown g_*(1 \otimes v) + (-1)^n \overline{O}_M \frown g_*(O_M \otimes v)),$$

where $\overline{O}_M$ is the dual of O_M. The first term vanishes unless $|v| = 0$ or n. The second term vanishes unless $|v| = 0$. Therefore, if $|v| = n$, then $L(g_v) = g_*(1 \otimes v)$; if $|v| = 0$, then $L(g_v) = 1 + (-1)^n \overline{O}_M \frown g_*(O_M \otimes 1)$. □

If $f : M \times M \to M$ is the multiplication of a compact Lie group, then D_f has a equilibrium [**12**, Example 2.3]. For more examples, see [**12**], [**25**], [**26**].

In this setting Corollary 4.2 reads as follows.

THEOREM 5.3. ***(Robustness of equilibria)*** *Suppose U is a manifold, and suppose that D_g has only one equilibrium state, $g(a,u) = a \in M \backslash \partial M$. Suppose* *condition (A) is satisfied for the equilibrium manifold* $F = \{u \in U : g(a,u) = a\}$ *and* $F \cap A = \varnothing$. *Then, if*

$$L(g_1) = \sum_k (-1)^k Trace(\overline{g}_{*k}) = 0,$$

where $\overline{g}(\cdot) = g(\cdot, u_0)$, *then there is an arbitrarily small perturbation of the discrete time system* D_g *which has no equilibria.*

In particular the trace condition is satisfied if $M = \mathbf{S}^n$, n odd, and $\overline{g} : \mathbf{S}^n \to \mathbf{S}^n$ has degree 1 (compare to condition (2) of the above corollary).

The Wecken type theorem due to Jezierski [**16**] provides a sufficient condition of removability of an equilibrium for $\dim U = 1$ without restrictions on n. However the conditions on f and g are hard to verify.

6. Controllability.

The results in Section 5 involve only a single application of the map while most problems of control theory deal with multiple iterations.

Suppose system D_f is given by $f : N = M \times U \to M$. We make two assumptions about the behavior of the system. First, $f(\partial M \times U) \subset \partial M$, i.e., the trajectories never leave the boundary ∂M. Second, there is some $U' \subset U$ such that $f(M \times U') \subset \partial M$, i.e., inputs from U' always take the system to the boundary ∂M. Thus f will be treated as a map of pairs, $f : (N, A) = (M, \partial M) \times (U, U') \to (M, \partial M)$.

The system D_f is called *controllable* if any state can be reached from any other state, i.e., for each pair of states $x, y \in M$ there are inputs $u_0, ..., u_s \in U$ such that $x_1 = f(u_0, x), x_2 = f(u_1, x_1), ..., y = f(u_s, x_s)$, notation $x \rightsquigarrow_f y$. This notion is generalized in two ways. First, we consider the possibility of an arbitrary state reached from a state in a particular subset of M. Second, we allow for (arbitrary) perturbations of f.

Definition 6.1. Given $L \subset M$, let $f' : (L, L') \times (U, U') \to (M, \partial M)$ be the restriction of f, where $L' = L \cap \partial M$. Then the system is called *robustly controllable from* L if there is $\varepsilon > 0$ such that for any map f_0 ε-homotopic to f', maps $f_1, ..., f_s$ ε-homotopic to f, and for each $y \in M$ there are $x \in L$ and inputs $u_0, ..., u_s \in U$, such that

$$x_1 = f_0(x, u_0), x_2 = f_1(x_1, u_1), ..., y = f_s(x_s, u_s).$$

The system is called *strongly robustly controllable from* L if this condition is satisfied for all ε, i.e., for all perturbations of f.

Theorem 6.2. ***(Sufficient condition of robust controllability)*** *Suppose that there are* $a_0 \in H_*(L, L')$, $v_0, ..., v_s \in H_*(U, U')$ *such that*

$$a_1 = f'_*(a_0 \otimes v_0), a_2 = f_*(a_1 \otimes v_1), ..., a_s = f_*(a_{s-1} \otimes v_{s-1}),$$

where $a_s = O_M \in H_n(M, \partial M)$, *the fundamental class of* M, *i.e.,* O_M *can be reached from any* $a_0 \in H_*(L, L')$ *by means of* f_*. *Then the discrete time system* D_f *is strongly robustly controllable from* L.

Proof. Define a map $F_s : (L, L') \times (U, U')^{s+1} \to (M, \partial M)$ for $s = 1, 2, ...$ by

$$F_s(x, u_0, ..., u_s) = f_s(...f_1(f_0(x, u_0), u_1), ..., u_s).$$

Then $x \rightsquigarrow_f F_s(x, u_0, ..., u_s)$. Therefore controllability from L means that $F_s : L \times U^s \to M$ is onto for some s. By Corollary 3.6 if

$$F_{s*} : H_n((L, L') \times (U, U') \times ... \times (U, U')) \to H_n(M, \partial M) = \mathbf{Q}$$

is nonzero then every map F'_s homotopic to F_s is onto. But F_{s*} is given by the composition

$$F_{s*} : H_*(L, L') \otimes H_*(U, U') \otimes ... \otimes H_*(U, U') \xrightarrow{f'_* \otimes Id}$$

$$H_*(M, \partial M) \otimes H_*(U, U') \otimes ... \otimes H_*(U, U') \xrightarrow{f_* \otimes Id} ....$$

Now if $f_*(...f_*(f'_*(a_0 \otimes v_0) \otimes v_2) \otimes ... \otimes v_s) \neq 0$ for some $a_0 \in H_*(L, L')$, $v_0, ..., v_s \in H_*(U, U')$ such that $|a_0| + |v_0| + ... + |v_s| = n$, then F_{s*} is nonzero. □

The theorem involves multiple iterations of f_* while it is preferable to have a condition involving only f_* itself. Let's consider a case when this is possible.

Consider first a simple example, $M_i = U = \mathbf{S}^1$, $f : \mathbf{S}^1 \times \mathbf{T}^n \to \mathbf{T}^n$, and let $f(u, x_1, ..., x_n) = (u, x_1, ..., x_{n-1})$. This may serve as a model for a robotic arm with n joints where only the first joint can be controlled directly and the next state of a joint is "read" from the current state of the previous joint. Then this system is controllable by the theorem. Indeed after n iterations with inputs $u_1, ..., u_n$ the system's state is $(u_n, ..., u_1)$.

More generally, suppose the state space M has the product structure, $M = K_1 \times ... \times K_s$, where K_i are manifolds of dimensions n_i. Suppose $f = (f_1, ..., f_s)$, where $f_i : U \times M \to K_i$. Suppose for $i = 1, ..., s$, maps $f_i^a : K_{i-1} \to K_i$, where $K_0 = U$, are given by $f_i^a(x_{i-1}) = f_i(a_0, ..., a_{i-2}, x_{i-1}, a_i, ..., a_s)$. If all f_i^a are onto then the system is controllable. According to Corollary 3.6 it suffices to require that all $f_{i*}^a : H_{n_i}(K_{i-1}) \to H_{n_i}(K_i)$ are nonzero, $i = 1, ..., s$. It is clear that what we have is the "finite time reachability", i.e., every state can be reached in a finite number, s, of steps and that number is common for all states.

The above theorem can be understood as follows. The restrictions of f, $f_0 : L \times U \to M_1, f_1 : M_1 \times U \to M_2, ..., f_s : M_{s-1} \times U \to M_s = M$, are onto, where $M_0, M_1, ...$ are submanifolds of M with $\dim M_i = |a_i|$. The robustness of each of these properties can be tested by means of Corollary 4.3.

7. Continuous systems.

Suppose M *is a compact orientable connected manifold,* $\dim M = n$, N *is a manifold*. Then $\dim TM = 2n$, where the tangent bundle TM on M.

A *continuous time control system* C_h [**22**, p. 16] is defined as a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{h} & TM, \\ \downarrow^{p} & \swarrow^{\pi_M} & \\ M & & \end{array}$$

where $p : N \to M$ is a fiber bundle over M and π_M is the projection. In other words we have a parametrized vector field on M.

Now $(x, u) \in N$ is an *equilibrium pair*, and x is an *equilibrium state*, if $h(x, u) = (x, 0) \in TM$. In other words, it is a coincidence of the pair h, g, where $g(x, u) = (x, 0)$ for all x.

A curve $c : [a, b] \to M$ is called a *trajectory* of the control system if there exists a *control* $c^U : [a, b] \to N$ in some space $\mathcal{A}$ of admissible controls satisfying: $pc^U = c$

and $\frac{d}{dt}c(t) = h(c^U)$. Assume that h is smooth so that the system C_h satisfies existence and uniqueness [**29**]. Then the following end point map is well defined and continuous. Given $t > 0$, define $f : M \times U \to M$ by $f(x_0, u) = c(t)$, where $c : [0,t] \to M$ is the trajectory of the system with *constant* control u and $c(0) = x_0$. Then D_f is a discrete time control system. Also for small enough t, the equilibrium pairs of D_f coincide with the ones of C_h. Therefore existence of equilibria of C_h can be established via Theorem 5.1 and their robustness via Theorem 5.3.

The system is called *controllable* if any state can be reached from any other state, i.e., for each pair of states $x, y \in M$ there are a trajectory $c : [a,b] \to M$ and a corresponding control $c^U \in \mathcal{A}$ such that $x = c(a), y = c(b)$. Now define $G : \mathcal{A} \to M$ as $G(c^U) = (c(a), c(b))$, the end points of the trajectory $c = pc^U$ corresponding to c^U.

Let $\mathcal{A}_h$ be the set of admissible controls $c^U \in \mathcal{A}$, $c^U : [a,b] \to N$, satisfying the above conditions. Let $\mathcal{A}'_h = \{\ c^U \in \mathcal{A}_h : c(a) \in \partial M$ or $c(b) \in \partial M\}$. Then $G : (\mathcal{A}_h, \mathcal{A}'_h) \to (M \times M, \partial(M \times M))$ is well defined.

Theorem 7.1. ***(Sufficient condition of controllability)*** *If*

$$G_* : H_{2n}(\mathcal{A}_h, \mathcal{A}'_h) \to H_{2n}(M \times M, \partial(M \times M)) = \mathbf{Q} \textit{ is non-zero}$$

then the continuous time system C_h is controllable.

Proof. By Corollary 3.6 G is onto. □

A similar condition is found in [**23**], where a boundary operator $l : AC([0,1], \mathbf{R}^n) \times L^\infty([0,1], \mathbf{R}^n) \to \mathbf{R}^p$ is considered instead of G. One of the conditions of controllability is $\deg l_0 \neq 0$, where l_0 is the restriction of l to some p-dimensional subspace and $\deg l_0$ its topological degree.

Similarly to the previous section we treat robust controllability via homotopies of h; however, we also assume that they are smooth.

Definition 7.2. Given a submanifold L of M. The system is called *robustly controllable from L* if there is $\varepsilon > 0$ such that for any map k smoothly ε-homotopic to h, for each $x \in L$ and each $y \in M$ there is $c^U \in \mathcal{A}_k$ such that $G(c^U) = (x, y)$.

Just as in Section 6 we make two assumptions about the behavior of D_f. First, $f(\partial M \times U) \subset \partial M$, i.e., the trajectories never leave the boundary ∂M. This happens when h is tangent to ∂M. Second, $f(M \times U') \subset \partial M$, i.e., inputs from U' always take the system to ∂M. This happens, for example, when $h(x,u)$ is large for all $x \in M$ and $u \in U'$. Thus f is a map of pairs, $f : (M, \partial M) \times (U, U') \to (M, \partial M)$.

Let $L' = L \cap \partial M$ and $f' : (L, L') \times (U, U') \to (M, \partial M)$ be the restriction of f.

Theorem 7.3. ***(Sufficient condition of robust controllability)*** *Suppose that there are $a_0 \in H_*(L, L')$, $v_0, ..., v_s \in H_*(U, U')$ such that*

$$a_1 = f'_*(a_0 \otimes v_0), a_2 = f_*(a_1 \otimes v_1), ..., a_s = f_*(a_{s-1} \otimes v_{s-1}) \in H_*(M, \partial M),$$

where $a_s = O_M$, the fundamental class of M. Then the continuous time system C_h is robustly controllable from L.

Proof. Suppose M' is a collar of ∂M in M. Then $f : (M, \partial M) \times (U, U') \to (M, M')$ is well defined. Also, for a small enough $\varepsilon > 0$ a smooth ε-homotopy of h produces a homotopy $H : [0,1] \times (M, \partial M) \times (U, U') \to (M, M')$ of f. Therefore without loss of generality we can assume that if k is ε-homotopic to h with small enough ε, then the corresponding end point map g of C_k is homotopic to f as a map of pairs, $g : (M, \partial M) \times (U, U') \to (M, \partial M)$. By Theorem 6.2 D_g is controllable. Therefore C_k is controllable. □

8. Robust stability.

Consider the Multivariable Nyquist Criterion of robust stability [**17**, Chapter 2]. Let Ω be the set of frequencies and D the set of uncertainties (perturbations). Let L be the transfer map. Assume that the closed loop system is stable for some $\Delta_0 \in D$. The Nyquist map $f : \Omega \times D \to \mathbf{C}$ is defined by $f(\omega, \Delta) = \det(I + L(j\omega)\Delta)$. However if there are open loop poles on the imaginary axis, f does not exist in this form. Under these circumstances the Nyquist map $\overline{f} : \Omega \times D \to \mathbf{S}^2$ is defined by $\overline{f} = \pi_s f$, where $\pi_s : C \to \mathbf{S}^2$ is the stereographic projection, and $\overline{f}(0, \Delta) = \infty$. It is continuous. For the closed loop system to be robustly stable (i.e., $f(\cdot, \Delta)$ is stable for all $\Delta \in D$) we must have $f(\omega, \Delta) \neq 0$ for all $\omega \in \Omega, \Delta \in D$. In particular, $\overline{f}$ cannot be surjective. Thus in light of Corollary 3.6, we have the following.

Theorem 8.1. ***(Necessary condition of robust stability)*** *If the closed loop system is robustly stable then*

$$\overline{f}_* : H_2(\Omega \times D) \to H_2(\mathbf{S}^2) = \mathbf{Q} \text{ is zero.}$$

In general, robust stability criteria consider “mapping of the uncertainty into a ‘performance evaluation’ space... and checking whether the image is in the correct subset” [**17**, p. 20]. This opens more possibilities for applications of results in Section 3.

Corollary 3.6 can be applied to other problems of robust stability discussed in [**17**]. In particular, the condition $f_* : H_1(\Omega \times D) \to H_1(\mathbf{S}^1) = \mathbf{Q}$ is zero replaces the requirement that the degree of the Nyquist map restricted to the 1-skeleton of $\Omega \times D$ is zero [**17**, Chapter 18].

Corollary 4.3 implies the following.

Theorem 8.2. ***(Stability of the Nyquist map)*** *Suppose Ω and D are manifolds. Let B be a disk neighborhood of $0 \in \mathbf{C}$. Suppose*

$$f_* : H_2(\Omega \times D, f^{-1}(\partial B)) \to H_2(B, \partial B) = \mathbf{Q} \text{ is zero.}$$

Then f is unstable in the sense that for any $\varepsilon > 0$ there is $g : K \to D$, where K is a neighborhood of the ”crossover” $f^{-1}(0)$ in $\Omega \times D$, such that $||f - g|| < \varepsilon$, $g|_{\partial K} = f|_{\partial K}$, and $g^{-1}(0) = \varnothing$.

9. Appendix: Preliminaries from algebraic topology.

In this paper we mostly follow Bredon [**2**]. Given topological spaces $N, M, A \subset N, B \subset M$, and a map $f : (N, A) \to (M, B)$ between them, let $H_k(N, A)$, $H_k(M, B)$, $k = 0, 1, 2...$, be the *homology groups* of M, N over $\mathbf{Q}$ or any other field, $H^k(N, A)$, $H^k(M, B)$, $k = 0, 1, 2...$, be the *cohomology groups* of M, N, $f_* : H_k(N, A) \to H_k(M, B)$ the *homology homomorphism* generated by f, and $f^* : H^k(M, B) \to H^k(N, A)$ the *cohomology homomorphism.* Also $H_k(M) = H_k(M, \varnothing)$.

If M is path connected, $H_0(M) = H^0(M) = \mathbf{Q}$. The generators of these groups are denoted by 1. If M is contractible, $H_k(M) = H^k(M) = 0$ for $k > 0$. If M is a compact orientable n-dimensional manifold with boundary ∂M then $H_n(M, \partial M) = H^n(M, \partial M) = \mathbf{Q}$, generated by the fundamental classes O_M and $\overline{O}_M$ of M, respectively. If $z \in H_k(M, B)$ then its degree is $|z| = k$.

Two maps $f, g : (N, A) \to (M, B)$ are called *homotopic,* $f \sim g$, if f can be continuously "deformed" into g, i.e., there is a map $F : [0,1] \times (N, A) \to (M, B)$ such that $F(0, \cdot) = f$ and $F(1, \cdot) = g$. An ε-homotopy is one satisfying $d(F(t, x), F(0, x)) < \varepsilon$. If f and g are homotopic then $f_* = g_*$. If f is homotopic to a constant map then f_* is trivial, i.e., $f_* : H_k(M) \to H_k(N)$ is zero for $k = 1, 2, ...$

By the Künneth Theorem, $H_k(M \times U) = \sum_{i+j=k} H_i(M) \otimes H_j(U), k = 0, 1, 2, ...$ The *cap product* is the homomorphism $\frown: (H^*(M) \otimes H_*(M))_k \to H_k(M)$ given on the chain level by $f \frown c = (1 \times f)\Delta c$, where Δ is a diagonal approximation. Then $a \in H_k(M)$ and $x \in H^k(M)$ are called *dual* if $x \frown a = 1$. In particular, O_M and $\overline{O}_M$ are dual.

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