Stability Criteria for Time-Periodic Systems via High-Order Averaging Techniques

Andrey Sarychev

Department of Mathematics, University of Aveiro, Portugal

Abstract. We study stability and asymptotic stability for time-periodic systems described by ODE's, in particular for systems with fast oscillating parameters. Since the discovery of stabilizing effect of vibration in the reverse pendulum example, there was much study regarding stability of such systems and design of fast-oscillating stabilizing feedback laws.

The approach we suggest is kind of high-order averaging procedure. It is based on a formalism of chronological calculus - a method of asymptotic analysis for flows generated by time-variant ODE. We apply this approach to study stability properties of linear and nonlinear systems. In particular we derive conditions of stability for reduced second-order linear ODE with periodic fast-oscillating coefficients, study the stability of reversed pendulum with fast oscillating suspension point, consider high-order averaging procedure for nonlinear systems under homogeneity assumptions.

1 Introduction and problem setting

We study averaging techniques and stability and stability properties for periodically time-varying systems $\dot{x} = f(t,x)$. In particular we deal with *fast oscillating* systems of the form

$$\dot{x} = f(\varepsilon^{-1}t, x, \varepsilon), \tag{1}$$

where $\varepsilon > 0$ is a small parameter and $f(t,x,\varepsilon)$ is 1-periodic in t, $f(t,0,\varepsilon) = 0, \ \forall t, \varepsilon$. For our objectives it suffices that $f(\tau,x,\varepsilon)$ is 'integrable' with respect to t and C^k-smooth (with k sufficiently large) with respect to x. To simplify the presentation we will consider $f(\cdot,\cdot,\cdot)$ to be continuous and $f(\tau,\cdot,\cdot)$ to be *real-analytic*. This will allow us to talk about converging series instead of asymptotic ones.

Differential equations with fast oscillating terms have been studied and averaging techniques have been developed since 18th century. What for the stability issues for periodic differential equations then substantial contributions have been already made by A.M.Lyapunov; further contributions have been made by O.Perron, N.Chetaev, I.Malkin, N.Erugin, E.Barbashin, R.Bellman, N.Krasovsky, J.LaSalle, J.Massera, L.Cesari among others.

In control theory and mechanics the discovery of stabilizing effect of vibration in the reverse pendulum example inspired study of time varying feedback laws. This area became much more active after discovery of obstructions (e.g.

R.Brockett's criterion) to time-invariant stabilizability and especially after the J.-M.Coron's ([6]) established general result on time-variant stabilizability for controllable nonlinear control systems.

In this paper we introduce high-order averaging method for time-varying systems with the objective of studying their stability properties. This method is useful for treating critical and nearly critical cases and is based on some results of *chronological calculus* - technique of asymptotic expansions for the flows generated by time-varying nonlinear vector fields. This calculus has been developed by A.A.Agrachev and R.V.Gamkrelidze in 70's ([1]).

We introduce the notion of *complete averaging* of a time-periodic system as a *logarithm* of the corresponding *monodromy map* (diffeomorphism). The notion of the *logarithm* of a flow appeared in [1]. In linear case it can be computed via ordinary matrix logarithm. In nonlinear case the logarithm of a diffeomorphism P is a time-invariant vector field Λ such that P is included into the flow of $\Lambda: \ e^{\Lambda t}\mid_{t=1} = P$.

The complete averaging can always be constructed as a formal Lie series; its terms are computed via Lie brackets of the vector fields $f(t, \cdot)$ for distinct t's. Its truncations correspond to *high-order averagings* of the system. The computation of these high-order averagings for an arbitrary time-varying vector field is technically rather involved. In Section 2.2 we collect the main constructions of *chronological algebras* (see [1–3] for details) utilized for this computation and provide formulae for the first three terms of the expansion of the logarithm. We also derive asymptotic stability of equilibrium for time-periodic system is implied by its asymptotic stability for the complete averaging.

The high-order averaging technique allows obtaining new (up to our knowledge) stability results for time-varying linear systems $\dot{x} = A(\varepsilon^{-1}t)x$ with fast oscillating coefficients. In particular we study stability issues for the reduced second-order linear differential equations with periodic fast-oscillating coefficients. We also study stability of reverse pendulum with oscillating point of suspension.

Nonlinear case is usually much more complex for treatment by most part of the techniques. In the scope of our approach we manage to handle the nonlinear case similarly to the linear one. We illustrate our method by establishing a local asymptotic stability condition for an equilibrium of time-periodic nonlinear vector field.

An obvious alternative to the introduced approach is the one which makes use of time-varying Lyapunov functions. One can find many interesting results of this kind in a series of publications by D.Aeyels, J.Peuteman, L.Moreau (see [12,10] and references therein).

We shall study the problem of asymptotic stability and/or of stability for the zero solution of such a system for small values of $\varepsilon > 0$. We study three types of stability properties, which can be formulated respectively as:
there exists $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$ the zero solution is

locally asymptotically stable (property $\mathbf{S}_{as}$*)*; respectively is *stable (property* $\mathbf{S}_0$*)* ; respectively is *unstable (property* $\mathbf{S}_{us}$*)*.

2 Method: high-order averaging for time-periodic systems - definition and computational tools.

2.1 High-order averagings for systems with fast-oscillating parameters

Although the problem has been set for nonlinear systems, we will first outline the method for linear systems with periodic coefficients. Obviously the notions of global and local asymptotic stability coincide for linear systems.

Consider the system

$$(dx/dt =)\,\dot{x}(t) = A(t/\varepsilon)x(t), \tag{2}$$

where $x \in \mathbf{R}^n$ or $\mathbf{C}^n$, $A(\tau)$ is $\mathbf{R}^{n\times n}$ or $\mathbf{C}^{n\times n}$-valued function continuous and 1-periodic with respect to τ, $\varepsilon > 0$ is a small parameter. Below $\mathbf{C}^-, \mathbf{C}^+$ are open left and right complex half-planes respectively.

What one can say about asymptotic stability of such a system?

An explicit but partial answer to this question is provided by the standard averaging result: *if all the eigenvalues of the corresponding averaged matrix* $\int_0^1 A(\tau)d\tau$ *are located in* $\mathbf{C}^-$*, then the system (2) is asymptotically stable for all sufficiently small* $\varepsilon > 0$. The question which persists is what happens in the critical cases where some of the eigenvalues of this averaged matrix are located on the imaginary axis (for example vanish)?

A complete but hardly verifiable stability criterion can be formulated in terms of *monodromy matrix* of the system. Introducing fast time variable $\tau = t/\varepsilon$ we arrive to the system $dx/d\tau = \varepsilon A(\tau)x(\tau)$ with 1-periodic coefficients. Consider the corresponding matrix differential equation

$$dX/d\tau = \varepsilon A(\tau)X(\tau),\ X \in \mathbf{R}^{n\times n} \text{ or, } X \in \mathbf{C}^{n\times n} \tag{3}$$

and the monodromy matrix $M_\varepsilon = X(1)$.

If all the eigenvalues of M_ε *(multiplicators) are located in the interior of the unit circle, then the system is asymptotically stable. If at least one of these eigenvalues lies outside the unit circle, then the system is unstable.*

The principal difficulty in utilizing this result is impossibility to compute the spectra of the matrix M_ε. The only feasible idea is dealing with expansion of M_ε in a power series with respect to ε.

Instead of M_ε one can deal with its *logarithm* [1] Λ_ε. The corresponding stability criterion becomes:

[1] The logarithm is a more suitable object. As it will be seen the notion can be generalized onto nonlinear case, where the monodromy map is a nonlinear diffeomorphism. In this latter case the 'logarithm' of this map, (if exists!) is a vector field - an element of a Lie algebra with natural linear structure suitable for expansions into series.

the system (2) (with fixed $\varepsilon > 0$) is asymptotically stable if all the eigenvalues of the logarithm Λ_ε are located in $\mathbf{C}^-$ and unstable if any of them is in $\mathbf{C}^+$.

The matrix Λ_ε admits a power series representation:

$$\Lambda_\varepsilon = \varepsilon \Lambda^{(1)} + \varepsilon^2 \Lambda^{(2)} + \cdots . \tag{4}$$

(It is easy to see that $\Lambda^{(1)} = A^1 = \int_0^1 A(\tau)d\tau$.)

Obviously the logarithm Λ_ε of the monodromy matrix M_ε provides a complete information about turnpike or averaged behaviour of the solution of the system (2). According to Floquet theorem this solution 'oscillates around' the trajectory $e^{t\Lambda_\varepsilon}$ of the system $\dot{X} = \Lambda_\varepsilon X, \ X(0) = I$. Therefore Λ_ε can be seen as 'complete averaging' of the system (3), while the truncations of its expansion can be seen as partial averagings of the system.

A major problem remains computation of the expansion for Λ_ε. In the Section 2.2 we will proceed with this computation in nonlinear setting.

Let us consider now the nonlinear system

$$\dot{x} = f(\varepsilon^{-1}t, x),$$

where $f(\tau, x)$ is 1-periodic with respect to τ. We assume $f(t, 0) = 0, \ \forall t$. Putting $\tau = t/\varepsilon$ we come to the equation $dx/d\tau = \varepsilon f(\tau, x)$. Let P_ε^τ be the corresponding flow - the family of diffeomorphisms satisfying the operator differential equation

$$dP_\varepsilon^\tau / d\tau = P_\varepsilon^\tau \circ \varepsilon f(\tau, \cdot), \ P_\varepsilon^0 = I.$$

Sometimes we will denote such a flow by $P_\varepsilon^\tau = \overrightarrow{\exp} \int_0^\tau f(\theta, \cdot)d\theta$ and call it, following [1], *right chronological exponential*. The diffeomorphism $M_\varepsilon = P_\varepsilon^1$ is a *time*-1-*map* or a *monodromy map*. Obviously $P_\varepsilon^{\tau+1} = M_\varepsilon \circ P_\varepsilon^\tau$ and the origin is a fixed point of the map M_ε.

The fact of time-invariant vector field Λ_ε being a logarithm of the diffeomorphism M_ε means that M_ε is included in the flow $e^{\Lambda_\varepsilon t}$ of this vector field: $M_\varepsilon = e^{\Lambda_\varepsilon t}\big|_{t=1} = e^{\Lambda_\varepsilon}$. Again we introduce the expansion (4) of Λ_ε into the power series (where $\Lambda^{(i)}$ are now time-invariant vector fields) and try to proceed in a similar way as in the linear case.

There are additional difficulties on this way. In contrast to the linear case *the logarithm Λ_ε may fail to exist.* Indeed as it is known the exponential map of the infinite-dimensional Lie algebra $\mathrm{Vect}\mathbf{R}^n$ of real analytic vector fields into the infinite-dimensional Lie group of real analytic diffeomorphisms $\mathrm{Diff}\mathbf{R}^n$ is *not locally onto.* This obstacle can be overcome. This exponential map turns out to be 'locally onto in a given direction':

for any 1-periodic real analytic vector field $f(\tau, x)$, $\exists \varepsilon_0 > 0$ such that for any positive $\varepsilon < \varepsilon_0$ logarithm Λ_ε of the diffeomorphism $\overrightarrow{\exp} \int_0^1 \varepsilon f(\theta, \cdot)d\theta$ exists.

In any case one is able to construct a formal power series (4) for Λ_ε, and even if this series diverges, still the exponentials of its truncations provide nice asymptotics for the diffeomorphism M_ε (see [1]).

Along this contribution we assume the logarithm Λ_ε to exist; obviously the origin is an equilibrium point for the vector field Λ_ε.

Coming back to the stability issue we formulate the following proposition.

Proposition 1 *The origin O is asymptotically stable for the system if it is an asymptotically stable fixed point for the map M_ε, i.e. $\forall \Delta > 0\ \exists \delta > 0$ such that $\rho(x, O) < \delta \Rightarrow \rho(M_\varepsilon^n(x), O) < \Delta$ and $\rho(M_\varepsilon^n(x), O) \to 0$.*

Thus the question is reduced to the one of stability of the origin for the diffeomorphism M_ε. Passing to the logarithm Λ_ε of the diffeomorphism M_ε or, just the same, to the complete averaging of the system we conclude.

Theorem 2 *The equilibrium point O of the system $\dot{x} = f(\varepsilon^{-1}t, x)$ is asymptotically stable if it is asymptotically stable equilibrium point for Λ_ε.*

As in linear case one can not conclude automatically property $\mathbf{S}_{as}$ for a nonlinear system basing on the stability properties of a given truncated series $\sum_{i=1}^N \varepsilon^i \Lambda^{(i)}$. We illustrate the phenomenon by the following example.

Example. Consider a system

$$\dot{x}_1 = -x_1 + x_2 \sin t/\varepsilon,\ \dot{x}_2 = x_1 \cos t/\varepsilon + x_2^3.$$

The first-order averaging of this system - the first term of the expansion of the corresponding logarithm - is the vector field $\Lambda^{(1)} = -x_1\partial/\partial x_1 + x_2^3\partial/\partial x_2$. The origin is an *unstable* equilibrium for $\Lambda^{(1)}$. Still in contrast to this fact *the system is asymptotically stable*, for sufficiently small $\varepsilon > 0$.

To prove the asymptotic stability we may consider the system as a perturbation of its linear part by the nonlinearity $x_2^3\partial/\partial x_2$. We will establish in the next section (see Example 2) that for a sufficiently small $\varepsilon > 0$ the linear part is asymptotically stable, i.e. spectrum of the corresponding averaging Λ_ε is in the open left half-plane. Since the linear system is time-periodic, it is reducible and hence *Lyapunov regular*, i.e. the sum of the characteristic numbers equals to the average of the trace of its coefficient matrix. Then by a standard result of Lyapunov the nonlinear system has an asymptotically stable equilibrium at the origin.

2.2 Computation of the high-order averagings

The complete averaging can be computed via 'formal Lie series'. The computation of the logarithm of a flow has been accomplished in [1]); in [3] another method of its computation has been provided. This latter is based on so called *chronological product* of time-variant vector fields (see [2,3]).

Definition 3 *For two time-varying vector fields $X_t(x), Y_t(x)$, which are absolutely continuous with respect to t, the chronological product $(X_. * Y_.)_.$ equals:*

$$(X_. * Y_.)_t = \int_0^t \left[X_\tau, \frac{\partial}{\partial \tau} Y_\tau\right] d\tau, \tag{5}$$

where $[\cdot,\cdot]$ stays for the Lie bracket of two vector fields.

This is a nonassociative product, satisfying the *chronological identity*: $X * (Y * Z) - Y * (X * Z) = (X * Y - Y * X) * Z$.

(In their recent paper [8] M.Kawski and H.J.Sussmann defined (right) chronological algebra by means of identity $u \cdot (v \cdot w) = (u \cdot v) \cdot w + (v \cdot u) \cdot w$.)

We will need another operation in the space of integrable functions:

$$(u \star v)(t) = (u \cdot v)(t) - (v \cdot u)(t) = v(t) \int_0^t u(\tau)d\tau - u(t) \int_0^t v(\tau)d\tau. \quad (6)$$

Using the chronological product (5) and the product (6) one is able to compute the expansion $\sum_{i=1}^{\infty} \varepsilon^i \Lambda^{(i)}$ of the logarithm Λ_ε of the diffeomorphism $M_\varepsilon = \overrightarrow{\exp} \int_0^\tau \varepsilon X_\theta d\theta$. The first terms of this series (expressed in terms of Lie brackets) are

$$\Lambda^1(X_\cdot) = \int_0^1 X_t dt, \qquad \Lambda^2(X_\cdot) = \frac{1}{2} \int_0^1 \left[\int_0^{t_1} X_{t_2} dt_2, X_{t_1} \right] dt_1, \quad (7)$$
$$\Lambda^3(X_\cdot) = -\frac{1}{2} \left[\Lambda^1(X_\cdot), \Lambda^2(X_\cdot) \right] + \frac{1}{3} \int_0^1 ad^2 \left(\int_0^{t_1} X_{t_2} dt_2 \right) X_{t_1} dt_1.$$

Here $\mathrm{ad}X\ Y = [X, Y]$, $\mathrm{ad}^i X\ Y = [X, \mathrm{ad}^{i-1} X\ Y]$, $i \geq 2$.

For a particular case of linear vector fields $A_1 x$ and $A_2 x$ the Lie bracket is a linear vector field equal to $[A_1 x, A_2 x] = -[A_1, A_2]x$, where $[A_1, A_2]$ is the commutator of matrices A_1 and A_2. Hence in the linear case the terms $\Lambda^{(i)}(A_\cdot)$ in the expansion (4) are (if we suppress x) matrices; to compute them one has to substitute $X_\cdot$ by $A_\cdot$ in (7) and multiply Lie monomials of even order by (-1).

We will now outline the method of computation of $\Lambda^{(i)}$.

Consider the right-chronological exponent $\Phi_\varepsilon^\tau = \overrightarrow{\exp} \int_0^\tau \varepsilon X_t dt$. Obviously $M_\varepsilon = \Phi_\varepsilon^1$. Differentiating Φ_ε^τ with respect to ε we obtain (see [3]):

$$(\partial/\partial\varepsilon)\Phi_\varepsilon^\tau = Z_\varepsilon^\tau \circ \Phi_\varepsilon^\tau,$$

where the Z_ε^τ is time-varying vector field depending absolutely continuously on time τ; for $\varepsilon = 0$, $Z_0^\tau = \int_0^\tau X_t dt$. The vector field Z_ε^τ has been called in [3] 'angular velocity' of the flow; Z_ε^τ satisfies the equation

$$(\partial/\partial\varepsilon) Z_\varepsilon^\tau = (Z_\varepsilon^\cdot * Z_\varepsilon^\cdot)^\tau = \int_0^\tau \left[Z_\varepsilon^\theta, \frac{\partial}{\partial\theta} Z_\varepsilon^\theta \right] d\theta, \quad (8)$$

where $*$ is the chronological product introduced in the Definition 3. The equation (8) allows us to compute recursively the terms of the expansion $Z_\varepsilon^\tau = Z_0 + \sum_{k=1}^{\infty} Z_{(k)}^\tau (\varepsilon^k/k!)$.

Various methods for computation of $Z_{(k)}^\tau$ were introduced in [3]. For example considering the free chronological algebra with a generator λ and a derivation D defined by: $D\lambda = \lambda * \lambda$ one can construct a sequence of polynomials with respect to the product $*$: $p_1(\lambda) = \lambda,\ p_{i+1}(\lambda) = Dp_i(\lambda),\ i \geq 1$. Then one can compute $Z^{\tau}_{(k)}$ as

$$Z^{\tau}_{(k)} = p_{k+1}\left(\int_0^{\cdot} X_t dt\right). \tag{9}$$

We yet need another construction to compute the terms of the expansion for the logarithm Λ_ε. To this end we differentiate the equality $e^{\Lambda_\varepsilon} = M_\varepsilon = \overrightarrow{\exp} \int_0^1 \varepsilon X_t dt$ with respect to ε, obtaining (see [1] for the details):

$$\int_0^1 e^{\tau \mathrm{ad}\Lambda_\varepsilon} d\tau \frac{\partial \Lambda}{\partial \varepsilon} \circ e^{\Lambda_\varepsilon} = Z^1_\varepsilon \circ \overrightarrow{\exp} \int_0^1 \varepsilon X_t dt.$$

and consequently

$$\int_0^1 e^{\tau \mathrm{ad}\Lambda_\varepsilon} d\tau \frac{\partial \Lambda}{\partial \varepsilon} = Z^1_\varepsilon.$$

Consider the function $\int_0^1 e^{\tau\xi} d\tau = (e^\xi - 1)/\xi$ and denote by $\phi(\xi) = \xi/(e^\xi - 1)$. Then the latter equality is equivalent to the formal equation

$$\frac{\partial \Lambda}{\partial \varepsilon} = \phi\left(\mathrm{ad}\ (\Lambda_\varepsilon)\right) Z^1_\varepsilon. \tag{10}$$

To obtain the expansion for Λ_ε from this latter equation we involve Taylor expansion $\phi(\xi) = \sum_{i=1}^n B_i(\xi^n/n!)$, where B_i are known to be Bernoulli numbers. The first Bernoulli numbers are: $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_{2k+1} = 0,\ k > 0,\ B_4 = -1/30, B_6 = 1/42, B_8 = -1/30$. From (9), (10) we derive the expressions for $\Lambda^{(i)}$ and in particular the expressions (7).

3 Applications: high-order averaging and stability properties

3.1 Asymptotic stability for linear systems: two examples

Here we provide two simple examples of linear systems, whose stability properties are not determined by the standard averaging but can be established on the basis of high-order averaging(s).

Example 1. Consider (a pair of) linear system(s)

$$\dot{x}_1 = -x_1 \pm x_2 \sin(2\pi/\varepsilon)t,\ \dot{x}_2 = x_1 \cos(2\pi/\varepsilon)t. \tag{11}$$

Its first-order averaging $\Lambda^{(1)} = \mathrm{diag}\{-1, 0\}$ is 'critical' - one of its eigenvalues vanishes. Computing $\Lambda^{(2)}$ we obtain

$$\Lambda_\varepsilon = \varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} + o(\varepsilon^2) = \begin{pmatrix} -\varepsilon \pm \varepsilon^2/4\pi & \mp\varepsilon^2/2\pi \\ 0 & \mp\varepsilon^2/4\pi \end{pmatrix} + o(\varepsilon^2).$$

Its trace equals $-\varepsilon$, while its determinant equals $\pm\varepsilon^3/4\pi+o(\varepsilon^3)$. The system (11) possesses property $\mathbf{S}_{as}$ provided that the sign '+' is chosen in the first equation and possesses property $\mathbf{S}_{us}$ otherwise. □

Example 2. The averaging of the system

$$\dot{x}_1=-x_1+\alpha x_2\sin(2\pi/\varepsilon)t,\ \dot{x}_2=x_1\cos(2\pi/\varepsilon)t+\varepsilon x_2/3 \tag{12}$$

is unstable for each $\alpha:\ \Lambda^{(1)}=\mathrm{diag}\{-1,\varepsilon/3\}$. Computing the terms of orders ε and ε^2 in the corresponding logarithm we obtain

$$\varepsilon\Lambda^{(1)}+\varepsilon^2\Lambda^{(2)}=\begin{pmatrix}-\varepsilon+\alpha\varepsilon^2/4\pi & b\\ 0 & \varepsilon^2/3-\alpha\varepsilon^2/4\pi\end{pmatrix}$$

conclude that the original system (12) possesses property S_{as}, provided that $\alpha>4\pi/3$, and possesses property $\mathbf{S}_{us}$, if $\alpha<4\pi/3$. □

3.2 Reduced second-order differential equation

Let us consider the reduced second-order equation

$$\ddot{x}+p(t/\varepsilon)x=0. \tag{13}$$

Here $q(t)\equiv 0$ and according to Liouville theorem zero solution of this equation is not asymptotically stable. From a standard averaging result we deduce that the stability property S_0 holds if $\bar{P}>0$ and the property $\mathbf{S}_{us}$ holds if $\bar{P}<0$. [2]

Much more interesting is the case where $\bar{P}=0$. Here three terms of the expansion for Λ_ε are needed for coming to a conclusion about stability. Indeed

$$\varepsilon\Lambda^{(1)}+\varepsilon^2\Lambda^{(2)}=\begin{pmatrix}\varepsilon^2P^{(1)} & \varepsilon\\ 0 & -\varepsilon^2P^{(1)}\end{pmatrix},$$

where $P^{(1)}=\int_0^1 tp(t)dt$. This matrix is trace-free while its determinant is negative. Still one can not conclude the instability property $\mathbf{S}_{us}$, because the determinant has order of smallness $O(\varepsilon^4)$ and the term of order ε^3 in the lower-left corner may affect the sign of the determinant. It does affect it in a radical way; as we will see in a minute the determinant of the matrix $\varepsilon\Lambda^{(1)}+\varepsilon^2\Lambda^{(2)}+\varepsilon^3\Lambda^{(3)}$ is always positive provided that $p(\tau)\not\equiv 0$.

Indeed direct computation involving the formulae (7) gives us

$$\varepsilon\Lambda^{(1)}+\varepsilon^2\Lambda^{(2)}+\varepsilon^3\Lambda^{(3)}=\begin{pmatrix}\varepsilon^2P^{(1)} & \varepsilon+O(\varepsilon^3)\\ -\varepsilon^3\int_0^1(\int_0^t p(\tau)d\tau)^2dt & -\varepsilon^2P^{(1)}\end{pmatrix}. \tag{14}$$

[2] The famous Lyapunov's integral criterion for stability of the second-order periodic equation must be mentioned. It regards coefficient $p(\cdot)$ of *arbitrary* period but under additional assumption: $p(\cdot)\geq 0$. We deal with $p(\cdot)$ of small period but with no sign assumption. For nonnegative $p(\cdot)$ of period ε the Lyapunov's criterion requires $0<\varepsilon\bar{P}\leq 4$ for stability. It holds trivially for sufficiently small $\varepsilon>0$.

Hence the determinant of Λ_ε equals $\varepsilon^4\left(-(P^{(1)})^2+\int_0^1(\int_0^t p(\tau)d\tau)^2dt\right)+o(\varepsilon^4)$. Integration by parts transforms $P^{(1)}=\int_0^1 tp(t)dt$ into $-\int_0^1\left(\int_0^t p(\tau)d\tau\right)dt$ (recall that $\bar{P}=\int_0^1 p(t)dt=0$). By virtue of Cauchy-Schwarz inequality

$$(P^{(1)})^2=\left(\int_0^1(\int_0^t p(\tau)d\tau)dt\right)^2\leq\int_0^1(\int_0^t p(\tau)d\tau)^2dt\int_0^1 dt.$$

Hence the determinant of Λ_ε is positive provided that $p(\tau)\not\equiv 0$ and $\varepsilon>0$ is sufficiently small. We have established the following property.

Theorem 4 *If $\bar{P}=0$, then the reduced second order equation (13) with fast-oscillating coefficient $p(t/\varepsilon)$ is not asymptotically stable. If $p(\tau)\not\equiv 0$, then the property $\mathbf{S}_0$ holds for it, i.e. the zero solution is stable if the rate of oscillation ε^{-1} is sufficiently large.*

3.3 Stabilization of equilibrium of reverse pendulum

As it is well known the upper position of a pendulum - the *reverse pendulum* - can be made stable if the suspension of the pendulum is subject to (sufficiently) fast harmonic oscillation (see [5]).

Here we shall treat the case where the *suspension is subject to fast oscillation of arbitrary form* $\delta s(kt)$ and will derive conditions for it to stabilize the upper position of the pendulum. We assume $\delta>0$ to be a small and $k>0$ to be a large parameter respectively, and $s(\tau)$ to be $1-$periodic C^2-function. Small oscillations of the pendulum in a neighborhood of upper equilibrium point are described by the equation

$$\ddot{x}=\left(\omega^2+\delta k^2\ddot{s}(kt)\right)x \tag{15}$$

where ω is proper frequency of the pendulum. Without loss of generality we may assume $\dot{s}(0)=0$. Putting $\dot{x}=y$ and proceeding with time substitution $\tau=kt$ we can rewrite the equation (15) as a system

$$dz/d\tau=\left(k^{-1}A+\delta kB_\tau\right)z, \tag{16}$$

where

$$z=(x,y)^T,\ A=\begin{pmatrix}0&1\\\omega^2&0\end{pmatrix},\ B_\tau=\begin{pmatrix}0&0\\\ddot{s}(\tau)&0\end{pmatrix}.$$

Invoking the variational formula of the chronological calculus (see [1] or [4]) we derive the formula (see the subsection 2.2 for the notation):

$$\overrightarrow{\exp}\int_0^1\left(k^{-1}A+\delta kB_\tau\right)d\tau=\overrightarrow{\exp}\int_0^1 C_\sigma d\sigma\circ e^{\delta k\int_0^1 B_\tau d\tau},$$

where C_σ is defined by the first equality in (17).

The second factor of the composition in the right-hand side is the identity. Indeed $\int_0^1 \ddot{s}(\tau)d\tau = \dot{s}(1) - \dot{s}(0) = 0$ and hence $\int_0^1 B_\tau d\tau = 0$. What for the first factor then direct computation gives us

$$C_\sigma = e^{\delta k \mathrm{ad} \int_0^\sigma B_\tau d\tau} k^{-1} A = k^{-1} A + \delta \mathrm{ad}\left(\int_0^\sigma B_\tau d\tau\right) A + \frac{\delta^2 k}{2} \mathrm{ad}^2\left(\int_0^\sigma B_\tau d\tau\right) A, \quad (17)$$

(the series for the exponential terminates on the second term due to nilpotency of B_τ).

Direct computation gives us

$$\mathrm{ad}\left(\int_0^\sigma B_\tau d\tau\right) A = \mathrm{diag}\{-\dot{s}(\sigma), \dot{s}(\sigma)\}, \ \mathrm{ad}^2(\int_0^\sigma B_\tau d\tau) A = \begin{pmatrix} 0 & 0 \\ -2\dot{s}^2(\sigma) & 0 \end{pmatrix}.$$

Therefore

$$C_\sigma = \begin{pmatrix} -\delta \dot{s}(\sigma) & k^{-1} \\ k^{-1}\omega^2 - \delta^2 k \dot{s}^2(\sigma) & \delta \dot{s}(\sigma) \end{pmatrix}.$$

The matrix C_σ is 1-periodic in σ; taking the corresponding monodromy matrix $\overrightarrow{\exp} \int_0^1 C_\sigma d\sigma$ we obtain for its logarithm the expansion, which starts with the (first-order) averaging

$$A^{(1)} = \int_0^1 C_\sigma d\sigma = \begin{pmatrix} 0 & k^{-1} \\ k^{-1}\omega^2 - \delta^2 k \int_0^1 \dot{s}^2(\sigma) d\sigma & 0. \end{pmatrix} \quad (18)$$

What for the (first-order) rest term of the expansion of $\ln \overrightarrow{\exp} \int_0^1 C_\sigma d\sigma$, then it admits an estimate $O(\delta^2 + k^{-2})$, as $\delta \to 0, k \to \infty$. As far as the logarithm is a trace-free matrix, then the stability of the system is defined by (the sign of) its determinant. According to the above estimates the determinant equals

$$-k^{-2}\omega^2 + \delta^2 \int_0^1 \dot{s}^2(\sigma) d\sigma + O(\delta^4 + k^{-4}).$$

This leads us to stability condition for the reverse pendulum.

Theorem 5 *For each $\varepsilon > 0$ there exist $\delta_0 > 0, k_0 > 0$ such that the equilibrium of the reverse pendulum is stable provided that $0 < \delta < \delta_0, \ k > k_0$ and $\delta^2 \int_0^1 \dot{s}^2(\tau)d\tau \geq \frac{\omega^2}{k^2} + \varepsilon$ and is unstable provided that $0 < \delta < \delta_0, \ k > k_0$ and $\delta^2 \int_0^1 \dot{s}^2(\tau)d\tau \leq \frac{\omega^2}{k^2} - \varepsilon$.*

3.4 High-order averaging and stability for time-varying nonlinear systems

Let us show how the technique works in periodic nonlinear non-fast-oscillating case.

Assume $X(t,x)$ to be a nonlinear 1-periodic time-varying vector field in $\mathbf{R}^n$, continuous together with all its partial derivatives with respect to x_i's; $X(t,0) = 0, \ \forall t$.

We will need to introduce some homogeneity. To this end let us consider an n-tuple $r=(r_1,\ldots,r_n)\geq 0$ and the dilation

$$\delta_\varepsilon^r:\mathbf{R}^n\to\mathbf{R}^n:\ \delta_\varepsilon^r(x_1,\ldots,x_n)=(\varepsilon^{r_1}x_1,\ldots,\varepsilon^{r_n}x_n),\ \varepsilon>0.$$

We define the weight of a monomial vector field $x_1^{\alpha_1}\cdots x_n^{\alpha_n}\partial/\partial x_\beta$ as $-r_\beta+\sum_{k=1}^n\alpha_k r_k$. For the sake of brevity let us say that vector field is of weight $\geq s$ if all the monomials in its Taylor expansion with respect to x are of weights $\geq s$ (see [3] for 'more invariant' definition).

We assume $X(t,x)$ to be of weight $\geq s$ for some positive s. Let us take *Fourier expansion* $X^0(x)+\sum_{n=1}^\infty(X^n(x)\cos 2\pi nt+Y^n(x)\sin 2\pi nt)$ of $X(t,x)$. Obviously $X^0=\int_0^1 X(t,x)dt$ coincides with the first-order averaging of X and all the vector fields X^n,Y^n are of weights $\geq s$. In addition we *assume* X^0 *to be of weight* $\geq 2s$. This includes the case $X^0=0$ (arbitrary weight can be assigned to vanishing vector field). Let us represent $X^i,\ i\geq 0,\ Y^i,\ i\geq 1$, as $X^i=\tilde{X}^i+\cdots, Y^i=\tilde{Y}^i+\cdots$, where $\tilde{X}^0$ is homogeneous of weight $2s$, $\tilde{X}^i,\tilde{Y}^i$ are homogeneous of weight s and the rest terms are of bigger weights.

Theorem 6 *If the equilibrium is locally asymptotically stable for the time-invariant homogeneous vector field*

$$\tilde{X}^0(x)+\sum_{n=1}^\infty(4\pi n)^{-1}[\tilde{X}^n,\tilde{Y}^n] \tag{19}$$

then it is asymptotically stable for the system $\dot{x}=X(t,x)$.

Remark. This result generalizes the one obtained by R.M'Closkey and P.Morin in [9]. They proved that if X^0 is homogeneous of weight $2s>0$ and X^1,Y^1 are homogeneous of weight $s>0$, then the origin is asymptotically stable for the system $\dot{x}=X^0(x)+X^1(x)\cos 2\pi t+Y^1(x)\sin 2\pi t$, provided that it is asymptotically stable for the time invariant system $\dot{x}=X^0(x)+(4\pi)^{-1}[X^1,Y^1]$.

We outline the proof of the theorem 6. Take the time-1 map for $X(t,x)$. Calculating the first two terms of the expansion of its logarithm (the logarithm may not exist, but the expansion exists as a formal series!) we obtain $\Lambda^{(1)}=X^0(x),\ \Lambda^{(2)}=\sum_{n=1}^\infty(4\pi n)^{-1}[X^n,Y^n]$. Denote by $\tilde{\Lambda}^{(2)}=\tilde{X}^0+\sum_{n=1}^\infty(4\pi n)^{-1}[\tilde{X}^n,\tilde{Y}^n]$. Then $\Lambda=\Lambda^{(1)}+\tilde{\Lambda}^{(2)}+\cdots$, where the omitted terms are of weights $>2s$. Therefore the homogeneous vector field $\Lambda^{(1)}+\tilde{\Lambda}^{(2)}$, appearing in (19), is principal part of Λ. If the origin is asymptotically stable for (19) and Λ exists (i.e. the respective formal series converges) then the origin is asymptotically stable for Λ by virtue of Massera-Hermes theorem ([7]) and hence is asymptotically stable for $X(t,x)$. If Λ only exists as a formal series still we are able to prove that the vector field (19) determines the asymptotics and stability of the trajectories of the system $\dot{x}=X(t,x)$.

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