Analysis and design of oscillatory control systems
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Statement of revision

1 Summary

We thank the editor and the reviewers for their thoughtful comments and suggestions on how to improve the paper. After a careful consideration of them, we revised the paper according to the following criteria:

1. highlight in clearer details the manuscript's contribution with respect to previous works (both Referees 1, 4 an Editor),
2. streamline the presentation and diminish the number of formal statements (as requested by Referee 4 and Editor),
3. correct the mathematical imprecisions (Referee 1 and Editor).

In what follows, we provide a detailed description of our changes and of their rationale. The following statement of revision includes responses to all the referees' points.

We have also made a number of improvements throughout the paper and slightly altered the title in such a way as to better reflect our understanding of the content.

2 Manuscript's contribution

2.1 With respect to previous work (Referee 4)

We have tried to reach the request by Referee 4 to clarify the contribution of the paper with respect to previous work. In addition to the already existing subsection 'Statement of contributions' in Section I, we have included several sentences along Sections III, IV, V and VI highlighting the novel developments. These include,

1. Proposition III.1 establishes sufficient conditions for the pull-back vector field to be periodic, thus enabling to perform the averaging procedure. Note that, in related work such as [37,38,39], this issue is overlooked.
2. Theorem III.2 presents a coordinate-free statement of the first-order averaging theorem for two time scales. Although somehow similar results to (i) and (ii) are known in the literature on dynamical systems [17], they are always presented in a coordinate-dependent way, and flows and vector fields are commonly mixed. The contribution here is the precise formulation of the result for general control systems. On the other hand, the use of (multinomial) iterated integrals and the refinement in (iii) are new.
3. Theorem III.3 is completely new. No such description of the pull-back vector field and its average has been reported before in the literature. The use in the series expansions of the notion of (multinomial) iterated integrals, and formulas such as those proved in Lemma II.1 is also new.
4. In Section IV, Proposition IV.1 is an extension of a result in [23], Proposition IV.3 is an extension of the classical result on Kapitsa's potential for multiple input systems with two time scales and Lemma IV.4 is a novel contribution to the setting of nilpotent-type systems.
5. With regards to design (Sections V and VI), we make appropriate explicit references to previous work just before, in or after the corresponding statement of the results.

2.2 With respect to Sarychev's papers (Referee 1)

We agree with Referee 1 on the importance of the two missing references he/she provided on the work on Sarychev (which we quote now). However, we strongly believe that both works do not overlap and are indeed rather complementary. On the one hand, Sarychev studies, as the titles of his works clearly state, stability issues for high frequency time-varying systems (linear, second-order and third-order systems) with a single time scale. To do so, he makes use of nonlinear Floquet theory and the so-called *logarithm of the time 1-map*. In order to compute the latter, he resorts to some tools of Chronological Calculus, specifically he deals with the *chronological product of time-varying vector fields*. He calls this a 'kind of higher-order averaging procedure'. He also devotes a section to the stabilisation of the unicycle example by means of time-varying smooth feedbacks.

On the other hand, the present work deals with a different class of systems, namely, *high amplitude* high frequency control systems with *two* time scales. We confine ourselves to the setting of *first-order averaging* (whereas Sarychev studies higher-order averaging), and the coordinate-free treatment relies on the use of the *variation of constants formula* (not the chronological product) from Chronological Calculus. Among the several important consequences presented of the analysis results, we should highlight the fact that previous work on vibrational control (which is also reviewed along the paper) by Bellman, Bentsman and Meerkov, is naturally understood in the light of the coordinate-free treatment. These results also apply to a broad family of systems (homogeneous, bilinear, polynomial, second-order, Hamiltonian and nilpotent-like), allowing us to establish nice relationships with known facts in physical phenomena and interesting extensions of previous work, for instance, in oscillatory mechanical control systems. We also present both *stabilisation* and *trajectory tracking* results for *underactuated* systems, without restricting our attention to any specific example.

As we said before, we think that the two works are complementary. Of course, both can be placed in the field of geometric control theory and both of them share the interest in stability issues. However, the type of systems considered, the technical approach used, and the results presented are substantially different.

3 Presentation and organization

We have changed the organization of the analysis results of the paper in an effort to comply with the observation made by Referee 4 to summarize clearly the exposition of the main technical tools and provide the reader with the useful information right-away. We have also shortened the paper by two pages. A detailed list of the changes concerning can be found in the following section. They can be summarized as follows,

1. We have deleted former Section II.A. Now, the paper directly deals with iterated integrals of multiple functions.
2. In Section II.B, we have removed former Lemma II.5, which is now introduced in Section III, in the proof of Theorem III.2.
3. We have gathered former Theorem III.2 and Proposition III.3 in the new Theorem III.2, which now accounts for the main averaging result of the paper stated in a coordinate-free setting.
4. We have gathered all the results of former Section III.A in the new Theorem III.3. After presenting the proof, we remark that the series expansions take simpler forms in the single-input setting.

5. We have unified for each example the presentation of the results obtained, by merging together former Sections IV and V.

4 Detailed list of changes

Below, we provide a detailed list of the changes that have been made in the revised version according to the reviewers' comments.

Section I

1. Subsection entitled 'Literature review'. We do now appropriate reference to the work by Sarychev when talking about the analysis of high frequency vibrations in mechanical and other types of systems. (Referee 1)
2. Subsection entitled 'Statement of contributions'. We have rephrased several sentences in both paragraphs of this section in order to clarify the contributions of the analysis part, and emphasize the results on trajectory tracking for second-order underactuated systems. (Referee 4)

Section II

1. We have deleted former Section II.A. The notation U_k is introduced here as a shortening for $U_{i_1,\dots,i_k}$ when $i_1 = \cdots = i_k = 1$, $m = 1$. (Referee 4)
2. Here, and throughout the paper, we have changed the notation for the multiindexes. Now, instead of $\{i_1, \dots, i_k\}$, we use $(i_1, \dots, i_k)$. (Referee 1)
3. We explictly say that we assume $u_1, \dots, u_m$ to be bounded *measurable* functions. (Referee 1)
4. We have improved the definition of the set $C_{k_1,\dots,k_m}(S)$ following the remarks made by Referee 1.
5. The first part of the proof of Lemma II.1 (former Lemma II.3) has been modified in order to clarify its general flow. We explain now that first we derive the right-hand side of equation (2), apply the induction hypothesis and finally integrate with respect to time, to get an expression which is an alternative definition of multinomial iterated integral. (Referee 1)
6. Proof of Lemma II.1. If the functions $U_{k_1,\dots,k_m}$ are T-periodic, then $\int_0^t u_i(\tau)d\tau$ are T-periodic. Since the derivative of a T-periodic function is also T-periodic, the $u_i(t)$ are T-periodic. They are also zero-mean since $\int_0^T u_i(\tau)d\tau = \int_0^0 u_i(\tau)d\tau$ (by the T-periodicity of $\int_0^t u_i(\tau)d\tau$) and the latter integral is 0. (Referee 4)

7. In Section II.B, we have removed former Lemma II.5, which is now introduced in Section III, in the proof of Theorem III.2. (Referee 4)

8. The definition of the flow map of a differential equation has been modified according to the observation made by Referee 1.

9. The introduction of complex variables in the end of Section II is necessary for the result in Proposition III.1 and for the proof of the convergence result of the series expansions in Theorem III.3. The use of complex functions is common in Chronological Calculus (see the work by Agrachev, Gramkrelidze, Sarychev, etc.), since one usually relies on the Cauchy estimates (see Section 2.3 in S.G. Krantz, *Function Theory of Several Complex Variables*, John Wiley and Sons, New York, 1982) to bound the various terms of the series, and these estimates are valid for complex analytic functions. (Referee 4)

Section III

1. First paragraph. We explain that in our averaging analysis we use tools from classical averaging theory (the first-order averaging theorem, cf. [17]) and differential geometry (variation of constants formula and pull-back vector fields [47,45]). (Referee 4)

2. We have renamed former Lemma III.1 as Proposition III.1 and have included the sentence 'Now, we give a novel sufficient condition to ensure that the pull-back vector field F' is T-periodic' to give more importance to the contribution presented in its statement. Also, we have included the necessary hypotheses on the vector fields $\{g_\tau,\ \tau \in [0,T]\}$ (uniformly integrable, analytic, admitting bounded analytical continuations over $B_\sigma(x_0)$, $\sigma > 0$, commutative, T-periodic and zero mean in their first argument) that guarantees that the Volterra series expansion can be used (cf. Proposition 2.1 in [45]). (Referees 1 and 4)

3. We have gathered former Theorem III.2 and Proposition III.3 in the new Theorem III.2, which now accounts for the main averaging result of the paper stated in a coordinate-free setting. We have also added a sentence explaining the relation with previous work. (Referee 4)

4. Proof of Theorem III.2, we have included the specific reference to the first-order averaging theorem in [17] and [48]. (Referee 4)

5. We have gathered all the results in Section III.A in the new Theorem III.3. (Referee 4)

6. Regarding the proof of Theorem III.3, 'it can be proven by induction that...', since we do not want to overload the paper with more technical details, we include the proof of this assertion here, and if Referee 4 feels that it should be added to the manuscript, we will do so.

Proof. We have to prove that, for the single input setting $g(\tau,t,x) = u(\tau,t)g(x)$, the following holds,

$$\operatorname{ad}_{g'(s_1,x')} f' = \left(0, u(s_1,t)\operatorname{ad}_{g(x)} f - \frac{\partial u}{\partial t}(s_1,t)g(x)\right) ,$$

$$\operatorname{ad}_{g'(s_k,x')} \dots \operatorname{ad}_{g'(s_1,x')} f' = \left(0, u(s_k,t)\dots u(s_1,t)\operatorname{ad}^k_{g(x)} f\right) ,$$

where $f'(x') = (1, f(t,x))$ and $g'(\tau,x') = (0, g(\tau,t,x))$, $x' = (t,x)$. Now, the expression for the first Lie bracket in terms of the original vector fields $f(t,x)$ and $u(s,t)g(x)$ is given by

$$\operatorname{ad}_{g'(s_1,t,x)} f'(t,x) = \left(0, u(s_1,t)\operatorname{ad}_{g(x)} f(t,x) - \frac{\partial u}{\partial t}(s_1,t)g(x)\right) ,$$

and for the second Lie bracket,

$$\begin{aligned}
&\operatorname{ad}_{g'(s_2,x')}\operatorname{ad}_{g'(s_1,x')} f'(x') \\
&\quad = \left(0, u(s_2,t)u(s_1,t)\operatorname{ad}^2_{g(x)} f(t,x) - \frac{\partial u}{\partial t}(s_1,t)u(s_2,t)\operatorname{ad}_{g(x)} g(x)\right) \\
&\quad = \left(0, u(s_2,t)u(s_1,t)\operatorname{ad}^2_{g(x)} f(t,x)\right) .
\end{aligned}$$

Reasoning inductively, if

$$\operatorname{ad}_{g'(s_{k-1},x')} \dots \operatorname{ad}_{g'(s_1,x')} f' = \left(0, u(s_{k-1},t)\dots u(s_1,t)\operatorname{ad}^{k-1}_{g(x)} f\right)$$

holds, then,

$$\begin{aligned}
\operatorname{ad}_{g'(s_k,x')} \dots \operatorname{ad}_{g'(s_1,x')} f' &= \operatorname{ad}_{g'(s_k,x')}\left(\operatorname{ad}_{g'(s_{k-1},x')} \dots \operatorname{ad}_{g'(s_1,x')} f'\right) \\
&= \operatorname{ad}_{g'(s_k,x')}\left(0, u(s_{k-1},t)\dots u(s_1,t)\operatorname{ad}^{k-1}_{g(x)} f\right) \\
&= \left(0, u(s_k,t)\dots u(s_1,t)\operatorname{ad}^{k}_{g(x)} f\right) ,
\end{aligned}$$

since $\operatorname{ad}_{(0,h_1)}(0,h_2) = (0, \operatorname{ad}_{h_1} h_2)$.

Section IV

We have changed the title of this section to "Extensions and applications" to highlight the fact that we deal here with *classes* of systems, not just specific examples (as we do for instance in the last section). We think that showing the particular form that the series expansions take for these classes of systems enables the reader to better understand the results. On the other hand, we have followed the observation made by Referee 4 in order to unify, for each example, the results obtained, by merging together former Sections IV and V.

1. Former Propositions V.I and V.II have been now included in Sections IV.B and IV.D, respectively. In this way the novel analysis results concerning bilinear and second-order systems are stated in a compact form. (Referee 4)
2. Former Lemma V.3 is now Lemma IV.4 in the new Section IV.F, devoted to systems with recurrence relations. (Referee 4)

Section V

Concerning the observation by Referee 4, '...many results are devoted to systems for which all vector fields vanish at the origin (for example bilinear systems). I am not convinced of the practical importance of these systems since they are not controllable...', there are two important things to be noted: first, these types of systems are a classical topic in nonlinear control theory (see, for instance, Section 2.4 on bilinear systems in Isidori's book [14], the discussion on the controllability properties of bilinear systems on Nijmeijer-van der Schaft's book [13], the treatment of time-varying output feedback stabilisation in R. Brockett, *A stabilization problem*, in V.D. Blondel, E.D. Sontag, M. Vidyasagar, J.C. Willems (Eds.), Open Problems in Mathematical Systems and Control Theory, Springer, London, 1998, pp. 75-78, or Section 6 in Sarychev's paper [28]). We understand that the referee means that these systems are not controllable *at the origin*, since in principle they can be controllable at any other point, see [13]. Indeed, vibrational control provides tools to stabilize any equilibrium. A great part of the literature on vibrational stabilization of nonlinear systems, which discusses practical applications to catalytic reactions, termochemical plants, eco-systems and problems with time-delays, among others, often deals with these types of systems (see, for instance, [38,39] or B. Lehman, J. Bentsman, *Vibrational stabilization and calculation formulas for nonlinear time-delay systems - linear multiplicative vibrations*, Automatica 30 (7) (1994), 1207-1211. Secondly, only *some* stabilisation results in the paper deals specifically with these systems: Lemma V.I (i), Proposition V.2, Corollary V.3 and Proposition V.4. On the other hand, Lemma V.I (ii), Proposition V.5 and Proposition VI.1 have control vector fields which do not vanish at the origin.

1. Lemma V.1. We have included what we mean by odd and even functions in this context. Since both f and g_j are vector fields on $\mathbb{R}^n$, they can be written in the form $f, g_j : \mathbb{R}^n \to T\mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$, $f(x) = (x, \tilde{f}(x))$, $g_j(x) = (x, \tilde{g}_j(x))$. Then, f is odd iff $\tilde{f}(x) = -\tilde{f}(-x)$, and g_j is even iff $\tilde{g}_j(x) = \tilde{g}_j(-x)$. (Referee 4)
2. Proposition V.4. We say now that '$x = 0$ is *locally* asymptotically stable'. The typo in equation (23) has been corrected. Also the wording of the end of the proof has been changed to make the argument more clear. (Referee 4)

References

We have included the references [27,28] provided by Referee 1 on the work by A.V. Sarychev.

We have also corrected some typos and other minor things throughout the paper.