Local Accessibility of Nonlinear Systems with an Uncontrollable Mode

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Abstract

Given a system with an uncontrollable linearization at the origin, we study the relationship between local accessibility and invariants of a nonlinear system. Simple sufficient conditions in terms of invariants are proved for local accessibility of systems with an uncontrollable mode. Necessary conditions of local accessibility are also proved for systems with a convergent normal form. The result proved in this paper is an application of invariants of nonlinear control systems, in addition to the applications to bifurcation control, controllability, and symmetry of nonlinear systems.

Keywords: Nonlinear systems, Invariants, Accessibility.

1 Introduction

Invariants of nonlinear systems are numbers associated with homogeneous nonlinear terms in a control system. Invariants of any degree of a control system cannot be changed by any homogeneous change of coordinates and feedback of the same degree. It was proved that invariants can be used to characterize qualitative nonlinear properties of a control system, such as bifurcation of equilibrium set, periodic solutions associated with Hopf bifurcation, and controllability. In this paper, we study the relationship between invariants and local accessibility of systems with a single uncontrollable mode. It is proved that a linearly uncontrollable system is locally accessible if an invariant in the system is nonzero. The same condition is also necessary if the system and its normal form are both analytic.

This paper is organized as follows. In Section 2, existing results on normal forms and invariants are briefly introduced without proof. In Section 3, the local accessibility of nonlinear systems is addressed. Results on both sufficient and necessary conditions based on invariants for local accessibility are introduced and proved. In Section 4, related results on necessary and sufficient conditions based on invariants for local controllability and stabilizability are introduced.

2 Normal Forms of Nonlinear Systems

Normal form is the tool used to prove the main theorem. In this section, normal form of nonlinear systems is introduced without proof. Details can be found in [11], [14] and [21]. In this paper, we consider a nonlinear system with a single input in the following form

$$\dot{x} = f(x, u) \tag{2.1}$$

where $x \in \Re^n$ is the state variable, and $u \in \Re$ is the control input. $f : \Re^{n+1} \to \Re^n$ is assumed to be C^k for sufficiently large k.

Definition 1 *A point $x_e \in \Re^n$ is an equilibrium or equilibrium point of (2.1) if and only if $\exists\, u_e \in \Re$ so that*

$$f(x_e, u_e) = 0. \tag{2.2}$$

System (2.1) is said to be linearly controllable at (x_e, u_e) if its linearization

$$\dot{\bar{x}} = F\bar{x} + Gu \tag{2.3}$$

is controllable where

$$F = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_e,u_e)}, \qquad G = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_e,u_e)}.$$

Without loss of generality, we assume $f(0,0)=0$.

Assumption 1 *We assume that $f(x,u)$ is C^k for sufficiently large k. We also assume that the linearization (F,G) at the origin $x_e=0$ has one uncontrollable mode with eigenvalue $\lambda \neq 0$.*

From Assumption 1, we adopt the following normal form for the linearization at $x_e=0$,

$$F=\begin{bmatrix} \lambda & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad G=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.4}$$

If the original linearization is not in this form, it is well known that it can be transformed to (2.4) by a linear change of coordinates and linear feedback. With the linearization (2.4), the nonlinear control system is in the following form,

$$\begin{array}{rcl} \dot{z} & = & \lambda z + f_1^{[2+]}(z,x,u) \\ \dot{x}_1 & = & x_2 + f_{2,1}^{[2+]}(z,x,u) \\ \dot{x}_2 & = & x_3 + f_{2,2}^{[2+]}(z,x,u) \\ & \vdots & \\ \dot{x}_{n-1} & = & u + f_{2,n-1}^{[2+]}(z,x,u) \end{array} \tag{2.5}$$

where the superscript in $f_1^{[2+]}(z,x,u)$ implies that the Taylor expansion of the function f_1 starts with quadratic or higher degree terms. Or equivalently,

$$f_1^{[2+]}(0,0,0)=0, \quad \frac{\partial f_1^{[2+]}}{\partial(z,x,u)}(0,0,0)=0 \tag{2.6}$$

The definition of the superscripts in $f_{2,j}^{[2+]}$ is similar.

To find the normal form for the nonlinear part of the system, we use nonlinear transformations of higher degrees. According to [14] and [21], a transformation consists of the following change of coordinates and feedback

$$\begin{aligned} \begin{bmatrix} \tilde{z} \\ \tilde{x} \end{bmatrix} &= \begin{bmatrix} z \\ x \end{bmatrix} - \sum_{k=2}^{d} \begin{bmatrix} \phi_1^{[k]}(z,x) \\ \phi_2^{[k]}(z,x) \end{bmatrix} \\ \tilde{u} &= u - \sum_{k=2}^{d} \alpha^{[k]}(z,x,u) \end{aligned} \tag{2.7}$$

where $\phi_1^{[k]}$ and $\alpha^{[k]}$ are homogeneous polynomials of degree k in its arguments, $\phi_2^{[k]}$ is a $n-1$-dimensional vector whose entries are homogeneous polynomials of degree k. The highest degree d is selected to be large enough so that adequate information about the local performance of a system can be extracted from the Taylor expansion. It was proved in [11], [14] and [21] that there exists a transformation under which the system (2.5) can be transformed to

$$\begin{aligned} \dot{z} &= \lambda z + \sum_{j=1}^{n} x_j^2 Q_j(z,\bar{x}_j) + x_1 S(z) + O(z,x,u)^{d+1} \\ \dot{x}_1 &= x_2 + \sum_{j=3}^{n} x_j^2 P_{1j}(z,\bar{x}_j) + O(z,x,u)^{d+1} \\ \dot{x}_2 &= x_3 + \sum_{j=4}^{n} x_j^2 P_{2j}(z,\bar{x}_j) + O(z,x,u)^{d+1} \\ &\vdots \\ \dot{x}_i &= x_{i+1} + \sum_{j=i+2}^{n} x_j^2 P_{ij}(z,\bar{x}_j) + O(z,x,u)^{d+1} \\ &\vdots \\ \dot{x}_{n-2} &= x_{n-1} + x_n^2 P_{n-2,n}(z,\bar{x}_n) + O(z,x,u)^{d+1} \\ \dot{x}_{n-1} &= u + O(z,x,u)^{d+1} \end{aligned} \tag{2.8}$$

where $\bar{x}_j = (x_1, x_2, \ldots, x_j)^T$ (we also denote $\bar{x}_{n-1} = x$ and $\bar{x}_1 = x_1$), and

$$\begin{aligned} Q_j(z,\bar{x}_j) &= Q_j^{[0]} + Q_j^{[1]}(z,\bar{x}_j) + \cdots + Q_j^{[d-2]}(z,\bar{x}_j) \\ S(z) &= S^{[1]}(z) + S^{[2]}(z) + \cdots + S^{[d-1]}(z) \\ P_{ij}(z,\bar{x}_j) &= P_{ij}^{[0]} + P_{ij}^{[1]}(z,\bar{x}_j) + \cdots + P_{ij}^{[d-2]}(z,\bar{x}_j). \end{aligned} \tag{2.9}$$

Once again, the variable x_n in (2.8) represents the control input u. To simplify the notation, we use (z, x) and u instead of $(\tilde{z}, \tilde{x})$ and $\tilde{u}$ as the state variable and the input in the normal form. According to [10], the computation of the normal form for a given system is equivalent to solving systems of linear algebraic equations. So, there is no fundamental obstacle toward the computation of the normal form. Therefore, the computation of the normal form can be carried out using the software equipped with linear algebraic equation solvers such as MAPLE, Mathematica, and Matlab.

Given a normal form up to degree $k-1$, the coefficients in the normal form of degree k can be computed by Lie bracket between the vector fields in the control system. It can be proved that these Lie bracket formulae are invariant under transformation of degree k. Therefore, the coefficients, $Q_j^{[k]}$, $S^{[k]}(z)$, and $P_{ij}^{[k]}$ are called invariants of degree k (see [11] and [14]).

3 Local Accessibility

Controllability and accessibility are fundamental properties of nonlinear control systems. It is proved that the accessibility of a control system is closely related to the dimension of the accessibility distribution. However, for systems with uncontrollable linearization, the computation of the dimension of the accessibility distribution is not straightforward, if it is possible. In this section, we prove a simple relationship between the normal form and its local accessibility for systems with a single nonzero uncontrollable mode. Based on this result, it is easy to check the local accessibility for systems in normal form.

In this section we consider affine systems of the following form

$$\Sigma \ : \ \dot{x} = f(x) + g(x)u, \qquad x(\cdot) \in \Re^n, \quad u(\cdot) \in \mathcal{U}, \tag{3.1}$$

where $\mathcal{U}$ is the space of piecewise continuous functions also called *admissible inputs*. The vector fields f and g are either smooth or analytic or of class C^k for sufficiently large k. Given a state x_0. Let V be a neighborhood of x_0. From [17], we denote $R^V(x_0, T)$ the reachable set from x_0 at time $T > 0$, following trajectories which remain for $t \leq T$ in V, and denote

$$R_T^V(x_0) = \cup_{\tau \leq T} R^V(x_0, \tau).$$

Definition 3.1 *([17]). The system Σ is locally accessible from x_0 if $R_T^V(x_0)$ contains a non-empty open set of $\Re^n$ for all neighborhood V of x_0 and all $T > 0$.*

Denote by $\mathcal{C}$ the smallest Lie algebra of vector fields on $\Re^n$ containing f and g. Let Δ be the involutive distribution generated by $\mathcal{C}$, that is,

$$\Delta(x) = \text{span}\,\{X(x), \quad X \in \mathcal{C}\} \qquad \text{for any } x \in \mathcal{M}.$$

It is well-known that Σ is locally accessible from x_0 if $\dim\Delta(x_0) = n$.

In the following, we assume that (3.1) is in the normal form defined by (2.8). Because (3.1) is affine in control, x_n do not appear in the nonlinear part of (2.8) (see [11]). In this section, the nonlinear normal form of degree less than or equal to m_0 is used, where m_0 is an integer to be specified later. Thus, (3.1) has the following form,

$$\begin{aligned} \dot{z} &= f_1(z,x) + g_1(z,x)u = \lambda z + \sum_{m=0}^{m_0} f_1^{[m]}(z,x) + O(z,x,u)^{m_0+1} \\ \dot{x} &= f_2(z,x) + g_2(z,x)u = Ax + Bu + \sum_{m=0}^{m_0} f_2^{[m]}(z,x) + O(z,x,u)^{m_0+1}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} f_1^{[m]}(z,x) &= \sum_{j=1}^{n-1} x_j^2 Q_j^{[m-2]}(z,\bar{x}_j) + x_1 S^{[m-1]}(z) \\ f_{2,i}^{[m]}(z,x) &= \sum_{j=i+2}^{n-1} x_j^2 P_{i,j}^{[m-2]}(z,\bar{x}_j). \end{aligned}$$

for $m \geq 2$ and $1 \leq i \leq n-3$.

Theorem 3.1 *The system (3.1) is locally accessible at the origin if its normal form (3.2) satisfies*

$$f_1^{[m]}(0,x) \neq 0 \tag{3.3}$$

for some positive integer $m \geq 2$.

Condition (3.3) is equivalent to the condition

$$Q_j^{[m-2]}(0,\bar{x}) \neq 0$$

for some positive integer $m \geq 2$ and $1 \leq j \leq n-1$. From the normal form (3.2), the condition (3.3) is equivalent to the existence of nonnegative integers $i_1, \cdots, i_n$, with $i_1 + \cdots + i_n \geq 2$, such that

$$\frac{\partial^{i_1+\cdots+i_n} f_1}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(0,0) \neq 0. \tag{3.4}$$

Let $m_0 = i_1 + i_2 + \cdots + i_n$ be the smallest positive integer that satisfies the condition (3.4). Following differential geometry, the vector fields f and g are also denoted by

$$\begin{aligned} f(z,x) &= f_1(z,x)\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot f_2(z,x) \\ g(z,x) &= g_1(z,x)\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot g_2(z,x), \end{aligned}$$

where

$$\frac{\partial}{\partial x} = \left[\begin{array}{cccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}, & \cdots, & \frac{\partial}{\partial x_{n-1}} \end{array} \right].$$

In $f_1^{[m_0]}(z,x)$, the terms independent of z form a new function, denoted by $f_1^{[m]}(x)$, i.e.,

$$f_1^{[m]}(x) = f_1^{[m]}(z,x)|_{z=0}.$$

Let $O_x(z)$ be any function of (z,x) satisfying $O_x(0) = 0$. Then, the function $f_1(z,x)$ has the following form

$$f_1(z,x) = f_1^{[m_0]}(x) + O_x(z) + O(z,x)^{m_0+1}. \tag{3.5}$$

Because

$$f_1^{[m_0]}(x) = \sum_{j=1}^{n-1} x_j^2 Q_j^{[m_0-2]}(0,\bar{x}_j) \neq 0,$$

let s be the largest positive integer so that $Q_s^{[m_0-2]}(0,\bar{x}_j) \neq 0$. Therefore,

$$\frac{\partial f_1^{[m_0]}(x)}{\partial x_j} = 0, \quad \text{if } j > s. \tag{3.6}$$

To prove Theorem 3.1, we must derive the formulae for the vectors $ad_f^k(g)$. Given

$$\begin{aligned} f &= (f_1^{[m_0]}(x) + O_x(z) + O(z,x)^{m_0+1})\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot (Ax + O(z,x)^2) \\ g &= \frac{\partial}{\partial x_{n-1}} + O(z,x)^{m_0+1}\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot O(z,x)^{m_0+1} \end{aligned} \tag{3.7}$$

and the equation (3.6), it is straightforward to prove the following equation

$$\begin{array}{rl} & (-1)ad_f(g) \\ = & [g, f] \\ = & (O_x(z) + O(z,x)^{m_0})\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial x_{n-2}} + \dfrac{\partial}{\partial x} \cdot O(z,x) \end{array} \tag{3.8}$$

if $s < n-1$. Once again, based on (3.8) and (3.6), it can be proved that

$$\begin{array}{rl} & ad_f^2(g) \\ = & [-ad_f(g), f] \\ = & (O_x(z) + O(z,x)^{m_0})\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial x_{n-3}} + \dfrac{\partial}{\partial x} \cdot O(z,x) \end{array}$$

provided $s < n-2$. In general, if $k < n-s$, we have

$$\begin{array}{rl} & (-1)^k ad_f^k(g) \\ = & (O_x(z) + O(z,x)^{m_0})\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial x_{n-k-1}} + \dfrac{\partial}{\partial x} \cdot O(z,x), \quad \text{for } k < n-s. \end{array} \tag{3.9}$$

When $k = n-s-1$, the vector field $ad_f^{n-s-1}(g)$ has a term $\dfrac{\partial}{\partial x_s}$. By the definition of s,

$$\frac{\partial f_1^{[m_0]}(x)}{\partial x_s} \neq 0.$$

Therefore, if $k = n-s$, we have

$$\begin{array}{rl} & (-1)^{n-s} ad_f^{n-s}(g) \\ = & (\dfrac{\partial f_1^{[m_0]}(x)}{\partial x_s} + O_x(z) + O(z,x)^{m_0})\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial x_{s-1}} + \dfrac{\partial}{\partial x} \cdot O(z,x). \end{array} \tag{3.10}$$

In general, for any positive integer k satisfying $n-2 > k > n-s$,

$$\begin{array}{rl} & (-1)^k ad_f^k(g) \\ = & (O(x)^{[m_0-1]} + O_x(z) + O(z,x)^{m_0})\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial x_{n-k-1}} + \dfrac{\partial}{\partial x} \cdot O(z,x). \end{array} \tag{3.11}$$

The formula of the vector field $ad_f^k(g)$ is summarized in the following lemma.

Lemma 3.1 *Define the vector fields $h_k = (-1)^k ad_f^k(g)$, for $k = 1, 2, \cdots, n-2$, then*

$$\begin{array}{rcll} h_k & = & (-1)^k ad_f^k(g) & \\ & = & (O_x(z) + O(z,x)^{m_0})\frac{\partial}{\partial z} + \frac{\partial}{\partial x_{n-k-1}} + \frac{\partial}{\partial x} \cdot O(z,x), & k < n-s, \\ h_{n-s} & = & (-1)^{n-s} ad_f^{n-s}(g) & \\ & = & (\frac{\partial f_1^{[m_0]}(x)}{\partial x_s} + O_x(z) + O(z,x)^{m_0})\frac{\partial}{\partial z} + \frac{\partial}{\partial x_{s-1}} + \frac{\partial}{\partial x} \cdot O(z,x), & \\ h_k & = & (-1)^k ad_f^k(g) & \\ & = & (O(x)^{m_0-1} + O_x(z) + O(z,x)^{m_0})\frac{\partial}{\partial z} + \frac{\partial}{\partial x_{n-k-1}} + \frac{\partial}{\partial x} \cdot O(z,x), & k > n-s. \end{array} \tag{3.12}$$

Proof of Theorem 3.1. Define the vector field $\hat{f}$ by the following equation,

$$\begin{array}{rcl} \hat{f} & = & [h_{n-s-1}, h_{n-s}] \\ & = & [(-1)^{n-s-1} ad_f^{n-s-1}(g), (-1)^{n-s} ad_f^{n-s}(g)]. \end{array} \tag{3.13}$$

From Lemma 3.1, we have

$$\hat{f} = (\frac{\partial^2 f_1^{[m_0]}(x)}{\partial x_s^2} + O_x(z) + O(z,x)^{m_0-1})\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot V(z,x), \tag{3.14}$$

where $V(z,x)$ is any vector field, which is not important for the derivation that follows. Suppose $i_1, i_2, \cdots, i_s$ is a sequence of nonnegative integers so that $i_1 + i_2 + \cdots + i_s = m_0$ and

$$\frac{\partial^{m_0} f_1^{[m_0]}(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_s^{i_s}} \neq 0. \tag{3.15}$$

From the definition of m_0 and the structure of normal form, we know that the sequence $\{i_1, i_2, \cdots, i_s\}$ exists and $i_s \geq 2$. From Lemma 3.1 and the equation (3.14), it is straightforward to derive the following equation,

$$\begin{array}{rl} & ad_{h_{n-j-1}}(\hat{f}) \\ = & [(-1)^{n-j-1} ad_f^{n-j-1}(g), \hat{f}] \\ = & (\frac{\partial^3 f_1^{[m_0]}(x)}{\partial x_j \partial x_s^2} + O_x(z) + O(z,x)^{m_0-2})\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot V(z,x), \end{array} \tag{3.16}$$

where $V(z,x)$ is any vector field. It is a different function from the vector V in (3.14). However, because its value is not important for the derivation, we keep the same notation, V, for the reason of simplicity.

In general, we have

$$\begin{array}{rl} & ad_{h_{n-2}}^{i_1}(ad_{h_{n-3}}^{i_2}(\cdots(ad_{h_{n-s-1}}^{i_s-2}(\hat{f}))\cdots)) \\ = & (\dfrac{\partial^{m_0} f_1^{[m_0]}(x)}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_s^{i_s}} + O_x(z) + O(z,x)^{m_0})\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial x}\cdot V(z,x), \end{array} \tag{3.17}$$

Denote this vector by h_{n-1}. At the origin $(z,x)=(0,0)$, the vectors h_{n-1}, h_{n-2}, $\cdots$, h_1, and g are

$$\frac{\partial^{m_0} f_1^{[m_0]}(x)}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_s^{i_s}}\frac{\partial}{\partial z} + \frac{\partial}{\partial x}\cdot V(0,0), \quad \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \cdots, \quad \frac{\partial}{\partial x_{n-1}},$$

respectively. From (3.15) and the definition of h_k, the dimension of the distribution Δ is n at the origin. therefore, the system is locally accessible from $(z,x)=(0,0)$. $\square$

The following remark is a partially converse result of Theorem 3.1.

Remark. Given a system in the form of

$$\begin{array}{rcl} \dot{z} & = & f_1(z,x) \\ \dot{x} & = & Ax + Bu + O(z,x)^2. \end{array} \tag{3.18}$$

Suppose

$$f_1(z,x)|_{z=0} = 0.$$

Then the system (3.18) is not locally accessible from the origin. $\triangleleft$

The proof of the remark is straightforward. If the initial condition satisfies $z=0$, then the trajectory, $(z(t),x(t))$, of the system satisfies $z(t)=0$ for all $t\geq 0$. Therefore, $R_T^V(0,0)$ is a subset of the subspace $\{(z,x)|z=0\}$, which does not contain an open set of $\Re^n$. Therefore, the system is not locally accessible from the origin.

As we know that the normal form (3.2) will have the form of (3.18) if we let $m\to\infty$. In this case, the normal form is a series with infinite number of terms. The result in the remark can be considered as a converse result of Theorem 3.1 only for analytic systems whose normal form is a convergent formal series.

As an illustrative example, let us consider the following system

$$\begin{aligned}
\dot{z} &= z + a_{01}x_1 z + (a_{12}x_1 + a_{22}x_2)x_2^2 \\
\dot{x}_1 &= x_2 + x_3^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u.
\end{aligned}$$

In this example $m_0 = 3$ and the integer $s = 2$. If $a_{12}^2 + a_{22}^2 \neq 0$, then (3.3) is satisfied for $m = 3$. Therefore, the system must be locally accessible from the origin, although the system is not linearly controllable at the origin.

4 Conclusion

Given a system with an uncontrollable linearization at the origin, we study the relationship between local accessibility and invariants of a nonlinear system. Simple sufficient conditions in terms of invariants are proved for local accessibility of systems with an uncontrollable mode. Necessary conditions of local accessibility are also proved for systems with a convergent normal form. The result proved in this paper is an application of invariants of nonlinear control systems, in addition to the applications to bifurcation control, controllability, and symmetry of nonlinear systems.

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