Time Optimal Synthesis for a $so(3)$-left-invariant Control System on a Sphere

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Abstract

We consider the time optimal problem on $SO(3)$ given by $\dot{x} = x(f + ug)$, $|u| \leq 1$, where $f, g \in so(3)$ define two perpendicular left-invariant vector fields on $SO(3)$, The purpose of this paper is to estimate, especially when α tends to zero, the quantity $N(\alpha)$ defined as the maximum number of switchings for time-optimal trajectories of (Σ). This is done by projecting (Σ) onto a control system (Σ_S) defined on the unit sphere S^2 by an adapted Hopf fibration and building the complete time-optimal synthesis of (Σ_S). This is one of the few examples of global two-dimensional time-optimal synthesis on manifolds with nontrivial topology. As a byproduct, we show that for the problem on $SO(3)$ we have $[\pi/2\alpha] \leq N(\alpha) \leq [\pi/2\alpha] + f(\alpha)$, where f is continuous and $f(\alpha)$ tends to three as α tends to zero. Applications to quantum systems and satellites are provided.

1 Introduction

Let Σ be the control system given by:

$$\dot{x} = x(f + ug), \tag{1}$$

where $x \in SO(3)$, $|u| \leq 1$ and $f, g \in so(3)$ give rise to left invariant vector fields (acting on the right) on $SO(3)$. The control space $Ad(\Sigma)$ is the set of admissible controls u i.e. the set of measurable functions $u : [a, b] \to [-1, 1]$, where a, b depend on u in general (see Jurdjevic [12] for a more general definition of control systems). A trajectory γ of Σ is an absolutely continuous curve $\gamma : J \to SO(3)$ with $J = [a, b]$ is a compact segment of $\mathbb{R}$, such that there exists an admissible control $u \in Ad(\Sigma)$ for which:

$$\dot{\gamma}(t) = \gamma(t)(f + ug), \tag{2}$$

holds a.e. in J. We then say (γ, u) defined as before is an admissible pair for Σ. The initial and terminal states of γ are $In(\gamma) := \gamma(a)$ and $Fi(\gamma) := \gamma(b)$. The *time along* γ, $T(\gamma) = b - a$, is the length of J. We use $Tj(\Sigma)$ and $Pa(\Sigma)$ resp. to denote the set of admissible trajectories and the set of admissible pairs of Σ.

Definition 1 *A trajectory γ of Σ is time optimal if $T(\gamma) \leq T(\gamma')$ for every trajectory $\gamma' \in Tj(\Sigma)$ with initial and terminal state $In(\gamma)$ and $Fi(\gamma)$.*

Using the Lie isomorphism between $(so(3), [\cdot, \cdot])$ and $(R^3, \times)$ denoted ϕ and given below, we can define the length $\|z\|$ of an arbitrary $z \in so(3)$. In this paper, we will assume that f and g are perpendicular and normalized so that $\|f\| = \cos(\alpha)$ and $\|g\| = \sin(\alpha)$, $\alpha \in (0, \pi/4)$. Here, we adopt the following notation used throughout the paper, $c_\alpha := \cos(\alpha)$, $s_\alpha := \sin(\alpha)$, $t_\alpha := \tan(\alpha)$ and $ct_\alpha := \cot(\alpha)$.

This case was already studied by Agrachev (cf. [1]): he showed that every time-optimal trajectory of (Σ) had at most $\pi/\alpha + 1$ switchings. That result was obtained by computing the index of a quadratic form introduced by Agrachev in that paper. Let $N(\alpha)$ be the maximum number of switchigs along a time-optimal trajectory of (Σ). Then Agrachev's results reads $N(\alpha) \leq \pi/\alpha + 1$. The purpose of the present paper is to refine Agrachev's result and in particular to provide a lower bound for $N(\alpha)$. We have the following result:

Theorem 1 *Let (Σ) be the control system defined in (1) with F, G perpendicular so that $\|F\| = c_\alpha$ and $\|G\| = s_\alpha$, $\alpha \in (0, \pi/4)$. Then, if $N(\alpha)$ is the maximum number of switchigs along a time-optimal trajectory of (Σ), we have*

$$[\pi/2\alpha] \leq N(\alpha) \leq [\pi/2\alpha] + f(\alpha), \tag{3}$$

where f is continuous and $f(\alpha)$ tends to three as α tends to zero.

Note that there exist general techniques to compute (at least in theory) upper bounds for $N(\alpha)$, such that the envelope theory (cf. Sussmann [20, 21]) or the index of the Agrachev quadratic form. However, there is nothing similar for estimating lower bounds for $N(\alpha)$. This is why getting the left inequality in (3) is the hardest part to establish Theorem 1 and the way to achieve it rests heavily the special features of the control system (Σ).

Our strategy consists of projecting (Σ) onto another control system (Σ_S) whose state space is the unit sphere S^2 and so that every time-optimal trajectory of (Σ_S) is a time-optimal trajectory of (Σ). The projection used here is the Hopf fibration along the direction $\phi(f)$. Therefore, if $N_S(\alpha)$ is used to denote the maximum number of switchings along a time-optimal trajectory of (Σ_S), we have $N_S(\alpha) \leq N(\alpha)$. The next step consists of computing $N_S(\alpha)$ and this is done by constructing the complete time-optimal synthesis for (Σ_S), which relies on the techniques introduced in [8]. Finally, the upper bound of (3) is obtained using the fact that the integral curves corresponding to the fiber of the Hopf fibration of f are those of f and therefore are admissible trajectories of (Σ).

The balance of the paper is organized as follows. Section 2 and 3 collect basic facts relative to the time-optimal trajectories of (Σ) and the time-optimal synthesis of (Σ_S) is computed in section 4. In Sections 5 applications to satellites and quantum systems are provided

2 General Properties of Optimal Trajectories

For several basic formulas, we refer to [9]. The Lie algebra $(so(3), [.,.])$ is isomorphic to the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ denotes the vector product. This isomorphism is realized by the map:

$$\phi_L : so(3) \to \mathbb{R}^3 \tag{4}$$

$$\phi_L\left(\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\right) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This isomorphism provides an inner product on $so(3)$:

$$< z_1, z_2 > := < \phi_L(z_1), \phi_L(z_2) >,$$

where $z_1, z_2 \in so(3)$ and the symbol $< .,. >$ in the right hand side stands for the Euclidean inner product of $\mathbb{R}^3$. With this definition it follows that:

$$< z_1, z_2 > := \frac{1}{2} Tr(z_1 z_2^t),$$

where z^t denotes the transposed of the matrix $z \in so(3)$. In the following, $\|z\| := \sqrt{< z, z >}$ and Id_3 is the 3×3 identity matrix. We will sometimes consider the 2×2 matrix corresponding to the planar rotation of angle β and we use R_β to denote it.

We define $h := [f, g] = fg - gf$ and Define:

$$X_+ := f + g, \quad X_- := f - g.$$

Note that $\|X_\varepsilon\| = 1$, with $\varepsilon = +, -$. For a vector field $z \in so(3)$, we use the exponential e^{tz} to denote the flow of z, acting on the right, so that $t \mapsto pe^{tz} \in SO(3)$ is the integral curve of z starting at p at time 0. Since z is linear, we have $e^{tz} = \sum_{n=0}^{\infty} \frac{(zt)^n}{n!}$.

We use ad_z to denote the operator $w \mapsto [z, w]$, acting on vector fields. If z, w are vector fields, then $e^{t\ \mathrm{ad}_z}(w) := e^{tz} w e^{-tz}$. The Lie bracket relations between f, g, h are

$$[f, g] = h, \quad [g, h] = s_\alpha f, \quad [h, f] = c_\alpha g.$$

One deduces from them:

$$\begin{aligned}
e^{t\ \mathrm{ad} X_\varepsilon}(f) &= (c_\alpha + s_\alpha \cos(t))f + \varepsilon c_\alpha (1 - \cos(t))g - \varepsilon \sin(t) h, && (5)\\
e^{t\ \mathrm{ad} X_\varepsilon}(f) &= \varepsilon s_\alpha (1 - \cos(t))f + (s_\alpha + c_\alpha \cos(t))g + \sin(t) h, && (6)\\
e^{t\ \mathrm{ad} X_\varepsilon}(h) &= \varepsilon s_\alpha \sin(t) f - c_\alpha \sin(t) g + \cos(t) h, && (7)\\
e^{t\ \mathrm{ad} X_\varepsilon}(X_{-\varepsilon}) &= (\cos(2\alpha) + 2 s_\alpha \cos(t))f + \varepsilon(\cos(2\alpha) - 2 c_\alpha \cos(t))g - 2\varepsilon \sin(t) h, && (8)\\
e^{tX_\varepsilon} &= Id_3 + \sin(t) X_\varepsilon + (1 - \cos(t)) X_\varepsilon^2. && (9)
\end{aligned}$$

2.1 The Pontryagin Maximum Principle and existence of optimal trajectories

In the rest of the paper, optimality means time-optimality. We say that a control system is *complete*, if for every measurable control function $\eta : [a, b] \to [-1, 1]$ and every initial state p there exists a trajectory γ for η which is defined on the whole interval $[a, b]$ and satisfies $\gamma(a) = p$. Since $SO(3)$ is compact and connected, the system (1) is complete. Note that (f, g) satisfies the Strong Bracket Generating Condition (cf. [16]) and the control set is compact and convex. Then, (cf. for instance [19]):

Proposition 1 *For each pair of points p and q belonging to $SO(3)$, there exists a time optimal trajectory joining p to q.*

Define the following real-valued map on $T^*SO(3) \times [-1, 1]$, called Hamiltonian,

$$\mathcal{H}(\lambda, x, u) := < \lambda, x(f + ug) > .$$

Set:

$$H(p, x) = \min_{v \in [-1,1]} \mathcal{H}(p, x, v). \qquad (10)$$

The Pontryagin Maximum Principle [15] asserts that, if $\gamma : [a, b] \to SO(3)$ is a time optimal trajectory corresponding to a control $u : [a, b] \to [-1, 1]$, then there exist *a nontrivial field of covectors along* γ, that is an absolutely continuous function $\lambda : t \in [a, b] \mapsto \lambda(t) \in T^*_{\gamma(t)} SO(3) \sim so(3)$ never vanishing, and a constant $\lambda_0 \geq 0$ such that:

i) $\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(\lambda(t), \gamma(t), u(t))$,

ii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$ for a.e. $t \in Dom(\gamma)$,

iii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = H(\gamma(t), \lambda(t))$ for a.e. $t \in Dom(\gamma)$.

In this case we say that the pair (γ, λ) is extremal. Moreover, if $\lambda_0 = 0$ then we say that (γ, λ) is an *abnormal extremal*. If γ is optimal we say that the pair (γ, λ) is optimal. Note that X_ε is periodic of period 2π and then if an optimal trajectory γ is a bang trajectory of the form $(X_\varepsilon)_t$, then $t < 2\pi$.

2.2 Notations for Trajectories and Switching functions

A control $\eta : [a,b] \to [-1,1]$ is said to be *bang-bang* if $\eta(t) \in \{-1,1\}$ a.e. in $[a,b]$. Moreover if $\eta(t) \in \{-1,1\}$ and $\eta(t)$ is constant for almost every $t \in [a,b]$ then η is called a bang control. A *switching* time of η is a time $t \in [a,b]$ such that for each for every $\varepsilon > 0$, η is not bang on $(t-\varepsilon, t+\varepsilon) \cup [a,b]$. A control with a finite number of switchings is called *regular bang-bang*. A trajectory of Σ is a bang trajectory, bang-bang trajectory, regular bang-bang trajectory respectively, if it corresponds to a bang control, bang-bang control, regular bang-bang control respectively. If $\eta(.)$ with $\mathrm{Dom}(\eta) = [a,b]$, then a switching time is a time $t \in [a,b]$ for which η is neither a.e. equal to 1 nor a.e. equal to -1 on any interval of the form $(t-\varepsilon, t+\varepsilon) \cap [a,b]$ for each $\varepsilon > 0$.

We define the switching functions:

$$\begin{aligned}
\varphi_1 &:= \; < \lambda, xf >, \\
\varphi_2 &:= \; < \lambda, xg >, \\
\varphi_3 &:= \; < \lambda, xh >,
\end{aligned}$$

as the Hamiltonian functions associated to the vector fields f, g, h (cf. Jurdjevic [12]). The switching functions are important because they determine where the controls may switch. Using the PMP one easily gets that a necessary condition for a time t to be a switching is that $\varphi_2(t) = 0$. Therefore on any interval where φ_2 has no zeros (respectively finitely many zeros) the corresponding control is bang (respectively bang-bang). On the other hand if φ_2 has a zero at t and $\dot\varphi_2(t)$ exists and it is different from zero then t is an isolated switching.

A trajectory γ of Σ defined on $[c,d]$ admitting an extremal lift Γ is said to be *singular* if the switching functions φ_2 vanishes on $[c,d]$.

In the sequel we will use the following convention. We use the letter B to refer to a bang trajectory and the letter S to refer a singular extremal trajectory. A concatenation of bang and singular trajectories will be labelled by the corresponding letter sequence, written in order from left to right. Sometimes we will use a subscript to indicate the duration of a trajectory so that, we use B_t to refer to a bang trajectory defined on an interval of length t and similarly S_t for a singular trajectory defined on an interval of length t.

2.3 The Adjoint System and Two Invariants

Let γ be an optimal trajectory for Σ. Let φ_i, $i = 1,2,3$ be the switching functions associated to γ by the PMP. Using the Lie bracket relations between f, g, h, one gets that $\varphi_1, \varphi_2, \varphi_3$ satisfy a system of differential equations called the adjoint system and given by:

$$\begin{aligned}
\dot\varphi_1 &= -u\varphi_3, && (11)\\
\dot\varphi_2 &= \varphi_3, && (12)\\
\dot\varphi_3 &= s_\alpha u \varphi_1 - c_\alpha \varphi_2. && (13)
\end{aligned}$$

From the fact that H is constant along the extremal trajectories one immediately gets for a.e. $t \in Dom(\gamma)$:

$$-\varphi_1(t) - u(t)\varphi_2(t) = -\varphi_1(t) + |\varphi_2(t)| = \lambda_0, \tag{14}$$

with λ_0 equal to zero or one. Moreover if (γ, u, λ) is an extremal lift of Σ, then there exists $K > 0$ such that for every t in the domain of γ:

$$c_\alpha \varphi_2^2 + s_\alpha \varphi_1^2 + \varphi_3^2 = K^2. \tag{15}$$

Let us assume first that γ is an abnormal extremal. Then $\lambda_0 = 0$. From (13), we get $\dot\varphi_3 = -(c_\alpha + u^2 s_\alpha)\varphi_2$. If γ contains a singular arc, we have along that arc $\varphi_2 \equiv 0$, $\varphi_1 \equiv 0$ from (14), $\varphi_3 \equiv 0$ from (12), which is then impossible. Then γ cannot contain a singular arc. Therefore, $|u^2| = 1$ a.e. $t \in Dom(\gamma)$ and the vector

(φ_3, φ_2) rotates with angular velocity equal to one. This implies γ is a regular bang-bang trajectory and the time duration between two consecutive switchings along γ is always equal to π.

For the rest of the subsection, γ is supposed to be a normal extremal.

put picture of the intersection of $|\varphi_2(t)| - \varphi_1(t) = 1$ and $c_\alpha \varphi_2^2 + s_\alpha \varphi_1^2 + \varphi_3^2 = K^2$. It is clear from that picture that the switching times are bigger than π.

3 Classification of optimal trajectories

In this section, we investigate the structure of optimal trajectories by analyzing the normal extremal flow defined in (11)-(13) subject to (14) and (15). We will characterize the inputs corresponding to a singular arc in an optimal trajectory and establish basic relations between the interior bang times.

3.1 Relations between the time durations corresponding to interior bang arcs along bang-bang trajectories

Let γ be a bang-bang trajectory of the type:

We have:

$$\varphi_2(t_0) = \varphi_2(t_0 + t_1) = \varphi_2(t_0 + t_1 + t_2) = \varphi_2(t_0 + t_1 + t_2 + t_3) = 0. \tag{16}$$

The three first equalities of equation (16) can be written as:

$$< \lambda(t_0 + t_1), p_2 e^{-t_1 adX_-}(g) >=< \lambda(t_0 + t_1), p_2 g >=< \lambda(t_0 + t_1), p_2 e^{t_2 adX_+}(g) >= 0, \tag{17}$$

and, proceeding similarly for the three last equalities, we get:

$$< \lambda(t_0 + t_1 + t_2), p_3 g >=< \lambda(t_0 + t_1 + t_2), p_3 e^{-t_2 adX_+}(g) >=< \lambda(t_0 + t_1 + t_2), p_3 e^{t_3 adX_-}(g) >= 0. \tag{18}$$

We need the following definition. If z_1, z_2, z_3 are (possibly time-varying) vector fields of $SO(3)$, the application $q \mapsto q(z_1 \wedge z_2 \wedge z_3)$ is the *field of 3-vectors* associated to the z_i's, where $q(z_1 \wedge z_2 \wedge z_3)$ is an element of $\bigwedge^3 T_q SO(3)$, the 3-fold exterior power of $T_q SO(3)$.

We now rewrite equations (17) and (18) by using fields of 3-vectors. We firstly obtain:

$$g \wedge e^{-t_1 adX_-}(g) \wedge e^{t_2 adX_+}(g) = 0, \tag{19}$$

and secondly:

$$g \wedge e^{-t_2 adX_+}(g) \wedge e^{t_3 adX_-}(g) = 0. \tag{20}$$

Thanks to equations (5) and (6), (19) and (20) are equivalent to $r(t_1, t_2) f \wedge g \wedge h = 0$ and $r'(t_2, t_3) f \wedge g \wedge h = 0$ for appropriate real-valued functions r and r'. After computations, we get $r(t_1, t_2) = \sin(\frac{t_1 - t_2}{2})$ and $r'(t_2, t_3) = \sin(\frac{t_3 - t_2}{2})$. This implies that $t_1 = t_2 = t_3$.

Recall that for an optimal trajectory associated to a abnormal extremal, the time duration along a nontrivial interior bang arc is π. Consider now an optimal trajectory of the form BB_TB, where B_T is a nontrivial interior bang arc associated to a normal extremal. Recall that $T < 2\pi$. Along such a bang arc, we deduce, from (12), (13) and (14), that the vector $z = (\varphi_3, \varphi_2 + us_\alpha)^T$ satisfies the differential equation

$$\dot{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z, \quad t \in (0, T),$$

with boundary conditions $z(0) = (\varphi_3(0), us_\alpha)$ and $z(T) = (\varphi_3(T), us_\alpha)$. If $R_t \in SO(2)$ is used to denote the rotation of oriented angle t, we have $z(T) = R_T z(0)$. Since $u\varphi_2 = -|\varphi_2|$ and $\varphi_2(0)\varphi_3(0) = \dot{\varphi}_2(0)\varphi_2(0) \geq 0$, we get

$$\tan(T/2) = -\frac{|\varphi_3(0)|}{s_\alpha}.$$

It implies that $T \in (\pi, 2\pi)$. In summary, we proved that

Proposition 2 *Let γ be a bang-bang optimal trajectory. Then the time duration T along an interior bang arc is the same for all interior bang arcs and verifies $\pi \leq T \leq 2\pi$.*

From the previous proposition, we deduce the following one:

Proposition 3 *Every optimal trajectory of Σ is a finite concatenation of bang and singular arcs.*

Proof of Proposition 3: Let $\gamma : [a,b] \to SO(3)$ be an optimal trajectory of Σ. Let S be the set of zeros of φ_2 such that if $t \in S$, then φ_2 does not vanish identically in some neighborhood of t. Clearly, S is the set of times t such that $\gamma(t)$ is the junction of two bang arcs or the the junction of a singular arc and a bang arc. The conclusion follows if S is finite. Reasoning by contradiction, then S has a limit point $\bar{t}$. Moreover $\bar{t} \in S$, otherwise φ_2 would vanish identically in a neighborhood of $\bar{t}$, contradicting the fact that $\bar{t}$ is a limit point of S. Note also that $\dot{\varphi}_2$ is continuous in an open (in $[a,b]$) neighborhood N of $\bar{t}$. By definition of S and $\bar{t}$, there exists a sequence (t_n) in N converging to $\bar{t}$ such that y$\varphi_2(t_n) \neq 0$. Pick n large enough, so that, if $[t'_n, t''_n] \subset N$ is the maximal subinterval containing t_n with $\varphi_2 \neq 0$ on (t'_n, t''_n). Then $\varphi_2(t'_n) = \varphi_2(t''_n) = 0$ γ is a bang arc on $[t'_n, t''_n]$ and $t''_n - t'_n$ tends to zero as n goes to infinity since t_n tends to $\bar{t}$. But, by Proposition 2, $t''_n - t'_n \geq \pi$ for n large enough. So we reached a contradiction and S is finite.

3.2 Optimal singular trajectories

Let γ be an optimal trajectory containing a singular arc S_t, $t > 0$. Then along S_t, $\varphi_2 = \varphi_3 \equiv 0$ and $|\varphi_1| \equiv 1$. From (13), we have $u \equiv 0$ along S_t. Then S_t is an integral curve of F.

Assume now that γ contains a singular arc and a nontrivial interior bang arc. Then we may assume that γ contains a piece of the type $S_s B_t$. Then we have $\varphi_2(s) = \varphi_3(s) = \varphi_2(s+t) = 0$. This translates to

$$g \wedge h \wedge e^{tadX_\varepsilon}(g) = 0. \tag{21}$$

It implies that $\cos(t) = 1$, i.e. $t = 2\pi$. The latter contradicts the time optimality of γ. Recall that $\|f\| = c_\alpha$.

Finally, notice that $e^{tadf}(g) = \cos(tc_\alpha)g + \frac{\sin(tc_\alpha)}{c_\alpha}h$. Then we get

$$e^{\frac{\pi}{c_\alpha}adf}(g) = -g.$$

We deduce that, for $t \geq 0$,

$$e^{\frac{\pi}{c_\alpha}f} e^{tX_\varepsilon} = e^{tX_{-\varepsilon}} e^{\frac{\pi}{c_\alpha}f}.$$

Therefore, for $t, s \geq 0$, we have

$$e^{sf} e^{tX_\varepsilon} = e^{(s-\frac{\pi}{c_\alpha})f} e^{tX_{-\varepsilon}} e^{\frac{\pi}{c_\alpha}f}.$$

Then, if $s > \frac{\pi}{c_\alpha}$ and $t > 0$ and taking into account what preceds, $e^{sf}e^{tX_{eps}}$ cannot be optimal. We summarize the results of the subsection in the next proposition:

Proposition 4 *Let γ be an optimal trajectory containing a singular arc. Then γ is of the type $B_t S_s B_{t'}$, with $s \leq \frac{\pi}{c_\alpha}$ if $t > 0$ or $t' > 0$ and $s < 2\frac{\pi}{c_\alpha}$ otherwise.*

4 The Time Optimal synthesis on the Sphere

In this section, we build explicitly the time optimal synthesis for the control problem down on the sphere, when $\alpha < \pi/4$. As we will see this condition guarantees that *there are not singular trajectories* on the sphere. Let $x_0 = (0,0,1)^T$ and $(0,0,-1)^T$ be respectively the north pole and the south pole of the sphere. We fix the following conventions for the vector fields of the problem:

$$f = c_\alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } g = s_\alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{22}$$

Then, using the Lie algebra isomorphism ϕ_L defined in (4), we have

$$\phi_L(f) := F = c_\alpha(0,0,1)^T \text{ and } \phi_L(g) := G = s_\alpha(1,0,0)^T. \tag{23}$$

Projecting the control system (1) along the Hopf fibration defined by the north pole of the sphere $x_0 = (0,0,1)^T$, (1) reduces to a control system on the sphere given by:

$$(\Sigma): \quad \dot{x} = F_S(x) + uG_S(x), \quad x \in S^2, \quad |u| \leq 1,, \tag{24}$$

where the vector fields F_S and G_S are respectively defined by

$$yF_S(x) = F \times x \text{ and } G_S(x) = G \times x. \tag{25}$$

Note that $F_S(x_0) = 0$. We are looking for the optimal synthesis from the point x_0, that is a way of associating a unique optimal trajectory (based in x_0) to every final point on the sphere.

Definition 2 (reachable set, optimal synthesis)

- *We call reachable set within time $T > 0$ the following set:*

$$\begin{aligned} \mathcal{R}(T) \quad := \quad & \{x \in \ S^2 \text{ such that there exists } b \in [0,T] \text{ and a trajectory} \\ & \gamma : [0,b] \to S^2 \text{ of (24) such that } \gamma(0) = x_0, \ \gamma(b) = x\}. \end{aligned} \tag{26}$$

$$\tag{27}$$

- *A synthesis in time T, for the control system (24), is a family $\Gamma = \{\gamma_x : [0,b_x] \mapsto S^2, \ x \in \mathcal{R}(T)$ of trajectories satisfying the following conditions:*
  - *for each $x \in \mathcal{R}(T)$ one has $\gamma_x(0) = x_0, \ \gamma_x(b_x) = x$;*
  - *if $y = \gamma_x(t)$, where $t \in Dom(\gamma_x)$, then $\gamma_y = \gamma_x|_{[0,t]}$.*
  - *every $\gamma_x \in \Gamma$ is time optimal that means the following. For every trajectory $\gamma : [0,b] \to S^2$ satisfying $\gamma(0) = 0, \gamma(b) = x$, one has $b \geq b_x$*
- *If the following condition is satisfied:*

$$\exists \ T_0 < \infty \text{ such that } \mathcal{R}(T_0) = S^2 \tag{28}$$

then we call the optimal synthesis in time T_0 simply the optimal synthesis.

- *A stratification of S^2 is a finite collection $\{M_i\}$ of connected embedded $\mathcal{C}^1$ submanifolds of S^2, called strata, such that the following holds. If $M_j \cap Clos(M_k) \neq \emptyset$ with $j \neq k$ then $M_j \subset Clos(M_k)$ and $dim(M_j) < dim(M_k)$. An optimal synthesis is called regular if there exists a stratification of S^2 such that $u(x) = u_x(t_x)$ is smooth on each stratum (assuming each u_x right continuous).*

Remark **1** In our specific case (24), condition (28) is satisfied since the system is completely controllable in finite time.

4.1 The PMP for (24)

The PMP for the system (24) says the following:

Theorem 2 (PMP) *Define for every $(x,p,u) \in T^*S^2 \times [-1,1]$, where T^*S^2 is the cotangent bundle of S^2, the following:*

$$\mathcal{H}_S(x,p,u) = < p, F_S(x) > +u \ < p, G_S(x) > \ \in \ \mathbb{R},$$

and:

$$H_S(x,p) = \max\{\mathcal{H}_S(x,p,u) : u \in [-1,1]\}. \tag{29}$$

If $\gamma : [0,b] \to S^2$ is a (time) optimal trajectory corresponding to a control $u : [0,b] \to [-1,1]$, then there exist a non trivial field of covectors along γ, that is a function $\lambda : t \in [0,b] \mapsto \lambda(t) \in T^*_{\gamma(t)}S^2$ never vanishing, and a constant $\lambda_0 \leq 0$ such that *i)* $\dot{\lambda}(t) = - < \lambda(t), (\nabla F_S + u(t)\nabla G_S)(\gamma(t)) >$,

ii) $\mathcal{H}_S(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$ *for almost every* $t \in Dom(\gamma)$,

iii) $\mathcal{H}_S(\gamma(t), \lambda(t), u(t)) = H_S(\gamma(t), \lambda(t))$ *for almost every* $t \in Dom(\gamma)$.

Definition 3 (extremals, optimals, abnormal extremals) *If a pair* (γ, λ) *satisfies the PMP (that is the conclusion of Theorem 2 holds), we say that the pair* (γ, λ) *is extremal. Moreover, if* $\lambda_0 = 0$ *then we say that* (γ, λ) *is an abnormal extremal. Finally, if* γ *is optimal we say that the pair* (γ, λ) *is optimal.*

Given the system Σ, it is possible to construct a synthesis Γ for Σ following the classical idea developed in [5, 6, 7, 8, 10, 11, 13, 14, 17, 18] of constructing extremal trajectories and deleting those trajectories that are not optimal. By construction this synthesis is optimal.

4.2 Basic Properties of Extremal Trajectories, Special curves and Loci

In the following we set:

$$X_S^+ = F_S + G_S \text{ and } X_S^- = F_S - G_S. \tag{30}$$

Let $\gamma : [t_1, t_2] \to S^2$ be a trajectory of (24). If γ corresponds to the constant control +1 (resp. -1) in $[t_1, t_2]$ we say that $\gamma|_{[t_1,t_2]}$ is a X^+–trajectory (resp. X^-–trajectory). Moreover we call $\gamma^\pm$ the trajectories exiting the point x_0 with respectively constant control $+1$ and -1. Let $t_{op}^\pm$ be the last times in which $\gamma^\pm$ are optimal. We define $\gamma_{op}^\pm = \gamma^\pm|_{[0,t_{op}^\pm]}$.

If $\gamma_1 : [a, b] \to S^2$ and $\gamma_2 : [b, c] \to S^2$ are trajectories of Σ, such that $\gamma_1(b) = \gamma_2(b)$, then the *concatenation* $\gamma_2 * \gamma_1$ is the trajectory:

$$(\gamma_2 * \gamma_1)(t) := \begin{cases} \gamma_1(t) \text{ for } t \in [a, b] \\ \gamma_2(t) \text{ for } t \in [b, c]. \end{cases}$$

Notice that in the notation $\gamma_2 * \gamma_1$, γ_1 comes first.

4.2.1 Frame Curves and Frame Points

In [14] (see also [6, 8]) it was proved that the control system (24) with M a smooth two-dimensional manifold and F_d and G_d smooth generic vector fields, admits an optimal regular synthesis in time T. Moreover it was proved that there exists a stratification of $\mathcal{R}(T)$ such that the optimal synthesis can be obtained from a feedback $u(x)$ satisfying:

- on strata of dimension 2, we have $u(x) = \pm 1$,
- on strata of dimension 1, called <u>frame curves</u> (in the following FCs), $u(x) = \pm 1$ or $u(x) = \varphi(x)$, where $\varphi(x)$ is a suitable feedback that can be explicitly computed in terms of F_S, G_S and the Lie bracket $[F_S, G_S]$.

The strata of dimension 0 are called <u>frame points</u> (in the following FPs). Every FP is an intersection of two FCs. A FP x that is the intersection of two frame curves F_1 and F_2 is called a (F_1, F_2) Frame Point. In [14] it is provided a complete classification of all types of FPs and FCs, under generic conditions. All the possible FCs are the following:

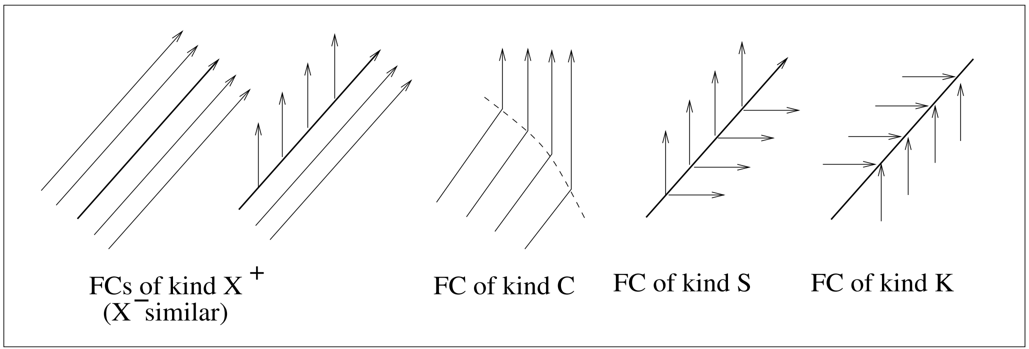
- FCs of kind X^+ (resp. X^-), that correspond to subsets of γ_{op}^+ (resp. subsets of γ_{op}^-),
- FCs of kind C, called *switching curves*, that are curves made of switching points,
- FCs of kind S, i.e. singular trajectories,
- FCs of kind K, called overlaps, reached optimally by two trajectories coming from different directions.
- FCs that are arcs of optimal trajectories that start at FPs. These trajectories "transport" special information. There are three kind of these curves:

– curves of kind γ_A that are abnormal extremals,

– curves of kind γ_k that are curves starting at the terminal point of an overlap

– curves of kind γ_0 that are the other arcs of optimal trajectories that start at FPs.

The FCs of kind X^+, C, S, K are depicted in the following figure.



There are seventeen topological equivalence classes of FPs. A detailed description can be found in [14, 6, 8].

As we will see, also in our (highly non generic) specific problem (24), there exists a regular optimal synthesis and FCs are of the kind described above.

4.2.2 Switching function, the $\Delta_A^{-1}(0), \Delta_B^{-1}(0)$ loci

As usual switchings are determined by the switching function:

Definition 4 (Switching Function) *Let $(\gamma, \lambda) : [0, \tau] \to \mathbb{R}^2$ be an extremal pair. The corresponding switching function is defined as $\phi(t) :=< \lambda(t), G_S(\gamma(t)) >$.*

From the PMP it immediately follows:

Lemma 1 *Let $(\gamma, \lambda) : [0, \tau] \to \mathbb{R}^2$ be an extremal pair and $\phi(.)$ the corresponding switching function. If $\phi(t) \neq 0$ for some $t \in]0, \tau[$, then there exist $\varepsilon > 0$ such that γ corresponds to a constant control $u = sgn(\phi)$ in $]t - \varepsilon, t + \varepsilon[$.*

If $\phi(t) \equiv 0$ in some interval $[a, b]$, then the corresponding trajectory is called singular in $[a, b]$.

In the following an essential role is played by the following:

Definition 5 *For each $x \in S^2$, define:*

$$\Delta_A(x) = \det\Big(F_S(x), G_S(x)\Big) = (F_S)_1(x)(G_S)_2(x) - (F_S)_2(x)(G_S)_1(x).$$
$$\Delta_B(x) = \det\Big(G_S(x), [F_S, G_S](x)\Big) = (G_S)_1(x)[F_S, G_S]_2(x) - (G_S)_2(x)[F_S, G_S]_1(x),$$

The locus $\Delta_A^{-1}(0)$ is essential for studying abnormal extremals while the locus $\Delta_B^{-1}(0)$ for studying singular trajectories. Moreover, in the regions where they are both different from zero, an optimal trajectory has at most one switching. These facts are described by the following propositions:

Proposition 5 *Let $\gamma : [0, \tau] \to \mathbb{R}^2$ be an extremal trajectory for the control problem (24), t_0 a time such that $\phi(t_0) = 0$, $G_S(\gamma(t_0)) \neq 0$. Then the following conditions are equivalent:*

(a) *γ is an abnormal extremal;*

(b) *$\gamma(t_0) \in \Delta_A^{-1}(0)$;*

(c) *$\gamma(t) \in \Delta_A^{-1}(0)$ for every time t such that $\phi(t) = 0$, $G_S(\gamma(t)) \neq 0$.*

Proposition 6 *Let $\gamma : [a, b] \to S^2$ be an extremal singular trajectory (that is $\phi(t) \equiv 0$ in $[a, b]$. Then $Supp(\gamma) \subseteq \Delta_B^{-1}(0)$.*

The following proposition describe what happen where Δ_A and Δ_B are different from zero. Before we need a definition and a lemma.

Definition 6 *A point $x \in \mathbb{R}$ is called an* ordinary point *if $x \notin \Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$. If x is an ordinary point we define the scalar function f_S, g_S to be the coefficients of the linear combination: $[F_S, G_S](x) = f_S,(x)F_S(x) + g_S,(x)G_S(x)$.*

The relation between $f_S,$, Δ_A and Δ_B is given by:

Lemma 2 *Let x an ordinary point then:*

$$f_S,(x) = -\frac{\Delta_B(x)}{\Delta_A(x)}. \tag{31}$$

Proposition 7 *Let $\Omega \in S^2$ be an open set such that every $x \in \Omega$ is an ordinary point. Then all extremal trajectories γ of $\Sigma|_\Omega$ are bang-bang with at most one switching. Moreover if $f_S, > 0$ throughout Ω then γ is a X_S^+, X_S^- or $X_S^+ * X_S^-$–trajectory (the X_S^- comes first). If $f_S < 0$ throughout Ω then γ_S is an X_S^+, X_S^- or $X_S^- * X_S^+$–trajectory.*

For our specific problem (24) we have:

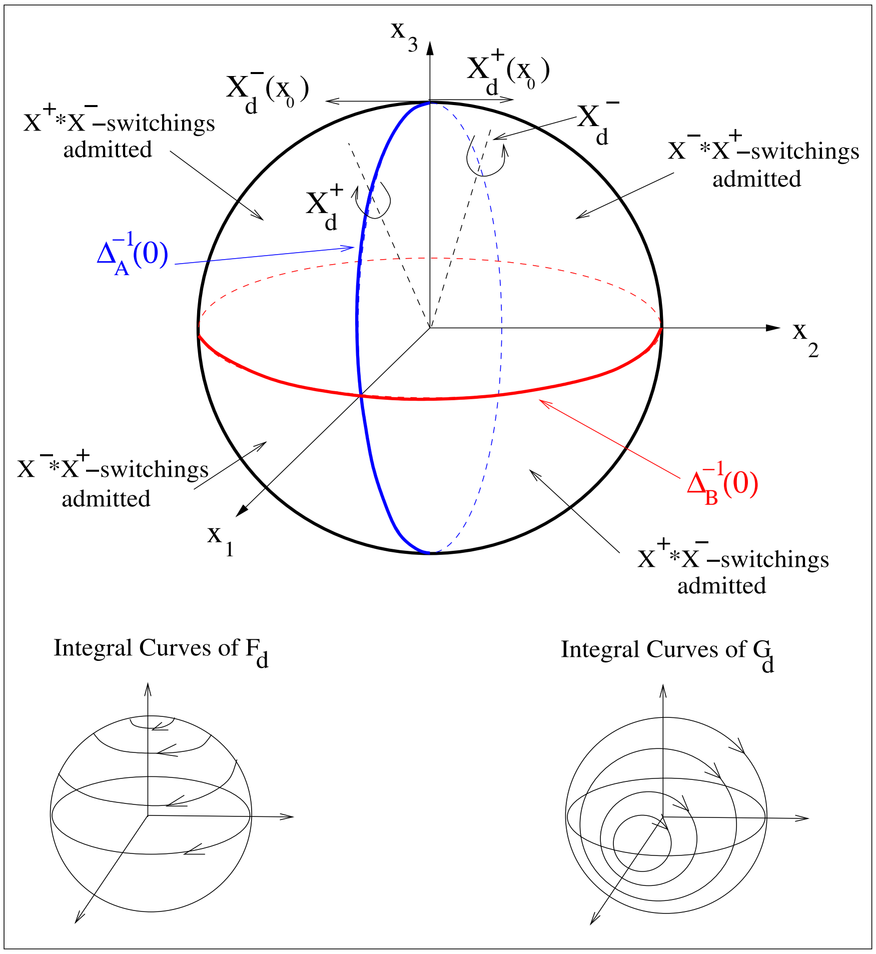
$$\Delta_A^{-1}(0) = \{(x_1, x_2, x_3)^T \in S^2 : x_1 = 0\}$$
$$\Delta_B^{-1}(0) = \{(x_1, x_2, x_3)^T \in S^2 : x_3 = 0\}$$

Then from Propositions 6, 7, it follows

Proposition 8 *Let $\gamma : [0, T] \to S^2$, $\gamma(0) = x_0$ be an optimal trajectory for the control system (24), (24). Then:*

- *γ has at most a $X^- * X^+$ switching in the region $\{x \in S^2 : x_3, x_2 > 0\}$. (That is if $Supp(\gamma|_{[a,b]}) \subset \{x \in S^2 : x_3, x_2 > 0\}$ then $\gamma|_{[a,b]}$ corresponds to one of the three following controls:*
  - *$u = +1$ in $[a, b]$*
  - *$u = -1$ in $[a, b]$,*
  - *there exists $c \in]a, b[$, such that $u = +1$ in $[a, c[$ $u = -1$ in $]c, b]$)*
- *γ has at most a $X_S^+ * X_S^-$ switching in the region $\{x \in S^2 : x_3 > 0, x_2 < 0\}$*
- *γ has at most a $X_S^+ * X_S^-$ switching in the region $\{x \in S^2 : x_3 < 0, x_2 > 0\}$*
- *γ has at most a $X_S^- * X_S^+$ switching in the region $\{x \in S^2 : x_3 < 0, x_2 < 0\}$*

The set $\Delta_B^{-1}(0)$ is called the *equator* and $\Delta_A^{-1}(0)$ *the meridian*. In the following figure the integral curves of F_S, G_S, X_S^+, X_S^- and the loci $\Delta_A^{-1}(0), \Delta_B^{-1}(0)$ are shown. Moreover the admitted switchings are indicated



4.2.3 Two crucial properties of extremal trajectories

The following two propositions are essential in the construction of the optimal synthesis.

Proposition 9 *Let $\gamma : [0, T] \to S^2$ be an optimal trajectory for the control system (24) and $0 < s < t_1 < t_2, \ldots$ be the sequence of the switching times. Then $t_1 = t_2 = \ldots\ldots = v(s)$, where:*

$$v(s) := 2\pi - 2 \arctan(\Big(\cos(s) + \cot(\alpha)^2\Big) \csc(s)) \tag{32}$$

Proof of Proposition 9: Consider $\gamma_H : [0, T] \to S0(3)$ the optimal trajectory that projects on γ through p_H. Thanks to Proposition 2, we have $\gamma_H = B_s B_{t_1} B_{t_1} \cdots$. Moreover, since that curve projects on an optimal trajectory for (24), we will establish a relation between s and t_1. We start from the relations $\varphi_2(s) = \varphi_2(s+t_1) = 0$, which can be written

$$< \lambda(s), G_S(\gamma(s)) >=< \lambda(s + t_1), G_S(\gamma(s + t_1)) >= 0. \tag{33}$$

Recall that $\lambda(s) = \lambda(0)e^{-sX_\varepsilon}$, $\lambda(s + t_1) = \lambda(0)e^{-sX_\varepsilon}e^{-t_1X_{-\varepsilon}}$ and $\gamma(s) = e^{sX_\varepsilon}\gamma(0)$, $\gamma(s + t_1) = e^{t_1X_{-\varepsilon}}e^{sX_\varepsilon}\gamma(0)$. Since γ is nontrivial, then $\lambda(0)$ is a nonzero line-vector of $\mathbb{R}^3$. Moreover, $\gamma(0) = x_0 = (0, 0, 1)^T$ Equation (33) can be written

$$\lambda(0)e^{-sX_\varepsilon}(G \times e^{sX_\varepsilon}\gamma(0)) = 0, \quad \lambda(0)e^{-sX_\varepsilon}e^{-t_1X_{-\varepsilon}}(G \times e^{t_1X_{-\varepsilon}}e^{sX_\varepsilon}\gamma(0)) = 0.$$

The previous equation can be transformed to

$$\det(e^{sX_\varepsilon}\lambda(0)^T, G, e^{sX_\varepsilon}\gamma(0)) = 0, \quad \det(e^{t_1X_{-\varepsilon}}e^{sX_\varepsilon}\lambda(0)^T, G, e^{t_1X_{-\varepsilon}}e^{sX_\varepsilon}\gamma(0)) = 0,$$

and then to

$$\det(e^{sX_\varepsilon}\lambda(0)^T, G, e^{sX_\varepsilon}\gamma(0)) = 0, \quad \det(e^{sX_\varepsilon}\lambda(0)^T, e^{-t_1X_{-\varepsilon}}G, e^{sX_\varepsilon}\gamma(0)) = 0.$$

Since $e^{sX_\varepsilon}\lambda(0)^T$ is not zero, we deduce that

$$\det(g, e^{sX_\varepsilon}\gamma(0), e^{-t_1X_{-\varepsilon}}G) = 0. \tag{34}$$

We end up with the relation

$$-s_\alpha\cos(s - t_1/2) = c_\alpha\cos(t_1/2). \tag{35}$$

Taking into account that $\pi \leq t_1 < 2\pi$, we can simplify the previous equation to get (32).

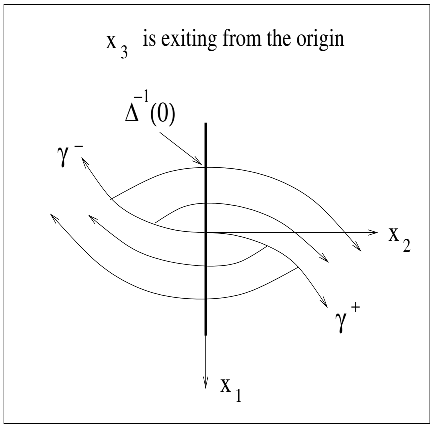
Proposition 10 *Let $\gamma : [0,T] \to S^2$ be an optimal trajectory containing a complete bang arc, then γ does not contain any singular arc.*

4.3 Construction of Time Optimal Synthesis

If $\Delta_B(x) \neq 0$ in a neighborhood of the origin (condition satisfied for our specific problem (24)), then it is well know (see for instance [14, 8]) that:

Proposition 11 *There exists T such that $\gamma : [0,T] \to \mathcal{R}(T)$ is optimal iff it is bang-bang with at most one switching.*

From Proposition 11, one gets the shape of the optimal synthesis in a neighborhood of the origin for the problem (24). See the following picture.



Consider now the trajectories $\gamma^\pm$. It holds:

Proposition 12 *Assume that there exists T such that:*

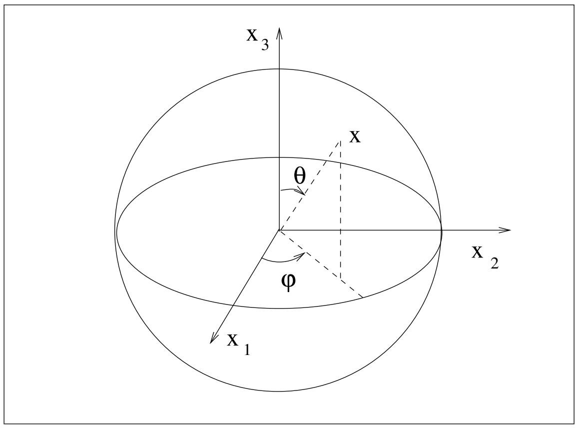
- $\gamma^+(t) \notin \Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$*, for every* $t \in]0,T[$,
- $\gamma^+(T) \notin \Delta_B^{-1}(0)$
- γ *crosses* $\Delta_A^{-1}(0)$ *at time* T,

then γ^+ is extremal up to time T. Moreover an extremal trajectory coinciding with γ^+ in $[0,T]$, has a $X_S^- * X_S^+$ switching at time T.

A similar result holds for γ^-. Since $\alpha < \pi/4$, this Proposition applies in our case with $T = \pi$. From Proposition 10 it follows that on the sphere every optimal trajectory does not contain any singular arc.

Let us use spherical coordinates (see the following figure):

$$\begin{cases} x_1 = \cos(\varphi)\sin(\theta) \\ x_2 = \sin(\varphi)\sin(\theta) \\ x_1 = \cos(\varphi), \end{cases} \tag{36}$$



and in this new system of coordinates set:

$$\begin{cases} P_0 = (0,0) \\ P_1^+ = (2\alpha, 0), \quad P_2^+ = (4\alpha, 0), \quad P_3^+ = (6\alpha, 0), ... \\ P_1^- = (-2\alpha, 0), \quad P_2^- = (-4\alpha, 0), \quad P_3^- = (-6\alpha, 0), ... \end{cases} \tag{37}$$

Assume first $\alpha < \pi/8$. This means that the X_S^--trajectory starting at the point P_1^+ will not intersect the locus $\Delta_B^{-1}(0)$, while it crosses the locus $\Delta_A^{-1}(0)$ after time π. From Proposition 5, 7 it follows that a switching curve of equation:

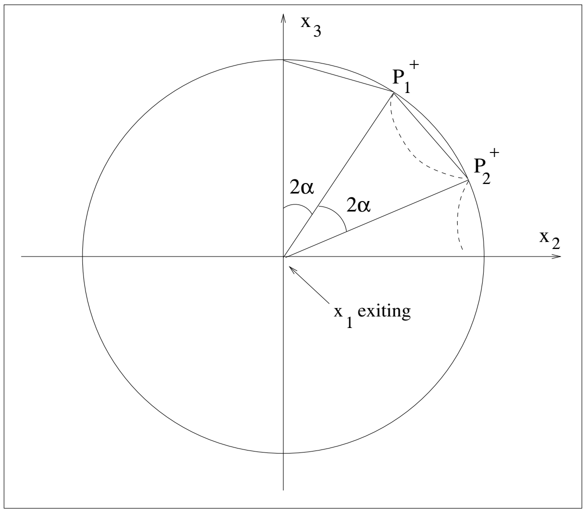
$$C_1^-(s) := e^{X_S^+ s} e^{X_S^- v(s)} x_0, \quad s \in [0, \pi] \tag{38}$$

joins the point P_1^- with the point P_2^-, and lies in the region $x_1 < 0, x_2 < 0, x_3 > 0$. The situation is exactly symmetric for $x_1, x_2 \to -x_1, -x_2$ and will continues similarly exactly as in the case of the pendulum (see for instance [15, 14]) up to when the switching curve starting from (some) $P_n^\pm$ intersects the equator. Notice that:

- all the trajectories switching exactly on the meridian are abnormal extremals;
- for every i, the switching curves starting and arriving at $P_i^\pm$ are tangent to the extremal trajectory switching in $P_i^\pm$. This is a general result for abnormal extremals studied in details in [7];
- if $\alpha \in [\pi/8, \pi/4[$, then the first switching curve intersects the equator.
- the switching curve intersecting the equator starts at $P_n^\pm$ where:

$$N_{eq} = \left[\frac{\pi/2}{2\alpha}\right]. \tag{39}$$

  This fact is clarified in the following picture:



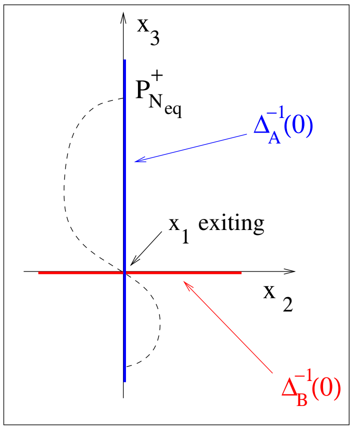
Let us study the switching curves starting at $P^{\pm}_{N_{eq}}$, that is the switching curves intersecting the equator. These curves has equation

$$C^{(\pm)}_{N_{eq}}(s) := e^{X_S^{\pm} s} e^{X_S^{\mp} v(s)} e^{X_S^{\pm} v(s)} ... x_0, \quad s \in [0, \pi] \tag{40}$$

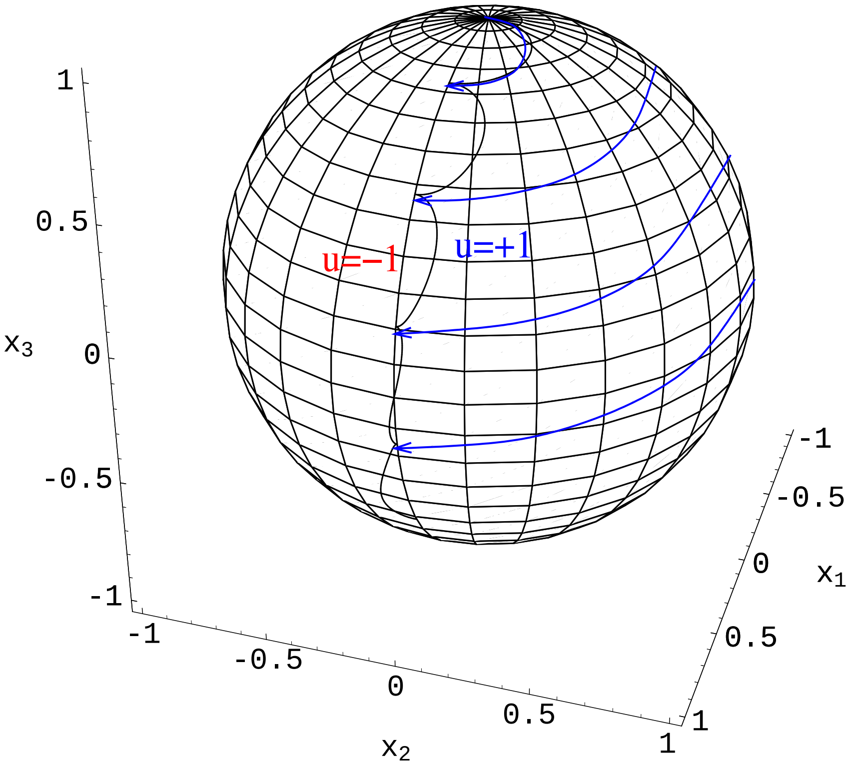
where after $e^{X_S^{\pm} s} x_0$ we have N_{eq} terms of kind $e^{X_S^{\pm} v(s)}$ or of kind $e^{X_S^{\mp} v(s)}$. The label $(\pm)$ in $C^{(\pm)}$ is chosen in such a way that $C^{(+)}$ starts at $P^{+}_{N_{eq}}$.

Proposition 13 *The switching curves $C^{(\pm)}_{N_{eq}}(s)$, $\quad s \in [0, \pi]$, intersect the equator at the points $\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0) = \{(\pm 1, 0, 0)\}^T$. More precisely $C^{(+)}_{N_{eq}}$ (resp. $C^{(-)}_{N_{eq}}$) intersects the equator at at the point $(1, 0, 0)^T$ (resp. $(-1, 0, 0)^T$) and lies in the region $\{x_3 > 0, x_2 > 0\} \cup \{x_3 < 0, x_2 < 0\}$ (resp. $\{x_3 > 0, x_2 < 0\} \cup \{x_3 < 0, x_2 > 0\}$.*

The switching curve $C^{(+)}_{N_{eq}}$ is drawn in the following figure.



The situation will continue similarly under the equator according to Proposition 7, and with switching curves starting at $P^{\pm}_{n}$ and ending at $P^{\pm}_{n+1}$ $(n = N_{eq} + 1, ...)$ but with switching curves lying on the other side (see figure).



Set:

$$N = \left[\frac{\pi}{2\alpha}\right].$$

Analyzing the evolution of the minimum time wave front in a neighborhood of the south-pole one can easily get:

Proposition 14 *It holds:*

R1 *For $T \le (N-1)\pi$, the synthesis built above is optimal.*

R2 *Every $x \in S^2$ is reached in time $T \le (N+1)\pi$.*

R3 *Every optimal trajectory has at most N switching.*

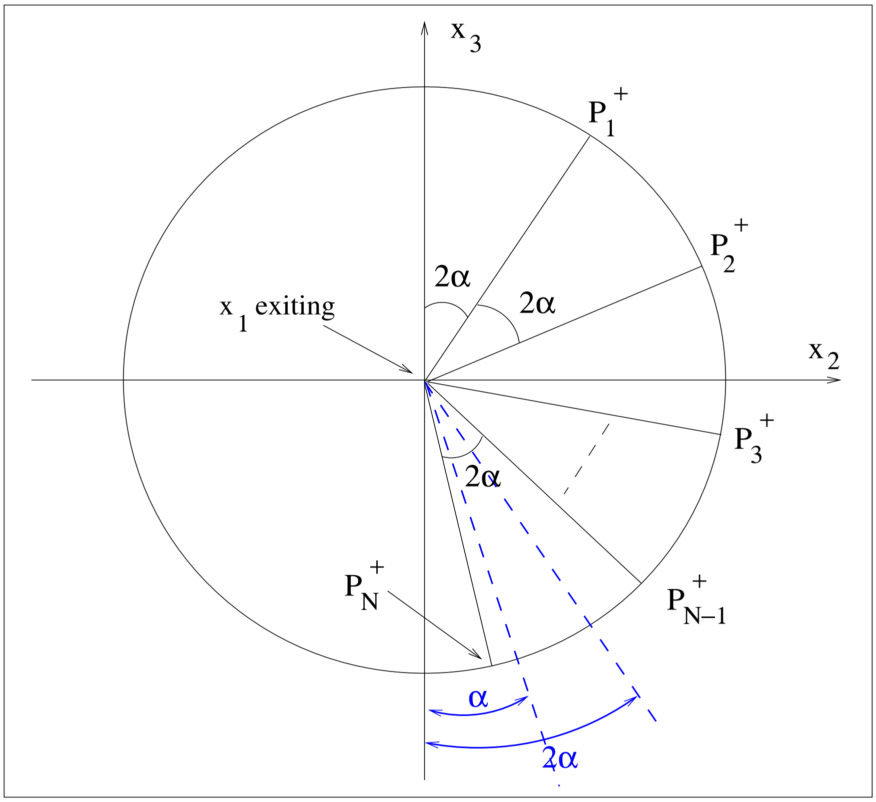
R4 *There exists an optimal trajectory having $N-1$ switchings*

The way in which the minimum time wave front "closes" the south pole is local, that means the following.

Proposition 15 *There exists $T > 0$ such that the minimum time wave front is always homeomorphic to a circle for every $t \in]0, T[$ and for time T it is the south pole (that is the last point reached.)*

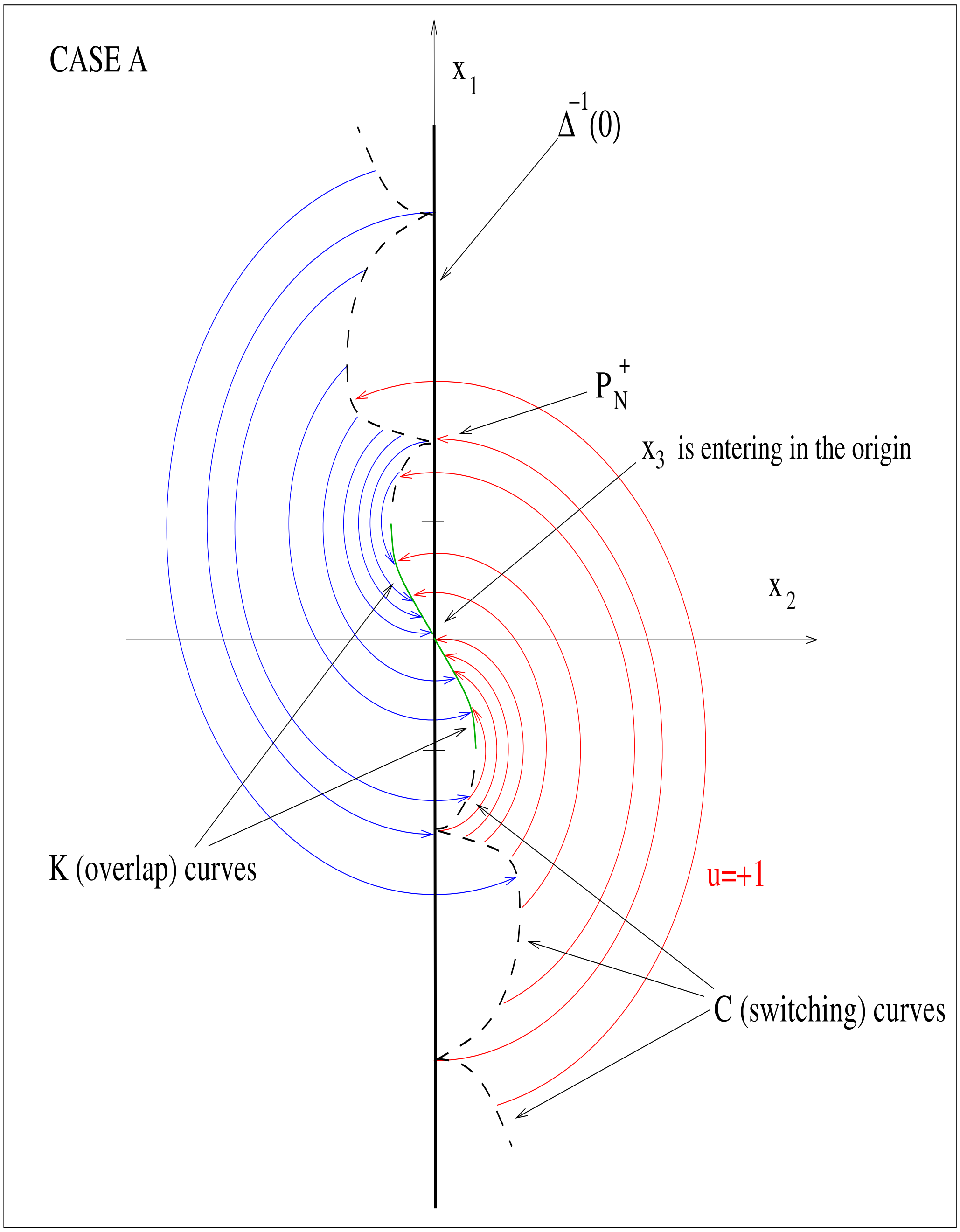
The shape of the optimal synthesis for time $T > (N-1)\pi$, depends on the rest (see figure):

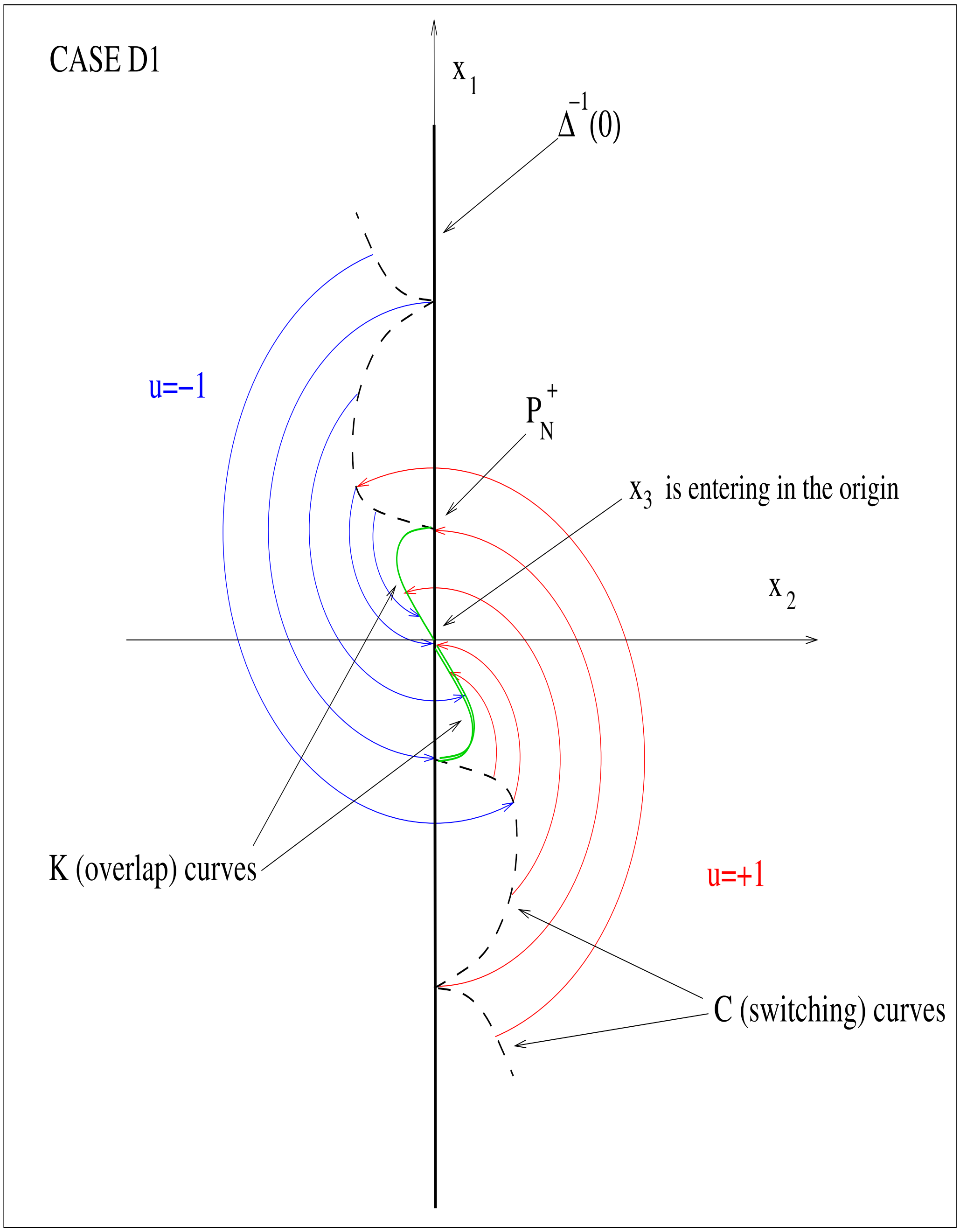
$$r = \pi - 2\alpha N = \pi - 2\alpha \left[\frac{\pi}{2\alpha}\right].$$

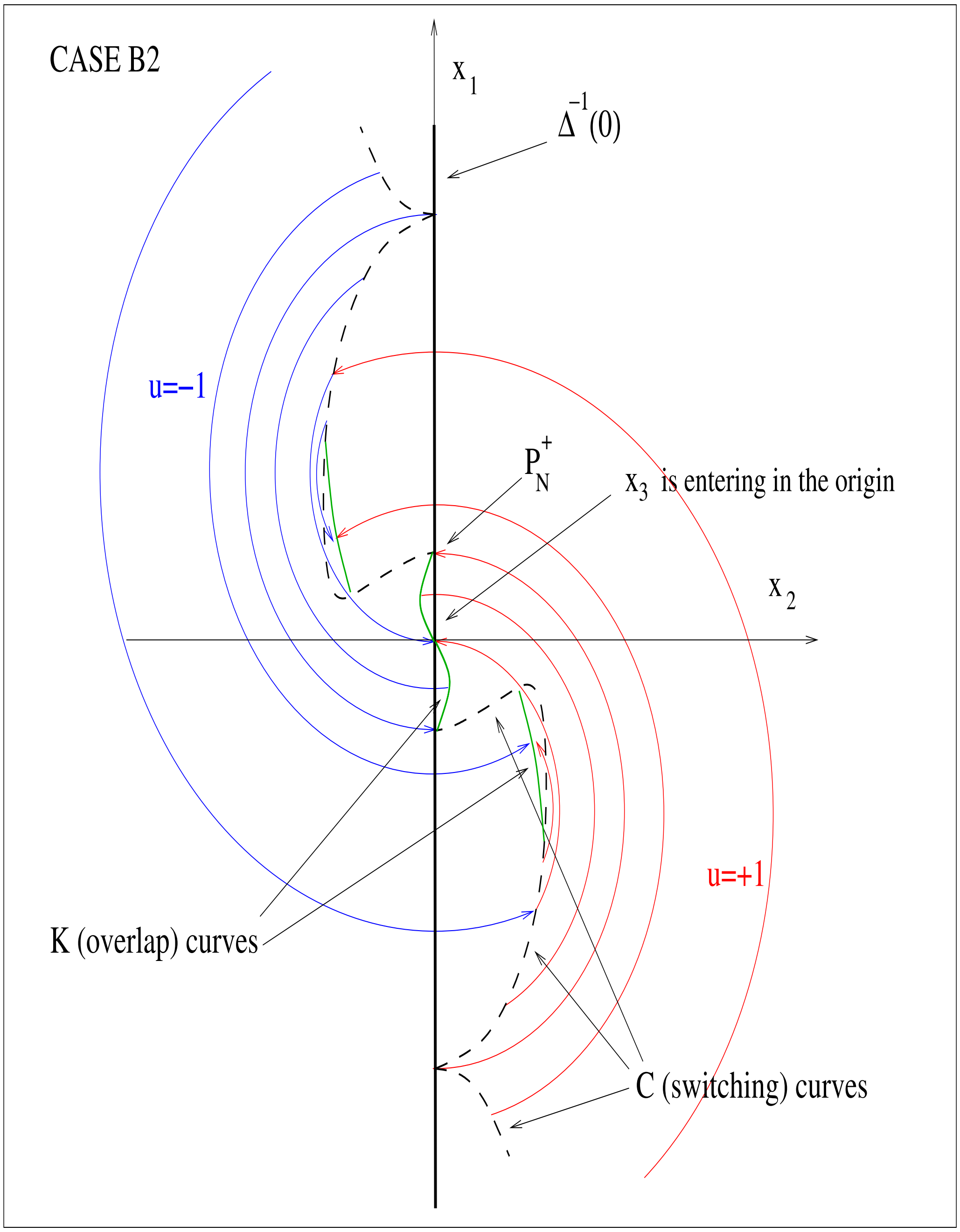


We have 4 different situation:

- CASE A: $r \in]\alpha, 2\alpha[$ In this case the switching curve starting at P_N^+ glues to an overlap curve that pass through the origin (see figure A).
- CASE D1 $r = \alpha$ In this degenerate case an overlap curve starts exactly at P_N^+ and pass through the origin (see figure D1).
- CASE B: $r \in]0, \alpha[$ We have two possible cases. There exists $\alpha_0 \in]0, \alpha[$ (depending from α) such that:
  - if $r \in]\alpha_0, \alpha[$ then the situation is exactly as in CASE D1.
  - if $r \in]0, \alpha_0[$ then the situation is more complicated and it is depicted in figure B2.
- CASE D2: $r = 0$ In this case the situation is exactly as in CASE D1.


CASE A
x_1
$\Delta^{-1}(0)$
P_N^+
x_3 is entering in the origin
x_2
K (overlap) curves
u=+1
C (switching) curves


CASE D1
x_1
$\Delta^{-1}(0)$
u=−1
P_N^+
x_3 is entering in the origin
x_2
K (overlap) curves
u=+1
C (switching) curves



From the analysis of this section one get easily Theorem 1.

5 Applications

The control system Σ can be seen as a model for reorienting a satellite: the vector fields f, g represent infinitesimal rotations of the satellite about two orthogonal fixed axis. One rotor is always turned on while the second is controlled by means of the parameter u. The problem consists of orienting the satellite from a given position to another one in minimum time.

On the other side the control system Σ projected on the sphere, describes a three-level quantum system driven by two external fields in resonance, in a model in which only closed levels are coupled and in which we control the amplitude of one of the two lasers.

A three-level quantum system driven by two external fields (in the rotating wave approximation) of arbitrary shape and frequency is described by the following time dependent Schrödinger equation (in a system of units such that $\hbar = 1$):

$$i\frac{d\psi(t)}{dt} = H\psi(t), \tag{41}$$

where $\psi(.) : \mathbb{R} \to \mathbb{C}^3$ and:

$$H = \begin{pmatrix} E_1 & \Omega_1 & 0 \\ \Omega_1^* & E_2 & \Omega_2 \\ 0 & \Omega_2^* & E_3 \end{pmatrix}. \tag{42}$$

Here $(^*)$ indicates the complex conjugation involution. The controls $\Omega_1(.), \Omega_2(.)$, that we assume to be different from zero only in a fixed interval $[0, T]$, are connected to the physical parameters by $\Omega_j(t) = \mu_j \mathcal{F}_j(t)/2$, $j = 1, 2$, with $\mathcal{F}_j$ the external pulsed field and μ_j the couplings (intrinsic to the quantum system) that we have restricted to couple only levels j and $j+1$ by pairs. If the lasers are assumed to be in resonance with the difference of between the energy levels that they are coupling:

$$\begin{aligned} &\Omega_j(t) = u_j(t)\ e^{i(\omega_j t + \alpha_j)}, \quad \omega_j = E_{j+1} - E_j, \\ &u_j(.) : \mathbb{R} \to \mathbb{R}, \quad \alpha_j \in [-\pi, \pi], \quad j = 1, 2. \end{aligned} \tag{43}$$

it is easy to see that this problem restrict to reals and it is equivalent to the problem described by equations (22), (24) (see [2, 3, 4]). For this problem going from the point $(0, 0, 1)$ to the point $(1, 0, 0)$ means precisely to go from the state number one to the state number 3 (the two nondirecly coupled levels) with two lasers in resonance and controlling the amplitude of one.

The time optimal synthesis built in section 4 could be very useful for applications. In fact the problem of design an efficient transfer of population between different atomic or molecular levels is crucial in many atomic-physics projects. Excitation or ionization is usually accomplished by using a sequence of laser pulses to drive state to state transitions The transfer should be as efficient as possible in order to minimize the effects of relaxation or decoherence that are always present.

References

[1] A.A. Agrachev and R.V. Gamkrelidze, "Symplectic Geometry for Optimal Control", in Nonlinear Controllability and Optimal Control (Hector J. Sussmann, ed.), Pure and Applied Mathematics, Vol. 133, Marcel Dekker, 1990.

"Symplectic methods for optimization and control", in Geometry of feedback and optimal control, 19–77, Monogr. Textbooks Pure Appl. Math., 207, Dekker, New York, 1998.

[2] U. Boscain, Grégoire Charlot, Jean-Paul Gauthier, Stéphane Guérin and Hans–Rudolf Jauslin, *Optimal Control in laser-induced population transfer for two- and three-level quantum systems*, Journal of Mathematical Physics, v.43, 2107-2132 (2002)

[3] U. Boscain, T. Chambrion, J.P. Gauthier, On the K+P Problem for a three-level quantum system: optimality implies resonance Journal of Dyn. and Contr. Systems, Vol. 8, No. 4, October 2002, 547572 (2002)

[4] U. Boscain, Gregoire Charlot "Resonance of Minimizers for n-level Quantum Systems" submitted to 41th IEEE Conference on Decision and Control.

[5] U. Boscain and B. Piccoli, " Extremal syntheses for generic planar systems", *Journal of Dynamical and Control Systems*, Vol.7 (2001) N.2, pp. 209–258.

[6] Ugo Boscain and Benedetto Piccoli, "Morse Properties for the Minimum Time Function on 2-D Manifolds" J. Dynam. Control Systems 7 (2001), no. 3, 385–423.

[7] Ugo Boscain and Benedetto Piccoli, "On Automaton Recognizability of Abnormal Extremals", *SIAM J. Control and Optimization,* Vol. 40, No. 5, pp. 1333 - 1357 (2002).

[8] Ugo Boscain and Benedetto Piccoli "Synthesis Theory for Generic Control Systems on 2-D Manifolds" book in preparation.

[9] Ugo Boscain and Yacine Chitour, On the minimum time problem for driftless left-invariant control systems on $SO(3)$, to appear

[10] A. Bressan and B. Piccoli, "Structural Stability for Time–Optimal Planar Syntheses", *Dynamics of Continuous, Discrete and Impulsive Systems*, Vol.3 (1997), pp. 335-371.

[11] A. Bressan and B. Piccoli, "A Generic Classification of Time Optimal Planar Stabilizing Feedbacks", *SIAM J. Control and Optimization,* Vol.36 No.1 (January 1998), pp. 12-32.

[12] Jurdjevic V., "Geometric Control Theory," Cambridge Studies in Adv. Math., Cam. Univ. Press, 1997.

[13] B. Piccoli, "Regular Time–Optimal Syntheses for Smooth Planar Systems," *Rend. Sem Mat. Univ. Padova*, Vol.95 (1996), pp. 59-79.

[14] B. Piccoli, "Classifications of Generic Singularities for the Planar Time-Optimal Synthesis", *SIAM J. Control and Optimization,* Vol.34 No.6 (1996), pp. 1914-1946.

[15] L.S. Pontryagin, V. Boltianski, R. Gamkrelidze and E. Mitchtchenko, "The Mathematical Theory of Optimal Processes", *John Wiley and Sons, Inc (1961).*

[16] Strichartz, R., "Sub-Riemannian Geometry," in *J. Diff. Geom.,* 24, pp.221-263, 1983.

[17] H.J. Sussmann, "The structure of time–optimal trajectories for single-input systems in the plane: the general real–analytic case," *SIAM J. Control and Opt.*, 25 (1987), pp. 868-904.

[18] H.J. Sussmann, "Regular synthesis for time optimal control of single–input real–analytic systems in the plane," *SIAM J. Control and Opt.*, 25 (1987), pp.1145-1162.

[19] Sussmann H.J. and Tang G., "Time-optimal Control of a Satellite with Two Rotors Attached Along Its Two Fixed Axes," preprint, 1995.

[20] H.J. Sussmann, "Envelopes, conjugate points, and optimal bang-bang extremals", in *Algebraic and geometric methods in nonlinear control theory*, 325–346, Math. Appl., 29, Reidel, Dordrecht, (1986).

[21] H.J. Sussmann, "Envelopes, higher–order optimality conditions and Lie Brackets", in *Proc. 1989 I.E.E.E, Conf. Decision and Control.*