Controllability by Low Modes Forcing of the Navier-Stokes Equation with Periodic Data

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Abstract

The aim of this work is to use the methods of geometric nonlinear control theory for studying controllability of the Galerkin approximation of the Navier-Stokes equation, controlled by degenerate (in few low modes) forcing.

Keywords: Navier-Stokes equation, controllability

1 Introduction and preliminary material

We study 2- and 3- dimensional Navier-Stokes equation with a controlled (nonrandom) forcing

$$\partial u/\partial t + (u \cdot \nabla)u + \nabla p = \nu \Delta u + F, \quad (1)$$
$$\nabla \cdot u = 0 \quad (2)$$

We assume the boundary conditions to be periodic, this means that $u(t,\cdot), p(t,\cdot)$ and $F(t,\cdot)$ are defined on a 2 or 3-dimensional torus $\mathbb{T}^k,\ k = 2,3$.

1.1 3D Navier-Stokes Equation

Consider the 3-dimensional Navier-Stokes equation (1)-(2).

To reduce this equation to an infinite-dimensional system of ordinary differential equations we will use "spectral algorithm" ([8]) invoking Fourier expansion of solution $u(t,x)$ with respect to the basis of eigenvectors (eigenfunctions) of the Laplacian operator on $\mathbb{T}^3$: $u(x,t) = \sum_k \underline{q}_k(t) e^{ik\cdot x}$. Here k is a 3-dimensional vector with integer components and $\underline{q}_k$ is *vector-valued* function. For u to satisfy the incompressibility condition the coefficients $\underline{q}_k$ have to be orthogonal to respective k : $\underline{q}_k \cdot k = 0$. Similarly we introduce the expansions for the pressure and the forcing:

$$p(x,t) = \sum_k \underline{p}_k(t) e^{ik\cdot x},\ F(x,t) = \sum_k \underline{v}_k(t) e^{ik\cdot x}.$$

We assume that the forcing has zero average ($\underline{v}_0 \equiv 0$) and then changing the reference frame (to the one uniformly moving with the center of mass) we may assume $\int u\, dx = 0$ and therefore $\underline{q}_0 = 0$. It is known that the pressure term can be separated from equations for $\underline{q}_k$ which take form of ODE:

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n) \Pi_k \underline{q}_n - \\ -\nu |k|^2 \underline{q}_k(t) + \underline{v}_k(t). \quad (3)$$

Here Π_k stays for the orthogonal projection of $\mathbb{R}^3$ onto the plane $k^\perp$ orthogonal to k. Formally we should also take the projection $\Pi_k \underline{v}_k(t)$ of the forcing, but the k-directed component of $\underline{v}_k$ can be taken into account by the pressure term.

Since $u(x,t), F(x,t)$ are real-valued we have to assume that $\underline{q}_k = \bar{\underline{q}}_{-k}, \underline{v}_k = \bar{\underline{v}}_{-k}$.

Important: in the equations (6) and (7) there is infinite number of terms under the summation sign and the components of Q enter these equations for the observed coordinates.

1.2 2D Navier-Stokes Equation

In the 2D case the reduction to the ODE form is easier. Introducing the vorticity $w = \nabla^\perp \cdot u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ and applying the operator $\nabla^\perp$ to the equation (1) we arrive to

$$\partial w/\partial t + (u \cdot \nabla)w = \nu \Delta w + df/dt, \quad (4)$$

where $f = \nabla^\perp \cdot F$.

Remark that: i) $\nabla^\perp \cdot \nabla p = 0$, ii) $\nabla^\perp$ and Δ commute as long as both are differential operators with

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constant coefficients; iii) $\nabla^\perp \cdot (u \cdot \nabla)u = (u \cdot \nabla)(\nabla^\perp \cdot u) + (\nabla^\perp \cdot u)(\nabla \cdot u) = (u \cdot \nabla)w$, for each u satisfying (2).

It is known that u satisfying (2) can be recovered in unique way (up to an additive constant) from w. From now on we will deal with the equation (4). Introduce again the Fourier expansion $w(t,x) = \sum_k q_k(t)e^{ikx}$, $f(t,x) = \sum_k v_k(t)e^{ikx}$. As far as w and v are real-valued, we get $\bar{w}_n = w_{-n}$, $\bar{v}_n = v_{-n}$. We assume $w_0 = 0, v_0 = 0$.

Then $\partial w/\partial t = \sum_k \dot{q}_k(t)e^{ikx}$ and after computing $(u\cdot\nabla)w$ we arrive to the (infinite-dimensional) system of equations for $q_k (k \in \mathbb{Z}^2)$:

$$\dot{q}_k(t) = \tag{5}$$
$$= \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k(t) + v_k(t).$$

2 Navier-Stokes equation controlled by a degenerate forcing. Problem setting

From now on we assume the forcing terms $\underline{v}_k$ in (3) and v_k in (5) to be controls at our disposal. Then (3) and (5) can be seen as infinite-dimensional control systems. We are going to study their controllability properties.

We will be specifically interested in the case where the controlled forcing is *degenerate*. This means that all but few v_k vanish identically, while these few can be chosen freely. From now on we fix a set of controlled modes $\mathcal{K}^1 \subset \mathbb{Z}^j$, $j = 2,3$ and assume $v_k \equiv 0$, $\forall k \notin \mathcal{K}^1$.

Further on we follow the dynamics of selected or *observed* modes indexed by a *finite* set $\mathcal{K}^{obs}$. We assume $\mathcal{K}^{obs} \supset \mathcal{K}^1$ and as we will see nontrivial controllability issues arise if $\mathcal{K}^1$ is a proper subset of $\mathcal{K}^{obs}$. We identify the space of observed modes with $\mathbb{R}^N$ and denote by Π^{obs} the operator of projection of solutions onto the space $\mathbb{R}^N$.

We can represent the 2D NS equation, controlled by degenerate forcing, in the following way:

$$\dot{q}_k(t) = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \\ -\nu|k|^2 q_k(t) + \dot{v}_k(t), \ k \in \mathcal{K}^1, \tag{6}$$
$$\dot{q}_k(t) = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n \\ -\nu|k|^2 q_k(t), \ k \in \mathcal{K}^{obs} \setminus \mathcal{K}^1, \tag{7}$$
$$\dot{Q}(t) = B_2(q,Q;q,Q) - \nu A_2 Q. \tag{8}$$

In the latter equation $-\nu A_2 Q$ stays for the dissipative term and $B_2(q,Q;q,Q)$ stays for nonlinear (bilinear) term.

Analogously 3D NS equation, controlled by degenerate forcing, can be written in the form:

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n)\Pi_k \underline{q}_n - \\ -\nu|k|^2 \underline{q}_k(t) + \underline{v}_k(t), \ k \in \mathcal{K}^1, \tag{9}$$
$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n)\Pi_k \underline{q}_n - \\ -\nu|k|^2 \underline{q}_k(t), \ k \in \mathcal{K}^{obs} \setminus \mathcal{K}^1, \tag{10}$$
$$\dot{Q}(t) = B_3(q,Q;q,Q) - \nu A_3 Q, \tag{11}$$

Let us introduce the Galerkin approximations of the 2D and 3D Navier-Stokes systems projecting these equations onto $\mathbb{R}^N$.

A simple way to see the Galerkin approximation of the 2D (respectively 3D) Navier-Stokes equation is just to eliminate the equation (8) (respectively (11)) and to put $Q = 0$ in (6)-(7) (resp. (9)-(10)). What results from this are the systems (6)-(7) (resp. (9)-(10)) under additional restriction on the modes:

$$k, m, n \in \mathcal{K}^G = \mathcal{K}^{obs}. \tag{12}$$

The systems (6)-(7)-(12) (resp. (9)-(10)-(12)) are control systems for ODE in finite-dimensional space of the observed modes.

Definition 2.1 *The Galerkin approximation of 2D (resp. 3D) Navier-Stokes equation is globally controllable if for any two points $\tilde{q}, \hat{q}$ in $\mathbb{R}^N$ there exists $T > 0$ and a control which steers in time $\leq T$ the solution of the system (6)-(7)-(12) (resp. (9)-(10)-(12)) from $\tilde{q}$ to $\hat{q}$. The system is time-T globally controllable if one can choose T the same for all $\tilde{q}, \hat{q}, \tilde{\varphi}$.*

In the next section we formulate sufficient conditions of global controllability for the Galerkin approximations of 2D and 3D Navier-Stokes systems.

Another question we are interested in is: under what conditions the NS equation is globally controllable in its observed component?

Definition 2.2 *The (2D or 3D) Navier-Stokes equation is globally controllable in its observed component if for any two points $\tilde{q}, \hat{q}$ in $\mathbb{R}^N$ and any $\tilde{\varphi} \in (\Pi^{obs})^{-1}(\tilde{q})$ there exist $T > 0$ and a control which steers in time $\leq T$ the solution of the controlled Navier-Stokes equation from $\tilde{\varphi}$ to some $\hat{\varphi}$ with $\Pi^{obs}(\hat{\varphi}) = \hat{q}$. The system is time-$T$ globally controllable if one can choose T the same for all $\tilde{q}, \hat{q}, \tilde{\varphi}$.*

This question is much more difficult because one has to study finite-dimensional projection of an infinite-dimensional dynamics.

We are able to prove that in 2D case the sufficient controllability condition for the Galerkin approximation is also sufficient for controllability in observed component of the (2D) Navier-Stokes equation.

3 Main results

The controllability criterion, we are going to formulate, is based on the evolution of the "sets of excited modes" $\mathcal{K}^j$ along the integer lattices $\mathbb{Z}^2$ or $\mathbb{Z}^3$ respectively.

Definition 3.1 *Let $\mathcal{K}^1$ be the set of forced modes. Define the sequence of sets $\mathcal{K}^j \subset \mathbb{Z}^i$, $(i = 2$ or $3;\ j = 2, \ldots)$, as:*

$$\mathcal{K}^j = \{m + n \mid m, n \in \mathcal{K}^{j-1} \bigwedge \|m\| \neq \|n\| \bigwedge m \wedge n \neq 0\ \}, \tag{13}$$

Remark 3.2 *The sets $\mathcal{K}^j$ are in general not disjoint.*

Theorem 1 *Let $\mathcal{K}^1$ be the set of controlled modes. Define sequence of sets $\mathcal{K}^j$, $j = 2, \ldots$ according to the Definition 3.1 and assume that $\bigcup_{j=1}^{M} \mathcal{K}^j$ contains all the modes: $\bigcup_{j=1}^{M} \mathcal{K}^j \supset \mathcal{K}^G = \mathcal{K}^{obs}$. Then for any $T > 0$ the Galerkin approximations of the 2D and 3D Navier-Stokes equations are time-T globally controllable.* □

Though there is an extensive literature regarding controllability of the NS systems we are not aware of any results on controllability by means of degenerate forcing. We would like to mention a publication of Weinan E and J.C.Mattingly [7] on ergodicity of Navier-Stokes system under degenerate forcing. From the control-theoretic viewpoint in [7] bracket generating property for finite-dimensional Galerkin approximation of the corresponding control system is established. This property guarantees accessibility property, which is nonvoidness of the interior of attainable set, but in general does *not* guarantee controllability.

Now we formulate the controllability-in-observed-component criterion for 2D Navier-Stokes equation.

Theorem 2 *Assume the conditions of the Theorem 1 to be fulfilled for a 2D Navier-Stokes system controlled by degenerate forcing. Then this system is globally controlled in observed component.*

4 Sufficient sets of forcing modes

We will call a set $\mathcal{K}^1$ of controlled forced modes *sufficient* if it satisfies the controllability criterion established by the Theorem 1, and moreover $\forall R > 0\ \exists j(R) \geq 1$ such that the union $\bigcup_{j=1}^{j(R)} \mathcal{K}^j$ of the sets, defined by the recurrent equation (13), contains the R-cube in $\mathbb{Z}^2$ (resp. $\mathbb{Z}^3$). An example of sufficient set is provided in the following Proposition.

Proposition 4.1 *The subset $\mathcal{K}^1 = \{k| \ |k_\alpha| \leq 3\}$, $\alpha = 1, 2$, (resp. $\alpha = 1, 2, 3$,) of $\mathbb{Z}^2$ (resp. $\mathbb{Z}^3$) is sufficient.*

5 Tools from geometric nonlinear control theory: controllability via reduction and completion

In this section we refer to some global controllability criteria obtained in the scope of geometric nonlinear control theory. We will be interested in criteria of *global controllability*.

We treat a real-analytic nonlinear control system $\dot{x} = f(x, u),\ u \in U$ as a collection $\mathcal{F}$ of real-analytic vector fields $f(\cdot, u)$ parameterized by $u \in U$. We will employ measurable essentially bounded functions of time as admissible controls.

Definition 5.1 *A point $\tilde{x}$ is attainable from $\hat{x}$ in time T for the system $\dot{x} = f(x, u)$ if there exists an admissible control $\tilde{u}(\cdot)$ such that the corresponding trajectory starting in $\hat{x}$ at $t = 0$ exists on the interval $[0, T]$ and attains $\tilde{x}$ at $t = T$. A point $\tilde{x}$ is attainable from $\hat{x}$ if it is attainable in some time $T \geq 0$. The set of points attainable from $\hat{x}$ in time T is called time-T attainable set from $\hat{x}$ and denoted by $\mathcal{A}^T_{\hat{x}}(\mathcal{F})$; the set of points attainable from $\hat{x}$ is called attainable set from $\hat{x}$ and denoted by $\mathcal{A}_{\hat{x}}(\mathcal{F})$. We say that the system is globally controllable from $\hat{x}$ (in time T) if its attainable set $\mathcal{A}_{\hat{x}}(\mathcal{F})$ (attainable set in time T $\mathcal{A}^T_{\hat{x}}(\mathcal{F})$) from $\hat{x}$ coincides with the whole state space.*

We describe (loosely following terminology of [10, Ch. 3]) some methods of *completion* or *extension* or *saturation* for control systems.

Definition 5.2 *The family $\mathcal{F}'$ of real analytic vector fields is an extension of $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$ and the closures of the attainable sets $\mathcal{A}_{\mathcal{F}}(\tilde{x})$ and $\mathcal{A}_{\mathcal{F}'}(\tilde{x})$ coincide.*

The inclusion $\mathcal{A}_{\mathcal{F}}(\tilde{x}) \subset \mathcal{A}_{\mathcal{F}'}(\tilde{x})$ is obvious as is the following

Lemma 5.3 *If an extension $\mathcal{F}'$ of a system $\mathcal{F}$ is globally controllable, then the attainable set $\mathcal{A}_{\tilde{x}}(\mathcal{F})$ of the original system is dense in the state space.*

The idea is to proceed with a series of extensions of a control system in order to come at the end to a system which is evidently controllable. It looks like we only may conclude "approximate controllability", which means that the *closure* of the attainable set of the original system coincides with the whole state space $\mathbb{R}^N$. In few moments we will indicate the conditions under which the approximate controllability implies controllability.

There are different ways of extension of a control system; we refer to [10] and the references therein for some of them. Here we will employ two methods: the first one is classical and underlies the theory of relaxed controls (see [9, 11]).

Let co$\mathcal{F}$ be the convex hull of $\mathcal{F}$, i.e. the set of vector fields of the form $\sum_{i=1}^{m} \beta_i f_i,\ f_i \in \mathcal{F},\ \beta_i \geq 0,\ \sum_{i=1}^{m} \beta_i = 1,\ i = 1, \ldots, m.$

Proposition 5.4 *For the systems co$\mathcal{F}$ and $\mathcal{F}$ the closures of their time-T attainable sets coincide.* $\square$

Another method arises from our previous work (see [4]) where it was called *reduction* of a *control-affine* system. Reduction sounds like something opposite to extension, but in fact it is the state space not the system which is reduced.

Consider control-affine nonlinear system:

$$\dot{q} = f(q) + G(q)v(t),\ q \in \mathbb{R}^N,\ v \in \mathbb{R}^r, \tag{14}$$

where $G(q) = \left(g^1(q), \ldots, g^r(q)\right)$, and $f(q), g^1(q), \ldots, g^r(q)$ are complete real-analytic vector fields in $\mathbb{R}^N$; $v(t) = (v_1(t), \ldots, v_r(t))$ is the control of the system.

The following result holds (see [2, 1] for the notation of chronological calculus used in its formulation):

Proposition 5.5 *Assume that the vector fields $g^1(q), \ldots, g^r(q)$ are mutually commuting: $\left[g^i, g^j\right] = 0,\ \forall i, j$. Then the flow of the system (14) can be represented as a composition of flows:*

$$\overrightarrow{\exp} \int_0^t \left(f(q) + G(q)\dot{v}(\tau)\right) d\tau =$$
$$\overrightarrow{\exp} \int_0^t e^{ad(GV(\tau))} f d\tau \circ e^{GV(t)}, \tag{15}$$

where $V(t) = \int_0^t v(s)ds$.

When studying the forced Navier-Stokes equation, we deal with *constant* controlled vector fields $g^1, \ldots, g^r$, for which the commutativity assumption holds automatically. From the Proposition 5.5 and the results of [4] it follows that one can reduce the study of the system 14 to the study of the control system

$$\dot{x} = e^{\mathrm{ad}(GV(\tau))} f(x), \tag{16}$$

on the quotient space $\mathbb{R}^N/\mathcal{G}$, where $\mathcal{G}$ is the linear span of the values of the constant vector fields $g^1, \ldots, g^r$.

The following result (see [4, Propositions 1 and 1']), based on the formula (15) will be instrumental in our reasoning.

Proposition 5.6 *Let $\pi_{\mathcal{G}}$ be the canonical projection of the quotient space $\mathbb{R}^N \to \mathbb{R}^N/\mathcal{G}$ and $\mathcal{A}_{red}(\pi_{\mathcal{G}}(\tilde{x})$ be the attainable set of the reduced system (16). Then the closures of the sets $\mathcal{A}(\tilde{x})$ and $\pi_{\mathcal{G}}^{-1}\left(\mathcal{A}_{red}(\pi_{\mathcal{G}}(\tilde{x})\right)$ in $\mathbb{R}^N$ coincide.* $\square$

One notices that the fact of system being control-affine is important for the validity of the formula (15) and therefore of the Proposition 5.6.

To eliminate the gap between *almost controllability* of a system (meaning that the *closure* of the attainable set coincides with the state space $\mathbb{R}^N$) and we invoke the well known necessary condition of accessibility which is a corollary of Nagano-Sussmann orbits theorem (see [10, 3]).

Proposition 5.7 *(Lie rank necessary condition) The following condition is necessary for the real-analytic system (14) to be globally controllable: the Lie rank of the system of vector fields $f, g^1, \ldots, g^r$ evaluated at each point $x \in \mathbb{R}^N$ must be complete, i.e. (iterated) Lie brackets of these vector fields evaluated at x^0 must span the whole $\mathbb{R}^N$.*

Under this additional condition "almost controllability" implies controllability (see [10, Ch.3, §1.1]).

Proposition 5.8 *If a system satisfies the Lie rank necessary condition holds at each point of the state space $\mathbb{R}^N$ and its attainable set is dense in $\mathbb{R}^N$, then this attainable set coincides with $\mathbb{R}^N$.* $\square$

6 Reduction+convexification for the controlled Navier-Stokes equation

6.1 2D case

We shall use the reduction and the convexification techniques surveyed in the previous section to establish controllability.

Let us start with the reduction of the control-affine system (6)-(7).

Consider the set $\mathcal{K}^1$ of controlled forcing modes. The controlled vector fields $g_k = \partial/\partial q_k,\ k \in \mathcal{K}^1$ are constant. Due to it for any vector field $Y(q)$ there holds: $e^{\mathrm{ad}(V_k g_k)}Y = Y(q + V_k e_k)$, where e_k is the (constant) value of g_k. Passing to the quotient space $\mathbb{R}^N/\mathcal{G}$, where $\mathcal{G} = \mathrm{span}\{g_k|\ k \in \mathcal{K}^1\}$ means that we can move freely along the directions $e_k, k \in \mathcal{K}^1$.

In the case of (6)-(8) the "drift" vector field f is quadratic+linear:

$$f = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu|k|^2 q_k.$$

Then the reduced system has form

$$\dot{q}_k = -\nu|k|^2 \left(q_k + \chi(k)V_k\right) + \quad (17)$$
$$+ \sum_{m+n=k} \frac{m \wedge n}{|m|^2} \left(q_m + \chi(m)V_m\right)\left(q_n + \chi(n)V_n\right)$$

where $\chi(\cdot)$ is the characteristic function of $\mathcal{K}^1 :\ \chi \equiv 1$ on $\mathcal{K}^1$ and vanishes outside $\mathcal{K}^1$.

The right-hand side of the reduced system (17) is a polynomial map with respect to (the components) of V with coefficients depending on q. Let us represent this polynomial map as $\mathcal{V}(V) = \mathcal{V}^{(0)} + \mathcal{V}^{(1)}V + \mathcal{V}^{(2)}(V)$ where $\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \mathcal{V}^{(2)}$ are the free, the linear and the quadratic terms respectively. Evidently $\mathcal{V}^{(0)}$ is the right-hand side of the projection of the unforced Navier-Stokes equation onto the quotient space.

We are not able to apply again the reduction to the system (17) as we would wish, because it is not control-affine anymore. Still we will be able to extend it and then extract from this extension a control-affine subsystem which is similar to (6)-(8).

First we establish that certain constant vector fields are contained in the image of the control-quadratic term $\mathcal{V}^{(2)}$.

Proposition 6.1 *Let $\mathcal{K}^{(2)}$ be the set of $k \in \mathbb{Z}^2$ for which there exist $m, n \in \mathcal{K}$ such that $m \wedge n \neq 0 \bigwedge \|m\| \neq \|n\| \bigwedge m + n = k$. Then the image of $\mathcal{V}^{(2)}$ contains all the vectors $\{\pm e_k|\ k \in \mathcal{K}^{(2)}\}$ from the standard base.* □

Proof. The projection of the vector-valued quadratic form $\mathcal{V}^{(2)}(V)$ onto e_k equals (see (17)) $\mathcal{V}^{(2)}_k(V) = \sum_{m+n=k} \frac{m \wedge n}{|m|^2}\chi(m)\chi(n)V_m V_n$. Grouping the coefficients of $V_m V_n$ and of $V_n V_m$ we can rewrite it as

$$\mathcal{V}^{(2)}_k(v) = \sum_{m+n=k,\ \|m\|<\|n\|,\ m,n\in\mathcal{K}^1} \gamma_{mn} V_m V_n. \quad (18)$$

where $\gamma_{mn} = (m \wedge n)\left(\frac{1}{\|m\|^2} - \frac{1}{\|n\|^2}\right)$. Note that for $\|m\| = \|n\|$ the corresponding coefficient γ_{mn} vanishes and the term $V_m V_n$ is lacking in the sum.

If $k \notin \mathcal{K}^{(2)}$ then there are no non-vanishing terms in the expression for $\mathcal{V}^{(2)}_k(V)$, and hence $\mathcal{V}^{(2)}_k \equiv 0$. If $k \in \mathcal{K}^{(2)}$, let us pick any $m, n \in \mathcal{K}$ such that $m+n = k$ and $(\|m\| < \|n\|)$. Construct two vectors V^+, V^- by taking $V^{\pm}_s = 0$ for $s \neq k \bigwedge s \neq m$, and then taking $V_m = V_n = 1$ for V^+ and $V_m = -V_n = 1$ for V^-.

A direct calculation shows that

$$\mathcal{V}^{(2)}(V^+) = -\mathcal{V}^{(2)}(V^-) =$$
$$(m \wedge n)\left(|m|^{-2} - |n|^{-2}\right) e_k.$$

Corollary 6.2 *The convex hull of the image of $\mathcal{V}^{(1)} + \mathcal{V}^{(2)}$ contains the (*independent of q*) linear space E^2 spanned by $\{e_k|\ k \in \mathcal{K}^{(2)}\}$.*

Proof. Indeed for each $k \in \mathcal{K}^{(2)}$ there exists v such that $\mathcal{V}^{(2)}(V) = e_k$, . Obviously $\mathcal{V}^{(2)}(-V) = e_k$, while $\mathcal{V}^{(1)}(V) = -\mathcal{V}^{(1)}(-V)$. Hence

$$\frac{1}{2}\left(\left(\mathcal{V}^{(1)} + \mathcal{V}^{(2)}\right)(V) + \left(\mathcal{V}^{(1)} + \mathcal{V}^{(2)}\right)(-V)\right) = e_k.$$

Therefore we come to the conclusion of the corollary.

Therefore the convex hull of the right-hand side (evaluated at q) of the reduced system (17) contains the affine space $\mathcal{V}^{(0)}(q) + E^2$. We consider this affine space as the right-hand side (evaluated at q) of a new control-affine system, which can be written as:

$$\dot{q}_k(t) = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n -$$
$$-\nu|k|^2 q_k(t) + v_k(t),\ k \in \mathcal{K}^2, \quad (19)$$
$$\dot{q}_k(t) = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n -$$
$$-\nu|k|^2 q_k(t),\ k \in \mathcal{K}^{obs} \setminus \left(\mathcal{K}^1 \bigcup \mathcal{K}^2\right). \quad (20)$$

Recall that we can move freely in the directions $e_k,\ k \in \mathcal{K}^1$.

If the image of the attainable set of this latter system under the canonical projection $\mathbb{R}^N \to \mathbb{R}^N/\mathcal{G}$ coincides with $\mathbb{R}^N/\mathcal{G}$ or, in other words, the (linear) sum of this attainable set with $\mathcal{G}$ coincides with $\mathbb{R}^N$, then according to the Proposition 5.6 the attainable set the original system will be dense in the state space and hence by Lemma 5.8 will coincide with the state space.

Therefore we managed to reduce the study of controllability of the system (6)-(8) to the study of a similar system with smaller (reduced) state space.

6.2 3D case.

We now consider the control system (9)-(11)

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n)\Pi_k \underline{q}_n - \nu|k|^2 \underline{q}_k(t) + \underline{v}_k(t),$$

with $v_(\cdot)$ vanishing for $k \notin \mathcal{K}^1$.

We use the same techniques as in the 2-dimensional case (see the previous section) to extend the set of controlled modes along the integer lattice $\mathbb{Z}^3$. The reasoning is the same as in 2D case.

7 Comments on the proofs of the main results

For the lack of space we are not able to provide proofs; let us just give some hints.

What for the controllability of the Galerkin approximations then proofs for the 2D and the 3D cases almost coincide. One proceeds by induction on M, where M is a number of sets $\mathcal{K}^j$ of modes appearing in the formulation of the Theorem 1.

If $M = 1$ then controllability of the Galerkin approximation is almost trivial fact. Actually we are not only able to attain arbitrary points but even to design arbitrary Lipschitzian trajectories.

Assume that we have proven the statement for all $M \leq \bar{M} - 1$. Then the transfer to $M = \bar{M}$ is fulfilled by application of the arguments of the Subsections 6.1,6.2 and by application of the Proposition 5.8.

What for the proof of the Theorem 2 then one has to proceed with the induction steps regarding the complete (nontruncated) 2D NS system. This requires rather heavy analytic estimates. The proof will be presented in a forthcoming paper [5].

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