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ON THE MINIMUM TIME PROBLEM FOR DRIFTLESS LEFT-INVARIANT CONTROL SYSTEMS ON $SO(3)$

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Abstract. In this paper, we investigate the structure of time-optimal trajectories for a driftless control system on $SO(3)$ of the type $\dot{x} =3D\, x(u_1 f_1 + u_2 f_2)$, $|u_1|, |u_2| \leq 1$, where $f_1, f_2 \in so(3)$ define two linearly independent left-invariant vector fields on $SO(3)$. We show that every time-optimal trajectory is a finite concatenation of at most five (bang or singular) arcs. More precisely, a time-optimal trajectory is, on the one hand, bang-bang with at most either two consecutive switchings relative to the same input or three switchings alternating between two inputs, or, on the other hand, a concatenation of at most two bangs followed by a singular arc and then two other bangs. We end up finding a finite number of three-parameters trajectory types that are sufficient for time-optimality.

1. Introduction and Statement of the Main Results.

We consider the time-optimal problem of the control system Σ given by:

$$\dot{x} =3D\, x(u_1 f_1 + u_2 f_2), \tag{1}$$

where $x \in SO(3)$, $|u_i| \leq 1$ and $f_i \in so(3)$ give rise to left invariant vector fields (acting on the right) on $SO(3)$. The control system Σ can be seen as a model for reorienting a satellite: the vector fields f_i represent infinitesimal rotations of the satellite about two non collinear fixed axis and the inputs u_i are scale multiples of the angular velocities of the rotors rotations. The problem consists of orienting the satellite from a given position to another one in minimum time.

The control system Σ is given as $\Big(SO(3), [-1,1]^2, Ad(\Sigma), (f_1, f_2)\Big)$ where $SO(3)$ is the state space, $[-1,1]^2$ is the control set, $Ad(\Sigma)$ is the control space (defined next) and f_1, f_2 are 3×3 skew-symmetric matrices, defining the dynamics of Σ by (1). The control space $Ad(\Sigma)$ is the set of admissible controls $u =3D(u_1, u_2)$ i.e. the set of measurable functions $u =3D(u_1, u_2) : [a,b] \to [-1,1]^2$, where a, b depend on u in general (see Jurdjevic [1] for a more general definition of control systems). A trajectory γ of Σ is an absolutely continuous curve $\gamma : J \to SO(3)$ with $J =3D\, [a,b]$

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is a compact segment of $\mathbb{R}$, such that there exists an admissible control $u \in Ad(\Sigma)$ for which:

$$\dot{\gamma}(t) =3D \gamma(t)(u_1(t) f_1 + u_2(t) f_2), \tag{2}$$

holds a.e. in J. We then say (γ, u) defined as before is an admissible pair for Σ. The initial and terminal states of γ are $In(\gamma) :=3D \gamma(a)$ and $Fi(\gamma) :=3D \gamma(b)$. The *time along* γ, $T(\gamma) =3D b - a$, is the length of J. We use $Tj(\Sigma)$ and $Pa(\Sigma)$ resp. to denote the set of admissible trajectories and the set of admissible pairs of Σ.

Definition 1. *A trajectory γ of Σ is time optimal if $T(\gamma) \leq T(\gamma')$ for every trajectory $\gamma' \in Tj(\Sigma)$ with initial and terminal state $In(\gamma)$ and $Fi(\gamma)$.*

Using the Lie isomorphism between $(so(3), [\cdot,\cdot])$ and $(R^3, \times)$, we can define the length $\|f\|$ of an arbitrary $f \in so(3)$. In the case where f_1 and f_2 are perpendicular and have the same length,Tang characterized in [7] the structure of time-optimal trajectories of Σ. He showed in particular that every time-optimal trajectory of Σ was either a concatenation of at most a bang, followed by a singular arc, and then another bang, or a bang-bang trajectory with at most three switchings alternating between u_1 and u_2. This results holds true when f_1 and f_2 have only the same length, case that was studied by Sussmann and Tang in an unpublished work [6].

In the present paper, the general case, i.e. f_1 and f_2 simply linearly independent, is completely solved. For this purpose, we use the techniques of modern geometric optimal control theory (see for instance the study of the shortest paths for the Reeds-Shepp car by Sussmann and Tang, [5]). We also develop a new technique (see Section 3.1), which mainly consists in a graphical visualization of the computations usually derived from the use of conjugate points (cf. [4]). After obvious normalizations ($\|f_1\| =3D 1$ and $a :=3D \|f_2\| \geq \|f_1\|$), the description of our results depends on the parameter $\rho :=3D f_1 \cdot f_2$ and it is decomposed in two cases $\rho \leq 1$ and $\rho > 1$. However, there is a global result given next:

Theorem 1. *For the system (2), assume that f_1 and f_2 are linearly independent. Then every time-optimal trajectory is a finite concatenation of at most four (bang or singular) arcs if $\rho \leq 1$ and five arcs if $\rho > 1$: it is either a concatenation of at most two bangs followed by a singular arc and then two other bangs, or a bang-bang trajectory with at most either two consecutive switchings of u_1 or three switchings alternating between u_1 and u_2.*

The strategy adopted here is standard: after showing the existence of time-optimal trajectories for any given pair of points of $SO(3)$, we consider the field of extremals given by the Pontryagin Maximum Principle. We only have to consider normal extremals and then certain periodic vector fields $X_\varepsilon :=3D f_1 + \varepsilon f_2$, $\varepsilon \in \{-1, 1\}$, $S_1 =3D -\rho f_1 + f_2$ and $S_2 =3D f_1 - \frac{\rho}{a^2} f_2$ corresponding to bang and singular arcs (see below the precise meaning of the word "singular" in this paper). Then, we progressively exclude certain types of bang and singular trajectories by applying three techniques briefly described below:

- $(T1)$**:** Envelope theory (cf. [4, 5]): Let γ be a bang-bang trajectory corresponding to a normal extremal. We start by making a variation Γ of γ such that $\Gamma =3D \{\gamma_l, l \in [l_1, l_2]\}$ is a one-parameter family of normal extremals starting at $In(\gamma)$ at time 0 with $\gamma_{l_2} =3D \gamma$. We next establish that the curve f defined on $[l_1, l_2]$ by $f(l) =3D Fi(\gamma_l)$ is an envelope of Γ and the concatenation $\bar{\gamma}$ of γ_{l_1} from $In(\gamma)$ to $f(l_1)$ and f from $f(l_1)$ to $f(l_2) =3D Fi(\gamma)$ steers $In(\gamma)$ to $Fi(\gamma)$

in the same time as γ does. We finally try to show that the trajectory $\bar{\gamma}$ is not optimal and so γ is not optimal either.

($T2$)**:** Embedding of γ in a one-parameter family of trajectories steering $In(\gamma)$ to $Fi(\gamma)$ (cf. [6]). Then we try to show that there exists a trajectory of the family steering $In(\gamma)$ to $Fi(\gamma)$ in a smaller time than γ does.

($T3$)**:** Replacing of some interior piece of γ by another one and then trying to show that the time duration along the new trajectory γ' is smaller than the one along γ.

The organization and presentation of the computations lead to consider two parameters c_+ and c_- defined by:

$$c_+ :=3\mathrm{D}\frac{1+\rho}{a^2+\rho} \quad c_- :=3\mathrm{D}\frac{1-\rho}{a^2-\rho}, \tag{3}$$

where a =3D $_2\|\|f$ The pair of parameters (c_+, c_-) belongs to an open region $\mathcal{R}$ defined by $0 \leq |c_-| < c_+ < 1$ together with the boundary point P :=3D $(1,1)$. Note that for $\|f_1\|$ =3D$_2\|\|f$ the corresponding pair (c_+, c_-) is equal to P, whatever $\alpha \in (0, \frac{\pi}{2}]$. In the sequel we refer to (c_+, c_-) =3D P as the degenerate case and to $(c_+, c_-) \in \mathcal{R}$ as the general case.

The most technical part of the work will focus on getting an upper bound on the number of switchings of an oscillatory bang-bang trajectory, i.e. with switchings alternating between u_1 and u_2. Depending on the values of those two parameters in $\mathcal{R}$, we use alternatively just one of the techniques or a combination of them in order to exclude from optimality oscillatory bang-bang trajectories with four switchings. The region $\mathcal{R}$ is then divided in subregions where some technique (or a combination of them) provides the desired upper bound. We end up finding a finite number of three-parameters families of trajectories whose union is sufficient for optimality.

The paper is organized as follows: Section 2 covers the definitions and notations used throughout the paper. In addition, we provide the basic properties of optimal trajectories. In Section 3, we start the classification of optimal trajectories by characterizing the singular and bang optimal trajectories, the relations between the times of the interior bangs and finally by providing the structure of all optimal trajectories except the oscillatory optimal trajectories. Section 4 is devoted to the presentation and application of the techniques (Ti), i =3D $1, 2, 3$ in order to show that any oscillatory bang-bang trajectories with four switchings is not optimal. We gather in Section 5 all the results and provide the three-parameters families of trajectories whose union is sufficient for optimality.

2. **Notations and Basic Properties of Optimal Trajectories.**

We recall that the Lie algebra $(so(3), [.,.])$ is isomorphic to the Lie algebra $(\mathbb{R}^3, \times)$ where × denotes the vector product. This isomorphism is realized by the map:

$$\phi : so(3) \to \mathbb{R}^3 \tag{4}$$

$$\phi\left(\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}\right) =3\mathrm{D} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This isomorphism provides an inner product on $so(3)$:

$$< Z_1, Z_2 >:=3\mathrm{D}< \phi(Z_1), \phi(Z_2) >,$$

where $Z_1, Z_2 \in so(3)$ and the symbol $< .,. >$ in the right hand side stands for the Euclidean inner product of $\mathbb{R}^3$. With this definition it follows that:

$$< Z_1, Z_2 > :=3\mathrm{D} \frac{1}{2} Tr(Z_1 Z_2^t),$$

where Z^t denotes the transposed of the matrix $Z \in so(3)$. In the following, $\|Z\| :=3\mathrm{D} \sqrt{< Z, Z >}$ and Id_3 is the 3×3 identity matrix. We will sometimes consider the 2×2 matrix corresponding to the planar rotation of angle β and we use R_β to denote it.

Here we consider the time-optimal problem of the control system Σ given by:

$$\dot{x} =3\mathrm{D}\ x(u_1 f_1 + u_2 f_2), \tag{5}$$

where $x \in SO(3)$, $|u_i| \leq 1$ and $f_i \in so(3)$ are left invariant vector fields (acting on the right) on $SO(3)$. In the sequel we always assume that f_1 and f_2 are linearly independent in the sense that $\phi(f_1)$ and $\phi(f_2)$ are not collinear.

Let us use h to denote the Lie bracket of f_1 and f_2:

$$h :=3\mathrm{D} [f_1, f_2] =3\mathrm{D} f_1 f_2 - f_2 f_1. \tag{6}$$

Up to time-reparameterization, we can always assume that $\|f_1\| =3\mathrm{D}\ 1$. Since the control problem is invariant by the transformation $f_1 \to -f_1$, we can assume that the non-oriented angle between f_1 and f_2 is $\alpha \in (0, \pi/\ D2]$. Finally we orient $\mathbb{R}^3$ so that the oriented (counterclockwise) angle between f_1 and f_2 is indeed α. Let $a :=3\mathrm{D} \|f_2\|$ and:

$$\rho :=3\mathrm{D} f_1 \cdot f_2 =3\mathrm{D}\ a \cos \alpha.$$

Define:

$$X_+ :=3\mathrm{D} f_1 + f_2, \quad X_- :=3\mathrm{D} f_1 - f_2, \quad \Gamma_+ ;=3\mathrm{D}\ \|X_+\|, \quad \Gamma_- :=3\mathrm{D}\ \|X_-\|.$$

We have:

$$\Gamma_\varepsilon =3\mathrm{D} \sqrt{a^2 + 1 + 2\varepsilon\rho},$$

where ε assumes the values $+1$ (or simply "+") and -1 (or simply "−"). We will use this notation throughout all the paper.

For a vector field $Z \in so(3)$, we use the exponential e^{tZ} to denote the flow of Z, acting on the right, so that $t \mapsto pe^{tZ} \in SO(3)$ is the integral curve of Z starting at p at time 0. Since Z is linear, we have $e^{tZ} =3\mathrm{D} \sum_{n=3\mathrm{D}0}^{\infty} \frac{(Zt)^n}{n!}$.

We use ad_Z to denote the operator $W \mapsto [Z, W]$, acting on vector fields. If Z, W are vector fields, we define $e^{t\,\mathrm{ad}_Z}(W) :=3\mathrm{D} e^{tZ} W e^{-tZ}$. From the Lie bracket relations between f_1, f_2, h one gets:

$$e^{t\,\mathrm{ad}_{X_\varepsilon}}(f_1) =3\mathrm{D} \frac{x_\varepsilon + c_\varepsilon}{1 + c_\varepsilon} f_1 + \varepsilon \frac{c_\varepsilon}{1 + c_\varepsilon}(1 - x_\varepsilon) f_2 - \varepsilon \frac{\sin \Gamma_\varepsilon t}{\Gamma_\varepsilon} h, \tag{7}$$

$$e^{t\,\mathrm{ad}_{X_\varepsilon}}(f_2) =3\mathrm{D}\ \varepsilon\frac{1 - x_\varepsilon}{1 + c_\varepsilon} f_1 + \frac{1 + c_\varepsilon x_\varepsilon}{1 + c_\varepsilon} f_2 + \frac{\sin \Gamma_\varepsilon t}{\Gamma_\varepsilon} h, \tag{8}$$

$$e^{t\,\mathrm{ad}_{X_\varepsilon}}(h) =3\mathrm{D}\ \varepsilon(a^2 + \varepsilon\rho) \frac{\sin \Gamma_\varepsilon t}{\Gamma_\varepsilon} f_1 - (1 + \varepsilon\rho) \frac{\sin \Gamma_\varepsilon t}{\Gamma_\varepsilon} f_2 + (\cos \Gamma_\varepsilon t), h, \tag{9}$$

$$e^{t\,\mathrm{ad}_{X_\varepsilon}}(X_{-\varepsilon}) =3\mathrm{D}\ (\cos \Gamma_\varepsilon t) X_{-\varepsilon} - (a^2 - 1) \frac{1 - \cos \Gamma_\varepsilon t}{\Gamma_\varepsilon^2} X_\varepsilon - 2\varepsilon \frac{\sin \Gamma_\varepsilon t}{\Gamma_\varepsilon} h, \tag{10}$$

$$e^{tX_\varepsilon} =3\mathrm{D}\ Id_3 + \frac{\sin \Gamma_\varepsilon t}{\Gamma_\varepsilon} X_\varepsilon + \frac{1 - \cos \Gamma_\varepsilon t}{\Gamma_\varepsilon^2} X_\varepsilon^2, \tag{11}$$

where we set $x_\varepsilon :=3\mathrm{D} \cos_\varepsilon t\Gamma$ We will sometimes use the notations $x =3\mathrm{D} \cos_- t\Gamma$ and $y =3\mathrm{D} \cos_+ t\Gamma$. Recall that the parameters c_ε were defined in (3). We also introduce:

$$k :=3\mathrm{D}\frac{\Gamma_+(a^2-\rho)}{\Gamma_-(a^2+\rho)} =3\mathrm{D}\frac{\Gamma_-(1+c_+)}{\Gamma_+(1+c_-)}. \tag{12}$$

Remark **1.** *To get relations (7) and (8), one can use the orthonormal generators of* $so(3)$ *obtained in the following way. Define* $\bar{f}_1$ *to be a unitary vector lying in the plane defined by* f_1, *and* f_2, *orthogonal to* f_1 *and such that* $\bar{f}_1 \cdot f_2 \geq 0$. *Let* $\bar{h} :=3\mathrm{D}_1[f\bar{f}_1]$; *we have:*

$$[\bar{f}_1, \bar{h}] =3\mathrm{D}_1 f, [\bar{h}, f_1] =3\mathrm{D}\bar{f}_1, \quad \bar{h} =3\mathrm{D}_1[f\bar{f}_1], \tag{13}$$

and $(f_1, \bar{f}_1, \bar{h})$ *are orthonormal. Useful relations are:*

$$h =3\mathrm{D}_1[f f_2] =3\mathrm{D}\, a\sin(\alpha)\bar{h}, [f_2, h] =3\mathrm{D}^3 a f_1 - \rho f_2, \quad [h, f_1] =3\mathrm{D} \neg_1\rho f f_2.$$

To get (11) we used the relation $X_\varepsilon^3 =3\mathrm{D}\ \frac{2}{\varepsilon}X X_\varepsilon$ *that one can prove as follows. By Cayley–Hamilton, we have* $X_\varepsilon^3 =3\mathrm{D}_1 a X_\varepsilon^2 + a_2X_\varepsilon + a_3Id_3$. *Since* $X_\varepsilon^3 - a_2X_\varepsilon$ *is skew symmetric and* $a_1X_\varepsilon^2 + a_3Id$ *is symmetric, the previous relation reduce to* $X_\varepsilon^3 =3\mathrm{D}_2 a X_\varepsilon$. *Defining* $M :=3\mathrm{D}_\varepsilon X_\varepsilon^t =3\mathrm{D}\ -\frac{2}{\varepsilon}, X$ *we get* $M^2 + a_2M =3\mathrm{D}\ 0$. *Since* M *is symmetric, positive semidefinite and* $rank(M) =3\mathrm{D}\ 2$ *(*$rank(M) =3\mathrm{D}\ rank(X_\varepsilon)$ *and* $X_\varepsilon \neq 3\mathrm{D}\ 0$*), it is diagonalizable with eigenvalues* $(0, -_2 a, -a_2)$, *so we have* $-2a_2 =3\mathrm{D}$ $Tr(M) =3\mathrm{D}\ Tr(X^t X)$ *It follows* $a_2 =3\mathrm{D}\ \frac{2}{\varepsilon}\Gamma$

Remark **2.** *As said previously, the parameters* c_+ *and* c_- *belong to the open region* $\mathcal{R}$ *given by* $0 \leq |c_-| < c_+ < 1$ *together with* $P =3\mathrm{D}\ (1,1)$. *Note that, for every point* (c_+, c_-) *of* $\mathcal{R}$, *there exists a pair* $(a, \alpha) \in (1, \infty) \times (0, \pi/\ \ D2]$ *such that (3) holds.*

2.1. **The Pontryagin Maximum Principle and existence of optimal trajectories.** In the rest of the paper, optimality means time-optimality. We say that a control system is *complete*, if for every measurable control function $\eta : [a,b] \to U$ and every initial state p there exists a trajectory γ for η which is defined on the whole interval $[a,b]$ and satisfies $\gamma(a) =3\mathrm{D}\ p$. Since $SO(3)$ is compact and connected, the system (5) is complete. Note that (f_1, f_2) satisfies the Strong Bracket Generating Condition (cf. [3]) and the control set is compact and convex. Then, (cf. for instance [6]):

Proposition 1. *For each pair of points* p *and* q *belonging to* $SO(3)$, *there exists a time optimal trajectory joining* p *to* q.

Moreover, thanks to the symmetries of the control set, the control system Σ is symmetric in the sense that every trajectory run backwards in time is also an admissible trajectory of Σ.

Define the following real-valued map on $T^*SO(3) \times [-1,1]^2$, called Hamiltonian,

$$\mathcal{H}(\lambda, x, u) :=3\mathrm{D}< \lambda, x(u_1 f_1 + u_2 f_2) > .$$

Set:

$$H(p, x) =3\mathrm{D}\min_{v\in[-1,1]^2}\mathcal{H}(p, x, v). \tag{14}$$

The Pontryagin Maximum Principle [2] asserts that, if $\gamma : [a,b] \to SO(3)$ is a time optimal trajectory corresponding to a control $u : [a,b] \to [-1,1]^2$, then there exist *a nontrivial field of covectors along* γ, that is an absolutely continuous function $\lambda : t \in [a,b] \mapsto \lambda(t) \in T^*_{\gamma(t)}SO(3) \sim so(3)$ never vanishing, and a constant $\lambda_0 \geq 0$ such that:

i): $\dot{\lambda}(t) =3D\frac{\partial\mathcal{H}}{\partial x}(\lambda(t), \gamma(t), u(t))$,
ii): $\mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 =3D 0$ for a.e. $t \in Dom(\gamma)$,
iii): $\mathcal{H}(\gamma(t), \lambda(t), u(t)) =3D H(\gamma(t), \lambda(t))$ for a.e. $t \in Dom(\gamma)$.

In this case we say that the pair (γ, λ) is extremal. Moreover, if $\lambda_0 =3D 0$ then we say that (γ, λ) is an *abnormal extremal*. If γ is optimal we say that the pair (γ, λ) is optimal.

Since the system (5) is subject to the Strong Bracket Generating Condition (cf. [3]), there is no nontrivial abnormal extremals i.e. for every nontrivial extremal trajectory we have $\lambda_0 \neq 3D\ 0$. In the following we always normalize $\lambda_0 =3D 1$.

2.2. **Notations for Trajectories and Switching functions.** Let us introduce some notations to describe different types of controls. Let:

$$V :=3D \{(-1,-1), (-1,1), (1,-1), (1,1)\},$$

be the set of vertices of $[-1,1]^2$. A control $\eta : [a,b] \to [-1,1]^2$ is said to be *bang-bang* if $\eta(t) \in V$ a. e. in $[a,b]$. Moreover if $\eta(t) \in V$ and $\eta(t)$ is constant for almost every $t \in [a,b]$ then η is called a bang control. A *switching* time of η is a time $t \in [a,b]$ such that for each for every $\varepsilon > 0$, η is not bang on $(t-\varepsilon, t+\varepsilon) \cup [a,b]$. A control with a finite number of switchings is called *regular bang-bang*. A trajectory of Σ is a bang trajectory, bang-bang trajectory, regular bang-bang trajectory respectively, if it corresponds to a bang control, bang-bang control, regular bang-bang control respectively. If $\eta(.) =3D (u_1(.), u_2(.))$, with $\mathrm{Dom}(\eta) =3D [a,b]$, then a u_i–*switching time* is a time $t \in [a,b]$ for which u_i is neither a.e. equal to 1 nor a.e. equal to -1 on any interval of the form $(t-\varepsilon, t+\varepsilon) \cap [a,b]$ for each $\varepsilon > 0$.

We define the switching functions:

$$\begin{aligned} \varphi_1 &:=3D < \lambda, xf_1 >, \\ \varphi_2 &:=3D < \lambda, xf_2 >, \\ \varphi_3 &:=3D < \lambda, xh >, \end{aligned}$$

as the Hamiltonian functions associated to the vector fields f_i and h (cf. Jurdjevic [1]). The switching functions are important because they determine where the controls may switch. Using the PMP one easily gets that a necessary condition for a time t to be a u_i-switching is that $\varphi_i(t) =3D 0$, $i =3D 1, 2$. Therefore on any interval where both φ_1 and φ_2 have no zeros (respectively finitely many zeros) the corresponding control is bang (respectively bang-bang). On the other hand if φ_i has a zero at t and $\dot{\varphi}_i(t)$ exists and it is different from zero then t is an isolated u_i-switching.

A trajectory γ of Σ defined on $[c,d]$ admitting an extremal lift Γ is said to be u_i*-singular* if the switching functions φ_i vanishes on $[c,d]$. From the fact that there are not nontrivial abnormal extremals one easily gets that there are no trajectories that are both u_1-singular and u_2-singular.

In the sequel we will use the following convention. We use the letter B to refer to a bang trajectory and the letter S to refer a u_i–singular extremal trajectory. A concatenation of bang and singular trajectories will be labelled by the corresponding letter sequence, written in order from left to right. Sometimes, it is desirable to specify which is the component of the control that switches. For instance a $Bu_1Bu_1Bu_2$-trajectory has its first and second switchings with u_1 and its third with u_2. Sometimes we will use a subscript to indicate the duration of a trajectory so that, we use B_t to refer to a bang trajectory defined on an interval of length t and similarly S_t for a singular trajectory defined on an interval of length t. A bang-bang trajectory is said to be *oscillatory* if the switchings alternate between u_1 and u_2.

Using relation (11) one gets:

Proposition 2. *Periodicity of X_ε: If an optimal trajectory γ is a bang trajectory of the form $(\pm X_\varepsilon)_t$, then $t \leq \pi/\ {}_{\varepsilon}D\Gamma$*

2.3. **The Adjoint System and Two Invariants.** Using the Lie bracket relations between f_1, f_2 and h, one gets that $\varphi_1, \varphi_2, \varphi_3$ satisfy a system of differential equations called the adjoint system and given by:

$$\dot{\varphi}_1 \quad =3D \quad -{}_2\varphi u_3, \tag{15}$$

$$\dot{\varphi}_2 \quad =3D \ {}_1\varphi u_3, \tag{16}$$

$$\dot{\varphi}_3 \quad =3D \ {}_1(\rho u + a^2 u_2)\varphi_1 - (u_1 + u_2\rho)\varphi_2. \tag{17}$$

From the fact that H is constant along the extremal trajectories one immediately gets:

$$|\varphi_1(t)| + |\varphi_2(t)| =3D\ 1. \tag{18}$$

Moreover if (γ, u, λ) is an extremal lift of Σ, then there exists $K > 0$ such that for every t in the domain of γ:

$$(\rho\varphi_1 - \varphi_2)^2 + (a^2 - \rho^2)\varphi_1^2 + \varphi_3^2 =3D^2K \tag{19}$$

From (18), it follows $K^2 =3D^2K(|\varphi_1(t)| + |\varphi_2(t)|)^2$ and, depending on the sign of $a - K$, one can define three types of curves as next,

Type-A $a > K$: we get:

$$|\varphi_2| \left((a^2 - 1)|\varphi_2| + 2|\varphi_1|(a^2 + \varepsilon\rho)\right) > \varphi_3^2. \tag{20}$$

As a consequence, φ_2 is never zero along a trajectory of type-A. Therefore, such a trajectory has no u_2-switching;

Type-B $a =3D$ K: then:

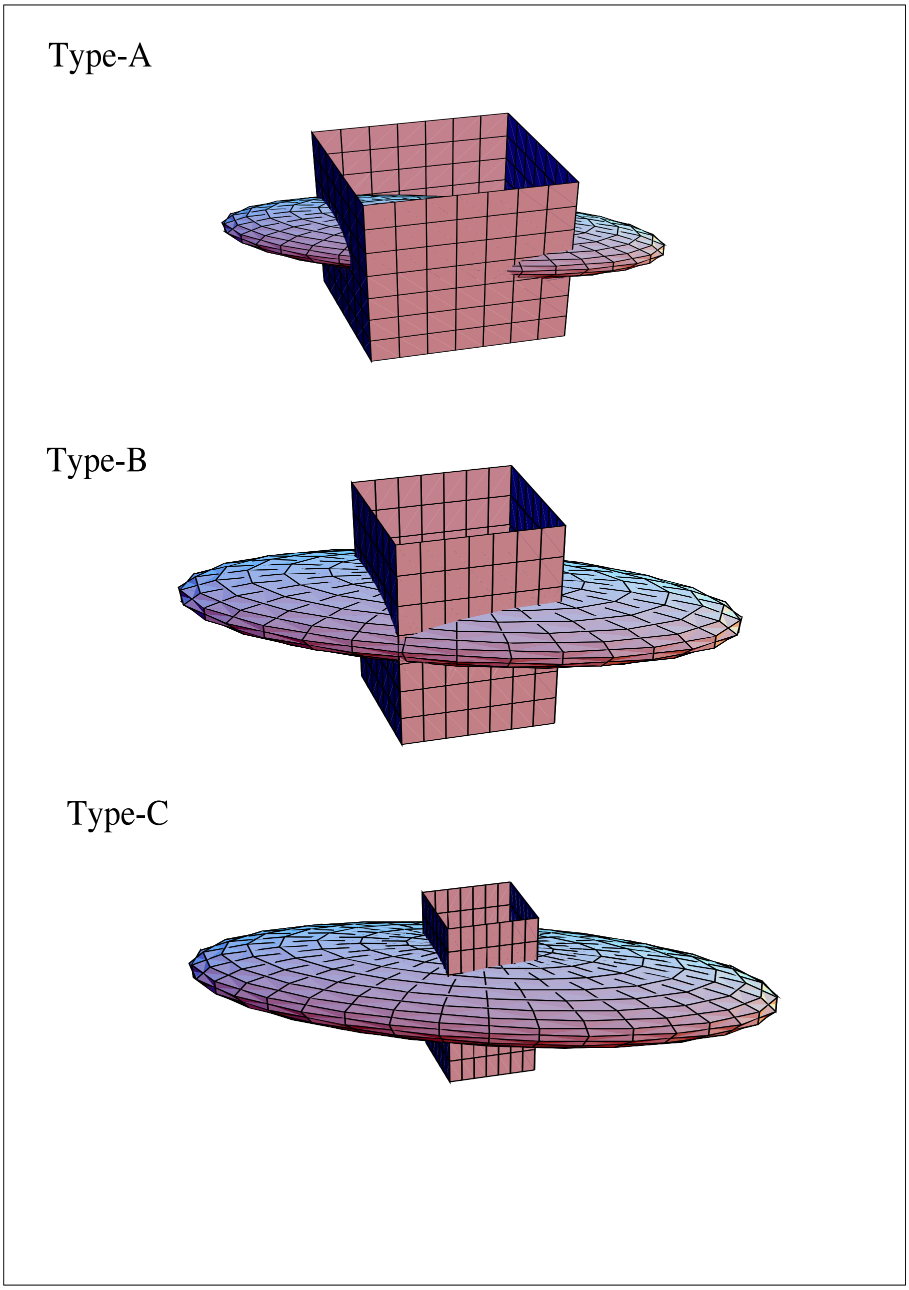
$$|\varphi_2| \left((a^2 - 1)|\varphi_2| + 2|\varphi_1|(a^2 + \varepsilon\rho)\right) =3D_3^2.\varphi \tag{21}$$

Along such a trajectory, φ_2 and φ_3 have the same zeros;

Type-C $a < K$: we have:

$$(a^2 - 1)\varphi_2^2 + 2|\varphi_1\varphi_2|(a^2 + \varepsilon\rho) < \varphi_3^2, \tag{22}$$

which implies that φ_3 is never zero.


Type-A
Type-B
Type-C

Lemma 1. *Assume that γ is a time-optimal bang-bang trajectory. If γ has two consecutive u_1-switchings, it is either of type-A and then there is no u_2-switching along γ or it is of type-B. In particular, a bang-bang normal extremal of type-A with three switchings u_1, u_1, u_2 or u_2, u_1, u_1 cannot be optimal.*

Proof of Lemma 1. There exist $t_0 < t_1$ such that $\varphi_1(t_0)$ =3D $\varphi_1(t_1)$ =3D 0 and $|\varphi_2| \geq 0$ on $[t_0, t_1]$. Since $|u_2|$ =3D 1, there exists by Rolle's theorem some $t_2 \in [t_0, t_1]$ such that $\varphi_3(t_2)$ =3D 0. Then type-C is excluded. If $\varphi_2(t_2) \neq$3D 0, γ has to be of type-A and, thanks to (20), $|\varphi_2| > 0$ along γ. If $\varphi_2(t_2)$ =3D 0, γ is of type-B.

3. Classification of optimal trajectories.

In this section, we investigate the structure of optimal trajectories by analyzing the normal extremal flow defined in (15)-(17) subject to (18) and (19). We will characterize the inputs corresponding to a singular arc in an optimal trajectory, determine when a bang-bang trajectory is oscillatory or not and establish basic relations between the interior bang times.

3.1. On the oscillation of switchings for optimal bang-bang trajectories.

We need another proposition regarding bang-bang optimal trajectories.

Proposition 3. *Assume that γ is an optimal bang-bang trajectory. Then,*

- (i): *if $\rho \leq 1$, γ is oscillatory;*
- (ii): *if $\rho > 1$, γ is either oscillatory or it has at most two consecutive u_1-switchings and on the bang arc between the two u_1-switchings, u_1u_2 =3D -1.*

Proof of Proposition 3: Consider $t_0 < t_1$ such that $\varphi_1(t_0)\varphi_2(t_0)$ =3D $\varphi_1(t_1)\varphi_2(t_1)$ =3D 0, $\varphi_1\varphi_2$ keeps a constant sign on the interval (t_0, t_1) with the possibility of being equal to zero only at a finite number of points. Define ε as the constant value (1 or -1) of u_1u_2 on (t_0, t_1) and consider V_ε :=3D $(\xi_\varepsilon, \Gamma_\varepsilon\varphi_3)^T$, where ξ_ε :=3D $\dot\varphi_3$. According to Proposition 2 we may assume that $0 < t_1 - t_0 \leq \frac{\pi}{\Gamma_\varepsilon}$. We also have:

$$\xi_\varepsilon(t) \text{ =3D } \varepsilon^2(1+\varepsilon\rho)|\varphi_1| - (\varepsilon+\rho)|\varphi_2|.$$

More generally, one can define ε :=3D u_1u_2 along a bang-bang trajectory as a piecewise constant function with values in $\{-1, 0, 1\}$ and V_ε as a piecewise C^1 vector-valued function. On the complement of the switching times, we have:

$$\dot V_\varepsilon \text{ =3D } \Gamma_\varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} V_\varepsilon. \tag{23}$$

Set V_{t_0} :=3D $\lim_{t\to t_0^+} V_\varepsilon(t)$ with first coordinate ξ_{t_0} and V_{t_1} =3D $\lim_{t\to t_1^-} V_\varepsilon(t)$ with first coordinate ξ_{t_1}. We have $V_\varepsilon(t)$ =3D $R_{\Gamma_\varepsilon(t-t_0)}V_{t_0}$, $\varphi_1\dot\varphi_1$ =3D $-\varepsilon\varphi_3|\varphi_1|$ and $\varphi_2\dot\varphi_2$ =3D $\varepsilon\varphi_3|\varphi_2|$. In particular V_{t_1} =3D $R_{\Gamma_\varepsilon(t_1-t_0)}V_{t_0}$.

Let us first show that γ cannot have two consecutive u_2-switchings at t_0 and t_1. Assume the contrary. Then, $\varphi_2(t_0)$ =3D $\varphi_2(t_1)$ =3D 0 and φ_1 has a constant sign on $[t_0, t_1]$ with the possibility of being equal to zero only at a finite number of points. We also have:

$$\xi_{t_0} \text{ =3D } \xi_{t_1} \text{ =3D } \varepsilon^2(1+\varepsilon\rho) \neq\text{3D } 0,$$

$\lim_{t\to t_0^+} \varphi_1\dot\varphi_1 \leq 0$ and $\lim_{t\to t_1^-} \varphi_1\dot\varphi_1 \geq 0$, since $|\varphi_1|$ admits a maximum in t_0 and t_1 and γ admits a bang arc on (t_0, t_1). Then V_{t_0} and V_{t_1} are symmetric with respect to the ξ-axis, the coordinates of V_{t_0} have the same sign and the coordinates of V_{t_1} have opposite sign. We then get that $\Gamma_\varepsilon(t_1 - t_0) > \pi$ (see the following picture), which is impossible.

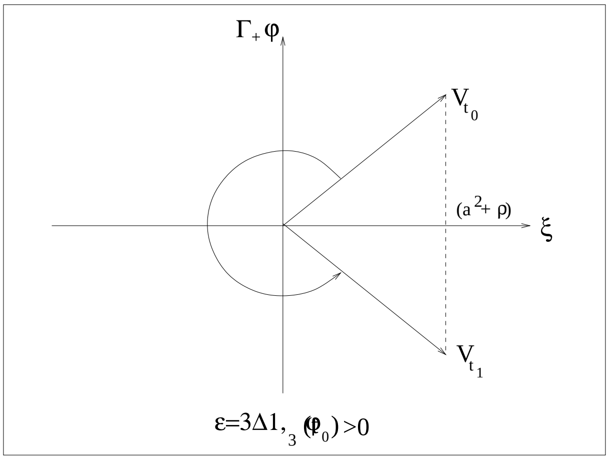


Fig. 3.1

Next, we show that γ may have two consecutive u_1-switchings at t_0 and t_1, only if $\rho > 1$ and $\varepsilon =3D -1$. For that purpose, assume that $\varphi_1(t_0) =3D \varphi_1(t_1) =3D 0$, φ_2 has a constant sign on $[t_0, t_1]$ with the possibility of being equal to zero only at a finite number of points. We also have:

$$\xi_{t_0} =3D \xi_{t_1} =3D -\varepsilon(1 + \rho\varepsilon),$$

$\lim_{t \to t_0^+} \varphi_2 \dot{\varphi}_2 \leq 0$ and $\lim_{t \to t_1^-} \varphi_2 \dot{\varphi}_2 \geq 0$, since $|\varphi_2|$ admits a maximum in t_0 and t_1.

If $\varepsilon =3D 1$, the same reasoning as above works since $1 + \rho\varepsilon \neq 3D\ 0$ and the coordinates of V_{t_0} have the same sign and the coordinates of V_{t_1} have opposite sign. We again have that $\Gamma_\varepsilon(t_1 - t_0) > \pi$ and then reach a contradiction. In this case, there cannot be two consecutive u_1-switchings.

Assume now that $\varepsilon =3D -1$. The discussion splits in three subcases

$\rho < 1$: same reasoning as before, and we get that there cannot be two consecutive u_1-switchings;

$\rho =3D 1$: we have $\xi_{t_0} =3D \xi_{t_1} =3D 0$. Recall that V_{t_1} cannot be identically equal to zero (otherwise $\varphi_2 \equiv 0$, impossible) and, since $\|V_{t_0}\| =3D \|V_{t_1}\|$, $\varphi_3(t_0) =3D \varphi_3(t_1)$ would imply that $V_{t_0} =3D V_{t_1}$ and then $\Gamma_-(t_1 - t_0) =3D 2\pi$, which is again impossible. Therefore, $\varphi_3(t_0) =3D -\varphi_3(t_1) \neq 3D\ 0$ and then $\Gamma_-(t_1 - t_0) =3D \pi$. The trajectory γ contains the piece $\gamma' :=3D e^{s(\pm X_+)} e^{\frac{\pi}{\Gamma_-}(\mp X_-)} e^{s'(\pm X_+)}$ with $s, s' \geq 0$, and from the relation $e^{-\frac{\pi}{\Gamma_-}X_-} =3D e^{\frac{\pi}{\Gamma_-}X_-}$, we have $\gamma' =3D e^{s(\pm X_+)} e^{\frac{\pi}{\Gamma_-}(\pm X_-)} e^{s'(\pm X_+)}$. Since γ' has two u_2-switchings, it cannot be optimal and therefore γ is not optimal;

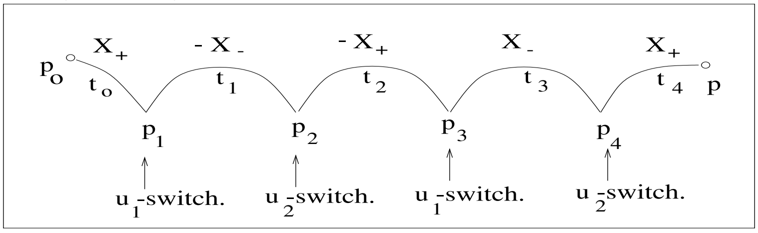
$\rho > 1$: γ has two u_1-switchings but not a third one thanks to lemma 1. Note that $\xi_{t_0} =3D \xi_{t_1} =3D -(\rho - 1) < 0$ and $\varphi_3(t_0) =3D -\varphi_3(t_1)$. This implies that $\varphi_3(t_0) \neq 3D\ 0$ otherwise $\Gamma_-(t_1 - t_0) =3D 2\pi$.

Proposition 4. *Every optimal trajectory of Σ is a finite concatenation of bang and singular arcs.*

Proof of Proposition 4: Let $\gamma : [a, b] \to SO(3)$ be an optimal trajectory of Σ. For $i =3D 1, 2$, let $S(\varphi_i)$ be the set of zeros of φ_i such that if $t \in S(\varphi_i)$, then φ_i does not vanish identically in some neighborhood of t. Clearly, $S(\varphi_1) \cup S(\varphi_2)$ is the set of times t such that $\gamma(t)$ is the junction of two bang arcs or the the junction of a singular arc and a bang arc. The conclusion follows if $S(\varphi_1) \cup S(\varphi_2)$ is finite. Reasoning by contradiction, then either $S(\varphi_1)$ or $S(\varphi_2)$ is infinite. Let S_0 be that set and S'_0 the other one. Similarly let φ_0 and φ'_0 be respectively the switching functions to which S_0 and S'_0 are associated to. Clearly, S_0 has a limit point $\bar{t}$. If $\bar{t}$ does not belong to S_0, then φ_0 would vanish identically in a neighborhood of $\bar{t}$, which would contradict the fact that $\bar{t}$ is a limit point of S_0. Then, $\bar{t} \in S_0$. From $|\varphi_0| + |\varphi'_0|$ =3D 1, we get $|\varphi'_0(\bar{t})|$ =3D 1 and then $|\varphi'_0| > 0$ on some open (in $[a, b]$) neighborhood N of $\bar{t}$. This implies that $\dot{\varphi}_0$ is continuous in N. By definition of S_0 and $\bar{t}$, there exists a sequence (t_n) in N converging to $\bar{t}$ such that $\varphi_0(t_n) \neq$3D 0. Pick n large enough, so that, if $[t'_n, t''_n] \subset N$ is the maximal subinterval containing t_n with $\varphi_0 \neq$3D 0 on (t'_n, t''_n). Then $\varphi_0(t'_n)$ =3D $\varphi_0(t''_n)$ =3D 0 and we may assume that $0 < t''_n - t'_n < \min(\frac{\pi}{\Gamma_1}, \frac{\pi}{\Gamma_-})$. Then γ is a bang arc on $[t'_n, t''_n]$. By Proposition 3, we have φ_0 =3D φ_1, $\rho > 1$ and ε =3D -1. Note that $\dot{\varphi}_1$ is continuous at t'_n and t''_n and $\dot{\varphi}_1(t'_n)$ =3D $-\dot{\varphi}_1(t''_n) \neq$3D 0. That implies that t'_n and t''_n are isolated zeros of φ_1 and then γ admits a bang arc for $t < t'_n$ and another bang arc for $t > t''_n$. If t'_n and t''_n are both different from $\bar{t}$, there would be at least three consecutive u_1-switchings, two of them at t'_n and t''_n, which is not possible. But if t'_n or t''_n is equal to $\bar{t}$, then $\bar{t}$ would be an isolated zero of φ_1, which is also impossible. We finally reached a contradiction and then Proposition 4 is proved.

3.2. **Relations between the time durations corresponding to interior bang arcs along oscillatory bang-bang trajectories.**

Let γ be an oscillatory bang-bang trajectory of the type:



We now rewrite equations (25) and (26) by using fields of 3-vectors. We firstly obtain:

$$f_2 \wedge e^{t_1 adX_-}(f_1) \wedge e^{-t_2 adX_+}(f_1) =3D\ 0, \tag{27}$$

and secondly:

$$f_1 \wedge e^{t_2 adX_+}(f_2) \wedge e^{t_3 adX_-}(f_2) =3D\ 0. \tag{28}$$

Thanks to equations (7) and (8), (27) and (28) are equivalent to $r(t_1, t_2, t_3) f_1 \wedge f_2 \wedge h =3D\ 0$ for appropriate real-valued functions $r_i(t_2, t_3)$. We thus get $r(t_1, t_2, t_3) =3D\ 0$ and we transform these equalities to get the following relations between t_1, t_2 and t_3:

$$k\frac{\sin \Gamma_+ t_2}{\sin \Gamma_- t_1} =3D \frac{\cos \Gamma_+ t_2 + c_+}{\cos \Gamma_- t_1 + c_-}, \tag{29}$$

and:

$$k\frac{\sin \Gamma_+ t_2}{\sin \Gamma_- t_3} =3D \frac{1 + c_+ \cos \Gamma_+ t_2}{1 + c_- \cos \Gamma_- t_3}, \tag{30}$$

where k was defined in (12). Define, for $\varepsilon =3D -1, 1$:

$$\bar{t}_\varepsilon :=3D \frac{\arccos(-c_\varepsilon)}{\Gamma_\varepsilon}.$$

Using the implicit function theorem, one can see that equations (29) and (30) are equivalent to $t_2 :=3D\ \delta(t_1)$ and $t_3 :=3D\ \eta(t_2)$ where:

$$\delta : [0, \frac{\pi}{\Gamma_-}] \to [0, \frac{\pi}{\Gamma_+}], \quad \eta : [0, \frac{\pi}{\Gamma_+}] \to [0, \frac{\pi}{\Gamma_-}].$$

These functions are continuously differentiable, strictly increasing and satisfy:

$$\delta(\bar{t}_-) =3D \bar{t}_+, \quad \eta(\bar{t}_+) =3D \bar{t}_-.$$

The functions δ and η satisfy respectively the following o.d.e. obtained by differentiating equations (29) and (30):

$$\dot{\delta}(t) =3D \frac{\Gamma_-^2 (a^2 + \rho)}{\Gamma_+^2 (a^2 - \rho)} \frac{1 + c_- \cos \Gamma_- t}{1 + c_+ \cos \Gamma_+ \delta(t)} \Big(\frac{\cos \Gamma_+ \delta(t) + c_+}{\cos \Gamma_- t + c_-} \Big)^2, \tag{31}$$

and:

$$\dot{\eta}(t) =3D \frac{\Gamma_+}{\Gamma_-} k \frac{\cos \Gamma_+ t + c_+}{\cos \Gamma_- \eta(t) + c_-} \Big(\frac{1 + c_- \cos \Gamma_- \eta(t)}{1 + c_+ \cos \Gamma_+ t} \Big)^2. \tag{32}$$

They realize bijections between their domains and their ranges. The following lemma establishes a key relation between δ and η, which will considerably simplifies all the subsequent computations,

Lemma 2. *With the notations above, we have* $\eta =3D \delta^{-1}$.

Proof of Lemma 2: It is enough to show that $t_1 =3D t_3$. First notice that since φ_2 has zeros, γ cannot be of type-A. We have $\varphi_2(t_0) =3D\ 1$ and simple computations show that:

$$\begin{aligned} \varphi_3(t_0) &=3D\ \psi_3(t_0 + t_1 + t_2) =3D \sqrt{K^2 - 1} > 0, \\ \varphi_3(t_0 + t_1) &=3D\ \psi_3(t_0 + t_1 + t_2 + t_3) =3D \sqrt{K^2 - a^2} \geq 0, \end{aligned}$$

and:

$$\lambda(t_0) =3D \begin{pmatrix} 0 \\ 1 \\ \sqrt{K^2 - 1} \end{pmatrix}, \ \lambda(t_0 + t_1) =3D \begin{pmatrix} -1 \\ 0 \\ \sqrt{K^2 - a^2} \end{pmatrix},$$

$$\lambda(t_0 + t_1 + t_2) =3D \begin{pmatrix} 0 \\ -1 \\ \sqrt{K^2 - 1} \end{pmatrix}, \ \lambda(t_0 + t_1 + t_2 + t_3) =3D \begin{pmatrix} 1 \\ 0 \\ \sqrt{K^2 - a^2} \end{pmatrix}.$$

Let $\mathcal{S}$ be the symmetry of $\mathbb{R}^3$ transforming (x_1, x_2, x_3) to $(-x_1, -x_2, x_3)$. For $t \in [0, t_1]$, define:

$$\psi(t) :=3D\ \mathcal{S}(\lambda(t_0 + t)).$$

Note that $\psi(0) =3D\ \lambda(t_0 + t_1 + t_2)$. Then, on $[0, \min(t_1, t_3)]$, $\psi(\cdot)$ and $\lambda(t_0 + t_1 + t_2 + \cdot)$ satisfy the same Cauchy problem:

$$\dot{Y} =3D \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ a^2 - \rho & 1 - \rho & 0 \end{pmatrix} Y, \ \ Y(0) =3D \begin{pmatrix} 0 \\ -1 \\ \sqrt{K^2 - 1} \end{pmatrix}.$$

Therefore they are identically equal on the whole interval $[0, \min(t_1, t_3)]$. Since $\psi(t_1) =3D\ \lambda(t_0 + t_1 + t_2 + t_3)$, it is easy to conclude that $t_1 =3D\ t_3$.

Corollary 1. *For* $t \in [0, \frac{\pi}{\Gamma_-}]$*, we have:*

$$(1 + c_- x_-)(x_+ + c_+) =3D\ (1 + c_+ x_+)(x_- + c_-), \tag{33}$$

where $x_- =3D\ \cos(\Gamma_- t)$ *and* $x_+ =3D\ \cos(\Gamma_+ \delta(t))$.

Proof of Corollary 1: By Lemma 2, we have $\dot{\eta}\dot{\delta} =3D\ 1$. Then the conclusion follows by multiplying equations (31) and (32). This leads to the following useful expression of x_- in terms of x_+, which is valid only when $(c_+, c_-) \in \mathcal{R}$:

$$x_- =3D \frac{c_+ - c_- + (1 - c_+ c_-)x_+}{1 - c_+ c_- + (c_+ - c_-)x_+}. \tag{34}$$

In the case where $(c_+, c_-) =3D\ (1, 1)$, Lemma 2 is immediate since $\dot{\delta} = \dot{\eta} =3D\ 1$ from (31) and (32), and equation (34) is a tautology. However, one gets a relation between x_+ and x_- directly from (29) by taking its square and simplifying. We get

$$k^2 \frac{1 - x_+}{1 - x_-} =3D \frac{1 + x_+}{1 + x_-}, \tag{35}$$

and

$$x_- =3D \frac{1 - k^2 + (1 + k^2)x_+}{1 + k^2 + (1 - k^2)x_+}.$$

Remark **3.** *We can extend the functions* δ *and* η *as odd functions on* $[-\bar{t}_-, \bar{t}_-]$ *and* $[-\bar{t}_+, \bar{t}_+]$ *respectively with the equations (31), (32) and (33) still holding. That procedure extends all the results of this section to oscillatory optimal trajectories with the same sequence of switchings as* γ *but starting with an* $(-X_+)$*. Moreover, the cases where the first bang arc is* $(\pm X_-)$ *are actually treated previously since the trajectory obtained by reversing time can be seen as a piece of a trajectory starting with a* $(\pm X_+)$*. The case where the sequence of switchings starts with a* u_2*-switching is also treated similarly since trajectories run backwards are still admissible.*

We can refine Proposition 3 to show that:

Proposition 5. *Along an optimal oscillatory trajectory, the time durations along an interior* X_- *arc and an interior* X_+ *arc respectively are less or equal than* $\bar{t}_-$ *and less or equal than* $\bar{t}_+$ *respectively. In addition, along a* $Bu_1Bu_1Bu_2$ *trajectory, the time duration between the two consecutive* u_1*-switchings is equal to* $2\bar{t}_-$ *and the time duration between the* u_1*-switching and the* u_2*-switchings is equal to* $\bar{t}_+$*.*

Proof of Proposition 5: All the possible situations reduce to the study of two cases:

Case 1: we have an $(X_-)_t$ arc with the sequence of switchings u_2, u_1;

Case 2: we have an $(X_-)_t$ arc with the sequence of switchings u_1, u_2.

We start with Case 1. Then we easily get:

$$\lambda(0) =3D \begin{pmatrix} 1 \\ 0 \\ -\sqrt{K^2-a^2} \end{pmatrix}, \quad \lambda(t) =3D \begin{pmatrix} 0 \\ -1 \\ -\sqrt{K^2-1} \end{pmatrix},$$

and deduce that:

$$V_{t_0} =3D \begin{pmatrix} -(a^2-\rho) \\ -\Gamma_-\sqrt{K^2-a^2} \end{pmatrix}, \quad V_{t_1} =3D \begin{pmatrix} 1-\rho \\ -\Gamma_-\sqrt{K^2-1} \end{pmatrix}, \quad V_{t_1} =3D R_{\Gamma_- t} V_{t_0}.$$

Then $1-\rho =3D -x(a^2-\rho) + \sin(\Gamma_- t)\Gamma_-\sqrt{K^2-a^2}$, which implies that $x + c_- =3D \frac{\sin(\Gamma_- t)\Gamma_-\sqrt{K^2-a^2}}{a^2-\rho} \geq 0$.

We treat Case 2 similarly. We have then:

$$\lambda(0) =3D \begin{pmatrix} 0 \\ -1 \\ \sqrt{K^2-1} \end{pmatrix}, \quad \lambda(t) =3D \begin{pmatrix} 1 \\ 0 \\ \sqrt{K^2-a^2} \end{pmatrix},$$

and deduce that:

$$V_{t_0} =3D \begin{pmatrix} 1-\rho \\ \Gamma_-\sqrt{K^2-1} \end{pmatrix}, \quad V_{t_1} =3D \begin{pmatrix} -(a^2-\rho) \\ \Gamma_-\sqrt{K^2-a^2} \end{pmatrix}, \quad V_{t_1} =3D R_{\Gamma_- t} V_{t_0}.$$

We multiply the first component of V_{t_1} by x, the second one by $\sin(\Gamma_- t)$ and add the two equations. We again end up with $x + c_- =3D \frac{\sin(\Gamma_- t)\Gamma_-\sqrt{K^2-a^2}}{a^2-\rho} \geq 0$. Note that equality can occur only along trajectories of type-B.

Assume now that we have a $Bu_1Bu_1Bu_2$ trajectory, let say $\gamma =3D (-X_-)_{t_1}(X_+)_{t_2}$. By lemma 1, γ is of type-B and then there exists $t_1' \in (0, t_1)$ such that $\varphi_2(t_1') =3D \varphi_3(t_1') =3D 0$. The piece of γ given by $\gamma' =3D (-X_-)_{t_1-t_1'}(X_+)_{t_2}$ can be embedded into an oscillatory extremal trajectory $Bu_2Bu_1Bu_2$ that has to be of type-B. Then, by what preceds, we get $t_1 - t_1' =3D \bar{t}_-$ and $t_2 =3D \bar{t}_+$. By proceeding similarly with $(-X_-)_{t_1'}$, we get $t_1' =3D \bar{t}_-$ and conclude. Note also that if γ is a bang optimal trajectory of the type $(\pm X_\varepsilon)_t$, then it follows from the previous computations on conjugate points that $t \leq \bar{t}_\varepsilon$).

3.3. **Optimal singular trajectories.**

Proposition 6. *Assume that an optimal trajectory contains a singular arc S_t, $t > 0$. Then,*

- (i)**:** *if S_t is u_1-singular, then S_t is the integral curve corresponding to the vector field $\pm S_1$, where $S_1 :=3D -\rho f_1 + f_2$. Note that S_t is admissible if and only if $\rho \leq 1$;*
- (ii)**:** *if S_t is u_2-singular, then S_t is the integral curve corresponding to the vector field $\pm S_2$, where $S_2 :=3D f_1 + \frac{\rho}{a^2} f_2$, which is always admissible.*

We also have $\|S_1\| =3D a\sin\alpha$ and $\|S_2\| =3D \sin\alpha$.

Note that, for (i), K has to be equal to 1 and then this is a trajectory of type-A and for (ii), K =3D a, i.e. this is a trajectory of type-B.

Proof of Proposition 6. Assume that S_t is u_1-singular. Then $\varphi_1 \equiv 0$ along S_t and φ_2 =3D $\mp\varepsilon$. From (16), we get $\varphi_3 \equiv 0$ and from (17), we obtain $u_1 + \rho u_2$ =3D 0. Then (i) follows at once and similarly for (ii).

Lemma 3. *Assume that an optimal trajectory contains the piece* γ' :=3D $S_s B_t$, $s, t >$ 0, *with B_t an interior bang arc corresponding to X_ε. Then $(c_+, c_-) \in \mathcal{R}$, S has to be u_2-singular and* t =3D $\frac{\pi}{\Gamma_\varepsilon}$.

Proof of Lemma 3: Let γ be the trajectory under consideration. It steers some point p_0 to $p_0 e^{sS} e^{tX_\varepsilon}$. Assume that S =3D $\pm S_1$. By optimality, we can associate a nontrivial covector λ to γ subject to the conclusion of the Pontryagin Maximum Principle, and then at the time s, we have $\varphi_1(s)$ =3D $\varphi_3(s)$ =3D 0, which can be written:

$$< \lambda(s), p_1 f_1 > \text{=3D} < \lambda(s), p_1 h > \text{=3D } 0,$$

where p_1 :=3D $p_0 e^{sS}$. There is another switching occurring at time $s + t$. If it were a u_1-switching, we would have two consecutive u_1-switchings with φ_3 =3D 0 at the switching times. That would imply that $\Gamma_- t$ =3D 2π, which is not possible (cf. the proof of Proposition 3). Then the switching occurring at time $s+t$ is a u_2-switching. Therefore $\varphi_2(t + s)$ =3D 0, which can be written $< \lambda(s), p_1 e^{tadX_\varepsilon} f_2 >$=3D 0. Since $\lambda(s)$ is not zero:

$$e^{tadX_\varepsilon} f_2 \wedge f_1 \wedge h \text{ =3D } 0.$$

Using (8), one is left with $1 \pm c_\varepsilon \cos(\Gamma_\varepsilon t)$ =3D 0, which is impossible unless (c_+, c_-) =3D $(1, 1)$ and t =3D $\frac{\pi}{\Gamma_\varepsilon}$. But, we have

$$e^{\frac{\pi}{\Gamma_\varepsilon} ad(X_\varepsilon)}(S_1) \text{ =3D } \varepsilon S_2$$

Then γ' =3D $S_s B_{\frac{\pi}{\Gamma_\varepsilon}}$ =3D $B_{\frac{\pi}{\Gamma_\varepsilon}}(\pm\varepsilon S_2)$. If ε =3D -1, γ' admits a covector with a switching time at $\frac{\pi}{\Gamma_\varepsilon}$ where φ_1 and φ_2 are both zero, which is impossible. If ε =3D 1, γ' is also equal to $B_{-\frac{\pi}{\Gamma_\varepsilon}}(\pm\varepsilon S_2)$ and again it admits a covector with a switching time where φ_1 and φ_2 are both zero. Therefore γ' and thus γ are not optimal. Since

$$e^{\frac{\pi}{\Gamma_\varepsilon} ad(X_\varepsilon)}(S_2) \text{ =3D } \varepsilon S_1$$

the same conclusion holds if (c_+, c_-) =3D $(1, 1)$ and S =3D $\pm S_2$.

Therefore $(c_+, c_-) \in \mathcal{R}$, S =3D $\pm S_2$ and the switching occurring at time $s + t$ is a u_1-switching. From the fact that $\varphi_2(s)$ =3D $\varphi_3(s)$ =3D 0 and $\varphi_1(t + s)$ =3D 0, one gets as above that $\cos(\Gamma_+ t)$ =3D $\mp c_+$ or $\cos(\Gamma_- t)$ =3D $\mp c_-$, i.e. t =3D $\frac{\pi}{\Gamma_\varepsilon}$.

Lemma 4. *We assume that $(c_+, c_-) \in \mathcal{R}$. For every $t > 0, t_1, t_2 \geq 0$, the trajectory* γ_+ :=3D $(X_+)_{t_1}(S_2)_t(X_+)_{t_2}$ *is not optimal if* $t_1 + t_2 \geq \frac{\pi}{\Gamma_+}$ *and the trajectory* γ_- :=3D $(X_-)_{t_1}(S_2)_t(X_-)_{t_2}$ *is not optimal either if* $t_1 + t_2 \geq \frac{\pi}{\Gamma_-}$.

Proof of Lemma 4: We start with γ_+. The terminal point of γ_+ can also be reached by the trajectory γ'_+ defined by:

$$\gamma'_+ \text{ :=3D } (-X_+)_{\frac{\pi}{\Gamma_+} - t_1}(\widetilde{S}_2)_t(-X_+)_{\frac{\pi}{\Gamma_+} - t_2},$$

where $\widetilde{S}_2$:=3D $e^{\frac{\pi}{\Gamma_+} ad(X_+)}(S_2)$ =3D $a_1 f_1 + a_2 f_2$, with:

$$a_1 \text{ =3D } 1\frac{4(c_+ - c_-)}{(1 + c_+)(2 - c_- - c_+)} < 1 \text{ and } a_2 \text{ =3D } 1\frac{2(1 - c_+)(1 - c_-)}{(1 + c_+)(2 - c_- - c_+)} < 1.$$

Note that:

$$a_1+1 =3D \frac{2(1-c_+)(2+c_++c_-)}{(1+c_+)(2-c_--c_+)} > 0 \text{ and } a_2+1 =3D \frac{2(1-c_+^2+2c_+(1-c_-))}{(1+c_+)(2-c_--c_+)} > 0.$$

Then $A :=3D \max(|a_1|, |a_2|) < 1$. The time duration $T(\gamma_+)$ along γ_+ is equal to t_1+t_2+t and the time duration $T(\gamma'_+)$ along γ'_+ is equal to $2(\frac{\pi}{\Gamma_+}-\frac{t_1+t_2}{2})+At$. We have:

$$T(\gamma_+)-T(\gamma'_+) =3D (1-A)t+2(t_1+t_2-\frac{\pi}{\Gamma_+}) > 0.$$

Then γ_+ is not optimal.

We proceed similarly. Let $\widetilde{S}'_2$ the vector defined by:

$$\widetilde{S}'_2 :=3D e^{\frac{\pi}{\Gamma_-}ad(X_-)}(S_2).$$

We have $\widetilde{S}'_2 =3D a'_1f_1+a'_2f_2$, with:

$$a'_1 =3D 1\frac{4(1-c_+)}{(1+c_-)(2-c_--c_+)} < 1 \text{ and } a'_2 =3D 1\frac{2(1-c_+)(1-c_-)}{(1+c_-)(2-c_--c_+)} < 1.$$

Note that

$$a'_1+1 =3D \frac{2(1-c_-)(c_++c_-)}{(1+c_-)(2-c_--c_+)} > 0 \text{ and } a'_2+1 =3D \frac{2(1+2c_--c_-^2-2c_-c_+)}{(1+c_-)(2-c_--c_+)} > 0.$$

Note that $A'_2 :=3D 1+2c_--c_-^2-2c_-c_+ =3D 1-c_-^2+2c_-(1-c_+) =3D 1+2c_--c_-(2c_++c_-)$. Then $A'_2>0$ if $c_-\geq -\frac{1}{2}$. If $c_-<-\frac{1}{2}$, $A'_2>1-c_-^2+2c_-(1+c_-) =3D (1+c_-)^2>0$. Therefore, $A' :=3D \max(|a'_1|, |a'_2|) < 1$. From that and $t_1+t_2\geq\frac{\pi}{\Gamma_-}$, we conclude as before that the time duration along:

$$(X_-)_{\frac{\pi}{\Gamma_-}-t_1}(\widetilde{S}'_2)_t(X_-)_{\frac{\pi}{\Gamma_-}-t_2},$$

is less than $T(\gamma_-)$ and then γ_- is not optimal.

Lemma 5. *For every $t\geq 0$, we have:*

$$e^{tS_2}e^{\bar{t}_+X_+}e^{-\bar{t}_-X_-} =3D e^{\bar{t}_+X_+}e^{-\bar{t}_-X_-}e^{-tS_2}. \tag{36}$$

Proof of Lemma 5: Let $\gamma(t) :=3D e^{-tS_2}e^{\bar{t}_+X_+}e^{-\bar{t}_-X_-}e^{tS_2}$. It is enough to show that $\frac{d\gamma}{dt} =3D 0$. Indeed:

$$\frac{d\gamma}{dt} =3D e^{-tS_2}e^{\bar{t}_+X_+}R(t)e^{-\bar{t}_-X_-}e^{tS_2},$$

where $R(t) :=3D -e^{\bar{t}_+ad(X_+)}(S_2)+e^{-\bar{t}_-ad(X_-)}(S_2)$. Using equations (7) and (8), one immediately gets that $R(t)\equiv 0$ and the lemma is proved.

3.4. **Useful expressions.** We need to consider relations useful for the next section. Here $x =3D \cos(\Gamma_+t)$ and $y =3D \cos(\Gamma_-t)$. Equation (27) says that, at the point p_2, the vector fields f_2, $e^{tadX_-}(f_1)$ and $e^{-\delta(t)adX_+}(f_1)$ are linearly dependent. Then there exist $\mu,\nu\in\mathbb{R}$ such that:

$$p_2e^{-\delta(t)adX_+}(f_1) =3D \mu p_2e^{tadX_-}(f_1)+\nu p_2f_2.$$

One gets, for $t\in(-\frac{\pi}{\Gamma_+},\frac{\pi}{\Gamma_+}]$, that:

$$\mu =3D \frac{e^{-\delta(t)adX_+}(f_1)\wedge f_2\wedge h}{e^{tadX_-}(f_1)\wedge f_2\wedge h}, \quad \nu =3D \frac{e^{-\delta(t)adX_+}(f_1)\wedge e^{tadX_-}(f_1)\wedge h}{f_2\wedge e^{tadX_-}(f_1)\wedge h}.$$

One deduces that:

$$\mu =3D=3D\frac{1+c_-}{1+c_+}\frac{y+c_+}{x+c_-},\quad \nu =3D\frac{c_+}{1+c_+}(1-y)+\frac{c_-}{1+c_-}(1-x)\mu. \tag{37}$$

Similarly, from equation (28), we get:

$$p_3e^{tadX_-}(f_2) =3D\beta p_3e^{\delta(t)adX_+}(f_2)+\sigma p_3f_1,$$

with:

$$\beta =3D\frac{1}{\mu},\quad \sigma =3D\Big(\frac{1-x}{1+c_-}+\frac{1-y}{1+c_+}\beta\Big). \tag{38}$$

Finally in terms of x and y, formula (34) becomes, when $(c_+,c_-)\in\mathcal{R}$:

$$x =3D\frac{c_+-c_-+(1-c_+c_-)y}{1-c_+c_-+(c_+-c_-)y}, \tag{39}$$

and, when $(c_+,c_-) =3D(1,1)$:

$$x =3D\frac{1-k^2+(1+k^2)y}{1+k^2+(1-k^2)y}.$$

4. **Bound on the Number of Bang Arc for Oscillatory Trajectories.** The purpose of this section is to prove the following theorem and proposition:

Theorem 2. *A time-optimal oscillatory trajectory has at most three switchings.*

Proposition 7. *Assume that $(c_+,c_-)\in\mathcal{R}$. The trajectory of type-B $(\mp X_-)_{2\bar{t}_-}(\pm X_+)_{\bar{t}_+}$ is not time-optimal.*

We will prove the theorem considering an oscillatory trajectory of the type:

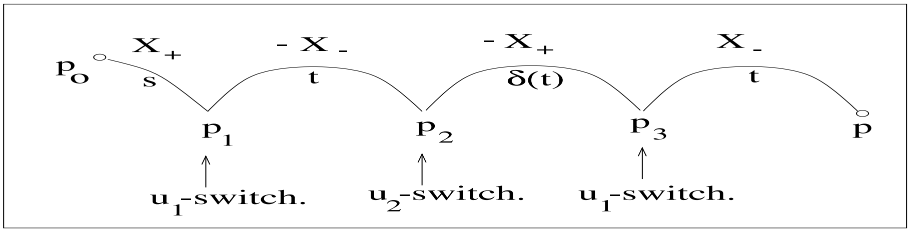
$$(X_+)_s(-X_-)_t(-X_+)_{\delta(t)}(X_-)_t.$$

All the other cases are either equivalent or similar.

The theorem will follow if we show that the trajectory γ given by:

$$p =3D p_0e^{sX_+}e^{-tX_-}e^{-\delta(t)X_+}e^{tX_-}, \tag{40}$$

is not time-optimal (see Fig. 4.1).



4.1. **Envelope Theory for 4 Bangs.** Let γ be defined in (40). Let us describe briefly the technique of envelope. We begin with a "variation" of γ by constructing a one-parameter family $\Gamma =3D\{\gamma_\tau\ \tau \in [c,d]\}$ of normal extremals starting at p_0 at time 0 such that $\gamma_d =3D \gamma$. We then show that the curve $\xi : \tau \in [c,d] \mapsto Term(\gamma_\tau)$ is an envelope of Γ such that $\xi(d) =3D p$, so that the concatenation $\bar\gamma$ of $\xi\gamma$ from p_0 to $\xi(c)$ and from $\xi(c)$ to p reaches $\bar p$ from p_0 in the same time as γ does. We finally manage to prove that such a $\bar\gamma$ is not optimal, so γ is not optimal either. We give the details of the proof in the following. First we want to construct a family of normal extremals with the fixed initial state p_0 and the fixed initial time 0 such that γ is embedded in this family. Recall that we have only to treat the case $t \in]0, \bar t_-]$. If $(c_+, c_-) =3D (1,1)$, we may even assume that $t < \bar t_- =3D \frac{\pi}{\Gamma_-}$ since, using $e^{\bar t_\varepsilon adX_\varepsilon}(X_{-\varepsilon}) =3D -X_{-\varepsilon}$, we can prove that γ is not optimal.

Let $D \subset \mathbb{R}^2$ be a small disc centered at (s,t) such that if $(a,b) \in D$, then $0 < a < \frac{\pi}{\Gamma_+}, 0 < b < \bar t_-$ and consider the following family of extremal trajectories:

$$\mathcal{T} =3D\{\gamma_{a,b} : (a,b) \in D,\ \gamma \text{ is a trajectory of the form } (X_+)_a(-X_-)_b(-X_+)_{\delta(b)}(X_-)_b \text{ starting from } p_0\}.$$

The terminal state of $\gamma_{a,b}$ is:

$$q(a,b) =3D p_0 e^{aX_+}e^{-bX_-}e^{-\delta(b)X_+}e^{bX_-} =3D \tilde p e^{-\delta(b)X_+}e^{bX_-}, \tag{41}$$

where we have defined $\tilde p :=3D p_0 e^{aX_+}e^{-bX_-}$. We would like to construct a nowhere vanishing smooth vector field E on D such that if $\zeta : [c,d] \to D$ is an integral curve of E, then $q(\zeta(.))$ is a trajectory of (5). This is equivalent to finding two smooth functions e_1 and e_2 such that:

$$E(a,b) =3D e_1(a,b)\frac{\partial}{\partial a} + e_2(a,b)\frac{\partial}{\partial b}, \tag{42}$$

and $dq(E)(a,b)$, the differential dq of q acting at the tangent vector $E(a,b)$ is a linear combination of the vectors $q(a,b)X_+$ and $q(a,b)X_-$. We have:

$$dq(E)(a,b) =3D e_1(a,b)\frac{\partial q}{\partial a} + e_2(a,b)\frac{\partial q}{\partial b} =3D \tilde p C(e_1, e_2)e^{-\delta(b)X_+}e^{bX_-}, \tag{43}$$

where:

$$C(e_1, e_2) =3D (e_1 + e_2\mu)e^{b\ \mathrm{ad}X_-}(X_+) + e_2(\mu - \nu - 1)X_- + (\nu - 1 - \dot\delta)e_2X_+. \tag{44}$$

Assume that $dq(E)(a,b) =3D q(a,b)(A_+X_+ + A_-X_-)$. Then we have the relation:

$$dq(E)(a,b) =3D \tilde p D(A_+, A_-)e^{-\delta(b)X_+}e^{bX_-}, \tag{45}$$

where one easily checks that:

$$\begin{aligned} D(A_+, A_-) \quad &=3D \ [A_+(1+\sigma) + A_-\mu]\, e^{b\ \mathrm{ad}X_+}(X_-) + \\ &[A_+((1+\sigma)(\mu-\nu) - \beta) + A_-(\mu-\nu)]\, X_- + \\ &[A_+((1+\sigma)\nu + (\beta - 1)) + A_-(\nu - 1)]\, X_+. \end{aligned} \tag{46}$$

Since $\Gamma_- b < \pi$, we have that $e^{b\ \mathrm{ad}X_-}(X_+) \wedge X_- \wedge X_+ =3D \frac{\sin \Gamma_- b}{\Gamma_-} h \wedge X_- \wedge X_+ \neq3D 0$ and we get:

$$\left\{ \begin{array}{rl} e_1 + e_2\mu & =3D \ A_+[\mu(1+\sigma)] + A_-\mu, \\ e_2(\mu - \nu - 1) & =3D \ A_+[(1+\sigma)(\mu-\nu) - \beta] + A_-(\mu - \nu), \\ e_2[\nu - 1 - \dot\delta] & =3D \ A_+[(1+\sigma)\nu + (\beta - 1)] + A_-(\nu - 1). \end{array} \right.$$

If we take $A_+ =3D\ 1$ and we use the relation $\dot{\sigma} =3D\ \mu$ (see formula (37)), then we obtain the system of linear equations:

$$M\ U =3D \begin{pmatrix} \mu(1+\sigma) \\ (1+\sigma)(\mu-\nu)-\beta \\ (1+\sigma)\nu+(\beta-1) \end{pmatrix},$$

where:

$$M =3D \begin{pmatrix} 1 & \mu & -\mu \\ 0 & \mu-\nu-1 & -(\mu-\nu) \\ 0 & \nu-1-\mu & -(\nu-1) \end{pmatrix} \text{ and } U =3D \begin{pmatrix} e_1 \\ e_2 \\ A_- \end{pmatrix}. \tag{47}$$

The determinant of M is:

$$\Delta =3D\ \nu-1-(\mu-\nu)\mu =3D \frac{2\mu}{1+c_-}[c_-(1-x_-)-(1+c_-x_-)\mu], \tag{48}$$

where to get the last expression we have used formulas (37). We have then:

$$A_- =3D \frac{N}{\Delta}$$

where:

$$N =3D\ (\mu-\nu+1)[(1+\sigma)(\mu-\nu)-\beta]+(\mu-\nu-1)[(1+\sigma)\nu+\beta-1]. \tag{49}$$

We recall that $x =3D\ \cos(\Gamma)$ and $y =3D\ \cos(\Gamma_+(t))$. We first assume that $(c_+,c_-) \in \mathcal{R}$. Expressing μ,ν,β,σ with formulas (37), (38) and x as a function of y with formula (39) we have that:

$$\begin{aligned} \Delta &=3D \frac{-2}{(1+c_+)^2(1-c_-)} P_\Delta(y), \\ N &=3D \frac{-4(1-y)}{(1+c_+^2)(1-c_+c_-+(c_+-c_-)y)} P_N(y), \end{aligned}$$

where:

$$\begin{aligned} P_\Delta(y) &=3D\quad 1 = c - c_+c_- + c_+^2c_- + y\,(2c_+(1-c_+c_-)) + y^2c_+(c_+-c_-), && (50) \\ P_N(y) &=3D\quad 1 + c_+^2 + c_+^3 - c_- - c_+c_- - c_+^3c_- + \\ &\qquad yc_+(3+c_+-c_--3c_+c_-) + y^2c_+(c_+-c_-). && (51) \end{aligned}$$

Finally:

$$A_- =3D \frac{2(1-y)(1-c_-)}{1-c_+c_-+(c_+-c_-)y} \frac{P_N(y)}{P_\Delta(y)}. \tag{52}$$

Notice that since $\frac{1-c_+c_-}{c_+-c_-} > 1$ we have:

$$1-c_+c_-+(c_+-c_-)y =3D (c_+-c_-)[y+\frac{1-c_+c_-}{c_+-c_-}] > 0 \text{ for } y \in [-1,1]. \tag{53}$$

We now assume that $(c_+,c_-) =3D\ (1,1)$. Then equations (48) and (49) become by using (37), (38),

$$\Delta =3D\ \frac{\frac{1+y}{2}}{1+x}(x+y) \text{ and } N =3D\ -(1-y)(1+2x+y).$$

Then $A_- =3D \frac{(1-y)(1+x)}{2(1+y)}\frac{1+2x+y}{x+y}$. Assume now that $A_- > 0$ (i.e. $\frac{1+2x+y}{x+y} > 0$ if $(c_+,c_-) =3D (1,1)$ and $\frac{P_N(y)}{P_\Delta(y)} > 0$ if $(c_+,c_-) \in \mathcal{R}$). We have then $A_-(b) > 0$. Set:

$$\bar A_- =3D \frac{A_-}{1+A_-},\ \bar A_+ =3D \frac{1}{1+A_-}.$$

Then $\bar A_- + \bar A_+ =3D 1$ and $\bar A_+ - \bar A_- =3D \frac{1-A_-}{1+A_-} \in (-1,1)$. Thus we can write:

$$dq(E)(a,b) =3D q(a,b)(\bar A_- X_- + \bar A_+ X_+) =3D q(a,b)[f_1 + (\bar A_+ - \bar A_-)f_2] =3D q(a,b)Z,$$

that is for any integral curve $\zeta : [c,d] \to D$ of the vector field E, $q(\zeta(.))$ is an integral curve of the vector field:

$$Z :=3D f_1 + (\bar A_+ - \bar A_-)f_2, \tag{54}$$

and it is an admissible curve with inputs $(1, \bar A_+ - \bar A_-)$. For a small $\varepsilon > 0$, let $\zeta : [-\varepsilon, 0] \to D$ be the integral curve of E such that $\zeta(0) =3D (s,t)$. For each $\tau \in [-\varepsilon, 0]$, we write $\zeta(\tau) =3D (a(\tau), b(\tau))$, and let $\gamma_{a(\tau),b(\tau)}$ be the extremal trajectory starting at p_0 at time 0, of the following type:

$$(X_+)_{a(\tau)}(-X_-)_{b(\tau)}(-X_+)_{\delta(b(\tau))}(X_-)_{b(\tau)}.$$

We now want to construct an envelope of the family of extremals Γ. Let $\xi : [-\varepsilon, 0] \to SO(3)$ be defined by $\xi(\tau) =3D q(\zeta(\tau))$ for all $\tau \in [-\varepsilon, 0]$. Then $\xi(0) =3D q(s,t) =3D p$, the terminal state of the trajectory γ. Let γ_τ be the concatenation of $\gamma_{a(\tau),b(\tau)}$ which goes from p_0 to $q(\zeta(\tau))$ and ξ from $q(\zeta(\tau))$ to p in time $a(\tau) + 2b(\tau) + \delta(b(\tau)) - \tau$. It is easily seen that $\xi : \tau \in [-\varepsilon, 0] \mapsto q(\zeta(\tau)) =3D \gamma_{a(\tau),b(\tau)}(T_\tau)$ where $T_\tau =3D a(\tau) + 2b(\tau) + \delta(b(\tau))$, is an envelope of the family of extremals Γ, because:

- **(i):** Γ is differentiable on $[-\varepsilon, 0]$,
- **(ii):** ξ is a trajectory of the system Σ corresponding to the control $\bar\eta =3D (\bar A_+ - \bar A_-)$,
- **(iii):** the identity:

$$H(\lambda_\tau(T_\tau), \gamma_{a(\tau),b(\tau)}(T_\tau), \eta_\tau(T_\tau)) =3D \varphi_1(T_\tau) =3D H(\lambda_\tau(T_\tau), \gamma_{a(\tau),b(\tau)}(T_\tau), \bar\eta(\tau)),$$

  holds for all $\tau \in [-\varepsilon, 0]$, where η_τ is the corresponding control for $\gamma_{a(\tau),b(\tau)}$.

Therefore by the general result of the theory of Envelopes (cf. [4]), the time along the trajectory $\gamma_{-\varepsilon}$ is exactly the same as the time along $\gamma =3D \gamma_{s,t}$.

We now want to prove non-optimality of γ. It is clear that the trajectory $\gamma_{-\varepsilon}$ steers p_0 to p in time $T(\gamma) =3D s + 2t + \delta(t)$. Thus $\gamma_{-\varepsilon}$ is optimal as γ is. Notice that since $\dot\xi(\tau) =3D \xi(\tau)(f_1 + (\bar A_+(b(\tau)) - \bar A_-(b(\tau)))f_2)$ it follows that ξ is a trajectory of Σ corresponding to the u_2-singular control $(1, \bar A_+(b(\tau)) - \bar A_-(b(\tau)))$. Thus $\gamma_{-\varepsilon}$ is a concatenation of a bang-bang trajectory with three switchings and a u_2-singular extremal.

By Lemma 3 we reach a contradiction in the case $x \neq 3D -c_+$ (that is $b \neq 3D \bar t_-$ where $\bar t_-$ is defined by $\cos\Gamma_-\bar t_- =3D -c_+$. In the case $b =3D \bar t_-$, it is enough to assume that $(c_+,c_-) \in \mathcal{R}$. Then $A_-(\bar t_-)$ (A_- is a function of b through y, cf. formula (52)) is always positive since $P_N(-c_+) =3D P_\Delta(-c_+)$:

$$A_-(\bar t_-) =3D \frac{2(1+c_+)(1-c_-)}{1 - c_+c_- + (c_+ - c_-)(-c_+)} =3D 2\frac{1-c_-}{1-c_+}. \tag{55}$$

It follows:

$$\bar{A}_+ - \bar{A}_- =3D \frac{1-A_-}{1+A_-} =3D \frac{2-\frac{1-c_+}{1-c_-}}{2+\frac{1-c_+}{1-c_-}}.$$

But we know that the control corresponding to a u_2-singular must be:

$$-\frac{\rho}{a^2} =3D \frac{1-\frac{1-c_+}{1-c_-}}{1+\frac{1-c_+}{1-c_-}}$$

and:

$$(\bar{A}_+ - \bar{A}_-) - (-\frac{\rho}{a^2}) =3D \frac{-2\omega}{(2+\omega)(1+\omega)} \neq 3D\ 0, \quad \text{with } \omega :=3D \frac{1-c_+}{1-c_-}.$$

Then Z defined in formula (54) is not an extremal arc. So we have proved the following:

Lemma 6. *For each $s,t>0$, if $\frac{P_N(y)}{P_\Delta(y)} > 0$ for $(c_+,c_-) \in \mathcal{R}$ and $x+y>0$ for $(c_+,c_-) =3D (1,1)$, then an oscillatory trajectory of the form $(B)_s(B)_t(B)_{\delta(t)}(B)_t$ is not optimal.*

4.2. **A One-Parameter Variation for 4 Bangs.** Let γ be an extremal trajectory of type $(X_+)_s(-X_-)_t(-X_+)_{\delta(t)}(X_-)_t$, with initial state p_0 and terminal state p:

$$p =3D p_0 e^{sX_+} e^{-tX_-} e^{-\delta(t)X_+} e^{tX_-}. \tag{56}$$

Let us consider a variation of γ in such a way the terminal point is fixed and can be expressed as:

$$q(\varsigma) =3D p_0 e^{t_1(\varsigma)X_+} e^{-t_2(\varsigma)X_-} e^{-t_3(\varsigma)X_+} e^{t_4(\varsigma)X_-},$$

where $t_1(\varsigma), t_2(\varsigma), t_3(\varsigma), t_4(\varsigma))$ are four functions defined in a neighborhood I of 0 and satisfying:

$$(t_1(0), t_2(0), t_3(0), t_4(0)) =3D\ (s, t, \delta(t), t),$$

(that is for $\varsigma =3D\ 0$ the variated trajectory coincides with the reference trajectory γ). These functions will be determined in such a way that $q(\varsigma) =3D\ q(0) =3D\ p$ for every ς in I (that is the terminal point is fixed). We have:

$$\dot{q} =3D p_0 e^{t_1X_+} e^{-t_2X_-} R e^{-t_3X_+} e^{t_4X_-},$$

where $R =3D \dot{t}_1 e^{t_2\ \mathrm{ad}X_-}(X_+) - \dot{t}_2X_- - \dot{t}_3X_+ + \dot{t}_4 e^{-t_3\ \mathrm{ad}X_+}(X_-)$. Now setting:

$$\begin{cases} \dot{t}_1 =3D \frac{\Gamma_-}{\Gamma_+}\frac{\sin\Gamma_+t_3}{\sin\Gamma_-t_2}, \\ \dot{t}_2 =3D \cos\Gamma_+t_3 + \frac{1-c_-}{1+c_-}(1-\cos\Gamma_-t_2)\frac{\Gamma_-}{\Gamma_+}\frac{\sin\Gamma_+t_3}{\sin\Gamma_-t_2}, \\ \dot{t}_3 =3D -(\cos\Gamma_-t_2)\frac{\Gamma_-}{\Gamma_+}\frac{\sin\Gamma_+t_3}{\sin\Gamma_-t_2} - \frac{1-c_+}{1+c_+}(1-\cos\Gamma_+t_3), \\ \dot{t}_4 =3D\ 1, \end{cases} \tag{57}$$

one easily checks that $R =3D\ 0$ (that is $\dot{q} =3D\ 0$) in I. Therefore we have constructed a family $\{\gamma_\varsigma\}_{\varsigma\in I}$ of trajectories steering p_0 to p in which $\gamma =3D \gamma_0$ is embedded. In the following we prove that, under suitable assumptions on c_+, c_-, t, the time in which γ_ς steers p_0 to p, $T(\varsigma) =3D \sum_{i=3D1}^4 t_i(\varsigma)$, has a strict local maximum at the reference trajectory γ. We have:

$$\dot{T} =3D \frac{dT}{d\varsigma} =3D 2\left(\frac{\cos\Gamma_+t_3 + c_+}{1+c_+} - \frac{\Gamma_-}{\Gamma_+}\frac{\sin\Gamma_+t_3}{\sin\Gamma_-t_2}\frac{\cos\Gamma_-t_2 + c_-}{1+c_-}\right). \tag{58}$$

Notice that from the relations:

$$\begin{cases} \dot t_1(0) =3D \frac{1+c_-}{1+c_+}\frac{y+c_+}{x+c_-}, \\ \dot t_2(0) =3D\ y\frac{1-c_-}{1+c_+}(1-x)\frac{y+c_+}{x+c_-}, \\ \dot t_3(0) =3D \frac{1+c_-}{1+c_+}x\frac{y+c_+}{x+c_-} - \frac{1-c_+}{1+c_+}(1-y), \\ \dot t_4(0) =3D\ 1, \end{cases}$$

one gets $\dot T(0)$ =3D 0, that is obvious since the reference trajectory γ is extremal. From (58) we have:

$$\ddot T =3D \frac{d^2T}{d\varsigma^2} =3D -2\Gamma_+\dot t_3 \left(\frac{\sin\Gamma_+ t_3}{1+c_+} + \frac{\Gamma_-}{\Gamma_+(1+c_-)}\frac{\cos\Gamma_+ t_3}{\sin\Gamma_- t_2}(\cos\Gamma_- t_2 + c_-)\right). \tag{59}$$

So the second derivative of T at zero is:

$$\ddot T(0) =3D \frac{2\Gamma_+(1+c_+y)}{(1+c_+)\sin\Gamma_+\delta(t)}\left(-\dot t_3(0) + \frac{1+c_-}{1+c_+}\ \frac{y+c_+}{x+c_-}\ \dot t_2(0)\right). \tag{60}$$

We deduce that, if (c_+,c_-) =3D $(1,1)$, we have $\ddot T(0)$ =3D $\frac{\Gamma_+(1+y)}{\sin\Gamma_+\delta(t)}(x+y)$, and if $(c_+,c_-)\in\mathcal{R}$, we get

$$\ddot T(0) =3D \frac{4\Gamma_+(1+c_+y)}{(1+c_+)^3(1-c_-)\sin\Gamma_+\delta(t)}P_\Delta(y), \tag{61}$$

where $P_\Delta(y)$ is defined in (50). Since:

$$\frac{4\Gamma_+(1+c_+y)}{(1+c_+)^3(1-c_-)\sin\Gamma_+\delta(t)} > 0,$$

we have that $T(\varsigma)$ has a local maximum at ς =3D 0 if $x+y<0$ when (c_+,c_-) =3D $(1,1)$ and if $P_\Delta(y)<0$ when $(c_+,c_-)\in\mathcal{R}$.

In the case where $\ddot T(0)$ =3D 0, we must compute the third derivative at 0, i.e. when $x+y$ =3D 0 or at the roots of P_Δ belonging to $(-1,-c_+]$, depending on (c_+,c_-). When (c_+,c_-) =3D $(1,1)$ $\frac{d^3T}{d\varsigma^3}(0)$ =3D $-2\Gamma_+^2(1+x)$. Then, if $\ddot T(0)$ =3D 0 and $\frac{d^3T}{d\varsigma^3}(0)$ =3D 0, we must have x =3D y =3D 0, which implies that k =3D 1, i.e. =3D Π_+ by equation (29). When $(c_+,c_-)\in\mathcal{R}$, we have

$$\frac{d^3T}{d\varsigma^3}(0) =3D \frac{-2\ (1+y\,c_+)}{(-1+y)^2\,(1+y)^2\,(-1+c_-)^2\,(1+c_+)^5}P_3(y), \tag{62}$$

where:

$$\begin{aligned} P_3(y)\quad :=3D\ & (c_- - c_+)\left(-1+c_-+5\,c_+-2\,c_-\,c_+-6\,c_-\,{c_+}^2+3\,c_-\,{c_+}^3\right)+ \\ & 2\left(1-{c_-}^2-2\,c_+-4\,c_-\,c_++{c_-}^2\,c_++4\,{c_+}^2+8\,c_-\,{c_+}^2+5\,{c_-}^2\,{c_+}^2-\right. \\ & \left.5\,{c_+}^3-6\,c_-\,{c_+}^3-8\,{c_-}^2\,{c_+}^3+6\,c_-\,{c_+}^4+{c_-}^2\,{c_+}^4\right)y+ \\ & \left(3\,c_-+3\,{c_-}^2+11\,c_+-4\,c_-\,c_++5\,{c_-}^2\,c_+-7\,{c_+}^2-34\,c_-\,{c_+}^2-\right. \\ & 7\,{c_-}^2\,{c_+}^2+13\,{c_+}^3+28\,c_-\,{c_+}^3+19\,{c_-}^2\,{c_+}^3-5\,{c_+}^4+ \\ & \left.-11\,c_-\,{c_+}^4-8\,{c_-}^2\,{c_+}^4\right)y^2+ \\ & 2\,c_+\left(-4\,c_-+{c_-}^2+8\,c_++7\,{c_-}^2\,c_+-{c_+}^2-18\,c_-\,{c_+}^2-2\,{c_-}^2\,{c_+}^2+\right. \\ & \left.3\,{c_+}^3+2\,c_-\,{c_+}^3+4\,{c_-}^2\,{c_+}^3\right)y^3+ \\ & (c_- - c_+)\,c_+\left(c_- - 7\,c_++c_-\,c_+-{c_+}^2+6\,c_-\,{c_+}^2\right)y^4. \end{aligned}$$

The resultant between $P_3(y)$ and $P_\Delta(y)$ is equal to:

$$-4\,(-1+c_-)^3\,{c_-}^2\,(c_- - c_+)^2\,(-1+c_+)^3\,{c_+}^2\,(1+c_+)^8\,\left(-1+c_- + c_+ + 3\,c_-\,c_+\right),$$

and is equal to zero in the region $\mathcal{R}$, if and only if:

$$Res :=3D\ c\!\!+ c_+ - 1 + 3c_-c_+, \tag{63}$$

is zero. We have proved the following:

Lemma 7. *For each $s, t > 0$, an oscillatory trajectory of the form:*

$$(B)_s(B)_t(B)_{\delta(t)}(B)_t,$$

is not optimal when

$(c_+, c_-) =3D\ (1,1)$**:** *if $x + y > 0$, or if $x + y =3D\ 0$ and $\!\!\!<\mathbf{\Gamma}_+$;*
$(c_+, c_-) \in \mathcal{R}$**:** *if $P_\Delta(y) < 0$ or if $P_\Delta(y) =3D\ 0$ and $Res \neq3D\ 0$.*

4.3. **Proof of Theorem 2.** We first prove Theorem 2 when $(c_+, c_-) \in \mathcal{R}$. Let us study in more details the polynomial $P_N(y)$ and $P_\Delta(y)$ in the region in which we are interested in, that is:

$$\bar{R} :=3D\ \{(\!c\!\!c_-, y) \in \mathbb{R}^3 :\ 0 \le |c_-| < c_+ < 1,\ y \in [-c_+, 1]\},$$

(cf. Proposition 5). Let us recall the expressions of $P_N(y)$ and $P_\Delta(y)$:

$$\begin{aligned} P_N(y) \quad &=3D\quad 1 \overset{2}{\underset{+}{+}} c\!\!+ c_+^3 - c_- - c_+c_- - c_+^3c_- + \\ &\qquad yc_+(3 + c_+ - c_- - 3c_+c_-) + y^2c_+(c_+ - c_-), \\ P_\Delta(y) \quad &=3D\quad 1 = c\!\!- c_+c_- + c_+^2c_- + y\,(2c_+(1 - c_+c_-)) + y^2c_+(c_+ - c_-). \end{aligned}$$

$P_N(y)$ and $P_\Delta(y)$ satisfy the following properties:

(P1): The two polynomials have the same coefficient of y^2 that is bigger than zero. Moreover $P_N(0) > 0$ and $P_\Delta(0) > 0$.

(P2): We have:

$$P_N(y) =3D_\Delta P\!\!(y) + c_+(1 + c_+)(1 - c_-)(y + c_+).$$

This means that in $\bar{R}$, we have:

$$\begin{aligned} P_N \quad &> \quad P_\Delta \text{ if } y > -c_+, \\ P_N \quad &=3D\ {}_\Delta P\!\!\text{if } y =3D\ {}_+\!\!-\!c \end{aligned}$$

(P3): We have:

$$\dot{P}_N(y) =3D_{\!+}\, c\!\!(3 - c_- - 2\,y\,c_- + c_+ + 2\,y\,c_+ - 3\,c_-\,c_+) > 0,$$

where we have used condition (53). Similarly:

$$\dot{P}_\Delta(y) =3D_{+}\!2(\!c\!1 - c_+c_- + (c_+ - c_-)y) > 0.$$

This means that both $P_N(y)$ and $P_\Delta(y)$ are strictly increasing and have at most one root in the interval $y \in [-c_+, 1)$. From **P1** it follows that if these roots exist, then they lye in the interval $y \in [-c_+, 0)$. Call these roots respectively y_N^+ and y_Δ^+.

Now using the properties **(P1)-(P3)**, Lemmas 6, 7 and Proposition 5, we prove Theorem 2. Set:

$$\bar{P} :=3D P_N(-c_+) =3D P_\Delta(-c_+) =3D (c_+ + 1)^2(1-c_+)(1-c_+-c_-).$$

We have the three cases (illustrated in the following picture): $\bar{P} > 0$ ($c_- < 1 - c_+$), $\bar{P} =3D 0$ ($c_- =3D 1 - c_+$) and $\bar{P} < 0$ ($c_- > 1 - c_+$).

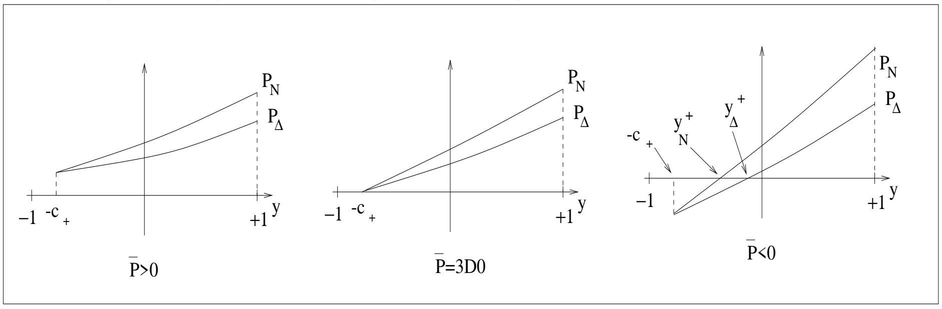


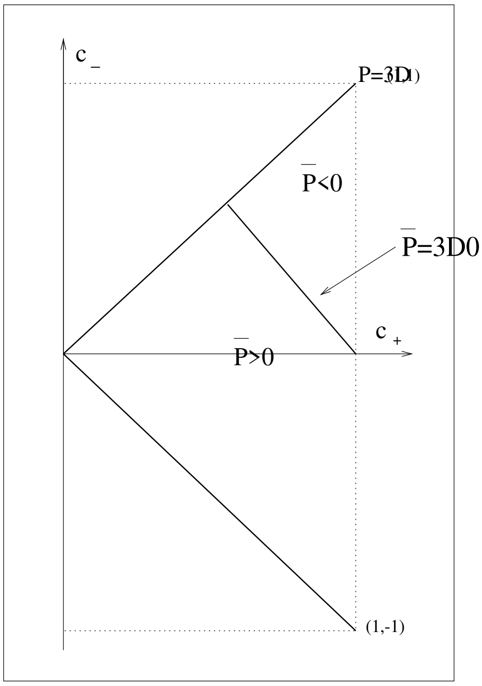
Fig. 4.2

In the first case, from **(P3)** we have $P_N(y), P_\Delta(y) > 0$ in $\bar{R}$, so from Lemma 6, the conclusion of Theorem 2 follows.
In the second case, we have $y_N^+ =3D y_\Delta^+$ and from **(P3)** we have $P_N(y), P_\Delta(y) > 0$ for $y \in (-c_+, 1)$ (subcase A) and $P_N(y) =3D P_\Delta(y) =3D 0$ for $y =3D -c_+$ (subcase B). In subcase A the conclusion follows from Lemma 6 while in subcase B, $R_{is} =3D 3c_+ + c_- > 0$ and the conclusion follows from Lemma 7.
In the third case, from **(P2)** and **(P3)** we have $y_N^+ < y_\Delta^+$. The conclusion follows from Lemma 6 for in $y \in (y_\Delta^+, 1)$ and from Lemma7 for $y \in [-c_+, y_\Delta^+]$. This concludes the proof when $(c_+, c_-) \in \mathcal{R}$.

The case where $(c_+, c_-) =3D (1,1)$ actually reduces to assume that $\Gamma_- =3D \Gamma_+ =3D \sqrt{2}$. But then, the statement of the theorem follows by taking into account the next equation:

$$e^{\frac{\pi}{2\Gamma_-}ad(X_{-\varepsilon})} e^{\frac{\pi}{2\Gamma_-}ad(X_\varepsilon)}(X_{-\varepsilon}) =3D X_\varepsilon$$

The following picture shows the projection of the region $\bar{R}$ on the (c_+, c_-)-plane.


c_-
P=3D(1,1)
$\bar{P}$<0
$\bar{P}$=3D0
c_+
$\bar{P}$>0
(1,-1)

Notice that from the relations, $\cos\Gamma_+ t_3(0) =3D \mp c$ and $\cos\Gamma_- t_2(0) =3D^2_- 2c - 1$, one gets $\dot{T}(0) =3D\ 0$ and:

$$\ddot{T}(0) =3D \frac{(1-c_+)(1-2c_-)}{2c_-^2(1-c_-)(1+c_+)}(2c_-^2 + 4c_-c_+ - c_- + c_+ - 1). \tag{65}$$

It is clear that the sign of $d^2T/\ D^2d(T)$ depends on that of:

$$\mathcal{E} :=3D^2_- 2c + 4c_-c_+ - c_- + c_+ - 1.$$

The conclusion of the proposition follows if we show that $\mathcal{E} < 0$. Recall that $0 < -c_- < c_+ < 1$. We have:

$$\mathcal{E} =3D\ A_+c + B,$$

with $A :=3D\ 1 + 4c_+$ and $B :=3D^2_- 2c - c_- - 1$. If $A > 0$, then $\mathcal{E} \geq 0$ would imply that $c_+ \geq \frac{-B}{A}$. But:

$$\frac{-B}{A} - 1 =3D \frac{-c_-(5+2c_-)}{A} > 0,$$

and we would have $c_+ > 1$, which is impossible. If $A =3D\ 0$, then $\mathcal{E} < 0$. If $A < 0$, then $\mathcal{E} \geq 0$ would imply that $c_+ \leq \frac{-B}{A}$. Finally:

$$\frac{-B}{A} + c_- =3D\ \frac{c_{-2}^2 + c_- + 1/\ D2}{-A} < 0,$$

implying that $c_+ < -c_-$, which is impossible.

5. **The structure of optimal trajectories.** In this section, we collect all the results obtained in the previous sections. Let γ be an optimal trajectory. Then, by Proposition 4, γ is a finite concatenation of bang and singular arcs. The discussion splits in two cases, whether γ has a singular arc or not.
In the case where γ does not contain a singular arc, there are two subcases whether γ is oscillatory or not. If γ is not oscillatory then, $(c_+, c_-) \in \mathcal{R}$ and thanks to Proposition 7, it has at most two consecutive u_1-switchings. Then γ is either equal to:

$$(\pm X_+)_s(\mp X_-)_{2\bar{t}_-}(\pm X_+)_t, \tag{66}$$

with $0 \leq s, t < \bar{t}_+$, or any piece of the previous trajectory. If γ is oscillatory, then thanks to Theorem 2, it has at most three switchings. Then γ (run in positive or negative time) is of the type:

$$B_s B_t B_{\delta(t)} B_{t'}, \tag{67}$$

with $0 \leq |\delta^{-1}(s)|, |t'| < |t| \leq \bar{t}_-$, or any piece of the previous concatenation. Note that, if a bang arc $B_t =3D\ (\pm X)_{\varepsilon t}$ is optimal, then $0 \leq t \leq \bar{t}_\varepsilon$.

Assume that γ has a singular arc. If $(c_+, c_-) =3D\ (1,1)$, then γ is of the type $B_s S_t B_{t'}$ with $s, t' < \pi/\ D_\varepsilon$ and $t < \frac{\pi}{\sin(\alpha)}$. Next we assume that $(c_+, c_-) \in \mathcal{R}$. If that singular arc is an S_1-arc, then it cannot be preceded or followed by an interior bang arc. Therefore, an optimal trajectory γ containing a S_1-arc is of the type:

$$(\pm X_\varepsilon)_s(\pm S_1)_t(\pm X_\varepsilon)_{t'}, \tag{68}$$

with $s + t' < \pi/\ D_\varepsilon$ and $t < \frac{\pi}{a\sin(\alpha)}$.

Assume now that the singular arc contained in γ is an S_2-arc. Suppose first that the singular arc is preceded and (or) followed by a $(\pm X_+)$ bang arc. Then γ contains a piece of the type $(\pm X_+)_{t_1}(\pm S_2)_t(\pm X_+)_{t_2}$. By Lemma 4, $t_1 + t_2 < \frac{\pi}{\Gamma_+}$. Assume that γ contains another arc. It has to be a $(\mp X_-)_s$ and $t_2 =3D\ \bar{t}_+$. Then $t_1 < \bar{t}_+$ and there is no arc before $(\pm X_+)_{t_1}$ We can assume that it follows the $(\pm X_+)_{t_2}$ arc. We claim that if $t_1 > 0$, then $s < \bar{t}_+$. Otherwise, by Lemma 5, we can replace γ by $(\pm X_+)_{t_1+\bar{t}_+}(\mp X_-)_{\bar{t}_-}(\mp S_2)_t$, which is not optimal since it contains $(\pm X_+)_{t_1+\bar{t}_+}$. If $t_1 =$3D 0, then s may be equal $\bar{t}_{\varphi}$. The next arc can be either a singular $(\mp S_2)_{t'}$ arc or a $(\mp X_+)_{s'}$ arc. The first option is possible only if $t + t' < \frac{\pi}{\sin\alpha}$ because:

$$(\pm S_2)_t(\pm X_+)_{\bar{t}_+}(\mp X_-)_{\bar{t}_-}(\mp S_2)_{t'} =\text{3D } (\pm S)_{2t+t'}(\pm X_+)_{\bar{t}_+}(\mp X_-)_{\bar{t}_-}.$$

Note that $(\mp S_2)_{t'}$ cannot be followed by a $(\mp X_-)$ thanks to Lemma 5, the above equation and the fact that a $(\mp X_-)_t$ arc with $t > \bar{t}_-$, is not optimal. The second option is that γ contains a piece:

$$(\pm S_2)_t(\pm X_+)_{\bar{t}_+}(\mp X_-)_{\bar{t}_-}(\mp X_+)_{s'}.$$

But it is then equal to $(\pm X_+)_{\bar{t}_+}(\mp X_-)_{\bar{t}_-}(\mp S_2)_t(\mp X_+)_{s'}$, which is not an extremal. In conclusion, if γ has a S_2-arc preceded and (or) followed by a $(\pm X_+)$ arc, it is of the type:

$$(\pm X_+)_{t_1}(\pm S_2)_t(\pm X_+)_{\bar{t}_+}(\mp X_-)_s, \tag{69}$$

with $t < \frac{\pi}{\sin\alpha}$, $t_1 + \bar{t}_+ < \frac{\pi}{\Gamma_+}$ and $s \leq \bar{t}_-$, the inequality being strict if $t_1 > 0$.

The last case to examine is when γ contains a S_2-arc, preceded and (or) followed by a $(\pm X_-)$ bang arc. We have a piece of the type: $(\pm X_-)_{t_1}(\mp S_2)_t(\pm X_-)_{t_2}$. The situation is similar to the previous one if $\bar{t}_- \geq \frac{\pi}{2\Gamma_-}$, i.e. $c_- \geq 0$, i.e. $\rho \leq 1$. We thus assume that $c_- < 0$. Then γ can contain a piece of the type $(\pm X_-)_{\bar{t}_-}(\mp S_2)_t(\pm X_-)_{\bar{t}_-}$. In this case, we claim that γ is necessarily of the type:

$$(\mp X_+)_{s'}(\pm X_-)_{\bar{t}_-}(\mp S_2)_t(\pm X_-)_{\bar{t}_-}(\mp X_+)_s, \tag{70}$$

with $s, s' < \bar{t}_+$ and $t \leq \frac{\pi}{\sin\alpha}$. Indeed, if s (or s') is equal to $\bar{t}_+$, then we use:

$$(\pm X_-)_{\bar{t}_-}(\mp S_2)_t(\pm X_-)_{\bar{t}_-}(\mp X_+)_{\bar{t}_+} =\text{3D } (\pm X)_{2\bar{t}_-}(\mp X_+)_{\bar{t}_+}(\pm S_2)_t,$$

to show that γ is not optimal. The last possibility is that γ contains a piece:

$$(\pm X_-)_{t_1}(\mp S_2)_t(\pm X_-)_{\bar{t}_-}(\mp X_+)_s,$$

with $t_1 < \bar{t}_-$. Then $s \leq \bar{t}_+$ with possible equality if $t_1 =$3D 0. In this case and for the reasons as above, no other arc follows $(\mp X_+)_{\bar{t}_+}$.

In conclusion, we have:

Theorem 3. *An optimal trajectory γ is equal or included in one of the following:*

1.: $(\pm X_+)_s(\mp X_-)_{2\bar{t}_-}(\pm X_+)_t$, *with* $0 \leq s, t < \bar{t}_+$;

2.: $B_sB_tB_{\delta(t)}B_{t'}$, *with* $0 \leq |\delta^{-1}(s)|, |t'| < |t| \leq \bar{t}_-$;

3.: $(\pm X_\varepsilon)_s(\pm S_1)_t(\pm X_\varepsilon)_{t'}$, *with* $s + t' < \pi/$ DΓ_ε*and* $t < \frac{\pi}{a\sin(\alpha)}$;

4.: $(\pm X_+)_{t_1}(\pm S_2)_t(\pm X_+)_{\bar{t}_+}(\mp X_-)_s$, *with* $t < \frac{\pi}{\sin\alpha}$, $t_1 + \bar{t}_+ < \frac{\pi}{\Gamma_+}$ *and* $s \leq \bar{t}_-$, *the inequality being strict if* $t_1 > 0$;

5.: $(\mp X_+)_{s'}(\pm X_-)_{\bar{t}_-}(\mp S_2)_t(\pm X_-)_{\bar{t}_-}(\mp X_+)_s$, *with* $s, s' < \bar{t}_+$ *and* $t \leq \frac{\pi}{\sin\alpha}$.

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