A dual to Lyapunov's second theorem

Anders Rantzer
Department of Automatic Control, Lund Institute of Technology
Box 118, S-221 00 Lund, Sweden, Phone: +46 46 222 03 62
Email: rantzer@control.lth.se

Submitted for publication in Systems and Control Letters
Revised April 13, 2000

Abstract

A convergence criterion for nonlinear systems is presented and can be viewed as a dual to Lyapunov's second theorem.

The criterion has a physical interpretation in terms of the stationary density of a substance that is generated in all points of the state space and flows along the system trajectories. If the stationary density is finite everywhere except at a singularity in the origin, then almost all trajectories tend towards the origin.

The new criterion differs from Lyapunov's theorem in two important respects. One is that the statement allows for an exceptional set of zero measure where the trajectories fail to converge. Another difference is a convexity property in synthesis of control laws.

Keywords

Stabilization, nonlinear systems, convex duality

1 Introduction

Lyapunov's second theorem has long been recognized as one of the most fundamental tools for analysis and synthesis of nonlinear systems. The importance of the criterion stems from the fact that it allows stability of a system to be verified without solving the differential equation explicitly.

The original work of Lyapunov in the late 19th century was devoted to problems from astronomy and fluid mechanics. In the 1950's, it was applied by Chetayev to aeronautical stability problems and by Lur'e and Letov for nonlinear control problems. The ideas were promoted in the 1960's by Kalman, Lefschetz and La Salle and have found widespread applications since then [3, 8, 16, 7, 5, 11].

Lyapunov functions play a role similar to potential functions and energy functions. Moreover, when asymptotic stability of an equilibrium has been proved using Lyapunov's theorem, input-output stability can often be proved using the Lyapunov function as a "storage function" [18].

It is surprising to find that Lyapunov's second theorem has a natural dual that has been neglected until present date. This is even more striking as the same kind of duality has been used since the 1940's for closely related problems in calculus of variations [6, 19, 17, 2].

The outline of this paper is as follows. In Section 2, the duality is explained from an intuitive viewpoint. The new convergence criterion in terms of a "density function" is then proved in Section 3 and followed by a few examples. The relationship between Lyapunov functions and density functions is discussed in Section 4.

In Section 5 we make a connection to more recent work on feedback control based on Lyapunov functions [4, 9, 10]. Some of the difficulties in stabilization of nonlinear systems can be associated with the fact that the set of "control Lyapunov functions" has a difficult structure. For some systems, it is not even connected. It is therefore interesting to note that the corresponding set for the new convergence criterion is convex. This convexity property has been the basis for the corresponding literature in calculus of variations.

The notation

$$\nabla V = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \qquad V : \mathbf{R}^n \to \mathbf{R}$$

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n} \qquad f : \mathbf{R}^n \to \mathbf{R}^n$$

will be used throughout the paper.

2 A viewpoint of duality

One way to interpret a Lyapunov function V for the globally stable dynamical system $\dot{x} = f(x)$ is to view $V(x_0)$ as the "cost to go" from the initial state x_0 to the equilibrium. In fact, existence of a Lyapunov function is often proved by introducing a penalty function $l(x) \geq 0$ and defining V by the formula

$$V(x_0) = \int_0^\infty l(x(t))dx$$

where $\dot{x}(t) = f(x(t))$, $x(0) = x_0$. The definition immediately implies that

$$\nabla V(x) \cdot f(x) = -l(x) \leq 0$$

Inspired by convexity arguments in optimal control [15, 19, 17], we consider the "flow" of trajectories generated by a system. Given the rate of generation $\psi(x) \geq 0$, the integral

$$\int_X V(x)\psi(x)dx$$

can be viewed as the total stationary cost per time unit, when the substance flows along the system trajectories towards the equilibrium. If the flow gives rise to the stationary density $\rho(x)$ in each point, another expression for the stationary cost per time unit is

$$\int_X \rho(x)l(x)dt$$

Equality between the two expressions follows from Gauss' theorem:

Proposition 1 *Given $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$, let $\psi, l \in \mathbf{C}(\mathbf{R}^n)$ and $\rho, V \in \mathbf{C}^1(\mathbf{R}^n)$ satisfy*

$$\nabla V \cdot f + l = 0 \qquad\qquad \nabla \cdot (f\rho) = \psi \tag{1}$$

in $X \subset \mathbf{R}^n$ while $V(x) = 0$ for x on the boundary of X. Then

$$\int_X V(x)\psi(x)dx = \int_X \rho(x)l(x)dx$$

Proof.

$$\int_X [V\psi - \rho l]dx = \int_X [V(\nabla \cdot (f\rho)) + \nabla V \cdot f\rho]dx = \int_X \nabla \cdot (Vf\rho)dx = 0$$

where the last equality is due to Gauss' theorem and the fact that $V(x) = 0$ on the boundary of X.

The duality between the V and ρ is apparent in Proposition 1. Since the first equality in (1) with $l(x) \geq 0$ is the basis for Lyapunov's second theorem, it should come as no surprise that a dual criterion can be stated based on the second equality in (1) with $\psi(x) \geq 0$. This is even more clear from the interpretation of ρ as stationary density of a substance generated with rate ψ. For a stable system, a finite stationary density can be achieved everywhere except at the equilibrium, where the substance will accumulate and the density will be infinite. Conversely, the existence of a stationary density indicates that almost all substance must accumulate at $x = 0$. This intuitive argument is formalized in the next section.

3 The main result

Theorem 1 *Given the equation $\dot{x}(t) = f(x(t))$, where $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$, suppose there exists a non-negative $\rho \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R})$ such that $\rho(x)f(x)/|x|$ is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$ and*

$$[\nabla \cdot (\rho f)](x) > 0 \qquad\qquad \text{for almost all } x \neq 0 \tag{2}$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ tends to zero as $t \to \infty$. Moreover, if $x = 0$ is a stable equilibrium, then the conclusion remains valid even if ρ takes negative values.

Remark 1 The assumptions of continuous differentiability of f and ρ are convenient, but more restrictive than necessary.

The proof of the theorem relies on the following lemma.

Lemma 1 *Let $D \subset \mathbf{R}^n$ be open and let $\rho \in \mathbf{C}^1(D, \mathbf{R})$ be integrable on $E \subset D$. For $f \in \mathbf{C}^1(D, \mathbf{R}^n)$ and $x_0 \in \mathbf{R}^n$, let $\Phi(x_0, t)$ be the solution of $\dot{x} = f(x)$, $x(0) = x_0$ and let $\Phi(X, t) = \{\Phi(x, t) \mid x \in X\}$. If $\Phi(X, t)$ is a measurable subset of E for every t, then*

$$\int_{\Phi(X,t)} \rho(x)dx - \int_X \rho(z)dz = \int_0^t \int_{\Phi(X,\tau)} [\nabla \cdot (\rho f)](x)dxd\tau$$

The lemma is proved in the appendix.

Proof of Theorem 1, second statement. Here it is assumed that $x = 0$ is a stable equilibrium, while ρ may take negative values. The proof for the other case is given in the appendix.

Given any $x_0 \in \mathbf{R}^n$, let $\Phi(x_0, t)$ be the solution of $\dot{x}(t) = f(x(t))$, $x(0) = x_0$. Assume first that ρ is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$ and $|f(x)|/|x|$ is bounded. Then Φ is well defined for all t. Given $r > 0$, define

$$X = \cap_{l=1}^{\infty} \{x_0 \ : \ |\Phi(x_0, t)| > r \text{ for some } t > l\} \tag{3}$$

Notice that X contains all trajectories with $\limsup_{t\to\infty} |x(t)| > r$. The set X, being the intersection of a countable number of open sets, is measurable. Moreover, $\Phi(X, t) = \{\Phi(x, t) \mid x \in X\}$ is equal to X for every t. By stability of the equilibrium $x = 0$, there is a positive lower bound ε on the norm of the elements in X, so Lemma 1 with $E = \{x \ : \ |x| \geq \varepsilon\}$ gives

$$0 = \int_{\Phi(X,t)} \rho(x)dx - \int_X \rho(z)dz = \int_0^t \int_{\Phi(X,\tau)} [\nabla \cdot (\rho f)](x)dxd\tau \tag{4}$$

By the assumption (2), this implies that X has measure zero. Consequently, $\limsup_{t\to\infty} |x(t)| \leq r$ for almost all trajectories. As r was chosen arbitrarily, this proves that $\lim_{t\to\infty} |x(t)| = 0$ for almost all trajectories.

When $|f(x)|/|x|$ is unbounded, there may not exist any nonzero t such that $\Phi(z, t)$ is well defined for all z. We then introduce

$$\rho_0(x) = \left[\frac{e^{-|x|}}{1 + |\rho(x)|} + \frac{|f(x)|}{|x|}\right] \rho(x) \qquad\qquad f_0(x) = \frac{f(x)\rho(x)}{\rho_0(x)}$$

Then $|f_0(x)|/|x|$ is bounded and ρ_0 is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$, so the argument above can be applied to f_0 together with ρ_0 to prove that $\lim_{t\to\infty} |x(t)| = 0$ for almost all trajectories of the system $\dot{x} = f_0(x)$. However, these trajectories are identical to the trajectories of the original system, modulo a transformation of the time axis

$$\tau = \int_0^t \frac{\rho_0(x(s))}{\rho(x(s))} ds$$

so the proof is complete.

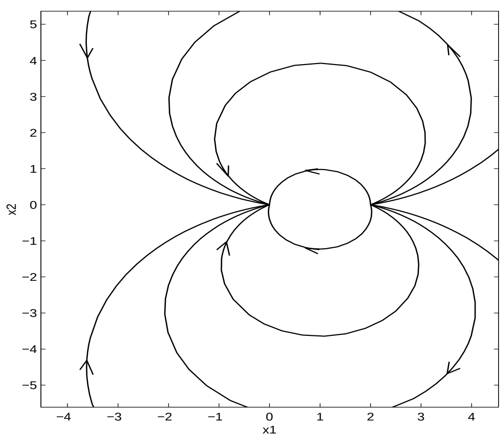

Figure 1: Phase plane plot for Example 3

Example 1 For scalar x, define

$$f(x) = x \qquad\qquad \rho(x) = -\frac{1}{x^4}$$

Then $[\nabla \cdot (f\rho)](x) = 3/x^4 > 0$, so all conditions of Theorem 1 hold except for non-negativity of ρ and stability of $x = 0$.

Example 2 With

$$f(x) = (x^2 - 1)x \qquad\qquad \rho(x) = \frac{1}{x^2}$$

we have $[\nabla \cdot (\rho f)](x) = 1 + x^{-2} > 0$, so all conditions of Theorem 1 hold except for the integrability of $\rho f/|x|$. In this case, all trajectories starting outside the interval $[-1, 1]$ have finite escape time.

Example 3 The system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -2x_2 + 2x_1x_2 \end{bmatrix}$$

has two equilibria $(0, 0)$ and $(2, 0)$. See Figure 1. Let $f(x)$ be the right hand side and let $\rho(x) = |x|^{-\alpha}$. Then

$$\begin{aligned}
[\nabla \cdot (\rho f)](x) &= \nabla\rho \cdot f + \rho(\nabla \cdot f) \\
&= -\alpha|x|^{-\alpha-2}x^T f + |x|^{-\alpha}(4x_1 - 4) \\
&= -\alpha|x|^{-\alpha-2}(x_1 - 2)|x|^2 + |x|^{-\alpha}(4x_1 - 4) \\
&= |x|^{-\alpha}\Big[(4 - \alpha)x_1 + 2\alpha - 4\Big]
\end{aligned}$$

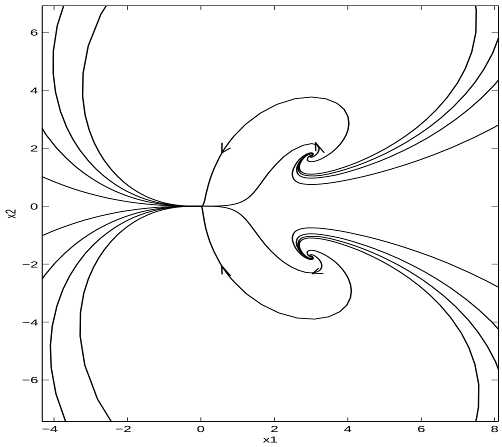

Figure 2: Phase plane plot for Example 4

With $\alpha = 4$ all conditions of Theorem 1 hold, so almost all trajectories tend to $(0,0)$ as $t \to \infty$. The exceptional trajectories turn out to be those that start with $x_1 \geq 2$, $x_2 = 0$.

Example 4 The system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -6x_2 + 2x_1x_2 \end{bmatrix} \tag{5}$$

has four equilibria $(0,0)$, $(2,0)$ and $(3, \pm\sqrt{3})$. See Figure 2. In this case, $\rho(x) = |x|^{-4}$ gives

$$\begin{aligned} [\nabla \cdot (\rho f)](x) &= -4|x|^{-6}x^T f + |x|^{-4}(4x_1 - 8) \\ &= -4|x|^{-6}\Big[(x_1 - 2)|x|^2 - 4x_2^2\Big] + |x|^{-4}(4x_1 - 8) \\ &= 16x_2^2|x|^{-6} \end{aligned}$$

so again Theorem 1 shows that almost all trajectories tend to $(0,0)$ as $t \to \infty$. The exceptional trajectories are the three unstable equilibria, the axis $x_2 = 0$, $x_1 \geq 2$ and the stable manifold of the equilibrium $(2,0)$, that spirals out from the equilibria $(3, \pm\sqrt{3})$.

4 Relation to Lyapunov functions

The fact that Lyapunov's theorem has a stronger implication than the convergence criterion of Theorem 1, suggests the possibility to derive a density function ρ from a Lyapunov function V. This can generally be done in the following way.

Proposition 2 *Let $V(x) > 0$ for $x \neq 0$ and*

$$\nabla V \cdot f < \alpha^{-1}(\nabla \cdot f)V \qquad \textit{for almost all } x$$

for some $\alpha > 0$. Then $\rho(x) = V(x)^{-\alpha}$ satisfies the condition (2).

In particular, if P is a positive definite matrix satisfying

$$A^T P + PA < (\alpha^{-1} \operatorname{trace} A)P$$

then $\rho(x) = (x^T Px)^{-\alpha}$ satisfies the condition (2) for the system $\dot{x} = Ax$.

Proof. With $\rho(x) = V(x)^{-\alpha}$, we get

$$\begin{aligned} \nabla \cdot (f\rho) &= (\nabla \cdot f)\rho + \nabla \rho \cdot f \\ &= (\nabla \cdot f)V^{-\alpha} - \alpha V^{-(\alpha+1)} \nabla V \cdot f \\ &= \alpha V^{-(\alpha+1)} \left[\alpha^{-1}(\nabla \cdot f)V - \nabla V \cdot f\right] \\ &> 0 \end{aligned}$$

With $V(x) = x^T Px$ and $f(x) = Ax$ the second statement follows since

$$\begin{aligned} \nabla V \cdot f &= x^T \left(A^T P + PA\right) x \\ \nabla \cdot f &= \operatorname{trace} A \end{aligned}$$

Transfer in the opposite direction, from density function to Lyapunov function, is generally not possible. The simple reason is that a density function may exist even if the system is not globally asymptotically stable. This was the situation in Example 3 and Example 4. However, with an additional assumption that $\nabla \cdot f \leq 0$, the following construction can be used.

Proposition 3 *Suppose for $x \neq 0$ that*

$$\nabla \cdot (f\rho) > 0 \qquad\qquad \nabla \cdot f \leq 0 \qquad\qquad \rho > 0$$

Then $V(x) = \rho(x)^{-1}$ satisfies $\nabla V \cdot f < 0$.

Proof.

$$\nabla V \cdot f = -\rho^{-2} \nabla \rho \cdot f = -\rho^{-2}[\nabla \cdot (f\rho) - (\nabla \cdot f)\rho] < 0$$

5 Convexity in nonlinear stabilization

An important application area for Lyapunov function is the synthesis of stabilizing feedback controllers. For a given system, the set of Lyapunov functions is convex. This fact is the basis for many numerical methods, most notably in computation of quadratic Lyapunov functions using linear matrix inequalities [1]. However, when the control law and Lyapunov function are to be found simultaneously, no such convexity property is at hand. In fact, the following example suggested by [13, 14] shows that the set of "control Lyapunov functions" (functions that can be used as Lyapunov functions for some stabilizing control law) may not even be connected.

Example 5 Every continuous stabilizing control law $u(x)$ for the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x,u) = \begin{bmatrix} -(x_2)^2 x_1 \\ u(x) \end{bmatrix}$$

must have the property that $u(x)$ has constant sign along the half line $x_1 > 0$, $x_2 = 0$. The reason is that a zero crossing would create a second equilibrium. A strictly decreasing Lyapunov function satisfies

$$0 > \nabla V \cdot f(x,u) = \frac{\partial V}{\partial x_2} u(x) \qquad\qquad \text{for } x_1 > 0,\ x_2 = 0$$

so also $\partial V / \partial x_2$ must have constant non-zero sign along the same half line.

The control law $u_l(x) = -x_2 + x_1$ is stabilizing with strictly decreasing Lyapunov function $V_l(x) = 2(x_1)^2 - x_1 x_2 + (x_2)^2/2 + (x_2)^4$. Apparently $\partial V_l / \partial x_2$ is negative along the half line.

Similarly, the control law $u_g(x) = -x_2 - x_1$ is stabilizing with Lyapunov function $V_g(x) = 2(x_1)^2 + x_1 x_2 + (x_2)^2/2 + (x_2)^4$, with $\partial V_g / \partial x_2$ positive along the half line.

In particular, we see that the two control Lyapunov functions V_l and V_g can not be connected by a continuous path without violating the sign constraint on $\partial V / \partial x_2$.

Given this negative example, it is most striking to find that a convexity result is easily available when instead of Lyapunov's theorem we consider the new convergence criterion. To make formal statement, we introduce

$$C = \{ f \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R}^n) \ : \ f(x)/|x| \text{ is bounded near } x = 0 \}$$

and state a modification of Theorem 1, relaxing the assumption on differentiability of f at $x = 0$.

Theorem 2 *Given the equation $\dot{x} = f(x)$, where $f \in C$, suppose there exists a non-negative $\rho \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R})$ such that $\rho(x) f(x)/|x|$ is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$ and*

$$[\nabla \cdot (\rho f)](x) > 0 \qquad\qquad \textit{for almost all } x \neq 0 \tag{6}$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ tends to zero as $t \to \infty$.

The proof Theorem 1 also gives Theorem 2. The convexity property can now be stated as follows.

Theorem 3 *Given $f_0 \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R}^n)$ and $g_0 \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R}^{n \times m})$ define $\mathcal{F}$ to be the set of all $f \in C$ that have the form $f = f_0 + g_0 u$ with $u \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R}^m)$. Let $\mathcal{R}$ be the set of all ρ such that the conditions of Theorem 2 hold for some $f \in \mathcal{F}$. Then $\mathcal{R}$ is convex.*

Proof of Theorem 3. Suppose that $\rho_1, \rho_2 \in \mathcal{R}$, while $\theta \in [0, 1]$. Then there exist $f_1 = f_0 + g_0 u_1$ and $f_2 = f_0 + g_0 u_2$ that satisfy the conditions of Theorem 2 together with ρ_1 and ρ_2 respectively. Let

$$\rho(x) = \theta \rho_1(x) + (1 - \theta)\rho_2(x)$$

$$u(x) = \frac{\theta \rho_1(x) u_1(x) + (1 - \theta)\rho_2(x) u_2(x)}{\rho(x)}$$

$$f(x) = f_0(x) + g_0(x) u(x)$$

Then $|f| \leq |f_1| + |f_2|$, so $f \in \mathcal{F}$. Furthermore,

$$f\rho = \theta f_1 \rho_1 + (1 - \theta) f_2 \rho_2$$

so both condition (6) and the integrability condition follow from the corresponding conditions for $f_1\rho_1$ and $f_2\rho_2$. Hence $\rho \in \mathcal{R}$ and the convexity is proved.

It is natural to ask what the implications of this result are in a case like Example 5, where the set of stabilizing controllers is known to be disconnected. Suppose that the controllers u_l and u_g have corresponding density functions ρ_l and ρ_g that prove convergence to the equilibrium in the two cases. The argument used in Theorem 3 connects ρ_l and ρ_g by a continuous path of control laws corresponding to convex combinations of ρ_l and ρ_g. In a case like Example 5, these controllers can not all be globally stabilizing. Nevertheless, by Theorem 3, they do give rise to systems such that $\lim_{t\to\infty} |x(t)| = 0$ for almost all initial conditions.

6 Concluding remarks

The new convergence criterion differs from Lyapunov's theorem in several important respects and therefore allows for new applications. In particular, it applies to examples where the system is not globally stable in the sense of Lyapunov.

Another important difference is a convexity property that appears in synthesis of stabilizing control laws. This convexity property is identical to the one that has been exploited for optimal control problems [19, 17].

In spite of the differences, many extensions to Lyapunov's theorem have analogs in terms of density functions. This includes convergence criteria for non-autonomous systems, inverse theorems and criteria for convergence to invariant sets. We hope to return to some of these issues in later publications.

Acknowledgement

The author is grateful to Andrej Ghulchak for inspiring suggestions and critical reading. The work has been supported by a Junior Individual Grant from the Swedish Foundation of Strategic Research.

Appendix — Proofs

Proof of Lemma 1 The differentiability of f gives that Φ is of class $\mathbf{C}^1$ in z and $\mathbf{C}^2$ in t [12, page 40]. The matrix function $M(t) = \frac{\partial \Phi}{\partial z}(z,t)$ satisfies

$$\begin{aligned}
\frac{d}{dt}\det M(t)\Big|_{t=0} &= \frac{d}{dt}\exp(\operatorname{trace}(\log M(t)))\Big|_{t=0} \\
&= \exp(\operatorname{trace}(\log M(t)))\cdot \frac{d}{dt}\operatorname{trace}(\log M(t))\Big|_{t=0} \\
&= \det M(t)\cdot \operatorname{trace}\frac{d}{dt}(\log M(t))\Big|_{t=0} \\
&= \det M(t)\cdot \operatorname{trace}[M(t)^{-1}M'(t)]\Big|_{t=0} \\
&= \operatorname{trace}[M'(0)] = \operatorname{trace}\left[\frac{\partial^2\Phi}{\partial t\partial z}(z,t)\right]\Big|_{t=0} = \operatorname{trace}\frac{\partial f}{\partial z} \\
&= \Big[\nabla\cdot f\Big](z)
\end{aligned}$$

Hence, with the notation $\rho_t(z) = \rho(\Phi(z,t))\left|\frac{\partial\Phi}{\partial z}(z,t)\right|$

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_t(z)\Big|_{t=0} &= \left[\frac{\partial}{\partial t}\rho(\Phi(z,t))\left|\frac{\partial\Phi}{\partial z}(z,t)\right| + \rho(\Phi(z,t))\frac{d}{dt}\det M(t)\right]_{t=0} \\
&= \nabla\rho\cdot f + \rho(\nabla\cdot f) \\
&= \Big[\nabla\cdot(f\rho)\Big](z)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_t(z)\Big|_{t=\tau} &= \frac{\partial}{\partial h}\left\{\rho_h(\Phi(z,\tau))\left|\frac{\partial\Phi}{\partial z}(z,\tau)\right|\right\}\Big|_{h=0} \\
&= \Big[\nabla\cdot(f\rho)\Big](\Phi(z,\tau))\left|\frac{\partial\Phi}{\partial z}(z,\tau)\right|
\end{aligned}$$

Let $\chi(\cdot)$ be the characteristic function of X. Then

$$\begin{aligned}
\int_{\Phi(X,t)} \rho(x)dx - \int_X \rho(z)dz &= \int_{\mathbf{R}^n} \rho(x)\chi(\Phi(x,-t))dx - \int_X \rho(z)dz \\
&= \int_{\mathbf{R}^n} \rho(\Phi(z,t))\chi(z)\left|\frac{\partial\Phi(z,t)}{\partial z}\right| dz - \int_X \rho(z)dz \\
&= \int_X [\rho_t(z) - \rho(z)]\, dz \\
&= \int_X \int_0^t [\nabla\cdot(\rho f)]\,(\Phi(z,\tau))\left|\frac{\partial\Phi}{\partial z}(z,\tau)\right| d\tau dz \\
&= \int_0^t \int_{\Phi(X,\tau)} [\nabla\cdot(\rho f)]\,(x)dxd\tau
\end{aligned}$$

Proof of Theorem 1, first statement. Here, $x = 0$ need not be a stable equilibrium, but ρ is assumed to be non-negative.

As in the previous case, we may assume without restriction that ρ is integrable for $|x| \geq 1$ and $|f(x)|/|x|$ is bounded by some constant C so $\Phi(x,t)$ is well defined for all x, t. The notation $\Phi(Z,T) = \{\Phi(z,t) \ : \ z \in Z,\ t \in T\}$ will be used.

Given any $r > 0$, define

$$X = \cap_{l=1}^{\infty} \{x_0 \ : \ |x_0| > r, |\Phi(x_0,t)| > r \text{ for some } t > l\}$$

Compared to (3), the condition $|x_0| > r$ has been added to guarantee a positive lower bound on the norm of the elements in X. Because of this difference, $\Phi(X,t)$ may be different from X for every $t > 0$, so the first equality in (4) does not hold. We will therefore make a partition of X into disjoint subsets X_k such that $\Phi(X_k,t)$ is approximately equal to X_k for some value of t.

Given any $\varepsilon > 0$, define for integers $k > 1/\varepsilon$

$$\begin{aligned}
X_k &= \{x \in X \ : \ \sup_{t\in[1,k\varepsilon]} |\Phi(x,t)| \leq r, \ \sup_{t\in[k\varepsilon,(k+1)\varepsilon]} |\Phi(x,t)| > r\} \\
Y_k &= \Phi(X_k, [0,\varepsilon])
\end{aligned}$$

These sets are all measurable, so Lemma 1 shows that

$$\begin{aligned}
\int_{\Phi(Y_k,k\varepsilon)} \rho(x)dx &= \int_{Y_k} \rho(z)dz + \int_0^{k\varepsilon} \int_{\Phi(Y_k,\tau)} [\nabla\cdot(\rho f)]\,(x)dxd\tau \\
&\geq \int_{X_k} \rho(z)dz + \int_0^1 \int_{\Phi(X_k,\tau)} [\nabla\cdot(\rho f)]\,(x)dxd\tau
\end{aligned}$$

Note that $\Phi(Y_k, k\varepsilon)$ are disjoint subsets of $\Phi(X,[-\varepsilon,\varepsilon])$ and $\cup_k X_k = X$. Summing over k therefore gives

$$\int_{\Phi(X,[-\varepsilon,\varepsilon])} \rho(x)dx - \int_X \rho(z)dz \geq \int_0^1 \int_{\Phi(X,\tau)} [\nabla\cdot(\rho f)]\,(x)dxd\tau$$

All elements of $\Phi(X, [-\varepsilon, \varepsilon])$ that are outside X must have a norm in the interval $[re^{-C\varepsilon}, r]$. Hence

$$\int_{re^{-C\varepsilon} \leq |x| \leq r} \rho(x) dx \geq \int_0^1 \int_{\Phi(X,\tau)} [\nabla \cdot (\rho f)] (x) dx d\tau$$

This holds for arbitrarily small ε, so X must, by assumption (2), have zero measure. Consequently, $\limsup_{t\to\infty} |x(t)| \leq r$ for almost all trajectories. Also r was arbitrary, so $\lim_{t\to\infty} |x(t)| = 0$ for almost all trajectories and the proof is complete.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, 1994.

[2] L. C. Evans. Partial differential equations and Monge-Kantorovich mass transfer. Department of Mathematics, University of California, Berkeley, Preprint.

[3] W. Hahn. *Theory and Applications of Lyapunov's Direct Method*. Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

[4] A. Isidori. *Nonlinear Control Systems*. Springer-Verlag, London, 1995.

[5] R.E. Kalman and J.E. Bertram. Control system design via the "second method" of Lyapunov, parts i and ii. *Journal of Basic Engineering*, 82:371–400, June 1960.

[6] L.V. Kantorovich. On a problem of Monge. *Uspekhi Mat. Nauk.*, 3:225–226, 1948.

[7] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ 07458, 1996. Second Edition.

[8] P.V. Krasovskii. *Stability of Motion*. Stanford University Press, Stanford, 1963.

[9] M. Krstic, I. Kanellakopoulos, and P. Kokotovich. *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York, 1962.

[10] Yuri S. Ledyaev and Eduardo D. Sontag. A Lyapunov characterization of robust stabilization. *Nonlinear Analysis*, 37:813–840, 1999.

[11] S. Lefschetz and J.P. La Salle. *Stability of Lyapunov's Direct Method*. Academic Press, New York, 1961.

[12] Solomon Lefschetz. *Differential Equations: Geometric Theory*. Dover Publications, New York, 1977.

[13] Laurent Praly. Personal communication.

[14] Christophe Prieur and Laurent Praly. Uniting local and global controllers. In *Proceedings of IEEE Conference on Decision and Control*, pages 1214–1219, Arizona, December 1999.

[15] A. Rantzer and M. Johansson. Piecewise linear quadratic optimal control. *IEEE Trans. on Automatic Control*, Scheduled to appear in April 2000.

[16] N. Rouche, P. Habets, and M. Laloy. *Stability Theory by Lyapunov's Direct Method*. Springer-Verlag, New York, 1977.

[17] R. Vinter. Convex duality and nonlinear optimal control. *SIAM J. Control and Optimization*, 31(2):518–538, March 1993.

[18] J.C. Willems. Dissipative dynamical systems, part I: General theory; part II: Linear systems with quadratic supply rates. *Arch. Rational Mechanics and Analysis*, 45(5):321–393, 1972.

[19] L. C. Young. *Lectures on the Calculus of Variations and Optimal Control Theory*. W. B. Saunders Company, Philadelphia, Pa, 1969.