Further Results on the Bellman Equation for Exit Time Optimal Control Problems with Nonnegative Lagrangians *

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Abstract

We study the Bellman equation for variable interest rate exit time control problems with nonnegative Lagrangians using the dynamic programming approach. We characterize the value functions for these problems as the unique viscosity solutions of the corresponding Bellman equations that satisfy appropriate side conditions. We allow problems whose dynamics, interest rates, Lagrangians, control sets, and targets are all unbounded, including cases where the Lagrangian and interest rate are not bounded below by positive constants and the value function and exit cost are discontinuous. These include undiscounted problems. As a consequence, we show that the value function of 'Fuller's Problem' is the unique viscosity solution of the corresponding Bellman equation which is null at the origin and bounded below. We also apply our results to degenerate eikonal equations from geometric optics, including cases where the target is unbounded, and to shape from shading equations from image processing.

Key Words. viscosity solutions, dynamical systems, optimal control

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1 Introduction

This note studies Hamilton-Jacobi-Bellman equations (HJBE's) for a large class of variable interest rate unbounded optimal control problems for fully nonlinear systems

$$\begin{cases} y'(t) \;=\; f(y(t),u(t)), \quad t \geq 0, \quad u(t) \in U \\ y(0) \;=\; x \end{cases} . \tag{1}$$

The objectives of these problems are the infimization of

$$\int_0^{t_x(\beta)} \ell\left[y_x(s,\beta),\beta(s)\right]\; e^{-\delta(s,\beta,x)} ds \;+\; g\left[y_x(t_x(\beta),\beta)\right]\; e^{-\delta(t_x(\beta),\beta,x)} \tag{2}$$

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over all $\beta \in \mathcal{U}(x)$ for each $x \in \mathbb{R}^N$, where

$$\mathcal{U} := \{ \text{ measurable functions } [0, +\infty) \to U \},$$

$y_x(\cdot, u)$ is the solution of (1) for each $u \in \mathcal{U}$, $\mathcal{T} \subseteq \mathbb{R}^N$ is a closed set which we will refer to as a target,

$$t_x(u) := \inf \{ t \geq 0 : y_x(t, u) \in \mathcal{T} \} \in [0, +\infty],$$

$\mathcal{U}(x)$ is the set of all $\beta \in \mathcal{U}$ for which $t_x(\beta) < \infty$, and

$$\delta(t, \beta, x) := \int_0^t h(y_x(r, \beta), \beta(r))\, dr.$$

Functions $\beta \in \mathcal{U}$ will be referred to as inputs or controls, U is called the control set, and ℓ is called the Lagrangian. The class of problems we consider includes the classical linear quadratic (LQ) case where

$$\ell(x, u) = x' R x + u' Q u,$$

including cases where Q is positive semidefinite but not positive definite and $\vec{0} \in U$ (cf. Remark 4.4). We also allow cases where f,ℓ, h, and $\mathcal{T}$ are unbounded, including cases where h and ℓ are not bounded below by positive constants and g is discontinuous (cf. §2 for the exact hypotheses on the data and §7 for generalizations).

The value function of (2) will be denoted by v, and $\mathcal{R}$ denotes the set of points that can be brought to $\mathcal{T}$ in finite time using the dynamics (1) and some control $\beta \in \mathcal{U}$. Therefore,

$$v(x) = \begin{cases} \inf_{\beta \in \mathcal{U}(x)} \left(\int_0^{t_x(\beta)} \ell\left[y_x(t, \beta), \beta(t)\right]\, e^{-\delta(t, \beta, x)} dt \;+\; g\left[y_x(t_x(\beta), \beta)\right]\, e^{-\delta(t_x(\beta), \beta, x)} \right), & x \in \mathcal{R} \\ +\infty, & x \notin \mathcal{R} \end{cases} \tag{3}$$

The value function v will sometimes be denoted by $v[g]$. Notice that since U is bounded, it need not be the case that $\mathcal{R}$ is all of $\mathbb{R}^N$, even in the classical case where the dynamics (1) has the form

$$\dot{x} = Ax + Bu$$

and (A, B) is a controllable pair (cf. [19]). The HJBE of (2) is

$$\sup_{u \in U} \{ -f(x, u) \cdot Dv(x) \;-\; \ell(x, u) \;+\; h(x, u) \cdot v(x) \} \;=\; 0\,. \tag{4}$$

We will study solutions of (4) on sets of the form $\Omega \setminus \mathcal{T}$, where $\Omega \subseteq \mathbb{R}^N$ is an open set which may or may not contain $\mathcal{T}$. Since the value function (3) will not in general be continuous (cf. §IV.3.1 of [1] and Example 2.2 below), we will study (4) in the framework of discontinuous viscosity solutions (cf. [1] and [10]). By discontinuous, we will mean "not necessarily continuous" (cf. Definition 2.1 below).

We prove a uniqueness theorem which characterizes v as the unique free boundary (discontinuous) viscosity solution of (4) on $\mathcal{R} \setminus \mathcal{T}$ that satisfies appropriate boundary conditions. The result applies to Fuller's Problem (FP) (cf. [17], [29], [32], and §4.1 below), degenerate eikonal and shape from shading equations (cf. [1], [24], and §§4.2-4.3 below), as well as problems with unbounded control sets. We also prove a local version of the theorem that shows that any viscosity solution of (4) on a more general open set of the form $\Omega \setminus \mathcal{T}$ which satisfies appropriate boundary conditions coincides with the value function on Ω (cf. §6). Our arguments are based on Zorn's Lemma and an extension of Barbălat's Lemma (cf. [9] and Lemma 2.8 below).

Value function characterizations of this kind have been studied and applied by many authors for a large variety of stochastic and deterministic optimal control problems and for differential games. Also, uniqueness characterizations are used to establish the convergence of many recently developed numerical schemes for approximating value functions. One proves the convergence of these schemes by showing that the value function v_n of the nth discretization of the problem converges uniformly on compact sets to a

function w which is a discontinuous viscosity solution of the HJBE as the discretization becomes finer, and then one invokes the characterization of the value function as the unique discontinuous viscosity solution of the equation to show that w is in fact the value function. Recent accounts of work in these areas may be found in [1], [3], [10], and [25].

For theorems characterizing v as the unique nonnegative viscosity solution of the HJBE (4) on $\mathcal{R} \setminus \mathcal{T}$ that satisfies appropriate boundary conditions, see [22]. The papers [4] and [20] give general uniqueness characterizations of the HJBE for exit time problems whose Lagrangians ℓ satisfy conditions of the form

$$(\forall \varepsilon > 0)\ (\exists C_\varepsilon > 0 \text{ s.t. } \forall a \in U\ \&\ \forall x \notin B(\mathcal{T}, \varepsilon)\ (\ell(x, a) \geq C_\varepsilon))\,, \tag{5}$$

where

$$B(S, \varepsilon) := \left\{ y \in \mathbb{R}^N : \inf \{||p - y|| : p \in S\} < \varepsilon \right\}$$

for all $S \subset \mathbb{R}^N$ and $||\cdot||$ is the usual norm. For cases where $\partial\mathcal{T}$ is compact, the papers [4] and [20] characterize the value function as the unique proper viscosity solution of the HJBE that agrees with the final cost on $\mathcal{T}$ (where by properness of a function w, we mean the condition that $w(x) \to +\infty$ as $||x|| \to \infty$). Since we allow $\ell(\cdot, a)$ to vanish outside $\mathcal{T}$ for some values of a, these earlier results cannot be applied to (4). Indeed, one easily finds HJBE's for exit time problems that have several proper viscosity solutions when the positive lower bound condition (5) fails. Here is an example where this occurs:

Example 1.1 Choose the dynamics

$$\dot{x}(t) \ = \ u(t) \ \in \ U := [-1, 1],$$

choose

$$\ell(x, a) \ := \ (x+2)^2\,(x-2)^2\,x^2\,(x+1)^2\,(x-1)^2,$$

and set $g \equiv 0$ and $h \equiv 0$. Let v_1 and v_2 denote the value functions for the associated problem (2) with the targets $\mathcal{T}_1 = \{0\}$ and $\mathcal{T}_2 = \{0, 2, -2\}$, respectively. One can easily check that v_1 and v_2 are both proper and that both are viscosity solutions of the associated HJBE

$$|Dv(x)| \ = \ (x+2)^2\,(x-2)^2\,x^2\,(x+1)^2\,(x-1)^2$$

on $\mathbb{R} \setminus \mathcal{T}$ with $\mathcal{T} := \mathcal{T}_1$. Moreover, with $\mathcal{T} = \{0\}$, the problem satisfies the hypotheses of the well-known theorems which characterize value functions of exit time control problems as the unique proper viscosity solutions of (4) which are zero on $\mathcal{T}$ (cf. §IV.4 of [1]) save for the fact that the positive lower bound requirement (5) on ℓ is not satisfied.

Remark 1.2 We will show that the ingredient missing from the preceding example is the extra condition

$$\left(x \notin \mathcal{T}, \beta \in \mathcal{U}, \int_0^\infty \ell(y_x(s, \beta), \beta(s))\, ds < \infty \right) \Rightarrow \left(\exists p \in \mathcal{T} \text{ s.t. } \lim_{s \to \infty} y_x(s, \beta) = p \right).$$

In Example 1.1, this condition of course fails, since the trajectory $x(t) \equiv 1$ gives zero running costs but does not reach the target $\mathcal{T} = \{0\}$. However, even in the classical case where f and ℓ are Lipschitz and h is a positive constant, this extra condition is not sufficient to guarantee uniqueness of solutions of the HJBE when g is discontinuous (cf. Example 7.3). This motivates our study our study of problems with discontinuous exit costs (cf. §7).

Remark 1.3 Another approach to guaranteeing that the value function is the unique viscosity solution of the HJBE is to assume the following condition:

$$(x \notin \mathcal{T}) \Rightarrow \left(\int_0^t \ell(y_x(s, \beta), \beta(s))\, ds > 0\ \forall t > 0\ \&\ \forall \beta \in \mathcal{U} \right) \tag{6}$$

For uniqueness characterizations for problems satisfying (6), see [13] and [14]. The results of [14] imply as a special case that the Fuller Problem value function is the unique viscosity solution of the corresponding HJBE which vanishes at the origin and satisfies a generalized version of properness which allows functions which are not bounded below (cf. Remark 4.3).

Our work is part of a larger project which generalizes uniqueness characterizations for HJBE's to versions that cover well-known optimization problems with non-Lipschitz dynamics and Lagrangians which violate the usual boundedness assumptions. Earlier work in this area can be found in [2] (which covers finite horizon LQ problems) and [8] (which covers infinite horizon LQ problems). For results on infinite horizon optimal control problems with non-Lipschitz dynamics, see [11]. The paper [16] gives a uniqueness characterization for the HJBE for a class of exit time problems whose dynamics are continuous but not Lipschitz continuous, including the Reflected Brachystochrone Problem (cf. [27] and [28]) and degenerate eikonal equations with non-Lipschitz speed functions. For a uniqueness characterization for solutions of (4) based on Clarke's proximal differentials (cf. [7]) which applies to problems with bounded dynamics, see [31].

This paper is organized as follows. In §2, we give the main assumptions and main lemmas. We state global versions of our main results in §3, which also explains how our results extend what was already known about the exit time HJBE (4). In §4, we recall Fuller's Problem and show how our results apply to this problem's HJBE. We also show how our results apply to degenerate eikonal equations from geometric optics and shape from shading equations from image processing. In §5, we prove our main results. We prove local versions of our main results in §6, and we close in §7 by showing how to extend our results to problems with unbounded control sets and discontinuous exit costs.

2 Definitions, Assumptions, and Main Lemmas

The value function v is not in general continuous (cf. [1] and Example 2.2 below), so we study solutions of the HJBE (4) in the discontinuous viscosity sense. By a discontinuous viscosity solution, we will mean a viscosity solution which is not necessarily continuous. We use the following 'test function' definition of viscosity solutions. For each $S \subseteq \mathbb{R}^N$ and each locally bounded function $w : S \to \mathbb{R}$, we define the **upper envelope** $w^\star$ and **lower envelope** $w_\star$ of w by

$$w^\star(x) = \limsup_{y \to x} w(y) \quad \text{and} \quad w_\star(x) = \liminf_{y \to x} w(y)$$

Notice that $w_\star$ is lower semicontinuous and $w^\star$ is upper semicontinuous. Letting $C^1(S)$ denote the set of all real-valued continuously differentiable functions on any subset S of a Euclidean space, the definition of viscosity solutions can then be stated as follows:

Definition 2.1 Assume $\mathcal{G} \subseteq \mathbb{R}^N$ is open, $S \supseteq \mathcal{G}$, $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous, and $w : S \to \mathbb{R}$ is locally bounded. We will say that w is a **(discontinuous) viscosity solution** of

$$F(x, w, Dw(x)) \;=\; 0$$

on $\mathcal{G}$ provided that the following conditions hold:

- (C_1) If $\gamma \in C^1(\mathcal{G})$ and $x_o \in \mathcal{G}$ is a local minimizer of $w_\star - \gamma$, then $F(x_o,\, w_\star(x_o),\, D\,\gamma(x_o)) \;\geq\; 0$.
- (C_2) If $\lambda \in C^1(\mathcal{G})$ and $x_1 \in \mathcal{G}$ is a local maximizer of $w^\star - \lambda$, then $F(x_1,\, w^\star(x_1),\, D\,\lambda(x_1)) \;\leq\; 0$.

By a denseness argument, the preceding definition is unchanged if we replace $C^1(\mathcal{G})$ with the set $C^\infty(\mathcal{G})$ of infinitely differentiable real-valued functions on $\mathcal{G}$ in (C_1) and (C_2). We also use the following equivalent definition of viscosity solutions based on the **superdifferentials** $D^+w^\star(x)$ and **subdifferentials** $D^-w_\star(x)$ of w. Let $\mathcal{G}$, S, F, and w be as in Definition 2.1, and define

$$D^+w^\star(x) \;:=\; \left\{p \,\in\, \mathbb{R}^N : \;\; \limsup_{\mathcal{G} \ni y \to x} \frac{w^\star(y) - w^\star(x) - p \cdot (y-x)}{||x-y||} \;\leq\; 0\right\}$$

$$D^-w_\star(x) \;:=\; \left\{p \,\in\, \mathbb{R}^N : \;\; \liminf_{\mathcal{G} \ni y \to x} \frac{w_\star(y) - w_\star(x) - p \cdot (y-x)}{||x-y||} \;\geq\; 0\right\}$$

One checks that conditions (C_1) and (C_2) are equivalent to

(C_1') $F(x, w_\star(x), p) \geq 0$ for all $x \in \mathcal{G}$ and $p \in D^- w_\star(x)$

(C_2') $F(x, w^\star(x), p) \leq 0$ for all $x \in \mathcal{G}$ and $p \in D^+ w^\star(x)$

respectively, so we equivalently define discontinuous viscosity solutions by saying that w is a viscosity solution of $F(x, w(x), Dw(x)) = 0$ on $\mathcal{G}$ provided conditions (C_1')-(C_2') hold. Set $\mathbb{R}_+ = [0, \infty)$ and

$$B_R(q) \ = \ \{p \in \mathbb{R}^N : ||p - q|| < R\}$$

for each $R > 0$ and $q \in \mathbb{R}^N$. For cases where the control set $U \subseteq \mathbb{R}^N$ is compact, we let U^r denote the set of Radon probability measures on U (i.e., the probability measures which are defined on the smallest σ-algebra on U containing all the open subsets of U) with the weak-$\star$ topology, $\mathcal{U}^r := \{\text{measurable functions } [0, \infty) \to U^r\}$, and

$$k^r(x, m) \ = \ \int_U k(x, u) dm(u), \ \ k = f, \ell, h, \ \ x \in \mathbb{R}^N, \ \ m \in U^r.$$

We call $\mathcal{U}^r$ the set of **relaxed controls** on U. Notice that $\mathcal{U}^r$ includes all controls in $\mathcal{U}$, which can be viewed as Dirac probability measure valued controls, and that any relaxed control can be approximated by standard controls in the weak-$\star$ topology (cf. [30]). For each relaxed control β, we let $y_x(\cdot, \beta)$ denote the unique solution of the initial value problem $\dot{y}(t) = f^r(y(t), \beta(t)), \ y(0) = x$ on $[0, \infty)$ when this problem has a unique solution, and we extend the interest rate δ by setting

$$\delta(t, \beta, x) \ := \ \int_0^t h^r(y_x(s, \beta), \beta(s)) \, ds$$

for all $t \geq 0$, $\beta \in \mathcal{U}^r$, and $x \in \mathbb{R}^N$. We define the exit times

$$t_q^S(\beta) := \inf\{t \geq 0 : y_q(t, \beta) \in S\}$$

for all $S \subset \mathbb{R}^N$, $q \in \mathbb{R}^N$, and $\beta \in \mathcal{U}^r$. We say that a function $\Gamma : \mathbb{R}^N \times U \to G$ into a subset G of a Euclidean space is **locally Lipschitz** provided that for each $R > 0$, there exists a constant $L_R > 0$ such that

$$||\Gamma(x, u) - \Gamma(y, u)|| \ \ \leq \ \ L_R \, ||x - y|| \ \ \forall \ \ x, y \in B_R(0) \ \ \text{ and } \ \ \forall \ \ u \in U. \tag{7}$$

If we can choose $L_R \equiv L > 0$ independently of $R > 0$ in (7), then we say that Γ is **Lipschitz (with Lipschitz constant** L). For cases where f is Lipschitz, the trajectories $y_x(\cdot, \beta)$ are defined on $[0, \infty)$ for all $\beta \in \mathcal{U}^r$ and all $x \in \mathbb{R}^N$ (cf. [1]). In what follows, $STC\mathcal{T}$ denotes the requirement that $\mathcal{T} \subset \text{Interior}\, \mathcal{R}(\varepsilon)$ for all $\varepsilon > 0$, where $\mathcal{R}(\varepsilon) = \big\{x \in \mathbb{R}^N : \exists t \in [0, \varepsilon) \ \text{ and } \ \beta \in \mathcal{U} \text{ s.t. } y_x(t, \beta) \in \mathcal{T}\big\}$. In much of what follows, we assume the following:

(A_0) The control set $U \subseteq \mathbb{R}^M$ is compact and nonempty.

(A_1) The mappings $h, \ell : \mathbb{R}^N \times U \to \mathbb{R}_+$ are continuous, and $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ is Lipschitz with Lipschitz constant L.

(A_2) The target $\mathcal{T} \subseteq \mathbb{R}^N$ is closed, $STC\mathcal{T}$ holds, and $g : \mathcal{R} \to \mathbb{R}$ is continuous.

(A_3) If $\beta \in \mathcal{U}^r$ and $x \in \mathbb{R}^N$, and if

$$\int_0^\infty e^{-\delta(s, \beta, x)} \, \ell^r(y_x(s, \beta), \beta(s)) \, ds < \infty,$$

then there exists a point $\bar{p} \in \mathcal{T}$ such that $\lim_{s \to \infty} y_x(s, \beta) = \bar{p}$.

(A_4) For each $q \in \mathcal{R}$, there exists a constant $\bar{h}_q < \infty$ so that

$$\int_0^{t_q^{\mathcal{R}}} h(y_q(s, \beta), \beta(s)) \, ds \leq \bar{h}_q$$

for all $\beta \in \mathcal{U}^r$ satisfying $t_q^{\mathcal{R}}(\beta) < \infty$.

We will also study cases where U is unbounded and g is discontinuous (cf. §7 below). In the classical case where $\mathcal{R} = \mathbb{R}^N$, (A_4) of course holds vacuously. Condition (A_3) has the intuitively appealing interpretation that trajectories which can be continued indefinitely at finite cost eventually reach the target. Henceforth, we write $t_x(\cdot)$ to mean $t_x^{\mathcal{T}}(\cdot)$. Our hypotheses are weaker than the usual ones since we do not require positive upper or lower bounds on h or on ℓ and since we allow unbounded dynamics and targets. Conditions (A_0)-(A_4) are not sufficient for the continuity of the value function. The following example of an exit time problem satisfying (A_0)-(A_4) whose value function is discontinuous comes from §IV.3.1 of [1]:

Example 2.2 Take $N = 2$, $U = \{1, 2\}$, $f(x, 1) = (0, 1)$, $f(x, 2) = (-1, 0)$, $h \equiv 0$, and $\ell \equiv 1$. Choose the target $\mathcal{T} = \{x \in \mathbb{R}^2 : x_1 \leq 0 \ \& \ x_2 \geq 0\}$ and the final cost

$$g(x) = \begin{cases} -2, & x_2 \geq 1 \\ -2x_2, & 0 \leq x_2 \leq 1 \\ 0, & x_2 \leq 0 \end{cases}$$

One checks (cf. [1]) that $v(x) = -x_2$ for $x_1 \leq 0$ and $x_2 \leq 0$ and that v is discontinuous on the halfline $\{(0, x_2) : x_2 \leq 0\}$. ♢

For cases where the sets

$$\{(f(x, a), \ell(x, a), h(x, a)) : a \in U\} \tag{8}$$

are convex for all $x \in \mathbb{R}^N$ (e.g., Fuller's Example), it follows from the Filippov Selection Theorem (cf. [30]) that each relaxed control $\beta \in \mathcal{U}^r$ admits a standard control $\gamma \in \mathcal{U}$ with the property that

$$\int_0^t k^r(y_x(s, \beta), \beta(s))\, ds \ = \ \int_0^t k(y_x(s, \gamma), \gamma(s))\, ds \ \ \forall t, \ \ k = h, \ell, f$$

For such cases, the results we give go through if one only requires the conditions (A_3)-(A_4) to hold for standard controls $\beta \in \mathcal{U}$ instead of for all relaxed controls $\beta \in \mathcal{U}^r$.

One easily checks that if (A_1) holds, then h^r and ℓ^r are also continuous and f^r is Lipschitz with Lipschitz constant L. It follows from assumption (A_1) that for each control $u \in \mathcal{U}^r$, the trajectory $y_x(\cdot, u)$ satisfies the conditions

$$||y_x(t, u) - y_z(t, u)|| \ \leq \ e^{Lt}||x - z|| \quad \text{for all} \quad t \geq 0 \tag{9}$$

and

$$||y_x(t, u) - x|| \ \leq \ M_x\, t \quad \text{for all} \quad t \ \in \ [0, 1/M_x], \tag{10}$$

where $M_x = \sup\{||f(z, a)|| : a \in U, ||z - x|| \leq 1\}$ if the supremum is nonzero and $M_x = 1$ otherwise (cf. Chapter 3 of [1]). By the estimate (9) and the $STC\mathcal{T}$ assumption, $\mathcal{R}$ is open. Note that $\mathcal{R}$ coincides with the relaxed reachability set

$$\mathcal{R}^r := \left\{p \in \mathbb{R}^N : t_p(\beta) < \infty \text{ for some } \beta \in \mathcal{U}^r\right\}$$

for cases where the sets (8) are convex (e.g., if f, ℓ, and h are affine in the controls).

Remark 2.3 Conditions (A_0)-(A_4) hold under standard hypotheses. The condition $STC\mathcal{T}$ is satisfied when suitable assumptions are made on the directions of the vector field f and its Lie brackets at $\partial\mathcal{T}$ (cf. [26] for a sufficient condition for $STC\mathcal{T}$ which generalizes Chow's Theorem from [12]). Condition (A_3) generalizes the convergence condition in Barbălat's Lemma (cf. [9]). For sufficient conditions for (A_3) to hold with $h \equiv 0$, see Lemma 2.8 below. Notice that we allow $\mathcal{T}$ to be unbounded. Notice too that (A_4) holds vacuously for all continuous nonnegative interest rates h in the classical case where $\mathcal{R}$ is all of $\mathbb{R}^N$. In particular, we do not require the interest rate h to be bounded above, nor do we require it to be bounded below by a positive constant. If conditions (A_0)-(A_2) hold, then we can assume that the last component in (1) has the form $\dot{\tau} = 1$ and that the target has the form $\mathcal{T} \times [0, \infty)$ (by adding the integrator $\dot{\tau} = 1$). Then (A_4) holds if h has the form $h(x, \tau, u) = \phi(\tau)$ with $\phi \in L^1(\mathbb{R}_+, \mathbb{R}_+) \cap C(\mathbb{R}_+)$, or, more generally, $h(x, \tau, u) = \phi(\tau b(x, \tau, u))$ for b bounded, continuous, and nonnegative if we also assume that ϕ is nondecreasing. ♢

We let v^r denote the value function for the exit time problem for the dynamics f^r, the Lagrangian ℓ^r, and the interest rate h^r, i.e.,

$$v^r(x) := \begin{cases} \inf_{\beta\in\mathcal{U}^r(x)} \left(\int_0^{t_x(\beta)} \ell^r\left[y_x(t,\beta),\beta(t)\right]\, e^{-\delta(t,\beta,x)}dt \;+\; g\left[y_x(t_x(\beta),\beta)\right]\, e^{-\delta(t_x(\beta),\beta,x)} \right), & x\in\mathcal{R}^r \\ +\infty, & x\notin\mathcal{R}^r \end{cases}$$

where for each $x\in\mathbb{R}^N$, we let $\mathcal{U}^r(x)$ denote the set of relaxed controls $\beta\in\mathcal{U}^r$ with the property that $y_x(\cdot,\beta)$ reaches $\mathcal{T}$ in finite time.

We will give conditions guaranteeing that if $\Omega\subseteq\mathbb{R}^N$ is open and $\omega_o\in\mathbb{R}\cup\{+\infty\}$, then the value function v^r is the unique viscosity solution of (4) on $\Omega\setminus\mathcal{T}$ in the class of functions w satisfying

$BC(\Omega,\omega_o)$ $w:\Omega\to\mathbb{R}$ is locally bounded and bounded below, $w_\star\geq g\geq w^\star$ on $\mathcal{T}\cap\Omega$, $w<\omega_o$ on Ω, and $\lim\limits_{x\to x_o} w(x)=\omega_o$ for all $x_o\in\partial\Omega$.

Notice that for cases where $\Omega=\mathbb{R}^N$, the limit condition in $BC(\Omega,\omega_o)$ is satisfied vacuously for any choice of ω_o. In particular, there are no growth restrictions on functions w satisfying the condition $BC(\mathbb{R}^N,+\infty)$. We will mainly study viscosity solutions of (4) on $\mathcal{R}\setminus\mathcal{T}$ which satisfy $BC(\mathcal{R},\omega_o)$, but see §6 for the case of general open sets Ω. Set

$$\operatorname{dist}(q,P) := \inf\{||q-p|| : p\in P\}$$

for all $q\in\mathbb{R}^N$ and $P\subseteq\mathbb{R}^N$. Our uniqueness results are based on the following representation lemma whose proof is a slight generalization of the proof of Theorem III.2.32 on p.122 of [1]:

Lemma 2.4 Let (A_0)-(A_2) hold, let $B\subseteq\mathbb{R}^N$ be a bounded open set, and let w be a viscosity solution of the HJBE (4) on B. Let $q\in B$, and set

$$T_\mu(q) = \inf_{u\in\mathcal{U}^r}\{t\geq 0 : \operatorname{dist}(y_q(t,u),\partial B)\leq\mu\}\;\;\forall\mu>0.$$

Then for all $\beta\in\mathcal{U}^r$ and $r\in[0,t_q^B(\beta))$, we have

$$w^\star(q) \;\leq\; \int_0^r e^{-\delta(s,\beta,q)}\ell^r\left[y_q(s,\beta),\beta(s)\right]\,ds \;+\; e^{-\delta(r,\beta,q)}w^\star\left[y_q(r,\beta)\right]. \tag{11}$$

Moreover, for all $\mu\in(0,\operatorname{dist}(q,\partial B))$, and $t\in[0,T_\mu(q))$, we have

$$w_\star(q)\geq\inf_{\beta\in\mathcal{U}^r}\left[\int_0^t e^{-\delta(s,\beta,q)}\ell^r\left[y_q(s,\beta),\beta(s)\right]\,ds + e^{-\delta(t,\beta,q)}w_\star\left[y_q(t,\beta)\right]\right]. \tag{12}$$

We remark for later reference that we can set $r=t_q^B(\beta)$ in (11) for cases where $t_q^B(\beta)<\infty$. We will be especially interested in cases where $\ell(x,a)=R(x)$ for continuous functions R having the property that if $\phi:[0,+\infty)\to\mathbb{R}$ has a Lipschitz derivative, then $\int_0^\infty R(\phi(s))\,ds<\infty$ implies $\lim_{s\to\infty}\phi(s)=0$. To study these cases, we introduce the following definition:

Definition 2.5 A continuous function $\alpha:\mathbb{R}\to\mathbb{R}_+$ is said to be of class $\mathcal{MK}$ provided that $\alpha(0)=0$ and α is even and strictly increasing on $[0,+\infty)$.

Remark 2.6 The results that follow stay true if we instead define $\mathcal{MK}$ functions to be the set of continuous, even, positive definite functions $\alpha:\mathbb{R}\to\mathbb{R}_+$ for which $\alpha(0)=0$ and which weakly increase on $[0,\infty)$. Notice that class $\mathcal{MK}$ functions need not be bounded, convex, or Lipschitz. Any function β of Sontag Class $\mathcal{K}$ gives rise to a function α of class of $\mathcal{MK}$ which we define by setting $\alpha(s)=\beta(|s|)$ (cf. [9]). In particular, for each $q>1$, the function $x\mapsto|x|^q$ is of class $\mathcal{MK}$.

The following lemma is easily shown:

Lemma 2.7 Let α be of class $\mathcal{MK}$ and let $\phi : [0, +\infty) \to \mathbb{R}$ be a Lipschitz function for which $\int_0^{+\infty} \alpha(\phi(s))\, ds < +\infty$. Then $\lim_{s\to\infty} \phi(s) = 0$.

We use the following consequence of Lemma 2.7:

Lemma 2.8 Let α be of class $\mathcal{MK}$, let $\phi : [0, +\infty) \to \mathbb{R}$ be differentiable, let ϕ' be Lipschitz, and let $\int_0^{+\infty} \alpha(\phi(s))\, ds < +\infty$. Then $\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \phi'(t) = 0$.

Proof. It suffices to check that $\lim_{t\to\infty} \phi'(t) = 0$ (since then ϕ is Lipschitz and we can use Lemma 2.7 to conclude). Suppose this were not the case. We could then find $1 \le t_k \uparrow \infty$ and a constant $\gamma \in (0, 1)$ such that for all k, either (i) $\phi'(t_k) \ge \gamma$ or (ii) $\phi'(t_k) \le -\gamma$. Passing to a subsequence, we can assume

$$t_{k+1} \;-\; t_k \;\;\ge\;\; 1 \quad \text{for all} \quad k. \tag{13}$$

Let $C > 1$ be any Lipschitz constant for ϕ'. We can assume (i) holds for all indices k in an infinite index set I_+. (Otherwise, apply the same argument to $-\phi$.) Choose $\delta \in (0,\ \frac{\gamma}{2C})$, and set $I_k = [t_k - \delta, t_k + \delta]$ for all $k \in \mathbb{N}$. This gives

$$\phi'(t) \;=\; \phi'(t) \;-\; \phi'(t_k) \;+\; \phi'(t_k) \;\ge\; -C\,|t - t_k| \;+\; \gamma \;\ge\; -C\,\delta \;+\; \gamma \;\ge\; \gamma/2$$

for all $t \in I_k$ and all $k \in I_+$. For each $k \in I_+$, set

$$v_k := \inf\{|\phi(t)| : t \in I_k\}$$

and pick $s_k \in I_k$ so that $|\phi(s_k)| = \; v_k$. We consider these subcases for $k \in I_+$:

(1) If $\phi(s_k) > 0$, then $s_k = t_k - \delta$ (since $\phi' > 0$ on I_k), so

$$\phi(t) \;\;\ge\;\; \gamma/2(t - (t_k - \delta)) \;\; \forall t \in I_k.$$

Since α is increasing on $(0, \infty)$, we get

$$\int_{I_k} \alpha[\phi(s)]\, ds \ge \int_{I_k} \alpha[\gamma/2(s - t_k + \delta)]\, ds \ge \int_0^{\frac{\gamma\delta}{4}} 2\alpha(u)/\gamma\, du. \tag{14}$$

(2) If $\phi(s_k) < 0$, then since $\phi' > 0$ on I_k, we get $s_k = t_k + \delta$, so

$$-\phi(t) = -\phi(t_k + \delta) + \int_t^{t_k+\delta} \phi'(u)du \ge\; \gamma/2(t_k + \delta - t) \;\; \forall t \in I_k.$$

Since α is even, this again gives $\int_{I_k} \alpha(\phi(s))\, ds \;\ge\; \frac{2}{\gamma} \int_0^{\frac{\gamma\delta}{4}} \alpha(u)\, du$.

(3) Assume $\phi(s_k) = 0$. Since $\phi' > 0$ on I_k, we can apply the method of subcase (1) to $[s_k, t_k + \delta]$ and the method of subcase (2) to $[t_k - \delta, s_k]$ to get

$$|\phi(t)| \;\;\ge\;\; \frac{\gamma}{2}\,|t - s_k| \quad \text{for all} \quad t \in I_k.$$

Since $[s_k - (t_k - \delta)] \vee [(t_k + \delta) - s_k] \ge \delta/2$, this gives

$$\begin{aligned} \int_{I_k} \alpha[\phi(s)]\, ds \;\;&\ge\;\; \frac{2}{\gamma} \int_0^{\frac{\gamma}{2}[s_k - (t_k - \delta)]} \alpha(u)\, du + \frac{2}{\gamma} \int_0^{\frac{\gamma}{2}[(t_k+\delta) - s_k]} \alpha(u)\, du \\ &\ge\;\; \frac{2}{\gamma} \int_0^{\frac{\gamma\delta}{4}} \alpha(u)\, du. \end{aligned} \tag{15}$$

Since $\delta < 1/2$ and (13) holds, the intervals I_k are disjoint. Since

$$\int_0^\infty \alpha(\phi(s))\,ds \;<\; \infty \qquad \text{and} \qquad \int_0^{\frac{\gamma\delta}{4}} \alpha(u)\,du \;>\; 0,$$

it follows from summing on $k \in I_+$ that I_+ is finite. This contradiction establishes the result. ∎

Remark 2.9 The preceding proof shows that the statement of Lemma 2.8 remains true if the hypothesis $\int_0^\infty \alpha(\phi(s))\,ds < \infty$ is replaced by the three requirements that $h : \mathbb{R} \to \mathbb{R}_+$ is continuous,

$$\int_0^\infty e^{-\int_0^s h(\phi(r))dr}\alpha(\phi(s))\,ds < \infty,$$

and ϕ satisfies the integral constraint $\int_0^\infty h(\phi(s))\,ds < \infty$.

3 Statement of Main Result and Remarks

We will prove the following theorem:

Theorem 1 *Assume that* (A_0)*-*(A_4) *hold and that* w *is a viscosity solution of (4) on* $\mathcal{R}\backslash\mathcal{T}$ *which satisfies* $BC(\mathcal{R}, \omega_o)$ *for some* $\omega_o \in \mathbb{R}\cup\{+\infty\}$*. If* $\omega_o \in [0, \infty]$*, then* $w \equiv v$ *on* $\{x : \mathcal{R} : w(x) < e^{-\bar{h}_x}\omega_o\}$*. If* $\omega_o < 0$*, then* $w \equiv v$ *on* $\mathcal{R}$*.*

Remark 3.1 As we will show in §5, the statement of this theorem remains true if v and $\mathcal{R}$ are replaced by v^r and $\mathcal{R}^r$, respectively.

Remark 3.2 Notice that the preceding theorem gives a uniqueness characterization on all of $\mathcal{R}^r$ for the undiscounted case where $\bar{h}_x \equiv 0$ and for discounted cases where $\omega_o = +\infty$. Moreover, if $\mathcal{R} = \mathbb{R}^N$, then $BC(\mathcal{R}, \omega_o)$ implies $BC(\mathcal{R}, +\infty)$, so we again get a uniqueness characterization on all of $\mathcal{R}$. Notice that for undiscounted cases where v satisfies $BC(\mathcal{R}^r, \nu_o)$ for some $\nu_o \in \mathbb{R}\cup\{+\infty\}$, this theorem becomes a uniqueness characterization for the value function of (2) which shows in particular that $v \equiv v^r$ (by Remark 3.1). This follows from the fact (cf. [1]) that for such cases, v is a viscosity solution of the HJBE (4) on $\mathcal{R}\setminus\mathcal{T}$. In particular, if $\mathcal{R} = \mathbb{R}^N$, then the theorem shows that v is the unique viscosity of the corresponding HJBE on $\mathcal{R}\setminus\mathcal{T}$ in a class of functions which includes functions which are not proper and which are not continuous. The assumptions that U is compact and g is continuous can be relaxed (cf. §7).

Remark 3.3 Recall that $t_x(\beta) := \inf\{t \geq 0 : y_x(t, \beta) \in \mathcal{T}\} \in [0, \infty]$. For the classical case where $\omega_o \geq 0$, our proof of Theorem 1 will show that if $p \in \mathcal{R}$ and $k \in [h_p, \infty)$ are such that $w(p) < e^{-k}\omega_o$, then each sequence $0 \leq m_j \uparrow +\infty$ admits an input $\alpha_p \in \mathcal{U}$ for which the following hold

(i) $w\left[y_p(m_j, \alpha_p)\right] < e^{-k+\delta(m_j,\alpha_p,p)}\omega_o$ for all j with $m_j < t_p(\alpha_p)$

(ii) $\lim_{s\to t_p(\alpha_p)} y_p(s, \alpha_p)$ exists and lies in $\mathcal{T}$.

This is an optimal control analog of global asymptotical stability from control systems theory (cf. [19]). Indeed, when $h \equiv 0$, $g \equiv 0$, and $\omega_o \neq \infty$, the theorem gives a trajectory which approaches the null set of w without leaving $\{x \in \mathcal{R} : w(x) < e^{-k}\omega_o\}$ along integral times, i.e., the equilibrium is approached with no large excursions.

Remark 3.4 Using the first part of Lemma 2.4, we can strengthen Theorem 1 by proving that if conditions (A_0)-(A_4) hold with $h \equiv 0$, if $\Omega \subset \mathbb{R}^N$ is an open set containing $\mathcal{T}$, and if w is a viscosity solution of the HJBE (4) on $\Omega\setminus\mathcal{T}$ which satisfies $BC(\Omega, \omega_o)$ for some $\omega_o \in \mathbb{R}_+\cup\{+\infty\}$, then $w \equiv v$ on Ω. In particular, we conclude that $\Omega \subseteq \mathcal{R}$, since $v \equiv +\infty$ outside $\mathcal{R}$. This more general version follows from Theorem 2 in §6 below, which also covers cases where $\mathcal{T} \not\subseteq \Omega$.

Remark 3.5 Notice that if $h \equiv 0$ and if the Lagrangian ℓ is such that

$$\forall\, \varepsilon > 0,\ \ \exists\, C_\varepsilon > 0 \quad \text{s.t.} \quad \ell(x,u) \ > \ C_\varepsilon \ \ \forall x \in \mathbb{R}^N \setminus B(\mathcal{T},\varepsilon) \ \ \&\ \ \forall u \in U, \tag{16}$$

then it follows from [20] that v is the only possible continuous viscosity solution of the HJBE (4) on $\mathcal{R} \setminus \mathcal{T}$ which satisfies $BC(\mathcal{R},\omega_o)$. On the other hand, in the general case we consider here, condition (16) could fail, even in the classical case where $\ell(x,u) = x'Rx + u'Qu$ with Q positive definite, since we allow $\vec{0} \in U$. Therefore, our uniqueness characterization does not follow from known results.

4 Physical Examples

Before turning to the proof of Theorem 1, we give some applications of the theorem. Our results apply to a variety of physical problems. In particular, we can use Theorem 1 to get uniqueness characterizations for the HJBE of Fuller's Example (cf. [32]). Also, the theorem gives uniqueness characterizations for the the eikonal equations in geometric optics for the propogation of light in media with unbounded speed functions. Our results also apply to shape from shading equations in image processing. This section discusses these applications.

4.1 Fuller's Example

This subsection shows how to apply Theorem 1 to get a uniqueness characterization for a class of problems which includes the Fuller Problem (FP) (cf. [32]). The problems have dynamics

$$\begin{cases} \dot{x}(t) \ = \ y(t), \ \ \dot{y}(t) \ = \ u(t) \ \in \ U := [-1,+1] \\ (x(0),y(0))' \ = \ p \end{cases} \tag{17}$$

and the solution trajectory of (17) for a given $u \in \mathcal{U}$ will be denoted by $t \mapsto (y_{1,p}(t,u), y_{2,p}(t,u))'$. We choose $g \equiv 0$ and $h \equiv 0$. To each function α of class $\mathcal{MK}$, we assign the exit time problem

$$\text{For each} \ \ p \in \mathbb{R}^2, \ \ \text{infimize} \ \ \int_0^{t_p(\beta)} \alpha\left[y_{1,p}(s,\beta)\right] ds \ \ \text{over all} \ \ \beta \in \mathcal{U}(p). \quad (P_\alpha)$$

with target $\mathcal{T} = \{0\} \subseteq \mathbb{R}^2$. The value function of P_α will be denoted by v_α. For the special case where $\alpha(s) = s^2$, we get the Fuller Problem, whose value function we denote by v_{FP}. By standard arguments, the dynamics (17) satisfies STC$\mathcal{T}$ (cf. [19]). The reachability set $\mathcal{R}$ for (17) is $\mathbb{R}^2$, and condition (A_3) holds, by Lemma 2.8. Theorem 1 and Remark 3.2 therefore give the following:

Corollary 4.1 For each function α of class $\mathcal{MK}$, v_α is the unique viscosity solution of

$$-\ y(Dw((x,y)'))_1 \ \ + \ \ |(Dw((x,y)'))_2| \ \ - \ \ \alpha(x) \ \ = \ \ 0 \tag{18}$$

on $\mathbb{R}^2 \setminus \{\vec{0}\}$ in the class of functions which are bounded below and null at $\vec{0} \in \mathbb{R}^2$.

In particular, there does not exist a viscosity solution w of (18) on $\mathbb{R}^2 \setminus \{\vec{0}\}$ which is bounded below, which vanishes at $\vec{0}$, and which takes negative values. Notice that Corollary 4.1 extends a result in [22] which characterizes the Fuller Problem value function as the unique viscosity solution of (18) in the class of nonnegative functions which vanish at the origin. This result can be generalized as follows. Let $M = [a_{ij}]$ be an $N \times N$ matrix with $a_{ij} = 0$ for $i \geq j$ and $a_{ij} \neq 0$ for $i < j$, so M is upper triangular with a null main diagonal and all entries above the main diagonal nonzero. Let α be any function of class $\mathcal{MK}$. Then the value function for the problem

$$\text{Infimize} \quad \int_0^{t_p(\beta)} \sum_{j=1}^{N-1} \alpha[x_j(t,\beta)]\, dt \quad \text{over all} \quad \beta \in \mathcal{U}(p)$$

$$\text{subject to} \quad \dot{x} = Mx + (0,0,\ldots,0,1)'u, \quad x(0) = p \in \mathbb{R}^N, \quad u \in U := [-1,+1]$$

with the target $\mathcal{T} = \{0\} \subset \mathbb{R}^N$ is the unique viscosity solution of the corresponding HJBE on $\mathbb{R}^N \setminus \{0\}$ which is bounded below and null at $0 \in \mathbb{R}^N$. This follows from the fact that the appropriate controllability matrix has full rank (which guarantees $STC\mathcal{T}$) and a repeated application of Lemma 2.8 which we omit.

Remark 4.2 The hypotheses of Corollary 4.1 are minimal, since for the FP case where $\alpha(x) = x^2$, the Bellman equation (18) admits the viscosity solution $w(x,y) \equiv -v_{FP}(-x,y)$ which is not bounded below. One way to check that w is a viscosity solution of (18) on $\mathbb{R}^2 \setminus \{\vec{0}\}$ is as follows. First recall (cf. [32]) that for each $x \in \mathbb{R}^2$, there exist a point $z \in \mathbb{R}^2$, an input $\alpha^\star : [0,\infty) \to U := [-1,1]$, and a $t > 0$ such that the corresponding FP trajectory $s \mapsto (y_{1,z}(s,\alpha^\star), y_{2,z}(s,\alpha^\star))$ satisfies

$$v_{FP}(z) = \int_0^{+\infty} [y_{1,z}(s,\alpha^\star)]^2 \, ds \quad \text{and} \quad x = y_z(t,\alpha^\star),$$

i.e., x is an 'optimal point'. Also, since v_{FP} is continuous, it follows from Soravia's backward dynamic programming principle (cf. [21] or §III.2.3 of [1]) that v_{FP} is a bilateral viscosity solution of the FP Bellman equation, i.e., for all $(p_1,p_2)' \in D^+ v_{FP}(x,y) \cup D^- v_{FP}(x,y)$ and all $(x,y)' \in \mathbb{R}^2 \setminus \{\vec{0}\}$, we have

$$-y\, p_1 \;+\; |p_2| \;-\; x^2 \;=\; 0$$

The fact that w is a viscosity solution of (18) on $\mathbb{R}^2 \setminus \{\vec{0}\}$ now follows since

$$(p_1,p_2) \in D^+ w(x,y) \;\;\Rightarrow\;\; (p_1,-p_2) \in D^- v_{FP}(-x,y)$$

and

$$(p_1,p_2) \in D^- w(x,y) \;\;\Rightarrow\;\; (p_1,-p_2) \in D^+ v_{FP}(-x,y)$$

for all $(x,y) \in \mathbb{R}^2 \setminus \{\vec{0}\}$. We leave the details to the reader.

Remark 4.3 The paper [14] proves a local uniqueness result for (4) which implies as a special case that the FP value function is the unique viscosity solution of the corresponding HJBE (18) on $\mathbb{R}^2 \setminus \{\vec{0}\}$ in the class of functions w which vanish at $\vec{0}$, are continuous on $\mathbb{R}^2$, and satisfy the following:

(*REG*) For each $x \in \mathbb{R}^2 \setminus \{\vec{0}\}$, there is a bounded open set $\Omega_x \subseteq \mathbb{R}^2 \setminus \{\vec{0}\}$ which contains x and an $\omega_{o,x} \in \mathbb{R} \cup \{+\infty\}$ such that

- (REG_1) $w(x) < \omega_{o,x}$
- (REG_2) $\liminf_{\Omega_x \ni p \to x_o} w(p) \geq \omega_{o,x}$ for all $x_o \in \partial(\Omega_x) \setminus \{\vec{0}\}$
- (REG_3) $\mathrm{STC}(\Omega_x^c)$.

Note that (REG_2) only restricts the values of w near the boundaries of the Ω_x's, so functions which are not bounded below can still satisfy (REG). In particular, [14] proves that the FP value function is the unique proper viscosity solution of (18) which is continuous in the plane and vanishes at the origin, where by properness we mean the condition $w(x) \to +\infty$ as $||x|| \to \infty$. The main argument of [14] is a generalization of the uniqueness arguments for free boundary problems from [4] and [20]. The results of the paper [14] also apply to problems which violate the asymptotical condition (A_3) and eikonal equations. The novelty of Corollary 4.1 above is that it characterizes the FP value function as a unique solution of the HJBE within a class of functions which includes functions that violate (*REG*) and which are not proper.

Remark 4.4 The preceding argument generalizes to the following LQ problem on $\mathbb{R}^{2N}$. Partition the state space to $x = (x_L, x_R) \in \mathbb{R}^N \times \mathbb{R}^N$, and pick $\mathcal{T} = \{0\} \subseteq \mathbb{R}^{2N}$ and any compact control set $U \subseteq \mathbb{R}^N$. Take $h \equiv 0$, the LTI system $f(x,u) = Ax + Bu$, and the quadratic cost $\ell(x,u) = x_L' R x_L + u'Qu$, where

$$A = \begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ B_1 \end{pmatrix}$$

for A_1 and B_1 of size $N \times N$, B_1 having all entries nonzero, and R and Q positive definite. (Similar results hold if Q is positive semidefinite.) Let f be such that $STC\mathcal{T}$ holds (e.g., $f(x,u) = u$). Since $\ell(x,u) \geq \lambda_{min}||x_L||^2$, where $\lambda_{min} > 0$ is the smallest eigenvalue of R, it follows that the corresponding exit time value function is the unique viscosity solution of the HJBE

$$-Dv(x)Ax - \frac{1}{4}Dv(x)BQ^{-1}B'\left[Dv(x)\right]' - x_L'Rx_L = 0$$

on $\mathbb{R}^{2N} \setminus \{0\}$ in the class of functions that vanish at 0 and are bounded below. Taking $N = 1$, $Q = 0$, and $R = A_1 = B_1 = 1$, we recover Corollary 4.1.

4.2 Degenerate Eikonal Equations

This subsection shows how Theorem 1 applies to Bellman equations for a class of exit time problems from geometric optics. The problems have the dynamics $f(x,y,a,b) = (a,b) \in \overline{B_1(0)} \subseteq \mathbb{R}^2$, the interest rate $h \equiv 0$, and the Lagrangians

$$\ell(x,y,a,b) \;=\; \left[1 + \sqrt{||(x,y)||}\right]^{-p},$$

where $p > 0$ will be further specified below. (The argument we are about to give also applies if we instead take the Lagangian $(1 + \sqrt{|x|})^p$ or $(1 + \sqrt{|y|})^p$, or if the state space and control set are in $\mathbb{R}^N$ for N arbitrary.)

We choose any closed target $\mathcal{T} \subseteq \mathbb{R}^2$, and we let $v_{e,p}$ denote the value function for the exit time problem we have defined for each $p > 0$. The corresponding HJBE is

$$||Dv(x,y)|| = \frac{1}{\left[1 + \sqrt{||(x,y)||}\right]^p}, \tag{19}$$

which is the eikonal equation of geometric optics for the propogation of light in a medium of speed

$$c(x,y) = \left[1 + \sqrt{||(x,y)||}\right]^p.$$

Viscosity solutions of eikonal equations have been studied extensively (cf. [1], which covers cases where the speed of the medium is bounded and also uniqueness questions for eikonal equation solutions on bounded sets subject to Dirichlet conditions, and the paper [24]). However, (19) is not covered by these results since c is unbounded and $\mathcal{T}$ may be unbounded. It is very easy to check that for $0 \leq p \leq 2$, the exit time problems for these data satisfy (A_0)-(A_4). Indeed, if $q \in \mathbb{R}^2$ and $\phi(s)$ is a trajectory for f starting at q, then we can find a $K > 0$ so that, for each $L > K$, we have

$$\int_0^L \frac{ds}{\left[1 + \sqrt{||\phi(s)||}\right]^p} \;\geq\; \int_0^L \frac{ds}{\left[1 + \sqrt{||q|| + s}\right]^p} \;\geq\; \frac{1}{2}\int_K^L \frac{ds}{s^{p/2}} \;\to\; \infty$$

as $L \to \infty$, so (A_3) is satisfied vacuously. By standard results from [1], $v_{e,p}$ is viscosity solution of (19) on $\mathbb{R}^2 \setminus \mathcal{T}$, so we conclude as follows:

Corollary 4.5 For each $p \in [0,2]$ and closed target $\mathcal{T} \subseteq \mathbb{R}^2$, $v_{e,p}$ is the unique viscosity solution of (19) on $\mathbb{R}^2 \setminus \mathcal{T}$ in the class of functions which are null on $\mathcal{T}$ and bounded below.

Remark 4.6 We emphasize that it was not necessary to assume that the target $\mathcal{T}$ is bounded. If $p > 2$, then Theorem 1 does not apply, since (A_3) could fail. For example, if $p = 4$, and if we take $\mathcal{T} = \{(x,0) \in \mathbb{R}^2 : x \leq -1\}$ and $\beta \equiv (1,0)$, then $\int_0^\infty \ell(y_{(0,0)}(s,\beta),\beta(s))\,ds < \infty$ but the trajectory does not approach the target. Moreover, the standard uniqueness characterizations for exit time HJBE's (e.g., Corollary IV.4.3 of [1]) would not apply, since the Lagrangian is not bounded below by a positive

constant. However, using the results of [15], one can show that the statement of Corollary 4.5 remains true even without the restriction $p \in [0,2]$. This is done by rewriting the HJBE (19) as

$$\left[1+\sqrt{||(x,y)||}\right]^p ||Dv(x,y)|| - 1 = 0, \;\; (x,y) \notin \mathcal{T} \tag{20}$$

and then viewing (20) as the HJBE for the exit time problem with the non-Lipschitz dynamics

$$\tilde{f}(x,y,a,b) = \left[1+\sqrt{||(x,y)||}\right]^p (a,b)$$

(with $(a,b) \in \overline{B_1(0)}$ as before) and Lagrangian $\tilde{\ell} \equiv 1$. For details, see §6.1 of [15].

4.3 Shape From Shading Equations

Our results also apply to equations of the form

$$I(x)\Psi(Du(x)) - b(x)\cdot Du(x) - h^2(x) = 0$$

for I nonnegative and Ψ a convex function with $\Psi(0) = 0$. This equation is studied in [22]. Taking the Legendre transform $\Psi^\star$ of Ψ, which is nonnegative, we can rewrite this equation as

$$\max_{a \in \text{domain}(\Psi^\star)} \left\{-(b(x) - I(x)a)\cdot Du(x) - \left[h^2(x) + I(x)\Psi^\star(a)\right]\right\} = 0.$$

A particular case of this equation is

$$I(x)\left[1+||Du(x)||^2\right]^{1/2} - 1 = 0, \;\; x \in \Omega \subseteq \mathbb{R}^2 \tag{21}$$

for open sets Ω, which in fact (cf. [22]) can be written as

$$\max_{||a||\le 1}\left\{I(x)a\cdot Du(x) - \left[1 - I(x)\left(1-||a||^2\right)^{1/2}\right]\right\} = 0. \tag{22}$$

The equation (22) arises in shape from shading models in image processing, where $I(x) \in [0,1)$ is the intensity of light reflected by an object (cf. [24]). The objective in image processing is to reconstruct the unknown function u, representing the height of the surface on some subset Ω of the plane, from the brightness of a single two-dimensional image of the surface. For the case of a Lambertian surface which is not self-shadowing and which is illuminated by a single distant vertical light source, the height u satisfies (22). Now pick any closed target $\mathcal{T} \subseteq \mathbb{R}^2 \setminus \{0\}$ and $\Omega = \mathbb{R}^2 \setminus \mathcal{T}$ and choose the intensity function

$$I(x) = \frac{||x||}{1+||x||}.$$

Then (22) is a Bellman equation for an exit time problem with the dynamics

$$f(x,u) = -I(x)u,$$

the control set $U = \overline{B_1(0)}$, the Lagrangian

$$\ell(x,u) = 1 - \frac{||x||}{1+||x||}(1-||u||^2),$$

$h \equiv 0$, and $g \equiv 0$. Since ℓ violates the positivity condition (5), the known results cannot be used to get uniqueness characterizations for (22). On the other hand, using the fact that $||y_q(s,\beta)|| \le ||q|| + s$ for all $\beta \in \mathcal{U}^r$, $s \ge 0$, and $q \in \mathbb{R}^2$, one can easily check that (A_0)-(A_4) hold. Therefore, we conclude that the value function is the unique viscosity solution of (22) on $\mathbb{R}^2 \setminus \mathcal{T}$ in the class of functions which vanish on $\mathcal{T}$ and which are bounded below.

5 Proof of Main Result

This section proves Theorem 1 for the case of viscosity solutions on $\mathcal{R}^r \setminus \mathcal{T}$ (cf. Remark 3.1). Let w and $\omega_o \in [0, \infty]$ satisfy the hypotheses, and let $\bar{x} \in \mathcal{R}^r \setminus \mathcal{T}$ satisfy

$$w(\bar{x}) \;<\; e^{-\bar{h}_{\bar{x}}}\omega_o$$

The case where $\omega_o < 0$ is handled by replacing $h_{\bar{x}}$ with 0 in what follows. The fact that $v^r(\bar{x}) \geq w^\star(\bar{x})$ is an easy consequence of the first part of Lemma 2.4. This argument is carried out in a more general context in §6 below (cf. Case A). Since $w_\star(\bar{x}) \leq w^\star(\bar{x})$, Theorem 1 will therefore follow once we show that $w_\star(\bar{x}) \geq v^r(\bar{x})$.

To simplify the notation, we will assume that w is lower semicontinuous (so $w_\star \equiv w$), and we drop the superscript r. Choose $\varepsilon \in (0, 1)$ such that

$$w(\bar{x}) \;\;<\;\; e^{-\bar{h}_{\bar{x}}}\omega_o \;-\; \varepsilon. \tag{23}$$

(We allow $\omega_o = +\infty$, in which case (23) holds for all $\varepsilon > 0$.) We drop the subscript on $\bar{h}$ in the sequel. We also set

$$\Delta(t, \gamma, \beta) \;:=\; \int_0^t h(\gamma(r), \beta(r))\, dr$$

and

$$J(t, \gamma, \beta) \;:=\; \int_0^t e^{-\Delta(s,\gamma,\beta)}\, \ell(\gamma(s), \beta(s))\, ds$$

wherever these quantities are defined. The inequality $w(\bar{x}) \geq v(\bar{x})$ will evidently follow if we can find an input $\bar{\beta} \in \mathcal{U}$ that drives $\bar{x}$ to $\mathcal{T}$ in finite time and a corresponding trajectory $\bar{\phi}$ which are such that

$$w(\bar{x}) \;\;\geq\;\; J(t_{\bar{x}}(\bar{\beta}), \bar{\phi}, \bar{\beta}) \;+\; e^{-\Delta(t_{\bar{x}}(\bar{\beta}),\bar{\phi},\bar{\beta})} g\left[\bar{\phi}(t_{\bar{x}}(\bar{\beta}))\right] \;-\; \varepsilon, \tag{24}$$

since that would give

$$w(\bar{x}) \;\;\geq\;\; v(\bar{x}) \;-\; \varepsilon,$$

and then we can let $\varepsilon \downarrow 0$. We now prove the existence of such a $\bar{\beta}$ by using Zorn's Lemma. In what follows, we define the functions $E_1, E_2, \ldots$ by

$$E_j(t) \;\;\equiv\;\; \frac{\varepsilon}{8}\left[e^{-(j-1)} - e^{-(t+j-1)}\right] \quad \text{for each} \quad t > 0.$$

Set

$$\mathcal{Z}_1 \;\;=\;\; \left\{ \begin{array}{l} (t, \gamma): \quad 0 \leq t \leq 1, \;\; \text{and } \gamma : [0, t] \to \mathcal{R} \;\; \text{is a trajectory} \\ \text{for } \dot{x} \;=\; f(x, \beta) \;\; \text{for some } \; \beta \in \mathcal{U} \;\; \text{with } \gamma(0) \;=\; \bar{x} \\ \text{so that } w(\bar{x}) \;\geq\; J(t, \gamma, \beta) + e^{-\Delta(t,\gamma,\beta)} w(\gamma(t)) - E_1(t). \end{array} \right\}.$$

Then $\mathcal{Z}_1$ is partially ordered by the relation

$$((t_1, \gamma_1) \;\;\sim\;\; (t_2, \gamma_2)) \quad\Longleftrightarrow\quad (\; t_1 \leq t_2 \;\;\&\;\; \gamma_2\lceil[0, t_1] \equiv \gamma_1 \;) \tag{25}$$

Moreover, one can use standard arguments to check that every totally ordered subset of $\mathcal{Z}_1$ has an upper bound in $\mathcal{Z}_1$.

To do this, let $\{(s_j, \mu_j)\}_j$ be a totally ordered subset of $\mathcal{Z}_1$, and set $s_o = 0$ and $\bar{s} = \sup_j s_j$. Assume $\bar{s} > 0$. Define the trajectory $\bar{\mu} : [0, \bar{s}) \to \mathbb{R}^N$ by $\bar{\mu}(p) = \mu_j(p)$ for any j so that $s_j > p$. (This is valid since the s_j's increase and μ_j extends μ_k for $k \leq j$.) Let $\beta_j : (s_{j-1}, s_j] \to U$ be a control for the trajectory $\mu_j\lceil(s_{j-1}, s_j]$, fix $\bar{u} \in U$, and define $\beta^\dagger : (0, \infty) \to U$ by

$$\beta^\dagger(s) = \left\{ \begin{array}{ll} \beta_j(s), & s_{j-1} \leq s < s_j, \;\; j \geq 1 \\ \bar{u}, & s \geq \bar{s} \end{array} \right. .$$

Then $\beta^\dagger$ is measurable and therefore admits a corresponding trajectory $y_{\bar{x}}(\cdot,\beta^\dagger)$ on $[0,\infty)$. Moreover, $y_{\bar{x}}(\cdot,\beta^\dagger)$ coincides with $\bar{\mu}$ on $[0,\bar{s})$. Setting $\bar{\mu}(\bar{s}) = y_{\bar{x}}(\bar{s},\beta^\dagger)$, it follows that $(\bar{s},\bar{\mu})$ is an upper bound for $\{(s_j,\mu_j)\}_j$ in the relation (25). Also, the choice of $\varepsilon > 0$ guarantees that $\bar{\mu}(\bar{s}) \in \mathcal{R}$, which implies that $(\bar{s},\bar{\mu}) \in \mathcal{Z}_1$.

Indeed, if $\bar{\mu}(\bar{s}) \notin \mathcal{R}$, then we would get

$$\mathcal{R} \ni \mu_j(s_j) \;\rightarrow\; \bar{\mu}(\bar{s}) \in \partial\mathcal{R} \quad \text{and} \quad w\left[\mu_j(s_j)\right] \;\rightarrow\; \omega_o.$$

Let α_j denote the control for μ_j for all j. Consider the case where $\omega_o \neq \infty$. Fixing an integer κ such that

$$w(\mu_j(s_j)) \;\geq\; \omega_o - \varepsilon\, e^{\bar{h}}/(2\kappa) \quad \text{and} \quad e^{-\Delta(s_j,\mu_j,\alpha_j)} \;<\; \kappa\, e^{-\bar{h}} \quad \text{for all} \quad j,$$

it would then follow from (A_4) and the nonnegativity of ℓ that

$$\begin{aligned} w(\bar{x}) &\geq \liminf_{j\to\infty}\left[J(s_j,\mu_j,\alpha_j) + e^{-\Delta(s_j,\mu_j,\alpha_j)}w(\mu_j(s_j)) - E_1(s_j)\right] \\ &\geq \liminf_{j\to\infty}\left[e^{-\Delta(s_j,\mu_j,\alpha_j)}\left[\omega_o - \frac{\varepsilon e^{\bar{h}}}{2\kappa}\right] - E_1(s_j)\right] \\ &\geq e^{-\bar{h}}\,\omega_o - \varepsilon \end{aligned}$$

for all j. This contradicts (23), so we conclude that $\bar{\mu}(\bar{s}) \in \mathcal{R}$. Since $0 < e^{-\bar{h}} \leq \lim_{j\to\infty} e^{-\Delta(s_j,\mu_j,\alpha_j)}$, the contradiction is clear from the limit in $BC(\mathcal{R},\omega_o)$ for cases where $\omega_o = +\infty$. Since w is lower semicontinuous at $\bar{\mu}(\bar{s}) \in \mathcal{R}$, it now follows that $(\bar{s},\bar{\mu}) \in \mathcal{Z}_1$ is an upper bound for $\{(s_j,\mu_j)\}$. If $\bar{s} = 0$, then (s_1,μ_1) is the desired upper bound.

It follows from Zorn's Lemma that $\mathcal{Z}_1$ contains a maximal element, which we will denote by $(\bar{t},\bar{\gamma})$. We can assume that $\bar{\gamma}$ does not reach $\mathcal{T}$ on $[0,\bar{t}]$ (since otherwise (24) is satisfied if we use the input for $\bar{\gamma}$). We will now show that $\bar{t} = 1$. First recall that the estimate (9) and the $STC\mathcal{T}$ assumption guarantee that $\mathcal{R}$ is open, and that the target $\mathcal{T}$ is closed. It follows that there exists an open set

$$B \;\ni\; \bar{\gamma}(\bar{t}) \;=:\; \hat{x}$$

whose closure lies in $\mathcal{R}\setminus\mathcal{T}$. Suppose that $\bar{t} < 1$. Taking $x = \hat{x}$ in the estimate (10), one easily checks that

$$\tau := \inf_{\beta\in\mathcal{U}^r}\{t : \operatorname{dist}(y_{\hat{x}}(t,\beta),\partial B) \leq \operatorname{dist}(\hat{x},\partial B)/2\} \;>\; 0.$$

Indeed, for each $\tilde{\delta} > 0$, we can use (10) to find a $\tilde{t} > 0$ so that $y_{\hat{x}}(t,\beta) \in B_{\tilde{\delta}}(\hat{x})$ for all $\beta \in \mathcal{U}$ and all $t \in [0,\tilde{t}]$. Choosing

$$\tilde{\delta} \;=\; 1/4\operatorname{dist}(\hat{x},\partial B),$$

it follows that

$$\begin{aligned} \operatorname{dist}(y_{\hat{x}}(t,\beta),\partial B) &\geq \operatorname{dist}(\hat{x},\partial B) - ||y_{\hat{x}}(t,\beta) - \hat{x}|| \\ &\geq \frac{3}{4}\operatorname{dist}(\hat{x},\partial B) \end{aligned}$$

for all $t \in [0,\tilde{t}]$ and $\beta \in \mathcal{U}$. In particular, $\tau \geq \tilde{t} > 0$.

By conclusion (12) of Lemma 2.4, it follows that there are $t \in (0, 1-\bar{t})$ and $\beta \in \mathcal{U}$ so that

$$\begin{aligned} w(\hat{x}) &\geq J(t, y_{\hat{x}}(\cdot,\beta),\beta) + e^{-\delta(t,\beta,\hat{x})}w(y_{\hat{x}}(t,\beta)) \\ &- E_1(\bar{t}+t) + E_1(\bar{t}) \end{aligned} \tag{26}$$

and so that $y_{\hat{x}}(\cdot,\beta)$ remains in $B \subseteq \mathcal{R}\setminus\mathcal{T}$ on $[0,t]$. Let $\tilde{\alpha}$ denote the control for $\bar{\gamma}$. Since $(\bar{t},\bar{\gamma}) \in \mathcal{Z}_1$, it follows that

$$w(\bar{x}) - e^{-\Delta(\bar{t},\bar{\gamma},\tilde{\alpha})}w(\hat{x}) \;\geq\; J(\bar{t},\bar{\gamma},\tilde{\alpha}) - E_1(\bar{t}). \tag{27}$$

Let $\beta^\sharp$ denote the concatenation of $\tilde{\alpha}\lceil[0,\bar{t}]$ followed by β. If we multiply (26) through by $e^{-\Delta(\bar{t},\bar{\gamma},\tilde{\alpha})}$, change variables, and add the result to (27), then we get

$$\begin{aligned} w(\bar{x}) &\geq J\left(\bar{t}+t, y_{\bar{x}}(\cdot,\beta^\sharp), \beta^\sharp\right) \\ &+ e^{-\delta(\bar{t}+t,\beta^\sharp,\bar{x})} w\left(y_{\bar{x}}(\bar{t}+t,\beta^\sharp)\right) - E_1(\bar{t}+t). \end{aligned} \tag{28}$$

Since t was chosen so that $\bar{t}+t<1$ and $B\subseteq\mathcal{R}$, we conclude from (28) that

$$\left(\bar{t}+t, y_{\bar{x}}(\cdot,\beta^\sharp)\right) \in \mathcal{Z}_1.$$

This contradicts the maximality of the pair $(\bar{t},\bar{\gamma})$. Therefore, $\bar{t}=1$. Moreover, the nonnegativity of ℓ guarantees that $w(\bar{\gamma}(1)) < e^{-\bar{h}+\Delta(1,\bar{\gamma},\tilde{\alpha})}\omega_o$.

We will now extend $\bar{\gamma}:[0,1]\to\mathcal{R}$ to get the desired trajectory. Set

$$\mathcal{Z}_2 = \left\{ \begin{array}{l} (t,\gamma): \quad 0\leq t\leq 1, \text{ and } \gamma:[0,t]\to\mathcal{R} \text{ is a trajectory} \\ \text{for } \dot{x} = f(x,\beta) \text{ for some } \beta\in\mathcal{U} \text{ with } \gamma(0) = \bar{\gamma}(1) \\ \text{so that } w(\bar{\gamma}(1)) \geq J(t,\gamma,\beta) + e^{-\Delta(t,\gamma,\beta)} w(\gamma(t)) - E_2(t). \end{array} \right\},$$

and partially order $\mathcal{Z}_2$ by the relation (25) as before. We can assume that

$$w(\bar{\gamma}(1)) < e^{-\bar{h}+\Delta(s,\bar{\gamma},\tilde{\alpha})}\omega_o - \varepsilon \tag{29}$$

If (29) does not hold, then choose $\varepsilon'>0$ so that

$$w(\bar{\gamma}(1)) < e^{-\bar{h}\,+\,\Delta(s,\bar{\gamma},\tilde{\alpha})}\omega_o - \varepsilon'.$$

Then the construction we are about to give produces a trajectory $\gamma:[0,1]\to\mathcal{R}$ and a corresponding input β with the property that

$$\begin{aligned} w(\bar{\gamma}(1)) &\geq J(1,\gamma,\beta) + e^{-\Delta(1,\gamma,\beta)} w(\gamma(1)) - \frac{\varepsilon'}{8}(e^{-1}-e^{-2}) \\ &\geq J(1,\gamma,\beta) + e^{-\Delta(1,\gamma,\beta)} w(\gamma(1)) - E_2(1), \end{aligned}$$

and then the procedure is iterated to get the desired extension of $\bar{\gamma}$. The fact that each totally ordered subset of $\mathcal{Z}_2$ has an upper bound in $\mathcal{Z}_2$ follows from the proof of the corresponding statement for $\mathcal{Z}_1$ and the definition of $\bar{h}$ from assumption (A_4). Indeed, let (s_j,γ_j) be a totally ordered subset of $\mathcal{Z}_2$, let α_j be the control corresponding to the trajectory γ_j for all j, and define $\beta^\dagger$, $\bar{\mu}$, and $\bar{s}$ as in the earlier proof. We can assume that $\omega_o\in[0,\infty)$, as before. If we suppose that $\bar{\mu}(\bar{s})\in\partial\mathcal{R}$, then we get

$$\begin{aligned} e^{-\bar{h}+\Delta(1,\bar{\gamma},\tilde{\alpha})}\omega_o - \varepsilon &> w(\bar{\gamma}(1)) \\ &\geq e^{-\Delta(s_j,\mu_j,\alpha_j)} w(\mu_j(s_j)) - \varepsilon \\ &\to e^{-\Delta(\bar{s},\bar{\mu},\beta^\dagger)}\omega_o - \varepsilon, \end{aligned} \tag{30}$$

so

$$e^{-\bar{h}+\Delta(1,\bar{\gamma},\tilde{\alpha})+\Delta(\bar{s},\bar{\mu},\beta^\dagger)}\omega_o > \omega_o.$$

Since the definition of $\bar{h}$ guarantees that $e^{-\bar{h}+\Delta(1,\bar{\gamma},\tilde{\alpha})+\Delta(\bar{s},\bar{\mu},\beta^\dagger)}<1$, this contradicts the assumption that $\omega_o\geq 0$. Therefore, it follows that $\bar{\mu}(\bar{s})\in\mathcal{R}$, so $\mathcal{Z}_2$ admits a maximal element. By the preceding, we know that the maximal element in $\mathcal{Z}_2$ has the form $(1,\tau_2)$. Let $u_2:[0,1]\to U$ denote the input for τ_2. We can assume that τ_2 does not reach $\mathcal{T}$ on $[0,1]$. Indeed, if there is an $s_2\in[0,1]$ such that $\tau_2(s_2)\in\mathcal{T}$, then we can satisfy the requirement (24) by setting

$$\bar{\alpha}(t) = \left\{ \begin{array}{ll} \tilde{\alpha}(t), & 0\leq t\leq 1 \\ u_2(t-1), & 1\leq t\leq 1+s_2 \end{array} \right. ,$$

where $\tilde{\alpha}$ is the control for $\bar{\gamma}$ on $[0,1]$. Let c_2 denote the concatenation of $\tilde{\alpha}\lceil[0,1]$ followed by $u_2\lceil[0,1]$, and let γ_2 denote the corresponding concatenated trajectory for c_2 from $\bar{x}$ to $\tau_2(1)$.

Now reapply the procedure to $\gamma_2(2)$ to get a longer trajectory that extends γ_2 and reaches $\mathcal{T}$ or runs for three time units without leaving $\mathcal{R}$. The procedure is iterated, and it results in a sequence of terminal points

$$p_n \;:=\; \phi_n(n) \;\in\; \{x \in \mathcal{R} : w(x) < e^{-\bar{h}+\Delta(n,\phi_n,c_n)}\omega_o\}$$

for trajectories ϕ_n which begin at $\bar{x}$, are defined on $[0,n]$, and remain in $\mathcal{R}\setminus\mathcal{T}$ without loss of generality. It follows that for all $n \in \mathbb{N}$, we have

$$w(\bar{x}) \;\;\geq\;\; J(n,\phi_n,c_n) \;+\; e^{-\Delta(n,\phi_n,c_n)}\, w(p_n) \;-\; \frac{\varepsilon}{8}(1-e^{-n}). \tag{31}$$

Setting $\hat{\alpha}(s) := c_n(s)$ for $n-1 < s \leq n$ and $n \in \mathbb{N}$, letting $\hat{\phi}$ denote the trajectory for $\hat{\alpha}$, and letting $b < 0$ denote a lower bound for w, a passage to the limit as $n \to \infty$ in (31) gives

$$\int_0^\infty e^{-\delta(s,\hat{\alpha},\bar{x})}\ell^r(y_{\bar{x}}(s,\hat{\alpha}),\hat{\alpha}(s))\,ds \;\;\leq\;\; w(\bar{x}) \;-\; b \;+\; \frac{\varepsilon}{2} \;<\; \infty,$$

since $\hat{\phi} \equiv \phi_n$ on $[0,n]$. It follows from the assumption (A_3) that $\bar{p} := \lim_{s\to\infty}\hat{\phi}(s)$ exists and lies in the target $\mathcal{T}$. Since U is compact and $\{p_n\}$ is bounded, and since ℓ is continuous, it follows from the estimate (10) that

$$\sup\{\ell(y_{p_n}(s,\beta),\beta(s)) : \;\; n \in \mathbb{N},\; \beta \in \mathcal{U},\; 0 \leq s \leq 1/M\} \;\;\leq\;\; M \;\;<\;\; +\infty$$

for some $M > 0$. Since condition $STC\mathcal{T}$ is satisfied, it follows that there is an input $\tilde{\beta}$ and an $m \in \mathbb{N}$ so that

$$J\left(t_{p_m}(\tilde{\beta}), y_{p_m}(\cdot,\tilde{\beta}), \tilde{\beta}\right) \;\bigvee\; \left[g\left(y_{p_m}(t_{p_m}(\tilde{\beta}),\tilde{\beta})\right) - w(p_m)\right] \;<\; \varepsilon/4. \tag{32}$$

Indeed, the estimates (9)-(10), the fact that $\{p_n\}$ converges to a point in $\mathcal{T}$, and the continuity of g guarantee the existence of a constant $\kappa \in (0, 1/M)$ such that $|g(p_n) - g(y_{p_n}(t,\beta))| < \varepsilon/8$ for all $\beta \in \mathcal{U}$ and $t \in [0,\kappa)$ and all large n. Also, the $STC\mathcal{T}$ condition and the estimate (10) ensure that we can drive p_n to $\mathcal{T}$ in time $< \varepsilon/[4(M+\kappa)] \wedge 1$ without leaving $\mathcal{R}$ when n is large enough. The estimate (32) now follows since

$$\{q \in \mathcal{R} : w(q) - g(q) > -\varepsilon/8\}$$

is open (by the assumption that w is lower semicontinuous). Fix $m \in \mathbb{N}$ for which there is a $\tilde{\beta}$ satisfying (32) and let $\bar{\alpha}$ denote the concatenation of the input $c_m\lceil[0,m]$ followed by the input $\tilde{\beta}$. We combine (31) and (32) to get

$$\begin{aligned}
w(\bar{x}) \;\;&\geq\;\; J(m,\phi_m,c_m) \;+\; e^{-\Delta(m,\phi_m,c_m)}\, w(p_m) \\
&+\; J\left(t_{p_m}(\tilde{\beta}), y_{p_m}(\cdot,\tilde{\beta}), \tilde{\beta}\right) \;-\; \frac{\varepsilon}{8}(1-e^{-m}) \;-\; \frac{\varepsilon}{4} \\
&\geq\;\; J(m,\phi_m,c_m) \;+\; J\left(t_{p_m}(\tilde{\beta}), y_{p_m}(\cdot,\tilde{\beta}), \tilde{\beta}\right) \\
&+\; e^{-\Delta(m,\phi_m,c_m)}\, g\left(y_{p_m}(t_{p_m}(\tilde{\beta}),\tilde{\beta})\right) \;-\; \frac{\varepsilon}{8}(1-e^{-m}) - \frac{\varepsilon}{2} \\
&\geq\;\; \int_0^{t_{\bar{x}}(\bar{\alpha})} e^{-\Delta(s,y_{\bar{x}}(\cdot,\bar{\alpha}),\bar{\alpha})}\,\ell\left(y_{\bar{x}}(s,\bar{\alpha}),\bar{\alpha}(s)\right)\,ds \\
&+\; e^{-\Delta(t_{\bar{x}}(\bar{\alpha}),y_{\bar{x}}(\cdot,\bar{\alpha}),\bar{\alpha})}\, g\left(y_{\bar{x}}(t_{\bar{x}}(\bar{\alpha}),\bar{\alpha})\right) \;-\; \frac{\varepsilon}{2}(2-e^{-m}).
\end{aligned}$$

Therefore, we can set $\bar{\beta} = \bar{\alpha}$ to satisfy the requirement (24). This proves Theorem 1. Our proof showed that

$$w(y_{\bar{x}}(j,\bar{\alpha})) \;\;<\;\; e^{-\bar{h}+\delta(j,\bar{\alpha},\bar{x})}\omega_o \quad \text{for all} \quad \mathbb{N} \ni j < t_{\bar{x}}(\bar{\alpha}).$$

Since we could have replaced $\bar{h}$ with any positive constant $k > \bar{h}$ in our proof, and since we could also have replaced $[0,1]$ with $[m_j, m_{j+1}]$ for any $m_j \uparrow +\infty$ in the definitions of the $\mathcal{Z}_j$'s, the conditions of Remark 3.3 hold also.

6 Local Uniqueness Characterizations

This section extends Theorem 1 to get a more general result that applies to solutions of (4) on $\Omega \setminus \mathcal{T}$ for general open sets $\Omega \subseteq \mathbb{R}^N$, including cases where $\mathcal{T} \not\subseteq \Omega$. Our notation is as before, so

$$\delta(s, \beta, x) := \int_0^t h^r(y_x(s, \beta), \beta(s))\, ds$$

for all $s \geq 0$, $\beta \in \mathcal{U}^r$, and $x \in \mathbb{R}^N$. We assume this local version of (A_4):

(A_4') $\quad (\forall q \in \Omega) \ \big((\exists \bar{h}_q < \infty) \ \big(y_q(t, \beta) \in \partial\Omega \Rightarrow \delta(t, \beta, q) \leq \bar{h}_q\big)\big)$.

We prove the following generalization of Theorem 1:

Theorem 2 *Assume (A_0)-(A_3) hold, $\Omega \subseteq \mathbb{R}^N$ is open, w is locally bounded on Ω, and one of the following:*

1. *$\mathcal{T} \subseteq \Omega$.*
2. *$\partial\Omega \subseteq \mathcal{T}$, and there is a neighborhood $\mathcal{N}$ of $\mathcal{T}$ such that $w^\star \leq g$ on $\mathcal{N} \cap \Omega$.*

Assume w is a viscosity solution of (4) on $\Omega \setminus \mathcal{T}$ satisfying $BC(\Omega, \omega_o)$. If $\omega_o \in [0, \infty]$ and (A_4') holds, then $w \equiv v^r$ on $\{x \in \Omega : w(x) < e^{-\bar{h}_x}\omega_o\}$. If $\omega_o < 0$, then $w \equiv v^r$ on Ω.

Remark 6.1 Note that we do not require Ω to be bounded in Theorem 2. If there exist Ω and w which satisfy the requirements of this theorem, then it must be the case that $\Omega \subset \mathcal{R}$, since $v^r \equiv +\infty$ on $\mathbb{R}^N \setminus \mathcal{R}$. The novelty of Theorem 2 is that it covers cases where $\mathcal{T} \not\subseteq \Omega \neq \mathcal{R}$. Note that Theorem 2 gives a uniqueness characterization on all of Ω for the undiscounted case where one can take $\bar{h} \equiv 0$. Moreover, our proof of Theorem 2 will show that for each $k \in [\bar{h}, \infty)$ and for each $p \in \Omega$ with $w(p) < e^{-k}\omega_o$, there exists a control $\alpha_p \in \mathcal{U}^r$ such that

$$w(y_p(j, \alpha_p)) \ < \ e^{-k+\delta(j,\alpha_p,p)}\omega_o \quad \text{for all} \quad \mathbb{N} \ni j \ < \ t_p(\alpha_p)$$

and such that

$$t_p(\alpha_p) \ < \ \infty \quad \text{or} \quad \lim_{s\to\infty} y_p(s, \alpha_p) \in \mathcal{T}.$$

We can also allow U to be closed but not necessarily compact and g to be discontinuous if we impose more regularity on the data (cf. §7).

Remark 6.2 Theorem 2 gives local versions of Corollary 4.1 on the FP and its generalizations (P_α), in the following sense. Let α be any convex function of class $\mathcal{MK}$, let $\xi > 0$ be given, and let v_α denote the value function for (P_α) as before. Pick

$$\Omega \ = \ \{x \in \mathbb{R}^2 : v_\alpha(x) < \xi\}.$$

Since v_α is convex and therefore continuous (cf. [18], p. 81), Ω is open. Also, v_α satisfies the condition $BC(\Omega, \xi)$. It follows from Theorem 2 that if w is a viscosity solution of the HJBE (18) on $\Omega \setminus \{\vec{0}\}$, and if there is a $\hat{\xi} > 0$ so that $\Omega = \{x \in \mathbb{R}^2 : w(x) < \hat{\xi}\}$ and w satisfies $BC(\Omega, \hat{\xi})$, then $w \equiv v_\alpha$ on Ω. In this way, we get a uniqueness characterization for the Bellman equation (18) on any sublevel set of v_α.

We now prove Theorem 2, assuming for notational simplicity that w is continuous (so $w \equiv w^\star \equiv w_\star$). Also, we assume that $\omega_o \in [0, \infty]$ and (A_4'). (The $\omega_o < 0$ case obtains by replacing $\bar{h}$ with 0 as before.) We drop the subscripts on $\bar{h}$ and the superscripts r as before. We indicate the changes needed in the proof of Theorem 1. The proof of the inequality "$w \geq v$" is exactly as before except that we replace $\mathcal{R}$

with Ω in the definitions of $\mathcal{Z}_1$ and $\mathcal{Z}_2$. To prove the reverse inequality, fix $x \in \Omega \setminus \mathcal{T}$ and $\varepsilon > 0$ such that $w(x) < e^{-\bar{h}}\omega_o - \varepsilon$, and let B be any bounded open set in Ω that contains x and is such that $\bar{B} \subseteq \Omega \setminus \mathcal{T}$. Then w is also a viscosity solution of the HJBE (4) on B. It follows from Lemma 2.4 that

$$w(x) \;\leq\; \int_0^t e^{-\delta(s,\beta,x)}\ell(y_x(s,\beta),\beta(s))\,ds \;+\; e^{-\delta(t,\beta,x)}\,w(y_x(t,\beta)) \tag{33}$$

for all $\beta \in \mathcal{U}$ and $t \in [0, t_x^B(\beta))$. Suppose that $w(x) > v(x)$. By the infimum definition, we would then be able to find a relaxed control $\tilde{\beta} \in \mathcal{U}(x)$ which is such that

$$\int_0^{t_x(\tilde{\beta})} e^{-\delta(s,\tilde{\beta},x)}\,\ell\left(y_x(s,\tilde{\beta}),\tilde{\beta}(s)\right)ds + e^{-\delta(t_x(\tilde{\beta}),\tilde{\beta},x)}g\left(y_x(t_x(\tilde{\beta}),\tilde{\beta})\right) < w(x). \tag{34}$$

We consider three cases.

Case A If $y_x(s,\tilde{\beta}) \in \Omega$ for $0 \leq s \leq t_x(\tilde{\beta})$, then $t_x(\tilde{\beta})$ is a limit of exit times from sets $B = B_k$ as above. For example, take $B_k \subseteq \Omega \setminus \mathcal{T}$ to be a tube around the restriction of $y_x(\cdot,\tilde{\beta})$ to $[0,\, t_x(\tilde{\beta}) - 1/k]$ for k large enough. In that case, we arrive at a contradiction with (34) once we put $\beta = \tilde{\beta}$ and $t = t_x^{B_k}(\tilde{\beta})$ in (33) and pass to the limit as $k \to \infty$.

Case B Assume there is an $s \leq t_x(\tilde{\beta})$ for which $y_x(\cdot,\tilde{\beta}) \notin \Omega$ and that $\mathcal{T} \subseteq \Omega$. Let $\hat{\tau}$ denote the supremum of the times in $(0, t_x(\tilde{\beta})]$ at which $y_x(\cdot,\tilde{\beta})$ is in $\partial\Omega$. Since $\mathcal{T} \subseteq \Omega$ and $\mathcal{T}$ is closed and Ω is open, it follows that $\hat{\tau} < t_x(\tilde{\beta})$ and that $y_x(\hat{\tau},\tilde{\beta}) \in \partial\Omega$. Apply the first part of Lemma 2.4 with the choices

$$q \;=\; z_n \;:=\; y_x(\hat{\tau} + 1/n, \tilde{\beta}), \qquad \beta(\cdot) \;=\; \alpha_n(\cdot) \;:=\; \tilde{\beta}(\cdot + \hat{\tau} + 1/n)$$

and

$$r \;=\; t_{z_n}(\tilde{\beta}(\cdot + \hat{\tau} + 1/n)),$$

with B chosen to be a tube in Ω which contains a portion of the trajectory $y_x(\cdot,\tilde{\beta})$ that runs from z_n to $y_x(t_x(\tilde{\beta}),\tilde{\beta}) \in \partial\mathcal{T}$ for n large. Lemma 2.4 would then give

$$\begin{aligned} w(x) \;&>\; \int_0^{\hat{\tau}+1/n} e^{-\delta(s,\tilde{\beta},x)}\,\ell\left(y_x(s,\tilde{\beta}),\tilde{\beta}(s)\right)\,ds \\ &+\; \left[\int_0^{t_x(\tilde{\beta})-\hat{\tau}-1/n} e^{-\delta(s,\alpha_n,z_n)}\,\ell\left(y_{z_n}(s,\alpha_n),\alpha_n(s)\right)\,ds\right. \\ &+\; \left. e^{-\delta(t_{z_n}(\alpha_n),\alpha_n,z_n)}\,w\left[y_{z_n}(t_{z_n}(\alpha_n),\alpha_n)\right]\right] e^{-\delta(\hat{\tau}+1/n,\tilde{\beta},x)} \\ &\geq\; \int_0^{\hat{\tau}+1/n} e^{-\delta(s,\tilde{\beta},x)}\,\ell\left(y_x(s,\tilde{\beta}),\tilde{\beta}(s)\right)\,ds \;+\; e^{-\delta(\hat{\tau}+1/n,\tilde{\beta},x)}\,w(z_n). \end{aligned}$$

Since ℓ is nonnegative, we get

$$w(x) > e^{-\delta(\hat{\tau}+1/n,\tilde{\beta},x)}w(z_n)$$

for large n. But

$$\Omega \;\ni\; z_n \;\;\to\;\; y_x(\hat{\tau},\tilde{\beta}) \;\in\; \partial\Omega$$

and

$$e^{-\delta(\hat{\tau}+1/n,\tilde{\beta},x)} \;\to\; e^{-\delta(\hat{\tau},\tilde{\beta},x)} \;\geq\; e^{-\bar{h}},$$

so the limit condition in $BC(\Omega,\omega_o)$ implies

$$e^{-\delta(\hat{\tau}+1/n,\tilde{\beta},x)}w(z_n) \;\;\to\;\; e^{-\delta(\hat{\tau},\tilde{\beta},x)}\omega_o \;\;\geq\;\; e^{-\bar{h}}\omega_o.$$

This contradicts the fact that $w(x) < e^{-\bar{h}}\omega_o$.

Case C Finally, assume that there is an $s \leq t_x(\tilde{\beta})$ for which $y_x(\cdot, \tilde{\beta}) \notin \Omega$ and that Condition 2 of Theorem 2 holds. Since $\partial\Omega \subseteq \mathcal{T}$, it follows that

$$\inf\left\{t \geq 0 : y_x(t, \tilde{\beta}) \notin \Omega\right\} \;=\; t_x(\tilde{\beta}).$$

Condition (34) and the limit condition in $BC(\Omega, \omega_o)$ guarantee the existence of a sequence $t_n \uparrow t_x(\tilde{\beta})$ and a constant k for which

$$\begin{aligned} e^{-\bar{h}}\omega_o - \varepsilon &\leq \liminf_{n\to\infty}\left\{e^{-\delta(t_n,\tilde{\beta},x)}\left[\omega_o - \frac{\varepsilon}{ke^{-\bar{h}}}\right]\right\} \\ &\leq \liminf_{n\to\infty}\left\{e^{-\delta(t_n,\tilde{\beta},x)}\, w(y_x(t_n,\tilde{\beta}))\right\} \\ &< w(x), \end{aligned}$$

which is a contradiction.

Therefore, $w(x) \leq v(x)$ for all $x \in \Omega$ such that $w(x) < e^{-\bar{h}}\omega_o$. As pointed out earlier, the case where $\omega_o < 0$ is handled by replacing $\bar{h}$ with 0 in the preceding argument. This establishes Theorem 2.

7 Unbounded Control Sets and Discontinuous Exit Costs

We close by showing how to relax our assumptions that the control set U is compact and that the exit cost g is continuous. To cover cases where g is discontinuous, we adapt the backward programming methods of [5]. Roughly speaking, we extend our results to cases where U is noncompact by replacing U with a compact set of vector field valued controls and a new dynamics which gives the same trajectories as the original dynamics f.

7.1 Discontinuous Exit Costs

This subsection shows how to generalize the arguments of the preceding sections to cases where the exit cost g is discontinuous. We assume conditions (A_0)-(A_3) and (A_4') hold except that instead of assuming that g is continuous, we assume that g is locally bounded and nonnegative. It follows that $g_\star : \mathcal{R} \to \mathbb{R}$ is lower semicontinuous, that $g^\star : \mathcal{R} \to \mathbb{R}$ is upper semicontinuous, and that there exist sequences $\{g_m^-\}$ and $\{g_m^+\}$ of continuous nonnegative functions such that

$$\sup_m g_m^- = g_\star, \quad \inf_m g_m^+ = g^\star, \quad g_k^- \leq g_{k+1}^- \;\forall k, \quad \text{and} \quad g_k^+ \geq g_{k+1}^+ \;\forall k \tag{35}$$

pointwise on $\mathcal{R}$. Let $\Omega \subset \mathcal{R}$ and w satisfy the hypotheses of Theorem 2. Let $USC(g)$ denote the set of upper semicontinuity points of g,

$$USC_o(g) = USC(g) \cap \{x : g(x) = 0\},$$

and w be a viscosity solution of (4) on $\Omega \setminus \mathcal{T}$ which satisfies the following relaxed analog of $BC(\Omega, \omega_o)$:

$BC_r(\Omega, \omega_o)$ $w : \Omega \to \mathbb{R}$ is bounded below, $\omega_o \in \mathbb{R} \cup \{+\infty\}$, $w_\star \geq g_\star$ and $w^\star \leq g^\star$ on $\mathcal{T} \cap \Omega$, $w < \omega_o$ on Ω, and $\lim_{x\to x_o} w(x) = \omega_o \;\forall x_o \in \partial\Omega$.

Following [5], we also assume that there is an outer field at each point along $\partial\mathcal{T}$, i.e., the conditions

(OF1) $d(\cdot, \partial\mathcal{T})$ is $C^{1,1}$ in some neighborhood of $\partial\mathcal{T}$.

(OF2) $\forall x \in \partial\mathcal{T}$, $\exists a \in U$ and $\delta > 0$ s.t. $-f(x,a) \cdot Dd(x, \partial\mathcal{T}) \geq \delta > 0$.

Condition $(OF2)$ says that at each point of $\partial\mathcal{T}$, the system can choose a direction pointing into $\mathcal{T}$. Finally, we assume the following variant of (A_3):

(A_3') If $\beta \in \mathcal{U}^r$ and $x \in \mathbb{R}^N$, and if $J(\infty, y_x(\cdot,\beta),\beta) < \infty$, then there is a point $\bar{p} \in \partial\mathcal{T} \cap USC_o(g)$ for which $\lim_{s\to\infty} y_x^r(s,\beta) = \bar{p}$.

Remark 7.1 A slight variant of what follows works if we replace $\bar{p} \in \partial\mathcal{T} \cap USC_o(g)$ in (A_3') with the conditions that $\bar{p} \in \partial\mathcal{T} \cap USC(g)$ and $g(\bar{p}) = g_{\min}$, where $g_{\min}$ is any lower bound on g (cf. Remark 7.7).

We prove the following consequence of Theorem 2:

Corollary 7.2 Let the preceding assumptions hold. If $\omega_o \in \mathbb{R}_+ \cup \{+\infty\}$, then the following inequalities hold on $\left\{x \in \Omega \setminus \mathcal{T} : v^r[g](x) < e^{-h_x}\omega_o\right\}$:

1. $w_\star \geq [v^r[g_\star]]_\star$
2. $w^\star \leq v^r[g^\star]$

If $\omega_o < 0$, then the previous two inequalities hold on all of $\Omega \setminus \mathcal{T}$.

In particular, $w \equiv v^r[g]$ is continuous if g is continuous and v^r is lower semicontinuous. Note that under the preceding hypotheses, it could be that $w \neq v^r[g]$ at points of discontinuity of g, even if w and v^r are both continuous. Here is an example from §3 of [5] where this occurs:

Example 7.3 Take $h \equiv \ell \equiv 1$, $f(x,a) \equiv a$, and $\mathcal{T} = U = \overline{B_1(0)} \subseteq \mathbb{R}^N$. Use $\Omega = \mathcal{R} = \mathbb{R}^N$. Set $x_o = (1,0)$ and

$$g = \mathbf{1}_{\partial(\overline{B_1(0)})\setminus\{x_o\}}.$$

One checks that $v^r[g](x) = 1 - e^{-||x-x_o||}$ on $\Omega \setminus \mathcal{T}$ and that $u_\xi(x) = 1 - \xi e^{-||x-x_o||}$ is also a viscosity solution of the HJBE on $\mathbb{R}^N \setminus \mathcal{T}$ for all $\xi \in]0,1[$ satisfying the correct boundary condition.

The proof of Corollary 7.2 is similar in spirit to the arguments of [5], except we use (A_3') and a weak-$\star$ convergence argument to replace the requirements in [5] that the discount factor h is a positive constant and that f and ℓ are globally bounded. Define $\tilde{J}$ by

$$\tilde{J}(x,\beta,t,\phi) = \int_0^t \ell^r(y_x(s,\beta),\beta(s))\, e^{-\delta(s,\beta,x)}\, ds + e^{-\delta(t,\beta,x)}\phi(y_x(t,\beta))$$

wherever the RHS is defined. Our strategy is to first show that $w_\star$ majorizes the lower envelope of the optional stopping value function

$$v_-(x) \;:=\; \begin{cases} \inf\left\{\tilde{J}(x,\beta,\theta,g_\star) :\ \beta \in \mathcal{U}^r,\ \theta \in [0,\infty),\ y_x^r(\theta,\beta) \in \mathcal{T}\right\}, & x \in \mathcal{R}^r \\ +\infty, & x \notin \mathcal{R}^r \end{cases}$$

Once this is done, the first condition of Corollary 7.2 will follow from the following lemma (whose proof is essentially the same as the proof of Lemma 2.15 from [5] except local bounds on f are used instead of global ones):

Lemma 7.4 Under the previous assumptions, let $\theta \in (0,\infty)$, $\mu \in \mathcal{U}^r$, and $x \in \Omega$ be given. Assume that $y_x(\theta,\mu) \in \partial\mathcal{T}$. Then there exist sequences $x_n \to x$ and $\mu_n \to \mu$ weak-$\star$ on $[0,\infty)$ such that $y_{x_n}(t_{x_n}(\mu_n),\mu_n) = y_x(\theta,\mu)$ and $t_{x_n}(\mu_n) = \theta$ for all n.

Let us show how the first conclusion of Corollary 7.2 would follow from $w_\star \geq (v_-)_\star$. Let $\varepsilon > 0$ be given, and choose $x \in \Omega \setminus \mathcal{T}$. We assume for simplicity that $\omega_o < 0$. (The $\omega_o \geq 0$ case follows easily from the argument we give.) Using the Lipschitz continuity of f, we would then be able to find $\bar{\theta} < \infty$, $\bar{\mu} \in \mathcal{U}^r$, and sequences $x_n \to x_o$, $\varepsilon_n \downarrow 0$, and $\mu_n \to \bar{\mu}$ weak-$\star$ such that

$$\begin{aligned} v_-(x_o) + \varepsilon &\geq \tilde{J}(x_o, \bar{\mu}, \bar{\theta}, g_\star) \\ &\geq \tilde{J}(x_n, \mu_n, t_{x_n}(\mu_n), g_\star) - \varepsilon_n \\ &\geq v^r[g_\star](x_n) - \varepsilon_n, \end{aligned}$$

which gives

$$v_-(x_o) \geq [v^r[g_\star]]_\star (x_o) \tag{36}$$

and the first inequality of the corollary once we take inferior limits. To prove that $w_\star \geq (v_-)_\star$, we need the following lemma:

Lemma 7.5 If $\bar{x} \in \Omega \setminus \mathcal{T}$ is such that $v^r(\bar{x}) < e^{-h_{\bar{x}}}\omega_o$, then

$$\liminf_{m\to\infty} v^r[g_m^-](\bar{x}) \geq [v^r[g_\star]]_\star (\bar{x})$$

The limit holds on all of $\Omega \setminus \mathcal{T}$ if $\omega_o < 0$.

Proof. We again assume that $\omega_o < 0$ for simplicity and choose $\bar{x} \in \Omega \setminus \mathcal{T}$. We can clearly assume that $\bar{x} \in \mathcal{R}^r$. We continue to use the notation of §5. Choose a sequence $\beta_n \in \mathcal{U}^r$ and a sequence of corresponding trajectories γ_n for f^r starting at $\bar{x}$ such that

$$v^r\left[g_m^-\right](\bar{x}) \geq -\frac{1}{m} + \tilde{J}(\bar{x}, \beta_m, t_{\bar{x}}(\beta_m), g_m^-) \ \ \forall m \in \mathbb{N} \tag{37}$$

By passing to a subsequence and using a diagonal argument based on the classical compactness lemma (cf. [1], p.354), we can assume that there is an input $\beta_o \in \mathcal{U}^r$ and a corresponding trajectory γ_o for the dynamics f^r starting at $\bar{x}$ so that $\beta_m \to \beta_o$ weak-* and $\gamma_m \to \gamma_o$ uniformly on compact intervals.

If $\{t_{\bar{x}}(\beta_m)\}$ is bounded, we can assume that $t_{\bar{x}}(\beta_m) \to t_o \in [0, \infty)$ and $\gamma_m(t_{\bar{x}}(\beta_m)) \to \gamma(t_o) \in \partial\mathcal{T}$ (since $\partial\mathcal{T}$ is closed). One easily checks (cf. [5]) that since the g_m^-'s are continuous, we have

$$\liminf_{m\to\infty} g_m^-\left[\gamma_m(t_{\bar{x}}(\beta_m))\right] \geq g_\star(\gamma(t_o)).$$

It follows from (36) and (37) that

$$\liminf_{m\to\infty} v^r[g_m^-](\bar{x}) \geq v_-(\bar{x}) \geq [v^r[g_\star]]_\star (\bar{x}).$$

On the other hand, if $\{t_{\bar{x}}(\beta_m)\}$ is unbounded, then we can assume that $t_{\bar{x}}(\beta_m) \uparrow +\infty$. It follows from the nonnegativity of ℓ and g, (35), and (37) that $0 \leq J(t, \gamma_o, \beta_o) \leq 1 + v^r[g](\bar{x}) < \infty$ for all $t \geq 0$ (bu picking m in (37) so that $t_{\bar{x}}(\beta_m) \geq t$). We conclude from the assumption (A_3') that there is a $\bar{p} \in \partial\mathcal{T} \cap USC_o(g)$ for which

$$\lim_{t\to\infty} y_{\bar{x}}(t, \beta_o) = \bar{p}. \tag{38}$$

Let $\varepsilon > 0$ be given. Using $STC\mathcal{T}$, it follows that there exist $M > 0$ and extensions γ_o^+ and β_o^+ of the restrictions $\gamma_o\lceil[0, M]$ and $\beta_o\lceil[0, M]$ so that γ_o reaches $\partial\mathcal{T}$ at some time $t^\star \geq M$ and which give

$$J(M, \gamma_o, \beta_o) \geq J(t^\star, \gamma_o^+, \beta_o^+) - \frac{\varepsilon}{3} \quad \text{and} \quad g^\star(\gamma_o^+(t^\star)) \leq \varepsilon/3, \tag{39}$$

where we use the upper semicontinuity of g at $\bar{p}$ and the estimate (10). On $[0, M]$, $\gamma_m \to \gamma_o$ uniformly and $\beta_m \to \beta_o$ weak-*, so (37) gives

$$\begin{aligned} v^r[g_m^-](\bar{x}) &\geq -\frac{1}{m} + J(M, \gamma_m, \beta_m) \\ &\geq -\frac{1}{m} - \frac{\varepsilon}{3} + J(M, \gamma_o, \beta_o) \\ &\geq -\varepsilon + J(t^\star, \gamma_o^+, \beta_o^+) \\ &\geq -2\varepsilon + v_-(\bar{x}) \end{aligned} \tag{40}$$

for large m, so $\liminf_{m\to\infty} v^r[g_m^-](\bar{x}) \geq v_-(\bar{x})$ and we conclude from (36) as before. ∎

We now prove Corollary 7.2. We assume that $\omega_o < 0$ and $x \in \Omega \setminus \mathcal{T}$ for simplicity as before. For each $m \in \mathbb{N}$, the argument of §5 and the fact that $w_\star \geq g_m^-$ on $\mathcal{T}$ for each m gives $w_\star(x) \geq v^r[g_m^-](x)$ for each m, so the first inequality of the corollary follows from Lemma 7.5. On the other hand, since $g \leq g_m^+$ for all m and the g_m^+'s are continuous, it follows from the arguments of §6 that $w^\star(x) \leq v^r[g_m^+](x)$ for each m. One easily checks that

$$\limsup_{m\to\infty} v^r[g_m^+](x) \leq v^r[g^\star](x),$$

so we get the second inequality of the lemma. This completes the proof of Corollary 7.2.

Remark 7.6 We remark that if $v^r[g]$ is lower semicontinuous, then the preceding proof gives $w_\star = w^\star = v^r[g]$ at all points in Ω where g is continuous, so if g is continuous on Ω, then there are in fact no viscosity solutions of the HJBE on $\Omega \setminus \mathcal{T}$ satisfying $BC_r(\Omega, \omega_o)$ which have discontinuities.

Remark 7.7 Similar results hold if we replace the requirement $g(\bar{p}) = 0$ in (A_3') with the requirement that $g(\bar{p}) = g_{\min}$, where $g_{\min}$ is any finite lower bound for g. One way to do this is as follows. We assume $f = (\tilde{f}, 1)$, $\Omega \subseteq \mathcal{R}$ is bounded, and that there is an integrable function ϕ for which

$$|h((x,s),a) - h((y,s),b)| \;\leq\; \phi(s)\left[||x-y|| + ||a-b|| + \bar{R}\right] \tag{41}$$

for all $x, y \in \mathbb{R}^{N-1}$ and $a, b \in U$, where $\bar{R}$ is constant. We also assume $g\lceil\mathcal{T}$ is bounded. The proof is as before up to the existence of $\bar{p} \in \partial\mathcal{T}$ in (38), except now $g(\bar{p}) = g_{\min}$. Choose $M > 0$ as before. We indicate the changes needed to complete the proof. The case where $\{t_{\bar{x}}(\beta_m)\}$ is bounded is handled exactly as before. By passing to a subsequence, it remains to consider the case where $t_{\bar{x}}(\beta_m) \to \infty$. First assume that $\int_0^\infty h(\gamma_o(s), \beta_o(s))\, ds < \infty$. By a diagonal argument, we can assume that

$$a_m(k) := \int_{k-1}^{k} h(\gamma_m(s), \beta_m(s))\, ds \quad \to \quad a(k) := \int_{k-1}^{k} h(\gamma_o(s), \beta_o(s))\, ds \quad \forall k \in \mathbb{N}.$$

Also, the boundedness of γ_o and the γ_m's in Ω and the estimate (41) guarantee that

$$\sum_{k=1}^{\infty} a_m(k) < \infty \;\; \forall m \in \mathbb{N}.$$

Therefore, we get

$$\int_0^\infty h(\gamma_m(s), \beta_m(s))\, ds \to \int_0^\infty h(\gamma_o(s), \beta_o(s))\, ds,$$

by dominated convergence. Notice that $g^\star(\bar{p}) = g(\bar{p})$. For possibly larger $M > 0$, (37) and elementary arguments which we leave to the reader therefore give

$$\begin{aligned} v^r[g_m^-](\bar{x}) &\geq -\frac{1}{2m} + J(M, \gamma_o, \beta_o) + e^{-\delta(t_{\bar{x}}(\beta_m), \beta_m, \bar{x})} g_m^-\left[\gamma_m(t_{\bar{x}}(\beta_m))\right] \\ &\geq -\frac{1}{2m} - \frac{\varepsilon}{3} + J(M, \gamma_o, \beta_o) + e^{-\delta(M, \beta_o, \bar{x})} g(\bar{p}) \\ &\geq -\varepsilon + J(t^\star, \gamma_o^+, \beta_o^+) + e^{-\delta(t^\star, \beta_o^+, \bar{x})} g^\star(\gamma_o^+(t^\star)) \\ &\geq -\varepsilon + v_-(\bar{x}) \end{aligned} \tag{42}$$

for some $t^\star \geq M$ depending on m (where we use the estimate (10) to ensure that $\gamma^+(t^\star)$ is near enough to $\bar{p}$ and assume that $g_m^- \geq g_{min}$ for all m for the third inequality). We then conclude as before. On the other hand, if

$$\int_0^\infty h(\gamma_o(s), \beta_o(s))\, ds = +\infty,$$

then one easily checks that $\lim_{m\to\infty}\int_0^{t_{\bar{x}}(\beta_m)} h(\gamma_m(s),\beta_m(s))\,ds = +\infty$, so

$$\lim_{m\to\infty} e^{-\delta(t_{\bar{x}}(\beta_m),\beta_m,\bar{x})} g_m^-\left[\gamma_m(t_{\bar{x}}(\beta_m))\right] = 0$$

since $g_\star\lceil\mathcal{T}$ is bounded, and

$$\lim_{M\to\infty} e^{-\delta(M,\beta_o,\bar{x})} g(\bar{p}) = 0.$$

The inequalities in (42) now follow as before.

7.2 Unbounded Control Sets

The assumption that U is compact can be relaxed to the requirement that U is closed if we add regularity conditions on that data. One way to do this is to assume that f and ℓ penalize large control set values and consider only viscosity solutions w of the Bellman equation on bounded sets for which the semidifferential sets $D^+w^\star(x)$ and $D^-w_\star(x)$ are locally bounded, in which case the infimization in the Bellman equation can be taken over bounded subsets of U (cf. [2], [8], and [14] for details).

Another way to do this is as follows. For simplicity, we assume $h \equiv 0$ and $\{(f(x,a),\ell(x,a)) : a \in A\}$ is convex for all $x \in \mathbb{R}^N$ (which implies that all trajectories for f^r are also trajectories for f, by the Filippov Selection Theorem). We give $C(\mathbb{R}^N,\mathbb{R}^N\times\mathbb{R})$ the topology of compact convergence. The additional assumptions we make are as follows:

- (NC_1) $\quad \sup\{\ell(0,u) : u \in U\} < \infty.$
- (NC_2) $\quad \{(f(\cdot,u),\ell(\cdot,u)) : u \in U\} \subseteq C(\mathbb{R}^N,\mathbb{R}^N\times\mathbb{R})$ is closed.
- (NC_3) $\quad f$ is bounded on $B_R(0)\times U$ for each $R > 0$.

These guarantee that the supremum in the HJBE is always finite. It follows from the Ascoli-Arzelá Theorem that

$$K := \{k_u(\cdot) := (f(\cdot,u),\ell(\cdot,u)) : u \in U\}$$

is a compact subset of the metric space $C(\mathbb{R}^N,\mathbb{R}^N\times\mathbb{R})$. Define the projection mappings π_j on K by

$$\pi_j(k_u(\cdot)) = \begin{cases} f(\cdot,u), & j=1 \\ \ell(\cdot,u), & j=2 \end{cases} \qquad \forall u \in U.$$

We now apply the method of our proofs to the exit time problem whose dynamics Φ, Lagrangian Λ, and control set $\tilde{\mathcal{U}}$ are defined by $\Phi(x,k) = (\pi_1\circ k)(x)$, $\Lambda(x,k) = (\pi_2\circ k)(x)$, and $\tilde{\mathcal{U}} := \{[0,\infty)\ni t\mapsto k_{\beta(t)} : \beta\in\mathcal{U}\}$. Let $\tilde{v}$ denote the value function of this problem. Since the trajectories of Φ with the controls $\tilde{\mathcal{U}}$ are exactly the trajectories of f with controls in $\mathcal{U}$, it follows that $\tilde{v} \equiv v \equiv v^r$. Moreover, the new problem satisfies conditions (A_0)-(A_3) (with the compact metric space K replacing U, and so forth). It follows from the proofs we gave that any viscosity solution of the HJBE on $\mathcal{R}\setminus\mathcal{T}$ which satisfies $BC(\mathcal{R},\omega_o)$ for some $\omega_o \in \mathbb{R}\cup\{+\infty\}$ coincides with the value function v of the original problem. Arguing in a similar way, we get analogs of Theorem 2 and Corollary 7.2 which apply to problems with noncompact control sets, since we can take weak-$\star$ limits of relaxed controls on the compact metric space K. We leave the details to the reader.

8 Conclusions

This paper presented uniqueness characterizations for discontinuous viscosity solutions for the Bellman equation of deterministic optimal control problems with exit times and fully nonlinear systems. We allow problems whose dynamics, interest rates, Lagrangians, control sets, and targets are all unbounded. Also, we allow cases where the Lagrangian and the interest rate are not bounded below by positive constants,

including the undiscounted case where the interest rate is identically zero, and we allow discontinuous value functions and discontinuous exit costs.

We gave conditions under which the value functions of these problems are the unique viscosity solutions of the corresponding Bellman equations that vanish on the target, are bounded below, and satisfy certain growth conditions. These growth conditions hold vacuously for cases where each point in the state space can be brought to the target in finite time. In particular, we showed that the value function for the Fuller Problem is the unique viscosity solution of the associated Bellman equation which is bounded below and which vanishes at the origin. We also gave a local uniqueness characterization which applies to solutions of exit time Bellman equations on neighborhoods which may or may not contain the target.

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