HYBRID FUNCTIONS METHOD FOR THE NONLINEAR VARIATIONAL PROBLEMS

ABSTRACT. A numerical technique for solving the nonlinear problems of the calculus of variations is presented. The method is based upon hybrid functions approximation. The properties of hybrid functions which consists of block-pulse functions plus Legendre polynomials are given. Two nonlinear examples are considered, in the first example the brachistochrone problem is formulated as a nonlinear optimal control problem, and in the second example a higher-order nonlinear problem is given. An operational matrix of integration is introduced and is utilized to reduce the calculus of variations problems to the solution of algebraic equations. The method is general, easy to implement and yields very accurate results.

Keywords: brachistochrone problem, Variational problems, Numerical methods, Hybrid functions.

1. INTRODUCTION.

There has been a considerable renewal of interest in the classical problems of the calculus of variations both from the point of view of mathematics and of applications in physics, engineering, and applied mathematics .

Finding the brachistochrone, or path of quickest decent, is a historically interesting problem that is discussed in virtually all textbooks dealing with the calculus of variations. In 1696, the brachistochrone problem was posed as a challenge to mathematicians by John Bernoulli. The solution of the brachistochrone problem is often cited as the origin of the calculus of variations as suggested in [1].

The classical brachistochrone problem deals with a mass moving along a smooth path in a uniform gravitational field. A mechanical analogy is the motion of a bead sliding down a frictionless wire.The solution to this problem was obtained by various methods such as the gradient method [2], successive sweep algorithm in [3-4] , the classical Chebyshev method [5] and multistage Monte Carlo method [6].

Orthogonal functions (OF's) have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem . The approach is based on converting the underlying differential equations into an integral equations through integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix of integration P, to eliminate the integral operations. The form of P depends on the particular choice of the orthogonal functions. Special attention has been given to applications of Walsh functions [7], block-pulse functions [8], Laguerre series [9], shifted Legendre polynomials [10] and shifted Chebyshev polynomials [11].

There are three classes of sets of OF's which are widely used. The first includes sets of piecewise constant basis functions (PCBF'S) (e.g., Walsh, block-pulse,etc.). The second consists of sets of orthogonal polynomials (OP's) (e.g., Laguerre, Legendre, Chebyshev, etc.). The third is the widely used sets of sine-cosine functions (SCF's) in Fourier series. While OP's and SCF's together form a class of continuous basis functions , PCBF's have inherent discontinuities or jumps. The inherent features(continuity or discontinuities) of a set of OF's largely determine their merit for application in a given situation. References [12] and [13] have demonstrated the advantages of PCBF spectral methods over Fourier spectral techniques. If a continuous function is approximated by PCBF's, the resulting approximation is piecewise constant. On the other hand if a discontinuous function is approximated by continuous basis functions the discontinuities are not properly modeled.

In the present paper we introduce a new numerical method to solve the nonlinear problems of the calculus of variations. Two examples are considered. In example 1, the brachistochrone problem is first formulated as an optimal control problem and in the second example a higher-order nonlinear problem is given. The method consist of reducing the calculus of variations problems to a set of algebraic equations by expanding the candidate function as hybrid functions with unknown coefficients. These hybrid functions, which consist of block-pulse functions and Legendre polynomials are given. The operational matrix of integration is then used to evaluate the coefficients of hybrid functions in such a way that the necessary conditions for extremization are imposed. The paper is organized as follows: In Section 2 we describe the formulation of the hybrid functions required for our subsequent development. Section 3 is devoted to numerical examples. In Section 3.1 the brachistochrone problem is considered and in Section 3.2 we consider a higher-order nonlinear problem. In both examples we demonstrate the accuracy of the proposed numerical scheme by comparing our numerical finding with the exact solutions.

2. Properties of Hybrid Functions.

2.1 Hybrid Functions of Block-pulse and Legendre Polynomials.

Hybrid functions $b_{nm}(t), n = 1, 2, \cdots, N, m = 0, 1, \cdots, M-1,$ have three arguments; n and m are the order of block-pulse functions and Legendre polynomials respectively, and t is the normalized time. They are defined on the interval $[0, t_f)$ as

$$b_{nm}(t) = \begin{cases} P_m(\frac{2N}{t_f}t - 2n + 1), & t \in [(\frac{n-1}{N})t_f, \frac{n}{N}t_f) \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Here, $P_m(t)$ are the well-known Legendre polynomials of order m which satisfy the following recursive formula.

$$P_\circ(t) = 1, \quad P_1(t) = t \quad (2)$$

$$P_{m+1}(t) = (\frac{2m+1}{m+1})tP_m(t) - (\frac{m}{m+1})P_{m-1}(t), \qquad m = 1, 2, 3, \cdots \quad (3)$$

2.2 Function Approximation.

A function $f(t)$, defined over the interval 0 to t_f may be expanded as

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} b_{nm}(t) = C^T B(t), \tag{4}$$

where

$$C = [c_{10}, \cdots, c_{1M-1}, c_{20}, \cdots, c_{2M-1}, \cdots, c_{N0}, \cdots, c_{NM-1}]^T, \tag{5}$$

and

$$B(t) = [b_{10}(t), \cdots, b_{1M-1}(t), b_{20}(t), \cdots, b_{2M-1}(t), \cdots, b_{N0}(t), \cdots, b_{NM-1}(t)]^T. \tag{6}$$

The integration of the vector $B(t)$ defined in Eq. (6) can be approximated by

$$\int_0^t B(t')dt' \simeq PB(t), \tag{7}$$

where P is the $MN \times MN$ operational matrix for integration and is given by

$$P = \begin{pmatrix} E & H & H & \cdots & H \\ 0 & E & H & \cdots & H \\ 0 & 0 & E & \cdots & H \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & E \end{pmatrix}, \tag{8}$$

where

$$H = \frac{t_f}{N} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$E = \frac{t_f}{2N} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M-1} & 0 \end{pmatrix}.$$

3. ILLUSTRATIVE EXAMPLES.

In this section two nonlinear problems of the calculus of variations are considered. Example 1 is the classical brachistochrone problem, whereas example 2 is a higher-order nonlinear problem taken from [14].

3.1 Example 1, The Brachistochrone Problem.

3.1.1 The Brachistochrone Problem as an Optimal Control Problem.

As an optimal control problem, the brachistochrone problem may be formulated as [15].

Minimize the performance index J,

$$J = \int_0^1 \left[\frac{1 + U^2(t)}{1 - X(t)} \right]^{\frac{1}{2}} dt, \tag{9}$$

subject to

$$\dot{X}(t) = U(t), \tag{10}$$

with

$$X(0) = 0, \quad X(1) = -0.5. \tag{11}$$

Eqs. (9), (10) and (11), describe the motion of a bead sliding down a frictionless wire in a constant gravitational field. The minimal time transfer expression (9) is obtained from the law of conservation of energy. Here X and t are dimensionless and they represent respectively the vertical and horizontal coordinates of the sliding bead.

As is well known the exact solution to the brachistochrone problem is the cycloid defined by the parametric equations

$$x = 1 - \frac{\beta}{2}(1 + \cos 2\alpha), \qquad t = \frac{t_0}{2} + \frac{\beta}{2}(2\alpha + \sin 2\alpha), \tag{12}$$

where

$$\tan \alpha = \frac{dX}{dt} = U,$$

with the given boundary conditions, the integration constants are found to be

$$\beta = 1.6184891, \quad t_0 = 2.7300631.$$

3.1.2 The Numerical Method.

Suppose, the rate variable $\dot{X}(t)$ can be expressed approximately as

$$\dot{X}(t) = C^T B(t) \tag{13}$$

using Eqs. (7) and (11), $X(t)$ can be represented as

$$\begin{aligned} X(t) =& \int_0^t \dot{X}(t')dt' + X(0) \\ =& C^T P B(t), \end{aligned} \tag{14}$$

also by using Eqs. (10) and (13) we have

$$U^2(t) = C^T B(t) B^T(t) C. \tag{15}$$

Equation (15) can be simplified by using the following property of the product of two hybrid function vectors

$$B(t)B^T(t)C \simeq \tilde{C}B(t), \tag{16}$$

where $\tilde{C}$ is a $MN \times MN$ product operational matrix. To illustrate the calculation procedure we choose $M = 3$ and $N = 4$. Thus we have

$$C = [c_{10}, c_{11}, c_{12}, \cdots, c_{40}, c_{41}, c_{42}]^T, \tag{17}$$

$$B(t) = [b_{10}(t), b_{11}(t), b_{12}(t), \cdots, b_{40}(t), b_{41}(t), b_{42}(t)]^T. \tag{18}$$

In Eq. (18) we have

$$\left.\begin{aligned} b_{10} &= 1 \\ b_{11} &= 8t-1 \\ b_{12} &= \frac{3}{2}(8t-1)^2 - \frac{1}{2} \end{aligned}\right\} 0 \le t < \frac{1}{4}, \qquad \left.\begin{aligned} b_{20} &= 1 \\ b_{21} &= 8t-3 \\ b_{22} &= \frac{3}{2}(8t-3)^2 - \frac{1}{2}, \end{aligned}\right\} \frac{1}{4} \le t < \frac{1}{2}, \tag{19}$$

and

$$\left.\begin{aligned} b_{30} &= 1 \\ b_{31} &= 8t-5 \\ b_{32} &= \frac{3}{2}(8t-5)^2 - \frac{1}{2} \end{aligned}\right\} \frac{1}{2} \le t < \frac{3}{4}, \qquad \left.\begin{aligned} b_{40} &= 1 \\ b_{41} &= 8t-7 \\ b_{42} &= \frac{3}{2}(8t-7)^2 - \frac{1}{2} \end{aligned}\right\} \frac{3}{4} \le t < 1. \tag{20}$$

We also get

$$B(t)B^T(t) = \begin{bmatrix} b_{10}b_{10} & b_{10}b_{11} & b_{10}b_{12} & \dots & b_{10}b_{42} \\ b_{11}b_{10} & b_{11}b_{11} & b_{11}b_{12} & \dots & b_{11}b_{42} \\ b_{12}b_{10} & b_{12}b_{11} & b_{12}b_{12} & \dots & b_{12}b_{42} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{42}b_{10} & b_{42}b_{11} & b_{42}b_{12} & \dots & b_{42}b_{42} \end{bmatrix}. \tag{21}$$

Using Eqs. (19) and (20) we have

$$\begin{aligned} b_{ij}b_{kl} &= 0 \quad \text{if} \quad i \neq k \\ b_{i0}b_{ij} &= b_{ij} \\ b_{i1}b_{i1} &= \frac{1}{3}b_{i0} + \frac{2}{3}b_{i2} \\ b_{i1}b_{i2} &= \frac{2}{5}b_{i1} + \frac{3}{5}b_{i3} \\ b_{i2}b_{i2} &= \frac{1}{5}b_{i0} + \frac{2}{7}b_{i2} + \frac{18}{35}b_{i4}. \end{aligned}$$

If we retain only the elements of $B(t)$ in Eq. (18), then using Eq. (21) we get

$$B(t)B^T(t) =$$

$$\begin{bmatrix} b_{10} & b_{11} & b_{12} & & & & \\ b_{11} & \frac{1}{3}b_{10} + \frac{2}{3}b_{12} & \frac{2}{5}b_{11} & & & & \\ b_{12} & \frac{2}{5}b_{11} & \frac{1}{5}b_{10} + \frac{2}{7}b_{12} & & \bigcirc & & \\ & & \bigcirc & \ddots & & & \\ & & & & b_{40} & b_{41} & b_{42} \\ & & & & b_{41} & \frac{1}{3}b_{40} + \frac{2}{3}b_{42} & \frac{2}{5}b_{41} \\ & & & & b_{42} & \frac{2}{5}b_{41} & \frac{1}{5}b_{40} + \frac{2}{7}b_{42} \end{bmatrix}. \quad \blacksquare$$

By using the vector C in Eq. (17) the 12×12 matrix $\tilde{C}$ in Eq. (16) is

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 & 0 & 0 \\ 0 & \tilde{C}_2 & 0 & 0 \\ 0 & 0 & \tilde{C}_3 & 0 \\ 0 & 0 & 0 & \tilde{C}_4 \end{bmatrix}$$

where $\tilde{C}_i, \;\; i = 1, 2, 3, 4$ are 3×3 matrices given by

$$\tilde{C}_i = \begin{bmatrix} c_{i0} & c_{i1} & c_{i2} \\ \frac{1}{3}c_{i1} & c_{i0} + \frac{2}{5}c_{i2} & \frac{2}{3}c_{i1} \\ \frac{1}{5}c_{i2} & \frac{2}{5}c_{i1} & c_{i0} + \frac{2}{7}c_{i2} \end{bmatrix}.$$

3.1.3 The Performance Index Approximation.

Using Eqs. (14)-(16) the performance index J can be approximated as follows:

$$J = \int_0^1 \left(\frac{1 + B^T(t)\tilde{C}C}{1 - C^T P B(t)} \right)^{\frac{1}{2}} dt.$$

Divide the interval $[0,1]$ into K equal subinterval, we have

$$J = \sum_{i=1}^{K} \int_{\frac{i-1}{K}}^{\frac{i}{K}} \left(\frac{1 + B^T(t)\tilde{C}C}{1 - C^T P B(t)} \right)^{\frac{1}{2}} dt. \tag{22}$$

In order to use Gaussian integration formula we transform the t-interval $(\frac{i-1}{K}, \frac{i}{K})$ into the τ interval $(-1,1)$ by means of the transformation

$$t = \frac{1}{2}(\frac{1}{K}\tau + \frac{2i-1}{K}). \tag{23}$$

The optimal control in Eqs. (9)-(11) is then restated as follows:

Minimize

$$J = \frac{1}{2} \int_{-1}^{1} \left[\frac{1 + u^2(\tau)}{1 - x(\tau)} \right]^{\frac{1}{2}} d\tau, \tag{24}$$

subject to

$$\frac{dx}{d\tau} = \frac{1}{2} u(\tau), \tag{25}$$

with

$$x(-1) = 0, \quad x(1) = -0.5 \tag{26}$$

Using Eqs. (22) and (23) we get

$$J = \sum_{i=1}^{K} \frac{1}{2K} \int_{-1}^{1} \left(\frac{1 + B^T(\frac{1}{2}(\frac{1}{K}\tau + \frac{2i-1}{K}))\tilde{C}C}{1 - C^T P B(\frac{1}{2}(\frac{1}{K}\tau + \frac{2i-1}{K}))} \right)^{\frac{1}{2}} d\tau. \tag{27}$$

Using Gaussian integration formula, Eq. (27) can be approximated as

$$J \approx \sum_{i=1}^{K} \frac{1}{2K} \sum_{j=0}^{s} \left(\frac{1 + B^T(\frac{1}{2}(\frac{1}{K}\tau_j + \frac{2i-1}{K}))\tilde{C}C}{1 - C^T P B(\frac{1}{2}(\frac{1}{K}\tau_j + \frac{2i-1}{K}))} \right)^{\frac{1}{2}} w_j, \tag{28}$$

where $\tau_j, j = 0, 1, \dots, s$ are the $s+1$ zeros of Legendre polynomials $P_{s+1}$and w_j are the corresponding weights, given in [16] . The idea behind the above approximation is the exactness of the Gaussian integration formula for polynomials of degree not exceeding $2s+1$.

3.1.4 Evaluating the Vector C.

The optimal control problem has been reduced to a parameter optimization problem which can be stated as follows.

Find c_{nm}, $n = 1, 2, \cdots, N$, $m = 0, 1, \dots, M-1$ that minimizes Eq. (28) subject to

$$x(-1) = 0, \quad x(1) = -0.5 \tag{29}$$

We now minimize Eq. (28) subject to Eq. (29) using the Lagrange multiplier technique. Suppose

$$J^* = J + \lambda_1 x(-1) + \lambda_2 [x(1) + 0.5].$$

The necessary conditions for minimum are

$$\frac{\partial J^*}{\partial c_{nm}} = 0 \quad n = 1, 2, \cdots, N, \quad m = 0, 1, \dots, M-1 \tag{30}$$

and

$$\frac{\partial J^*}{\partial \lambda_1} = 0, \qquad \frac{\partial J^*}{\partial \lambda_2} = 0. \tag{31}$$

Eqs. (30) and (31) give $(NM+2)$ non-linear equations which can be solved for c_{nm}, λ_1 and λ_2 using Newton's iterative method. The initial values required to start Newton's iterative method have been chosen by taking $x(\tau)$ as linear function between $x(-1) = 0$ and $x(1) = -0.5$. In Table 1 the results for Hybrid functions approximation with $K = 2$, $N = 4$, $s = 5$ and $M = 2, 4, 5$ together with $K = 2$, $N = 4$, $s = 8$ and $M = 5$ are listed, we compare the solution obtained using the proposed method with other solutions in the literature together with the exact solution.

Methods	$x(1)$	$u(-1)$	J
Dynamic programming gradient method[2]	-0.5	-0.7832273	0.9984988
Dynamic programming successive sweep method[3,4]	-0.5	-0.7834292	0.9984989
Chebyshev solutions[5]			
M = 4	-0.5	-0.7844893	0.9984982
M = 7	-0.5	-0.7864215	0.99849815
M = 10	-0.5	-0.7864406	0.9984981483
Hybrid Functions, K=2, N=4, s=5			
M = 2	-0.5	-0.7852418	0.9985049
M = 4	-0.5	-0.7864397	0.9984980
M = 5	-0.5	-0.7864402	0.9984981
Hybrid Functions K=2, N=4, s=8 and M=5	-0.5	-0.7864408	0.99849814829
Exact Solution[4]	-0.5	-0.7864408	0.99849814829

Table 1. The hybrid functions and other solutions in the literature for Example 1.

3.2 Example 2, Higher-Order Nonlinear Problem.

Consider the functional

$$J(X) = \int_0^1 (\frac{1}{3}e^{-t}\dot{X}_1^3(t) + \frac{1}{2}\dot{X}_2^2(t) + \frac{1}{4}\dot{X}_3^4(t) + \frac{1}{2}e^{-2t}X_1^4(t)$$
$$+ 48e^{6t}X_3(t) - e^{-t}X_3(t)X_1(t) - t\dot{X}_3(t) - X_2(t)sint)dt \tag{32}$$

subject to the following boundary conditions

$$X(0) = (1,0,1)^T \quad , \quad X(1) = (e, sin1, e^2)^T. \tag{33}$$

The problem is to find the minimum of Eq. (32) subject to Eq. (33). The exact solution to this problem is

$$X(t) = (X_1(t), X_2(t), X_3(t))^T = (e^t, sint, e^{2t})^T.$$

Here we solve this problem with hybrid functions. Let

$$\dot{X}_1(t) = C_1^T B(t), \quad \dot{X}_2(t) = C_2^T B(t), \quad \dot{X}_3(t) = C_3^T B(t), \tag{34}$$

using Eqs. (7) and (33) we have

$$X_1(t) = C_1^T PB(t) + 1, \quad X_2(t) = C_2^T PB(t), \quad X_3(t) = C_3^T PB(t) + 1. \tag{35}$$

Using Eqs. (32), (34) and (35) we have

$$\begin{aligned} J(X) = \int_0^1 & (\frac{1}{3}e^{-t}(C_1^T \Psi(t))^3 + \frac{1}{2}(C_2^T B(t))^2 + \frac{1}{4}(C_3^T B(t))^4 + \frac{1}{2}e^{-2t}(C_1^T PB(t) + 1)^4 \\ & + 48e^{6t}(C_3^T PB(t) + 1) - e^{-t}(C_3^T PB(t) + 1)(C_1^T PB(t) + 1) - t(C_2^T B(t)) - C_2^T PB(t) sint) dt \end{aligned} \tag{36}$$

Eq. (36) is solved similarly to Example 1. In Table 2, a comparison is made by using hybrid functions approximations for $K = 2$, $N = 4$, $s = 5$ and $M = 4$ together with the exact solutions.

t	Hybrid Functions	Exact
0.0	$(1, 0, 1)^T$	$(1, 0, 1)^T$
0.1	$(1.10513, 0.09982, 1.22142)^T$	$(1.10517, 0.09983, 1.22140)^T$
0.2	$(1.22142, 0.19867, 1.49186)^T$	$(1.22140, 0.19866, 1.49182)^T$
0.3	$(1.34982, 0.29553, 1.82215)^T$	$(1.34986, 0.29552, 1.82212)^T$
0.4	$(1.49185, 0.38942, 2.22558)^T$	$(1.49182, 0.38941, 2.22554)^T$
0.5	$(1.64874, 0.47942, 2.71825)^T$	$(1.64872, 0.47942, 2.71828)^T$
0.6	$(1.82215, 0.56462, 3.32016)^T$	$(1.82212, 0.56464, 3.32012)^T$
0.7	$(2.01379, 0.64422, 4.0555)^T$	$(2.01375, 0.64421, 4.0552)^T$
0.8	$(2.22553, 0.71733, 4.95306)^T$	$(2.22554, 0.71735, 4.95303)^T$
0.9	$(2.65962, 0.78331, 6.04962)^T$	$(2.45960, 0.78332, 6.04965)^T$
1	$(2.71828, 0.84147, 7.38906)^T$	$(2.71828, 0.84147, 7.38906)^T$

Table 2. Estimated and exact values of $X_i(t)$, $\quad i = 1, 2, 3$, for Example 2.

4. CONCLUSION.

The aim of present work is to develop an efficient and accurate method for solving nonlinear problems of the calculus of variations. The problem has been reduced to solving a system of nonlinear algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. The advantages of using the hybrid functions method are:

(1) The operational matrix P contains many zeros which plays an important rule in simplifying the performance index.

(2) The Gaussian integration formula is exact for polynomials of degree not exceeding $2s+1$

(3) Only a small number of K, N, s and M are needed to obtain a very satisfactory results.

REFERENCES.

1. V.M. Tikhomirov, *Stories about maxima and minima,* American Mathematical Society, Providence, RI, (1990).

2. R. Bellman, *Dynamic programming*, Princeton University Press , NJ, (1957).

3. A. V. Balakrishnan and L.W. Neustadt, *Computing methods in optimization problems*, Academic Press, New York, (1964).

4. A.E. Bryson and Y.C. Ho, *Applied optimal control*, Blaisdell Waltham, (1969).

5. J. Vlassenbroeck and R. Van Dooren, A new look at the Brachistochrone problem, *Journal of Applied Mathematics and Physics (ZAMP)* **31**, 785–790, (1980).

6. D.S. Szarkowicz, Investigating the brachistochrone with a multistage Monte Carlo method *Interna- tional Journal of Systems Science*, **26**, 233–243, (1995).

7. C.F., Chen, and C.H. Hsiao, C. H., A Walsh series direct method for solving variational problems, *Journal of the Franklin Institute*, **300**, 265–280, (1975).

8. C.F. Chen, Y. T., Tsay, and T.T. Wu, Walsh operational matrices for fractional calculus and their application to distributed-parameter systems , *Journal of*

the Franklin Institute, **503**, 267–284, (1977).

9. C. Hwang, and Y.P. Shih, Solution of integral equations via Laguerre polynomials , *Journal of Computer and Electrical Engineering*, **9**, 123–129 (1982).

10. R.Y. Chang, and M.L. Wang, Shifted Legendre direct method for variational problems, *Journal of Optimization Theory and Applications*, **39**, 299–307, (1983).

11. S.C. Sinha and D.H. Wu, An efficient computational scheme for the analysis of periodic systems, *Journal of Sound and Vibration*, **151**, 91-117, (1991).

12. T.H. Moulden and M.A. Scott, Walsh Spectral analysis for ordinary differential equations: Part1-Initial value problems *IEEE Trans. Circuits Syst.* **35**, 742-745, (1989).

13. M. Razzaghi and J. Nazarzadeh, Optimum pulse-width modulated patterns in induction motors using Walsh functions *Electric Power Systems Research* **35**, 87–91, (1995.

14. J. Gregory and R.S. Wang, Discrete variable methods for the m-dependent variable nonlinear extremal problems in the calculus of variations, *SIAM J. Numerical Analysis*, **2** 470–487 (1990).

15. P. Dyer and S.R. Mcreynolds, *The computational and theory of optimal control*, Academic Press : New York, (1970).

16. A. Constantinides, *Applied numerical methods with personal computers*, McGraw-Hill , New York, (1987).