Output feedback tracking for nonlinear systems

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Abstract: *In this paper, we firstly present a tracking-error observer for a class of nonlinear systems based on the input output linearization. While the previous result presented observer for nonlinear systems of full relative degree, we proposed a procedure for the design of nonlinear tracking-error observer which do not require the hypothesis of full relative degree. Assuming that there exists a global tracking-error observer for internal dynamics and that some functions are globally Lipschitz, we can design a globally convergent observer with strong robustness. Then, we address the problem of output feedback tracking of single-input-single-output nonlinear plants. The proposed approach is based on continuous sliding manifold. The resulting controller, a continuously high-gain sliding mode controller, exhibits strong robustness properties, chattering phenomena can be avoided. Moreover, the error between plant output and referential signal converges rapidly due to introducing a perturbation parameter.*

Key words: *tracking-error observer, nonlinear system, sliding mode, chattering phenomena, internal dynamics*

1. Introduction

In output tracking for nonlinear system, the error and its derivatives up to order r between the plant output and referential signal play important roles. The problem of observing the error between the plant output and referential signal of a nonlinear system has been considered in the literature. Some sufficient conditions for the existence of an observer have been established, and computational algorithms for construction of the observer have been presented. The observer problem is to design a dynamical system which asymptotically estimates the state of a given plant using the input and output information of the plant. In contrast to the case for linear systems, the nonlinear observer problem has not yet been fully solved in the general sense, but several design methods have been proposed for particular classes of nonlinear systems. We known the well-known approach of linearized error dynamics (Krener & Isidori, 1983; Banaszuk & Sluis, 1997; Hou & Pugh, 1999), where the nonlinearities of the plant are canceled out in the error dynamics so that the applicable class of systems is quite restricted, While Shim, Seo and Teel (2003) proposed a method directly handles those nonlinearities in the plant. Bestle and Zeitz (1983) introduced a nonlinear observer canonical form in which system nonlinearities depend only on the input and output of the original system. To broaden the class of nonlinear systems for which a state observer exists, Keller (1987) presented an observer design based on a transformation into a generalized observer canonical form that depends on the first n time derivatives of the input variables. Since afore-mentioned approaches require quite restrictive conditions on coordinate transformation, the problem of deriving approximate observers has been also studied (Baumann & Rugh, 1986; Nicosia, Tomei & Tornambe, 1989; Zeitz, 1987). On the other hand, a nonlinear observer is not robust in general to measurement disturbances in the sense that arbitrarily small disturbance may result in a blowup of error state.

Sliding manifold approaches have been using for years in control application. There are many reasons for the successful application of these strategies, among which we cite their ability to deal with the control of nonlinear plants and their strong robustness properties, with respect to unmodeled dynamics and exogenous unknown disturbances. The key idea is to formulate closed-loop system performances in terms of a desired behavior of the system state and to represent this behavior as a set of constraints to satisfied. The method of linearization is adopted for nonlinear systems (Sastry, S. S. & A. Isidori.,1989; Byrnes, C. I., & Isidori, A., 1991), some controlling tools are used thereafter (Chunjiang Qian, & Wei Lin, 2001; V. O. Nikiforov, 2001;

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Kokotovic, P. & M. Arcak, 2001). Seungrohk OH and Hassan K. Khalil (1997) present a nonlinear output feedback tracking using high-gain observer and variable structure control. Usual observers can't observe the external and internal dynamics at the same time. Some sliding mode controllers lead to the phenomenon of chattering (C. Edwards, & S.K. Spurgeon, 1998). With the introduction of saturation functions, the phenomenon of chattering can be avoided, however, the convergent velocity and precision decrease (Kokotovic, P. & M. Arcak, 2001)。

In this paper, we firstly propose a global nonlinear tracking-error observer that guarantees the estimation error to converge to zero asymptotically with strong robustness. It is based on the input output linearization technique and utilizes the error transformation into the normal form, the proposed condition is reduced to that the zero dynamics have a locally exponentially stable equilibrium at the origin. At the same time, a continuous sliding controller is designed based on the property of the solution a one-order differential equation reaching zero in a finite time and keeping invariant, and a perturbation parameter is introduced in the defined sliding variable, therefore, the controller can make the error between plant output and referential signal converge to zero rapidly. This paper is organized as follows. In section 2, the problem is precisely formulated. In section 3, tracking-error observer design and analysis of stability are obtained. In section 4, the sliding mode controller is designed. In section 5, conclusion is presented, and in section 6, simulation is given.

2. Problem statement

We consider a class of nonlinear systems, represented by

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1}$$

where, $x \in R^n$ is state variable, $u \in R^1$ is input, $y = h(x) \in R^1$ is output, $f(x)$ and $g(x)$ are smooth vectors.

Assumption 1: System (1) has uniform relative degree $r \le n$, i.e.,

$$L_g L_f^{i-1} h(x) = 0, i = 1, \cdots, r-1$$

$$L_g L_f^{r-1} h(x) \ne 0 \tag{2}$$

There exist *n-r* functions $\eta_i(x), 1 \le i \le n-r$, such that

$$\chi(x) := \begin{bmatrix} h(x) & L_f h(x) & \cdots & L_f^{r-1} h(x) & \chi_1(x) & \cdots & \chi_{n-r}(x) \end{bmatrix}^{\mathrm{T}} = [\xi \quad \eta]^T = [\xi_1 \quad \cdots \quad \xi_r \quad \eta_1 \quad \cdots \quad \eta_{n-r}]^{\mathrm{T}} \tag{3}$$

is a global diffeomorphism with $\chi(x(0)) = 0$, and it transforms (1) into

$$\begin{cases} \dot{\xi} = A\xi + B[a(\xi,\eta) + b(\xi,\eta)u] \\ \dot{\eta} = q(\xi,\eta) \\ y = C^{\mathrm{T}}\xi \end{cases} \tag{4}$$

where,

$$a(\xi,\eta) = L_f^r h(\chi^{-1}(\xi,\eta)), b(\xi,\eta) = L_g L_f^{r-1} h(\chi^{-1}(\xi,\eta)) \ne 0 \tag{5}$$

$$A=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&0\end{bmatrix}_{r\times r},\quad B=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}_{r\times 1},\quad C=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}_{r\times 1} \tag{6}$$

Let $e=[e_1\cdots e_r]^{\mathrm{T}}=\left[\xi_1-y_d\cdots\xi_r-y_d^{(r-1)}\right]^{\mathrm{T}}$, $Y_d=\left[y_d\quad\cdots\quad y_d^{(r-1)}\right]^{\mathrm{T}}$, y_d is referential signal, and $y_d,\dot{y}_d,\cdots,y_d^{(r)}$ are measurable.

Here, we introduce an item of disturbance $d(t)$. Therefore, the error system between plant output and referential signal is written as

$$\begin{cases}\dot{e}=Ae+B\left[a(e+Y_d,\eta)-y_d^{(r)}+b(e+Y_d,\eta)u+d(t)\right]\\ \dot{\eta}=q(e+Y_d,\eta)\end{cases} \tag{7}$$

where, $|d(t)|\le l_4$, l_4 is a positive constant.

Assumption 2: There exist a positive definite matrix $P_2\in R^{(n-r)\times(n-r)}$ and a positive constant k_0 such that

$$v^{\mathrm{T}}P_2\left\{\frac{\partial q}{\partial\eta}(e+Y_d,\eta)\right\}v\le -k_0\|v\|^2\quad \forall(e+Y_d,\eta,v)\in R^r\times R^{n-r}\times R^{n-r} \tag{8}$$

If the zero dynamics of (1) are locally exponentially stable, then Assumption 2 is fulfilled at least locally. In fact, suppose that the origin of system

$$\dot{\eta}=q(0,\eta)$$

is locally exponentially stable. Then, it follows from Lyapunov converse theorem (Hahn, 1967) that

$$\frac{\partial q(0,0)}{\partial\eta}$$

is a Hurwitz matrix. Thus, there exist a positive constant k and a positive definite matrix P such that

$$v^{\mathrm{T}}P_2\left\{\frac{\partial q}{\partial\eta}(0,0)\right\}v=-k\|v\|^2$$

By the continuity of $(\partial q/\partial\eta)(e+Y_d,\eta)$, there exists a $\delta>0$ such that

$$v^{\mathrm{T}}P_2\left\{\frac{\partial q}{\partial\eta}(e+Y_d,\eta)\right\}v\le -\frac{1}{2}k\|v\|^2,\ \forall\|e+Y_d\|<\delta,\|\eta\|<\delta,v\in R^{n-r}$$

Assumption 3: The functions $a(e+Y_d,\eta)$ and $b(e+Y_d,\eta)$ are globally Lipschitzian, and u is bounded, i.e., there exist positive constant l_1 and l_2 such that

$$\|a(\tau_1+Y_d,\upsilon_1)+b(\tau_1+Y_d,\upsilon_1)u-a(\tau_2+Y_d,\upsilon_2)-b(\tau_2+Y_d,\upsilon_2)u\|\le l_1\|\tau_1-\tau_2\|+l_2\|\upsilon_1-\upsilon_2\| \tag{9}$$

for all $\tau_1, \tau_2 \in R^r, \upsilon_1, \upsilon_2 \in R^{n-r}$.

Assumption 4: $q(e+Y_d,\eta)$ is globally Lipschitz in $(e+Y_d)$, uniformly in η, i.e., there exists a positive constant l_3 such that

$$\left\|q(\tau_1+Y_d,\eta)-q(\tau_2+Y_d,\eta)\right\| \le l_3\left\|\tau_1-\tau_2\right\|,\ \forall \tau_1,\tau_2 \in R^r, \eta \in R^{n-r} \tag{10}$$

3. Error observer design and analysis of stability

Theorem 1: If the tracking-error observer for the system (7) is selected as

$$\begin{cases} \dot{\hat{e}} = A\hat{e} + B\left[a(\hat{e}+Y_d,\hat{\eta}) - y_d^{(r)} + b(\hat{e}+Y_d,\hat{\eta})u\right] + K(\varepsilon)\left(e_1 - C^{\mathrm{T}}\hat{e}\right) \\ \dot{\hat{\eta}} = q(\hat{e}+Y_d,\hat{\eta}) \end{cases} \tag{11}$$

then, there exists ε, for all $0<\varepsilon<\varepsilon^*$, such that

$$\left\|\hat{e}-e\right\| \le 8\varepsilon \cdot l_4\left\|P_1\right\|,\ \left\|\hat{\eta}-\eta\right\| \le \frac{4\sqrt{\varepsilon}}{\sqrt{3k_0}} l_4 \left\|P_1\right\| \quad \text{as } t\to\infty \tag{12}$$

where

$$\varepsilon^* = \min\left(\frac{1}{4l_1\left\|P_1\right\|}, \frac{k_0}{16\left(l_2\left\|P_1\right\|+l_3\left\|P_2\right\|\right)^2}\right) \tag{13}$$

$$K(\varepsilon) = \begin{bmatrix} \dfrac{k_1}{\varepsilon} & \cdots & \dfrac{k_r}{\varepsilon^r} \end{bmatrix}^{\mathrm{T}} \tag{14}$$

P_1 is the solution of

$$A_1^T P_1 + P_1 A_1 = -I_r \tag{15}$$

$A_1 = \begin{bmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -k_r & 0 & 0 & \cdots & 0 \end{bmatrix}$, $s^r + k_1 s^{r-1} + \cdots + k_r = 0$ is a Hurwitz polynomial.

Proof:

Define

$$z_1 = \hat{e} - e, z_2 = \hat{\eta} - \eta \tag{16}$$

Then, it follows from (7) and (11) that

$$\dot{z}_1 = \left(A - K(\varepsilon)C^{\mathrm{T}}\right)z_1 + B\left[a(\hat{e}+Y_d,\hat{\eta}) + b(\hat{e}+Y_d,\hat{\eta})u - a(e+Y_d,\eta) - b(e+Y_d,\eta)u - d(t)\right]$$
$$\dot{z}_2 = q(\hat{e}+Y_d,\hat{\eta}) - q(e+Y_d,\eta)$$

which can be written as

$$\begin{aligned} \dot{z}_1 &= A_1(\varepsilon)z_1 + B\left[a(\hat{e}+Y_d,\hat{\eta}) + b(\hat{e}+Y_d,\hat{\eta})u - a(e+Y_d,\eta) - b(e+Y_d,\eta)u - d(t)\right] \\ \dot{z}_2 &= q(\hat{e}+Y_d,\hat{\eta}) - q(\hat{e}+Y_d,\eta) + q(\hat{e}+Y_d,\eta) - q(e+Y_d,\eta) \end{aligned} \tag{17}$$

where

$$A_1(\varepsilon) = A - K(\varepsilon)C^{\mathrm{T}} = \begin{bmatrix} -k_1/\varepsilon & 1 & 0 & \cdots & 0 \\ -k_2/\varepsilon^2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -k_r/\varepsilon^r & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{18}$$

Define

$$\Xi(\varepsilon) := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \varepsilon & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon^{r-1} \end{bmatrix} \tag{19}$$

Then, we can obtain that

$$A_1(\varepsilon) = \varepsilon^{-1}\Xi(\varepsilon)^{-1} A_1 \Xi(\varepsilon) \tag{20}$$

From (15), we define

$$P_1(\varepsilon) := \Xi(\varepsilon)^{\mathrm{T}} P_1 \Xi(\varepsilon) \tag{21}$$

therefore, we have

$$A_1^{\mathrm{T}}(\varepsilon)P_1(\varepsilon) + P_1(\varepsilon)A_1(\varepsilon) = -\varepsilon^{-1}\Xi(\varepsilon)^{\mathrm{T}}\Xi(\varepsilon) \tag{22}$$

Define Lyapunov function candidate

$$V(z_1, z_2) = \varepsilon^{(2-2r)} z_1^{\mathrm{T}} P(\varepsilon) z_1 + z_2^{\mathrm{T}} P_2 z_2 \tag{23}$$

where, P_2 is given by (8). Then, taking (8) and (22) into consideration, we have

$$\begin{aligned}
\dot{V} &= \varepsilon^{(2-2r)}\left\{z_1^{\mathrm{T}} A_1^{\mathrm{T}}(\varepsilon) + B^{\mathrm{T}}\left[a(\hat{e}+Y_d,\hat{\eta}) + b(\hat{e}+Y_d,\hat{\eta})u - a(e+Y_d,\eta) - b(e+Y_d,\eta)u - d(t)\right]\right\}P_1(\varepsilon)z_1 \\
&+ \varepsilon^{(2-2r)} z_1^{\mathrm{T}} P_1(\varepsilon)\left\{A_1(\varepsilon)z_1 + B\left[a(\hat{e}+Y_d,\hat{\eta}) + b(\hat{e}+Y_d,\hat{\eta})u - a(e+Y_d,\eta) - b(e+Y_d,\eta)u - d(t)\right]\right\} \\
&+ \left\{q(\hat{e}+Y_d,\hat{\eta}) - q(\hat{e}+Y_d,\eta) + q(\hat{e}+Y_d,\eta) - q(e+Y_d,\eta)\right\}^{\mathrm{T}} P_2 z_2 \\
&+ z_2^{\mathrm{T}} P_2 \left\{q(\hat{e}+Y_d,\hat{\eta}) - q(\hat{e}+Y_d,\eta) + q(\hat{e}+Y_d,\eta) - q(e+Y_d,\eta)\right\} \\
&\le \varepsilon^{(2-2r)} z_1^{\mathrm{T}}\left(A_1^{\mathrm{T}}(\varepsilon)P_1(\varepsilon) + P_1(\varepsilon)A_1(\varepsilon)\right)z_1 \\
&+ 2\varepsilon^{(2-2r)} B^{\mathrm{T}} P_1(\varepsilon) z_1\left[a(\hat{e}+Y_d,\hat{\eta}) + b(\hat{e}+Y_d,\hat{\eta})u - a(e+Y_d,\eta) - b(e+Y_d,\eta)u - d(t)\right] \\
&- k_0\|z_2\|^2 + 2\left[q(\hat{e}+Y_d,\eta) - q(e+Y_d,\eta)\right]^{\mathrm{T}} P_2 z_2 \\
&= -\varepsilon^{(1-2r)} z_1^{\mathrm{T}}\Xi(\varepsilon)^{\mathrm{T}}\Xi(\varepsilon)z_1 \\
&+ 2\varepsilon^{(2-2r)} B^{\mathrm{T}}\Xi(\varepsilon)P_1\Xi(\varepsilon)z_1\left[a(\hat{e}+Y_d,\hat{\eta}) + b(\hat{e}+Y_d,\hat{\eta})u - a(e+Y_d,\eta) - b(e+Y_d,\eta)u - d(t)\right] \\
&- k_0\|z_2\|^2 + 2\left[q(\hat{e}+Y_d,\eta) - q(e+Y_d,\eta)\right]^{\mathrm{T}} P_2 z_2
\end{aligned}$$

Set $\varsigma_1 = \Xi(\varepsilon)z_1$, using $\left\|B^{\mathrm{T}}\Xi(\varepsilon)\right\| = \varepsilon^{r-1}$ together with (9) and (10), we have

$$\dot{V} \le -\varepsilon^{(1-2r)}\|\varsigma_1\|^2 + 2\varepsilon^{(2-2r)}\varepsilon^{(r-1)}\|P_1\|\|\varsigma_1\|\left(l_1\|z_1\| + l_2\|z_2\| + l_4\right) - k_0\|z_2\|^2 + 2l_3\|z_1\|\|P_2\|\|z_2\|$$
$$= -\varepsilon^{(1-2r)}\|\varsigma_1\|^2 + 2\varepsilon^{(1-r)}l_1\|P_1\|\|\varsigma_1\|\|z_1\| + 2\varepsilon^{(1-r)}l_2\|P_1\|\|\varsigma_1\|\|z_2\| + 2\varepsilon^{(1-r)}l_4\|P_1\|\|\varsigma_1\| - k_0\|z_2\|^2 + 2l_3\|z_1\|\|P_2\|\|z_2\|$$

Because of $\left\|\Xi(\varepsilon)^{-1}\right\| = \varepsilon^{1-r}$, and $z_1 = \Xi(\varepsilon)^{-1}\varsigma_1$, it follows that $\|z_1\| \le \varepsilon^{1-r}\|\varsigma_1\|$, which leads to

$$\dot{V} \le -\varepsilon^{(1-2r)}\|\varsigma_1\|^2 + 2\varepsilon^{(2-2r)}l_1\|P_1\|\|\varsigma_1\|^2 + 2\varepsilon^{(1-r)}l_2\|P_1\|\|\varsigma_1\|\|z_2\| + 2\varepsilon^{(1-r)}l_4\|P_1\|\|\varsigma_1\| - k_0\|z_2\|^2$$
$$+ 2\varepsilon^{(1-r)}l_3\|P_2\|\|\varsigma_1\|\|z_2\|$$

Since $\varepsilon \le 1/\left(4l_1\|P_1\|\right)$, we have

$$\dot{V} \le -\varepsilon^{(1-2r)}\|\varsigma_1\|^2 + 2\varepsilon^{(2-2r)}l_1\frac{1}{4l_1\varepsilon}\|\varsigma_1\|^2 + 2\varepsilon^{(1-r)}l_2\|P_1\|\|\varsigma_1\|\|z_2\| + 2\varepsilon^{(1-r)}l_4\|P_1\|\|\varsigma_1\| - k_0\|z_2\|^2$$
$$+ 2\varepsilon^{(1-r)}l_3\|P_2\|\|\varsigma_1\|\|z_2\|$$
$$= -\frac{1}{2}\varepsilon^{(1-2r)}\|\varsigma_1\|^2 + 2\varepsilon^{(1-r)}\left(l_2\|P_1\| + l_3\|P_2\|\right)\|\varsigma_1\|\|z_2\| - k_0\|z_2\|^2 + 2\varepsilon^{(1-r)}l_4\|P_1\|\|\varsigma_1\|$$
$$= -\frac{3}{4}k_0\|z_2\|^2 - k_0\left(\frac{1}{2}\|z_2\| - \frac{2}{k_0}\varepsilon^{(1-r)}\left(l_2\|P_1\| + l_3\|P_2\|\right)\|\varsigma_1\|\right)^2$$
$$-\left(\frac{1}{2}\varepsilon^{(1-2r)} - \frac{4}{k_0}\varepsilon^{(2-2r)}\left(l_2\|P_1\| + l_3\|P_2\|\right)^2\right)\|\varsigma_1\|^2 + 2\varepsilon^{(1-r)}l_4\|P_1\|\|\varsigma_1\|$$
$$\le -\frac{3}{4}k_0\|z_2\|^2 - \left(\frac{1}{2}\varepsilon^{(1-2r)} - \frac{4}{k_0}\varepsilon^{(2-2r)}\left(l_2\|P_1\| + l_3\|P_2\|\right)^2\right)\|\varsigma_1\|^2 + 2\varepsilon^{(1-r)}l_4\|P_1\|\|\varsigma_1\|$$

Moreover, since $\varepsilon \le \dfrac{k_0}{16\left(l_2\|P_1\| + l_3\|P_2\|\right)^2}$, we have

$$\dot{V} \le -\frac{3}{4}k_0\|z_2\|^2 - \frac{1}{4}\varepsilon^{(1-2r)}\|\varsigma_1\|^2 + 2\varepsilon^{(1-r)}l_4\|P_1\|\|\varsigma_1\| \tag{24}$$

therefore, we can transform (24) into

$$\dot{V} \le -\frac{3}{4}k_0\|z_2\|^2 - \frac{1}{4}\varepsilon^{(1-r)}\|\varsigma_1\|\left(\frac{\|\varsigma_1\|}{\varepsilon^r} - 8l_4\|P_1\|\right) \tag{25}$$

If $\dfrac{\|\varsigma_1\|}{\varepsilon^r} > 8l_4\|P_1\|$, then $\dot{V} < 0$. Therefore, $\|\varsigma_1\| \le 8\varepsilon^r l_4\|P_1\|$ as $t \to \infty$. Since $\|z_1\| \le \varepsilon^{1-r}\|\varsigma_1\|$, we have

$\|z_1\| = \|\hat{e} - e\| \le 8\varepsilon \cdot l_4\|P_1\|$ as $t \to \infty$.

At the same time, we can rewrite (24) by another way, i.e.,

$$\dot{V} \le -\frac{1}{4}\varepsilon^{(1-2r)}\left(\|\varsigma_1\| - 4\varepsilon^r l_4\|P_1\|\right)^2 + 4\varepsilon \cdot l_4^2\|P_1\|^2 - \frac{3}{4}k_0\|z_2\|^2 \tag{26}$$

If $\|z_2\| > \frac{4\sqrt{\varepsilon}}{\sqrt{3k_0}} l_4 \|P_1\|$, then $\dot{V} < 0$. Therefore, $\|z_2\| = \|\hat{\eta} - \eta\| \le \frac{4\sqrt{\varepsilon}}{\sqrt{3k_0}} l_4 \|P_1\|$ as $t \to \infty$.

Which can conclude the proof. □

Remark:

From the theorem above, we can find that if ε is sufficiently small, then $\|\hat{e} - e\| \to 0$, $\|\hat{\eta} - \eta\| \to 0$ as $t \to \infty$.

On the other hand, if $d(t) = 0$, then $l_4 = 0$, and from (24), we have

$$\dot{V} \le -\frac{3}{4} k_0 \|z_2\|^2 - \frac{1}{4} \varepsilon^{(1-2r)} \|\varsigma_1\|^2 \tag{27}$$

Therefore, when $0 < \varepsilon < \varepsilon^*$, $\varepsilon^* = \min\left(\frac{1}{4l_1 \|P_1\|}, \frac{k_0}{16\left(l_2 \|P_1\| + l_3 \|P_2\|\right)^2} \right)$, we can obtain

$$\|\hat{e} - e\| \to 0, \|\hat{\eta} - \eta\| \to 0 \quad \text{as} \quad t \to \infty$$

If $\dot{y}_d, \cdots, y_d^{(r)}$ are not measurable, we estimate $\dot{y}_d, \cdots, y_d^{(r)}$ by a linear observer (see for example, Khalil, 1994) as follow.

$$\dot{\hat{Y}}_{d+1} = A(\varepsilon)\hat{Y}_{d+1} + L(\varepsilon) y_d \tag{28}$$

where

$$\hat{Y}_{d+1} = \begin{bmatrix} \hat{y}_{d_1} & \cdots & \hat{y}_{d_r} & \hat{y}_{d_{r+1}} \end{bmatrix}^{\mathrm{T}} \tag{29}$$

$$A(\varepsilon) = \begin{bmatrix} -l_1/\varepsilon & 1 & 0 & \cdots & 0 \\ -l_2/\varepsilon^2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -l_{r+1}/\varepsilon^{r+1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad L(\varepsilon) = \begin{bmatrix} l_1/\varepsilon \\ \vdots \\ l_{r+1}/\varepsilon^{r+1} \end{bmatrix} \tag{31}$$

$s^{r+1} + l_1 s^r + \cdots + l_{r+1} = 0$ is a Hurwitz polynomial. □

4. Design of controller

Consider the differential equation below.

$$\dot{\sigma}(t) = -k_\alpha \sigma(t) - k_\beta \sigma(t)^{\frac{q}{p}} \tag{32}$$

Where, $k_\alpha, k_\beta > 0$, $p > q > 0$, and p, q are all odd numbers. The solution of (32) is

$$k_\alpha \sigma(t)^{\frac{p-q}{p}} + k_\beta = c \exp\left(-\frac{k_\alpha (p-q)}{p} t \right).$$

Where, $c = k_\alpha \sigma(t_0)^{\frac{p-q}{p}} + k_\beta$. When $\sigma(t_s) = 0$, we can obtain that $t_s = \frac{p}{k_\alpha(p-q)} \ln \frac{k_\alpha \sigma(t_0)^{\frac{p-q}{p}} + k_\beta}{k_\beta} + t_0$.

A conclusion can be drawn that $\sigma \equiv 0$ for $t \geq t_s$.

By the conclusion above, let the sliding variable be

$$\sigma(t) = \hat{e}_r + a_{r-1}\hat{e}_{r-1} + \cdots + a_2\hat{e}_2 + a_1\hat{e}_1 \tag{33}$$

where, $s^{r-1} + a_{r-1}s^{r-2} + \cdots + a_2 s + a_1 = 0$ is Hurwitz with respect to *s*, and sliding surface is selected as (32). We give a theorem in the following.

Theorem 2 For system (7), if the observer is selected as (11), and the controller is

$$\begin{aligned} u = & -b(\hat{e} + Y_d, \hat{\eta})^{-1}\Bigg[a(\hat{e} + Y_d, \hat{\eta}) + \left(\frac{k_r}{\varepsilon^r} + a_{r-1}\frac{k_{r-1}}{\varepsilon^{r-1}} + \cdots + a_2\frac{k_2}{\varepsilon^2} + a_1\frac{k_1}{\varepsilon^1} \right)(e_1 - \hat{e}_1) + \\ & (a_{r-1}\hat{e}_r + \cdots + a_2\hat{e}_3 + a_1\hat{e}_2) - y_d^{(r)} + k_\alpha\sigma(t) + k_\beta(\sigma(t))^{\frac{q}{p}} \Bigg] \end{aligned} \tag{34}$$

Then,

$$\|e(t)\| \leq k_p\sqrt{\varepsilon} \quad \text{as} \quad t \to \infty \tag{35}$$

where, k_p is a positive constant.

Proof:

$$\begin{aligned} \dot{\sigma} = & \dot{\hat{e}}_r + a_{r-1}\dot{\hat{e}}_{r-1} + \cdots + a_2\dot{\hat{e}}_2 + a_1\dot{\hat{e}}_1 \\ = & a(\hat{e} + Y_d, \hat{\eta}) + b(\hat{e} + Y_d, \hat{\eta})u - y_d^{(r)} + \frac{k_r}{\varepsilon^r}(e_1 - \hat{e}_1) + a_{r-1}\left(\hat{e}_r + \frac{k_{r-1}}{\varepsilon^{r-1}}(e_1 - \hat{e}_1) \right) + \cdots \\ & a_2\left(\hat{e}_3 + \frac{k_2}{\varepsilon^2}(e_1 - \hat{e}_1) \right) + a_1\left(\hat{e}_2 + \frac{k_1}{\varepsilon^1}(e_1 - \hat{e}_1) \right) \\ = & a(\hat{e} + Y_d, \hat{\eta}) + b(\hat{e} + Y_d, \hat{\eta})u - y_d^{(r)} + \left(\frac{k_r}{\varepsilon^r} + a_{r-1}\frac{k_{r-1}}{\varepsilon^{r-1}} + \cdots + a_2\frac{k_2}{\varepsilon^2} + a_1\frac{k_1}{\varepsilon^1} \right)(e_1 - \hat{e}_1) + \\ & a_{r-1}\hat{e}_r + \cdots + a_2\hat{e}_3 + a_1\hat{e}_2 = -k_\alpha\sigma(t) - k_\beta(\sigma(t))^{\frac{q}{p}} \end{aligned} \tag{36}$$

Therefore, there exists $t_s = \frac{p}{k_\alpha(p-q)} \ln \frac{k_\alpha \sigma(t_0)^{\frac{p-q}{p}} + k_\beta}{k_\beta} + t_0$ such that $\sigma = 0$ for $t \geq t_s$, then

$$\hat{e}_r + a_{r-1}\hat{e}_{r-1} + \cdots + a_2\hat{e}_2 + a_1\hat{e}_1 = 0 \tag{37}$$

From (7), (37), and let $z_1 = [z_{1,1}, \cdots, z_{1,r}]^{\mathrm{T}}$, we have

$$\begin{aligned} \dot{e}_{r-1} &= e_r = \hat{e}_r - z_{1r} = -(a_{r-1}\hat{e}_{r-1} + \cdots + a_2\hat{e}_2 + a_1\hat{e}_1) - z_{1,r} \\ &= -\{a_1(e_1 + z_{1,1}) + \cdots + a_{r-1}(e_{r-1} + z_{1,r-1})\} - z_{1,r} \\ &= -a_1e_1 - \cdots - a_{r-1}e_{r-1} - a_1z_{1,1} - \cdots a_{r-1}z_{1,r-1} - z_{1,r} \end{aligned} \tag{38}$$

Therefore, from (7) and (38), we have

$$\dot{\tilde{e}} = \tilde{A}\tilde{e} + Hz_1 \tag{39}$$

where, $\tilde{e} = [e_1, \cdots, e_{r-1}]^{\mathrm{T}}$, $\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -a_1 & -a_2 & \cdots & -a_{r-1} \end{bmatrix}, H = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -a_1 & \cdots & -a_{r-1} & 1 \end{bmatrix}$.

Because $\tilde{A}$ and $A - K(\varepsilon)C^{\mathrm{T}}$ are Hurwitz, for given matries Q_1, Q_2, there exist positive-defined matries P_3, P_4 such that

$$P_3\tilde{A} + \tilde{A}^{\mathrm{T}}P_3 = -Q_1, P_4A + A^{\mathrm{T}}P_4 = -Q_2 \tag{40}$$

Define $\Phi(\tilde{e}, z_1) = \tilde{e}^{\mathrm{T}}P_3\tilde{e} + z_1^{\mathrm{T}}P_4z_1$, and taking derivative along the solutions of (17) and (39) for $\Phi(\tilde{e}, z_1)$, we have

$$\dot{\Phi}(\tilde{e}, z_1) < -\eta_1\Phi(\tilde{e}, z_1), \quad \Phi(\tilde{e}, z_1) > r_1\varepsilon \tag{41}$$

where, η_1, r_1 are positive constants. Let $r_2 > r_1$, and define

$$\Omega = \{(\tilde{e}, z_1) | \Phi(\tilde{e}, z_1) \le r_2\varepsilon\} \tag{42}$$

Therefore, the exists a finite time t_1, for $t > t_1$, such that $\Phi(\tilde{e}, z_1) \in \Omega$. From (38) and $\|z_1\| \le 8\varepsilon \cdot l_4\|P_1\|$ in (12), we have $\lim_{t\to\infty}\|e(t)\| \le k_p\sqrt{\varepsilon}$, where k_p is a positive constant, which can conclude the proof. □

5. Simulation

Consider the nonlinear system as follow.

$$\dot{x} = \begin{bmatrix} x_1x_2 - x_1^3 \\ x_1 \\ -x_3 \\ x_1^2 + x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 + 2x_3 \\ 0 \\ 0 \end{bmatrix}u + M(x)w$$

$y = h(x) = x_4$, $M(x)$ is gain function of disturbance.

We can obtain

$$\frac{\partial h}{\partial x} = [0 \quad 0 \quad 0 \quad 1], L_g h(x) = 0, L_f h(x) = x_1^2 + x_2, L_g L_f h(x) = 2(1 + x_3),$$

$$L_f^2 h(x) = 2x_1^2x_2 - 2x_1^4 + x_1$$

The relative degree is 2.

Let $\xi_1 = h(x) = x_4$, $\xi_2 = L_f h(x) = x_2 + x_1^2$. In addition, we have $\eta_1 = x_3$, $\eta_2 = x_1$.

So the Jaccio matrix $\frac{\partial}{\partial x}\chi(x)=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 2x_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is not singular, and conserve transformation is

$$x_1 = \eta_2,\ x_2 = \xi_2 - \eta_2^2,\ x_3 = \eta_1,\ x_4 = \xi_1$$

The referential output $y_d = 2e^{-05t}\sin 0.5t + \cos t$, $|d(t)| \le 0.02$. let $e = e_1 = y - y_d$ .The error system is

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = \eta_2 + \eta_2\left(\eta_2\left(e_2 - \eta_2^2\right) - \eta_2^3\right) + (2 + 2\eta_1)u - y_d^{(2)} + d(t) \\ \dot{\eta}_1 = -\eta_1 \\ \dot{\eta}_2 = -2\eta_2^3 + e_2\eta_2 \end{cases}$$

Observer is selected as

$$\begin{cases} \begin{bmatrix} \dot{\hat{e}}_1 \\ \dot{\hat{e}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left[\hat{\eta}_2 + \hat{\eta}_2\left(\hat{\eta}_2\left(\hat{e}_2 - \hat{\eta}_2^2\right) - \hat{\eta}_2^3\right) - y_d^{(2)} + (2 + 2\hat{\eta}_1)u\right] + \begin{bmatrix} 1/0.01 \\ 2/0.01^2 \end{bmatrix}\left(e_1 - \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix}\right) \\ \begin{bmatrix} \dot{\hat{\eta}}_1 \\ \dot{\hat{\eta}}_2 \end{bmatrix} = \begin{bmatrix} -\hat{\eta}_1 \\ -2\hat{\eta}_2^3 + \hat{e}_2\hat{\eta}_2 \end{bmatrix} \end{cases}$$

Sliding variable is selected as $\sigma(t) = \hat{e}_2 + \hat{e}_1$. Sliding surface is adopted as (12).where, $k_\alpha = 2, k_\beta = 0.5$ $q = 3, p = 7$. The controller designed is

$$u = -(2 + 2\hat{\eta}_1)^{-1}\left[\left(\hat{\eta}_2 + \hat{\eta}_2\left(\hat{\eta}_2\left(\hat{e}_2 - \hat{\eta}_2^2\right) - \hat{\eta}_2^3\right)\right) + \left(\frac{2}{0.01^2} + \frac{1}{0.01}\right)(e_1 - \hat{e}_1) + \right.$$
$$\left. \hat{e}_2 - y_d^{(r)} + 2\sigma(t) + 0.5(\sigma(t))^{\frac{3}{7}}\right]$$

The curve of error between factual output and referential output is shown is figure 1.

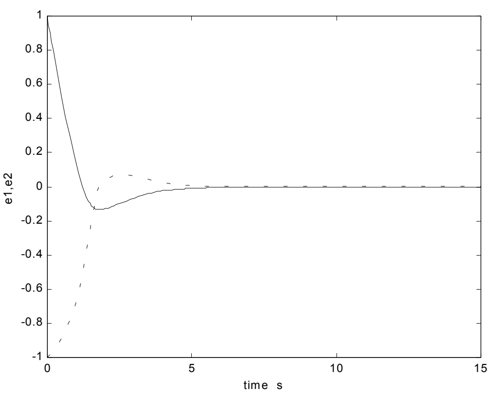

Fig. 1 The curve of the error and the its derivate between the output and the referential signal

6. Conclusion

The problem of designing global tracking-error observer for a class of nonlinear systems has been discussed. We proposed a procedure for the design of nonlinear tracking-error observer which do not require the hypothesis of full relative degree. As far as local observation problem is concerned, the exponential stability of zero dynamics is sufficient to guarantee that the output of the proposed observer converges to true error. Moreover, the tracking-error observer designed has a strong robustness. From the analysis above, the sliding mode controller designed has a strong robustness and rapidly convergent velocity.

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