Controllability of systems on solvable Lie groups: the generic case

Dirk Mittenhuber

August 8, 2000

Abstract

A Lie group G with Lie algebra $\mathfrak{g}$ is called SID-controllable if there exist $A, B \in \mathfrak{g}$ such that the (Single Input with Drift) control system $\dot{g} = g(A + uB)$, $u \in \mathbb{R}$ is controllable. This is tantamount to saying that the semigroup $S(A, \pm B)$ generated by $\exp(\mathbb{R}^+ A) \cup \exp(\mathbb{R}B)$ is all of G.

This definition is due to Sachkov who also classified SID-controllable solvable Lie algebras, cf. [7, 8, 9, 10]. It turns out that SID-controllability is actually a property of the Lie algebra (rather than of a control system): if a solvable $\mathfrak{g}$ is SID-controllable, then a generic SID-system will be controllable.

In this paper we generalize this result to systems with multiple inputs and drifts: G is $I_n D_m$-controllable if there exist "inputs" $B_1 \dots B_n$ and "drifts" $A_1 \dots A_m$ such that $S(A_1, \dots, A_m; \pm B_1, \dots, B_n) = G$. A Lie algebra is called $I_n D_m$-controllable if the corresponding simply connected group has this property.

We will show that every solvable Lie algebra has a *generic controllability rank* $r \in \mathbb{N}$ and a *generic controllability type* $(i, d) \in \mathbb{N} \times \mathbb{N}_0$ such that:

(GCR) $\mathfrak{g}$ is not I_{r-1}-controllable, $\mathfrak{g}$ is I_r-controllable, and the latter is generic. In particular, $\mathfrak{g}$ is $I_n D_m$ controllable if $n \geq r$ or $n + m > r$.

(GCT) $\mathfrak{g}$ is $I_i D_d$-controllable, a generic $I_i D_d$ system is controllable, $i + d = r$, and i is minmimal.

Determination of these invariants is one of our goals. Our major tools will be reduction arguments which are of independent interest.

AMS Subject Classification: 22E15, 22E25, 93B05

1 Introduction

The question of controllability of systems on solvable Lie groups was settled, in principle, by Lawson's classification of their maximal subsemigroups. Roughly speaking, a system is controllable if and only if its set of generators $U \subseteq \mathfrak{g}$ is not contained in a halfspace of $\mathfrak{g}$ whose boundary hyperplane is a subalgebra. As hyperplane subalgebras of real Lie algebras are classified and well understood, cf. [4] it seems as if there were nothing, or little, left to find. For an individual system on a particular group this is true, but for certain classes of systems some interesting questions remain as Sachkov's work shows.

In this paper we will show that every solvable Lie algebra has two generic controllability invariants: the generic controllability rank, say $\mathrm{GCR}(\mathfrak{g}) = r \in \mathbb{N}$, and a generic controllability type $\mathrm{GCT}(\mathfrak{g}) = (i, d) \in \mathbb{N} \times \mathbb{N}_0$ such that:

(GCR) Every rank $r - 1$ distribution on $\mathfrak{g}$ is noncontrollable, a generic rank r distribution is controllable.

(GCT) A generic system with i inputs and d drifts is controllable, $i + d = r$, and (i, d) is minimal with this property.

Giving it a little thought, the existence of these invariants is not surprising: r is simply the minimal number of elements of a Lie generating set $U \subseteq \mathfrak{g}$. And since a generic $(r, 0)$-system is controllable, there also exists a minimal i such that a generic $(i, r-i)$-system is controllable. But this leaves the problem of determining $\mathrm{GCR}(\mathfrak{g}), \mathrm{GCT}(\mathfrak{g})$ open. In the sequel we will see that, roughly speaking, this is as hard as finding the Jordan normal form of a inear map.

As mentioned above, an essential tool in the solution of our problem is the classification of maximal subsemigroups with nonempty interior in solvable Lie groups which is due to Lawson, cf. [5], or [2, V.5.42]. An abridged version that suffices for our purposes is:

Theorem 1.1 *Let $\mathfrak{g}$ be a solvable Lie algebra, G the corresponding simply connected group, and $U \subseteq \mathfrak{g}$ an arbitrary subset. Then the following statements are equivalent:*

(i) $\langle \exp(\mathbb{R}^+ U) \rangle = G$,

(ii) *U generates $\mathfrak{g}$ as a Lie algebra, and U is not contained in a halfspace $\mathfrak{h}^+ \subseteq \mathfrak{g}$ whose bounding hyperplane $\mathfrak{h}$ is a subalgebra.*

Our strategy is to apply reduction arguments. We want to pass to quotients $\mathfrak{g}/\mathfrak{i}$ where the ideal $\mathfrak{i} \trianglelefteq \mathfrak{g}$ is irrelevant for controllability (IFC). We will show first that every solvable Lie algebra contains a maximal (IFC)-ideal. Passing to the quotient we obtain a controllability normal form (CNF) of $\mathfrak{g}$. This result is of its own interest because it applies to general systems defined by an arbitrary subset

$U \subseteq \mathfrak{g}$ rather than sets of the form "subspace + polyhedral cone," $\mathrm{span}\{B_k\} + \sum_j \mathbb{R}^+ A_j$.

A Lie algebra in (CNF) has a particularly simple structure. We will show how the (GCR) and (GCT)-data can be read off the (CNF) of $\mathfrak{g}$.

2 Ideals irrelevant for controllability

Definition 2.1 Let $\mathfrak{g}$ be a Lie algebra, G the corresponding simply connected group. A subset $U \subseteq \mathfrak{g}$ is called *controllable* if the semigroup $\langle \exp(\mathbb{R}^+ U)\rangle$ is all of G. The subalgebra of $\mathfrak{g}$ generated by U is denoted $\langle\langle U\rangle\rangle$.

Next let $\mathfrak{i} \trianglelefteq \mathfrak{g}$ be an ideal, and $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ the canonical projection. We call $\mathfrak{i}$ an *(IFC)-ideal* of $\mathfrak{g}$ if for every subset $U \subseteq \mathfrak{g}$: $U \subseteq \mathfrak{g}$ controllable $\iff$ $\pi(U) \subseteq \pi(\mathfrak{g})$ controllable.

Finally we say that $\mathfrak{i}$ is an *(NCA)-ideal* if there is no proper subalgebra complementary to $\mathfrak{i}$, i.e., if $\mathfrak{h} \leq \mathfrak{g}$ is a subalgebra and $\mathfrak{h} + \mathfrak{i} = \mathfrak{g}$, then $\mathfrak{h} = \mathfrak{g}$.

As already mentioned hyperplane subalgebras play a crucial role in our investigation. From [4] we recall the notion of the Δ-radical:

$$\Delta(\mathfrak{g}) = \bigcap \{\mathfrak{h} \mid \mathfrak{h} \text{ hyperplane subalgebra in } \mathfrak{g}\}$$

This is a fully characteristic ideal. A halfspace $\mathfrak{h}^+$ whose boundary is a hyperplane subalgebra will be called a *halfspace semialgebra.*

An immediate consequence of Lawson's Theorem is:

Corollary 2.2 *Let $\mathfrak{g}$ be solvable and $U \subseteq \mathfrak{g}$ such that $\langle\langle U\rangle\rangle = \mathfrak{g}$. Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\Delta(\mathfrak{g})$ denote the canonical projection. Then*

$$U \subseteq \mathfrak{g} \text{ controllable} \quad \iff \quad \pi(U) \subseteq \pi(\mathfrak{g}) \text{ controllable.}$$

In general $\Delta(\mathfrak{g})$ may be too large for factorization: $\langle\langle \pi(U)\rangle\rangle = \pi(\mathfrak{g}) \not\Rightarrow \langle\langle U\rangle\rangle = \mathfrak{g}$. This is the reason why we introduced (IFC)-ideals. They can be characterized as follows:

Proposition 2.3 *Let $\mathfrak{i} \trianglelefteq \mathfrak{g}$ be an ideal in a solvable Lie algebra $\mathfrak{g}$. Then $\mathfrak{i}$ is an (IFC)-ideal if and only if $\mathfrak{i}$ is an (NCA)-ideal. In particular, $\mathfrak{i} \subseteq \Delta(\mathfrak{g})$ must hold.*

Proof. Assume first that $\mathfrak{i}$ is an (IFC)-ideal. Then it must be an (NCA)-ideal, for otherwise take a proper subalgebra $\mathfrak{h}$ with $\mathfrak{h} + \mathfrak{i} = \mathfrak{g}$. Then $\pi(\mathfrak{h}) \subseteq \mathfrak{g}/\mathfrak{i}$ is controllable, but $\mathfrak{h} \subseteq \mathfrak{g}$ is not.

Conversely, let $\mathfrak{i}$ be an (NCA)-ideal. Let $\mathfrak{h}$ be a hyperplane subalgebra. As $\mathfrak{i}$ is (NCA), $\mathfrak{h} + \mathfrak{i} \neq \mathfrak{g}$, hence $\mathfrak{i} \subseteq \mathfrak{h}$. Since $\mathfrak{h}$ was arbitrary, $\mathfrak{i} \subseteq \Delta(\mathfrak{g})$ follows.

Now consider a subset $U \subseteq \mathfrak{g}$. We must show that U is controllable $\iff$ $\pi(U) \subseteq \mathfrak{g}/\mathfrak{i}$ is controllable. The implication $\Rightarrow$ is trivial. Conversely, let $\pi(U)$ be controllable. Since

$$\pi(\langle\langle U\rangle\rangle) = \langle\langle \pi(U)\rangle\rangle = \mathfrak{g}/\mathfrak{i},$$

we obtain $\langle\langle U\rangle\rangle + \mathfrak{i} = \mathfrak{g}$. Since $\mathfrak{i}$ is (NCA), $\langle\langle U\rangle\rangle = \mathfrak{g}$ follows, so U is Lie-generating in $\mathfrak{g}$. Now assume, by contradiction, that U is not controllable. Then, by Lawson's Theorem 1.1, we can find a halfspace semialgebra $\mathfrak{h}^+ \subseteq \mathfrak{g}$ containing U, hence $\pi(U) \subseteq \pi(\mathfrak{h}^+)$. But the latter is a halfspace semialgebra in $\mathfrak{g}/\mathfrak{i}$ because of $\mathfrak{i} \subseteq \Delta(\mathfrak{g}) \subseteq \mathfrak{h}$. Thus $\pi(U)$ is not controllable! This contradiction proves our claim. □

Proposition 2.4 *If $\mathfrak{g}$ is solvable and $\mathfrak{i}_1, \mathfrak{i}_2$ are (IFC)-ideals, then $\mathfrak{i}_1 + \mathfrak{i}_2$ is an (IFC)-ideal, too. In particular, every solvable Lie algebra contains a maximal (IFC)-ideal.*

Proof. By the previous proposition we must show that $\mathfrak{i}_1 + \mathfrak{i}_2$ is an (NCA)-ideal. So let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra such that $\mathfrak{h} + \mathfrak{i}_1 + \mathfrak{i}_2 = \mathfrak{g}$. As $\mathfrak{i}_1$ is an ideal $\mathfrak{h} + \mathfrak{i}_1$ is a subalgebra. Since $\mathfrak{i}_2$ is (NCA), $(\mathfrak{h} + \mathfrak{i}_1) + \mathfrak{i}_2 = \mathfrak{g}$ implies $\mathfrak{h} + \mathfrak{i}_1 = \mathfrak{g}$. Now $\mathfrak{i}_1$ being (NCA) too yields $\mathfrak{h} = \mathfrak{g}$. Hence $\mathfrak{i}_1 + \mathfrak{i}_2$ is an (NCA)-ideal. As $\mathfrak{g}$ is finite-dimensional, there exists a maximal (NCA)-ideal which is obviuosly proper. □

Thus we have proved the existence of a maximal (IFC)-ideal $\mathfrak{i} \trianglelefteq \mathfrak{g}$. We call the factor algebra $\mathfrak{g}/\mathfrak{i}$ the controllability normal form (CNF) of $\mathfrak{g}$. All controllability questions (in $\mathfrak{g}$) can be answered looking at its (CNF). Our next goal is to describe (NCA)-ideals and, eventually, the structure of a solvable $\mathfrak{g}$ in (CNF).

Lemma 2.5 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a subalgebra such that $\mathfrak{h} + \mathfrak{g}' = \mathfrak{g}$. Then $(\forall k \in \mathbb{N})$ $\mathfrak{h} + \mathcal{C}^k\mathfrak{g} = \mathfrak{g}$, and with $\mathcal{C}^\infty\mathfrak{g} = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k\mathfrak{g}$: $\mathfrak{g} = \mathfrak{h} + \mathcal{C}^\infty\mathfrak{g}$*

Proof. For $k = 1$ this is our assumption. Now we prove for $k \in \mathbb{N}$ arbitrary that

$$(\mathfrak{h} + \mathcal{C}^1\mathfrak{g} = \mathfrak{g}) \quad \text{and} \quad (\mathfrak{h} + \mathcal{C}^k\mathfrak{g} = \mathfrak{g}) \quad \Rightarrow \quad (\mathfrak{h} + \mathcal{C}^{k+1}\mathfrak{g} = \mathfrak{g}).$$

First we observe that

$$\mathcal{C}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{h} + \mathcal{C}^k\mathfrak{g}, \mathfrak{h} + \mathcal{C}^k\mathfrak{g}] \subseteq [\mathfrak{h}, \mathfrak{h}] + \mathcal{C}^{k+1}\mathfrak{g} \subseteq \mathfrak{h} + \mathcal{C}^{k+1}\mathfrak{g}.$$

Hence $\mathfrak{g} = \mathfrak{h} + \mathcal{C}^1\mathfrak{g} \subseteq \mathfrak{h} + \mathcal{C}^{k+1}\mathfrak{g}$ follows, whence $\mathfrak{h} + \mathcal{C}^{k+1}\mathfrak{g} = \mathfrak{g}$. □

Theorem 2.6 *For nilpotent $\mathfrak{g}$ the maximal (IFC)-ideal is $\mathfrak{g}'$. Its (CNF) is the abelian Lie algebra $\mathbb{R}^r$. The generic controllability rank of $\mathfrak{g}$ is $r = \dim(\mathfrak{g}/\mathfrak{g}')$, and its generic controllability type is $(r, 0)$.*

Proof. For nilpotent $\mathfrak{g}$ we have $\Delta(\mathfrak{g}) = \mathfrak{g}'$. Thus every (NCA)-ideal is contained in $\mathfrak{g}'$, but the latter is an (NCA)-ideal by the previous lemma. thus our first claim follows. $\square$

This observation has already been made in [6]. The next step, too, can be found there.

Lemma 2.7 *For solvable $\mathfrak{g}$ the second derived algebra $D^2\mathfrak{g} = [\mathfrak{g}', \mathfrak{g}']$ is an (NCA)-ideal.*

Proof. Since $\mathfrak{g}$ is solvable, $\mathfrak{g}' =: \mathfrak{n}$ is nilpotent, and $D^2\mathfrak{g} = \mathfrak{n}'$. From $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}'$ and $\mathfrak{n}' \subseteq \mathfrak{n}$ we deduce $\mathfrak{n} = (\mathfrak{h} \cap \mathfrak{n}) + \mathfrak{n}'$. Since $\mathfrak{n}'$ is an (NCA)-ideal in $\mathfrak{n}$, $\mathfrak{h} \cap \mathfrak{n} = \mathfrak{n}$ follows, hence $\mathfrak{h} \supseteq \mathfrak{n} \supseteq \mathfrak{n}' = D^2\mathfrak{g}$, whence $\mathfrak{h} \supseteq \mathfrak{h} + D^2\mathfrak{g} = \mathfrak{g}$. Thus our claim is proved. $\square$

Thus the (CNF) of a solvable $\mathfrak{g}$ will always be metabelian, and its maximal (IFC)-ideal is trivial ($\{0\}$). We will determine and analyze the structure of the (CNF) in the next section.

3 Metabelian Lie algebras and the controllability normal form

Our goal in this section is to prove:

Theorem 3.1 *Let $\mathfrak{g}$ be solvable and in (CNF). Then $\mathfrak{g}$ has a root decomposition of the following form:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}^\lambda \oplus \bigoplus_{\omega \in \Omega} \mathfrak{g}^\omega.$$

Here $\mathfrak{h} \subseteq \mathfrak{g}$ is an abelian Cartan subalgebra, $\Lambda \subseteq \mathrm{Hom}(\mathfrak{h}, \mathbb{R}) \setminus \{0\}$ is the set of real roots, $\Omega \subseteq \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ a set of nonreal roots (pairwise nonconjugate); the root spaces are eigenspaces, i.e.,

$$(\forall \lambda \in \Lambda,\ H \in \mathfrak{h},\ X \in \mathfrak{g}^\lambda)\ \ \mathrm{ad}(H)X = \lambda(H)X,$$

as well as

$$(\forall \omega \in \Omega,\ H \in \mathfrak{h},\ X \in \mathfrak{g}^\omega)\ \big(\mathrm{ad}(H)^2 - 2\,\mathrm{Re}(\omega(H))\,\mathrm{ad}(H) + |\omega(H)|^2\,\mathbf{1}\big)X = 0,$$

and $(\forall \alpha, \beta \in \Lambda \cup \Omega)\,[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = 0$. *Moreover* $\overline{\Omega} \cap \Omega = \emptyset$.

Speaking in terms of linear (matrix) groups, a solvable $\mathfrak{g}$ in (CNF) is isomorphic to the direct sum of its center $\mathfrak{z}(\mathfrak{g})$ and a semidirect product of a vector space $\mathbb{R}^n$ and an abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{gl}(n, \mathbb{R})$ such that the elements $H \in \mathfrak{a}$ are simultaneously diagonalizable (over $\mathbb{C}$);

$$\mathfrak{g} \cong \mathfrak{z}(\mathfrak{g}) \oplus \left\{ \begin{pmatrix} H & v \\ 0 & 0 \end{pmatrix} \mid H \in \mathfrak{a},\ v \in \mathbb{R}^n \right\} \subseteq \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{gl}(n+1, \mathbb{R}), \quad \mathfrak{g} \cong \mathfrak{z}(\mathfrak{g}) \oplus (\mathbb{R}^n \rtimes \mathfrak{a}),$$

and $\mathfrak{g}' = \mathcal{C}^\infty \mathfrak{g} \cong \mathbb{R}^n$.

In order to prove this theorem we start with an arbitrary solvable Lie algebra $\mathfrak{g}$ and pass to the quotient $\mathfrak{g}/D^2\mathfrak{g}$ thus obtaining a metabelian Lie algebra. Now we follow the path outlined in [1, p.28f].

For a real Lie algebra $\mathfrak{g}$ we choose a Cartan subalgebra of $\mathfrak{h}$. So $\mathfrak{h}$ is nilpotent, self-normalizing, and maximal with this property. Passing to the complexification $\mathfrak{g}_\mathbb{C} = \mathbb{C} \otimes \mathfrak{g}$, it is standard (cf. [1, p.17ff]) that $\mathfrak{h}_\mathbb{C} = \mathbb{C} \otimes \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$, and one obtains a root decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\mathbb{C}^\alpha,$$

where Δ is the set of roots, a root $\alpha : \mathfrak{h}_\mathbb{C} \to \mathbb{C}$ is a nonzero linear form, and

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g}_\mathbb{C} \mid (\exists n \in \mathbb{N})(\forall H \in \mathfrak{h}_\mathbb{C})(\mathrm{ad}(H) - \alpha(H)\mathbf{1})^n X = 0\}$$

is the corresponding generalized root space.

For general $\mathfrak{g}$ one obtains that $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$. But since we assume that $\mathfrak{g}$ is metabelian, we obtain from $\mathfrak{g}^\alpha \subseteq [\mathfrak{h}, \mathfrak{g}^\alpha] \subseteq \mathfrak{g}'$ that

$$(\forall \alpha, \beta)\ [\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq [\mathfrak{g}', \mathfrak{g}'] = D^2\mathfrak{g} = 0.$$

Considering $\alpha(\mathfrak{h}) \subseteq \mathbb{C}$ the roots fall into two different classes, the real roots: $\Lambda = \{\alpha \mid \alpha(\mathfrak{h}) \subseteq \mathbb{R}\}$ and the nonreal roots $\Delta \setminus \Lambda$. The other roots are called nonreal. For every root $\alpha \in \Delta$ there is a conjugate root $\bar{\alpha}$. Now $\alpha = \bar{\alpha}$ iff $\alpha \in \Lambda$. Thus the nonreal roots appear in conjugate) pairs. We let Ω denote a selection of nonreal roots such that $\Delta = \Lambda \cup \Omega \cup \overline{\Omega}$ is a disjoint union. For $\lambda \in \Lambda$ and $\omega \in \Omega$ the associated generalized real root spaces are:

$$\mathfrak{g}^\lambda = \mathfrak{g} \cap \mathfrak{g}_\mathbb{C}^\lambda, \quad \mathfrak{g}^\omega = \mathfrak{g} \cap (\mathfrak{g}_\mathbb{C}^\omega + \mathfrak{g}_\mathbb{C}^{\bar{\omega}}).$$

Then the root decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}^\lambda \oplus \bigoplus_{\omega \in \Omega} \mathfrak{g}^\omega.$$

We note that, in general, $\mathfrak{h}$ need not be abelian and the spaces $\mathfrak{g}^\lambda, \mathfrak{g}^\omega$ need not be eigenspaces. We need another reduction argument to arrive at Theorem 3.1.

First we need two more definitions. For $\lambda \in \Lambda$ and $\omega \in \Omega$ we define ideals $\mathfrak{i}_\lambda, \mathfrak{i}_\omega$ as follows:

$$
\begin{aligned}
\mathfrak{i}_\lambda &= \sum_{H\in\mathfrak{h}} (\mathrm{ad}(H) - \lambda(H)\mathbf{1})\,\mathfrak{g}^\lambda,\\
\mathfrak{i}_\omega &= \sum_{H\in\mathfrak{h}} \Big(\mathrm{ad}(H)^2 - 2\,\mathrm{Re}(\omega(H))\,\mathrm{ad}(H) + |\omega(H)|^2\mathbf{1}\Big)\,\mathfrak{g}^\omega.
\end{aligned}
$$

Lemma 3.2 *The following ideals are (IFC)-ideals:*

$$\mathfrak{h}', \quad \mathfrak{i}_\lambda, \mathfrak{i}_\omega, \quad \lambda\in\Lambda, \omega\in\Omega.$$

Proof. Let $\mathfrak{a}$ be a subalgebra and $\mathfrak{a}+\mathfrak{h}'=\mathfrak{g}$. Then $(\mathfrak{a}+\mathfrak{h}')\cap\mathfrak{h}=\mathfrak{h}$ follows. As $\mathfrak{h}'$ is an (NCA)-ideal of $\mathfrak{h}$, $\mathfrak{a}\supseteq\mathfrak{h}\supseteq\mathfrak{h}'$, whence $\mathfrak{a}=\mathfrak{g}$.

The proofs for $\mathfrak{i}_\lambda$ and $\mathfrak{i}_\omega$ require a little bit more work. For $\lambda\in\Lambda$, $\omega\in\Omega$, and $n\in\mathbb{N}$ we let

$$
\begin{aligned}
\mathfrak{i}_\lambda^n &= \sum_{H\in\mathfrak{h}} (\mathrm{ad}(H) - \lambda(H)\,\mathbf{1})^n\,\mathfrak{g}^\lambda,\\
\mathfrak{i}_\omega^n &= \sum_{H\in\mathfrak{h}} (\mathrm{ad}(H)^2 - 2\,\mathrm{Re}(\omega(H))\,\mathrm{ad}(H) + |\omega(H)|^2\,\mathbf{1})^n\,\mathfrak{g}^\omega.
\end{aligned}
$$

Thus $\mathfrak{i}_\lambda^1=\mathfrak{i}_\lambda$, $\mathfrak{i}_\omega^1=\mathfrak{i}_\omega$, and $\mathfrak{i}_\lambda^n=0$, $\mathfrak{i}_\omega^n=0$ for $n>\dim\mathfrak{g}^\lambda$, resp. $n>\dim\mathfrak{g}^\omega$.

Now we show that $\lambda\in\Lambda$ $\mathfrak{i}_\lambda$ is an (IFC)-ideal. So let $L\leq\mathfrak{g}$ be a subalgebra such that $L+\mathfrak{i}_\lambda=\mathfrak{g}$. We show inductively that $L+\mathfrak{i}_\lambda^n=\mathfrak{g}$ holds for all $n\in\mathbb{N}$. In order to show that, we prove that $\mathfrak{i}_\lambda^n\subseteq L+\mathfrak{i}_\lambda^{n+1}$.

So let $X\in\mathfrak{i}_\lambda^n$ be arbitrary but fixed. Since $n\geq 1$ we can find $H\in\mathfrak{h}$ and $Y\in\mathfrak{i}_\lambda^{n-1}$ such that $[H,Y]=\lambda(H)Y+X$. Since $L+\mathfrak{i}_\lambda^n=\mathfrak{g}$ we find $v,w\in\mathfrak{i}_\lambda^n$ such that $H+v, Y+w\in L$. Thus we obtain

$$[H+v,Y+w]=[H,Y]+[H,w]=\lambda(H)Y+X+\lambda(H)w+\underbrace{(\mathrm{ad}(H)-\lambda(H)\mathbf{1})w}_{\in\mathfrak{i}_\lambda^{n+1}},$$

Thus

$$X=[H+v,Y+w]-\lambda(H)(Y+w)-(\mathrm{ad}(H)-\lambda(H)\mathbf{1})w\in[L,L]+L+\mathfrak{i}_\lambda^{n+1}.$$

Since L is a subalgebra, $\mathfrak{i}_\lambda^n\subseteq L+\mathfrak{i}_\lambda^{n+1}$, whence $\mathfrak{g}=L+\mathfrak{i}_\lambda^n=L+\mathfrak{i}_\lambda^{n+1}$ is proved.

We proceed similarly to prove that $\mathfrak{i}_\omega$ is an (IFC)-ideal for all $\omega\in\Omega$. So let $L+\mathfrak{i}_\omega^1=\mathfrak{g}$. We claim that $\mathfrak{i}_\omega\subseteq L+\mathfrak{i}_\omega^n$ for all $n\in\mathbb{N}$. For $X\in\mathfrak{i}_\omega^n$ we find $H\in\mathfrak{h}$, $Y\in\mathfrak{i}_\omega^{n-1}$ such that

$$\Big(\mathrm{ad}(H)^2-2\,\mathrm{Re}(\omega(H))\,\mathrm{ad}(H)+|\omega(H)|^2\Big)\,Y=X.$$

Now pick $v,w\in\mathfrak{i}_\omega^n$ such that $H+v, Y+w\in L$, then

$$\mathrm{ad}(H+v)(Y+w)=\mathrm{ad}(H)(Y+w), \quad\text{and}\quad \mathrm{ad}(H+v)^2(Y+w)=\mathrm{ad}(H)^2(Y+w),$$

whence

$$\begin{aligned}
X &= (\mathrm{ad}(H)^2 - 2\,\mathrm{Re}(\omega(H))\,\mathrm{ad}(H) + |\omega(H)|^2\,\mathbf{1})(Y + w - w) \\
&= (\mathrm{ad}(H+v)^2 - 2\,\mathrm{Re}(\omega(H))\,\mathrm{ad}(H+v) + |\omega(H)|^2\,\mathbf{1})(Y + w) \\
&\quad - \Big(\mathrm{ad}(H)^2 - 2\,\mathrm{Re}(\omega(H))\,\mathrm{ad}(H) + |\omega(H)|^2\,\mathbf{1}\Big)\, w \\
&\in L + \mathfrak{i}_\omega^{n+1}.
\end{aligned}$$

Thus $\mathfrak{i}_\omega^n \subseteq L + \mathfrak{i}_\omega^{n+1}$ follows. Hence $L + \mathfrak{i}_\omega^n = \mathfrak{g} \;\Rightarrow\; L + \mathfrak{i}_\omega^{n+1} = \mathfrak{g}$ is proved, too. □

The ideal $\bigoplus_\lambda \mathfrak{g}^\lambda$ is also called the *base ideal.* The hyperplane subalgebras in $\mathfrak{g}$ can be described as follows, cf. [4, p.218]:

Proposition 3.3 *Let* $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda\in\Lambda} \mathfrak{g}^\lambda \oplus \bigoplus_{\omega\in\Omega} \mathfrak{g}^\omega$ *be in (CNF). Then* $\Delta(\mathfrak{g}) = \bigoplus_\omega \mathfrak{g}^\omega$*, and every hyperplane subalgebra, say* $\mathfrak{a}$*, has the form*

$$\begin{aligned}
\mathfrak{a} &= \mathfrak{a}_0 \oplus \bigoplus_{\lambda\in\Lambda} \mathfrak{g}^\lambda \oplus \Delta(\mathfrak{g}), \quad \textit{with } \mathfrak{a}_0 \textit{ a hyperplane in } \mathfrak{h}, \textit{ or} \\
\mathfrak{a} &= \mathfrak{h} \oplus \mathfrak{a}_{\lambda_0} \bigoplus_{\lambda\neq\lambda_0} \mathfrak{g}^\lambda \oplus \Delta(\mathfrak{g}), \quad \textit{with } \mathfrak{a}_{\lambda_0} \textit{ a hyperplane in } \mathfrak{g}^{\lambda_0}.
\end{aligned}$$

4 Generic controllability rank and type

Our next goal is to determine the generic controllability rank and type of the (CNF) of a solvable $\mathfrak{g}$. Of course this is something one can read off its root decomposition.

For $\lambda \in \Lambda$ we let $d_\lambda = \dim_{\mathbb{R}} \mathfrak{g}^\lambda$. For $\omega \in \Omega$ we let $d_\omega = \dim_{\mathbb{C}} \mathfrak{g}^\omega = \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{g}^\omega$. Furthermore we set

$$d_\Lambda := \max_{\lambda\in\Lambda} d_\lambda, \quad \text{and} \quad d_\Omega := \max_{\omega\in\Omega} d_\omega.$$

Theorem 4.1 *Suppose that* $\mathfrak{g}$ *is in (CNF). Let*

$$\begin{aligned}
r &= \max\{\dim \mathfrak{h}, 1 + d_\Lambda, 1 + d_\Omega\}, \\
i &= \max\{\dim \mathfrak{h}, 1 + d_\Lambda\}, \\
d &= r - i = \max\{0, 1 + d_\Omega - i\}.
\end{aligned}$$

Then $\mathrm{GCR}(\mathfrak{g}) = r$ *and* $\mathrm{GCT}(\mathfrak{g}) = (i, d)$*.*

The proof is carried out in several steps below. We must prove the following statements:

(GCR-N) Every I_{r-1}-system is noncontrollable.

(GCR-Y) There exist $B_1, \ldots, B_r$ with $\langle\langle B_1, \ldots, B_r \rangle\rangle = \mathfrak{g}$.

(GCT-Y) A generic (i,d)-system is controllable.

(GCT-N) For $n<i$ I_nD_{r-n}-controllability is not generic.

First we deal with the (GCR)-part. We use four types of test algebras:

(1) Abelian algebras: $\mathfrak{g}\cong\mathbb{R}^n$.

(2) Almost abelian but nonabelian Lie algebras, i.e.,

$$\mathfrak{g}\cong\left\{\begin{pmatrix} t\,\mathbf{1}_n & x\\ 0 & 0\end{pmatrix}\mid x\in\mathbb{R}^n,\ t\in\mathbb{R}\right\}\subseteq\mathfrak{gl}(n+1,\mathbb{R}).$$

(3) $\mathfrak{g}=\mathbb{C}^n\rtimes_\alpha\mathbb{R}$, where $\alpha\in\mathbb{C}\setminus\mathbb{R}$ is arbitrary but fixed:

$$\mathfrak{g}\cong\left\{\begin{pmatrix} \alpha t\,\mathbf{1}_n & v\\ 0 & 0\end{pmatrix}\mid v\in\mathbb{C}^n,\ t\in\mathbb{R}\right\}\subseteq\mathfrak{gl}(n+1,\mathbb{C}).$$

(4) $\mathfrak{g}=\mathbb{C}^n\rtimes\mathbb{C}$, i.e.,

$$\mathfrak{g}\cong\left\{\begin{pmatrix} z\,\mathbf{1}_n & v\\ 0 & 0\end{pmatrix}\mid v\in\mathbb{C}^n,\ z\in\mathbb{C}\right\}\subseteq\mathfrak{gl}(n+1,\mathbb{C}).$$

Proposition 4.2 *For an almost abelian Lie algebra* $\mathfrak{g}$ *one has* $\mathrm{GCR}(\mathfrak{g})=\dim\mathfrak{g}$, *and*

$$\mathrm{GCR}(\mathbb{C}^n\rtimes_\alpha\mathbb{R})=\mathrm{GCR}(\mathbb{C}^n\rtimes\mathbb{C})=n+1.$$

Proof. The first statement is clear since in an almost abelian Lie algebra every vector subspace is a subalgebra. Now $\mathbb{C}\rtimes_\alpha\mathbb{R}=\langle\langle\,(0,1),(e_1,0),\ldots,(e_n,0)\rangle\rangle$ and $\mathbb{C}\rtimes\mathbb{C}=\langle\langle\,(0,i),(e_1,0),\ldots,(e_n,0)\rangle\rangle$. Thus $\mathrm{GCR}(\mathbb{C}\rtimes_\alpha\mathbb{R}),\mathrm{GCR}(\mathbb{C}\rtimes\mathbb{C})\le n+1$.

Next, consider $B_i=(v_i,t_i)\in\mathbb{C}^n\rtimes_\alpha\mathbb{R}$, $i=1,\ldots,n$. If $t_i=0$ for all i, then $\langle\langle B_1,\ldots,B_n\rangle\rangle\subseteq\mathfrak{g}'$. Otherwise assume w.l.o.g $t_1\neq 0$. Let $X=\left(\frac{1}{\alpha t_1}v_1,0\right)$. Then $[B_1,X]=(v_1,0)$ and $e^{\operatorname{ad}X}B_1=B_1+[X,B_1]=(0,t_1)$. Let $e^{\operatorname{ad}X}B_k=(w_k,t_k)$ for $k=2,\ldots,n$. Then

$$e^{\operatorname{ad}(X)}\langle\langle B_1,\ldots,B_n\rangle\rangle=\mathbb{R}(0,1)+(\mathbb{C}w_2+\cdots+\mathbb{C}w_n)\times\{0\}.$$

Hence $\dim_\mathbb{R}\langle\langle B_1,\ldots,B_n\rangle\rangle\le 2n-1<2n+1=\dim\mathfrak{g}$, whence $\mathrm{GCR}(\mathfrak{g})\ge n+1$.

For $\mathbb{C}^n\rtimes\mathbb{C}$ we proceed similarly: $B_i=(v_i,z_i)$, $i=1,\ldots,n$. If $z_i=0$ for all i, then $\langle\langle B_i\rangle\rangle\neq\mathfrak{g}$ is clear. Otherwise, assume w.l.o.g. $z_1\neq0$ and let $X=(\frac{1}{z_1}v_1,0)$. Then $[X,B_1]=(0,z_1)$, and with $e^{\operatorname{ad}X}B_k=(w_k,z_k)$ we obtain

$$e^{\operatorname{ad}(X)}\langle\langle B_1,\ldots,B_n\rangle\rangle\subseteq\mathbb{C}(0,1)+\mathbb{C}(w_2,0)+\cdots+\mathbb{C}(w_n,0)\neq\mathfrak{g},$$

hence $\mathrm{GCR}(\mathbb{C}\rtimes\mathbb{C})\ge n+1$. □

Lemma 4.3 *Every I_{r-1}-system in $\mathfrak{g}$ is noncontrollable, so* $\mathrm{GCR}(\mathfrak{g}) \geq r$.

Proof. First we observe that $\mathrm{GCR}(\mathfrak{g}) \geq \mathrm{GCR}(\mathfrak{g}/\mathfrak{i})$ holds for every ideal $\mathfrak{i} \trianglelefteq \mathfrak{g}$. The choice $\mathfrak{i} = \mathfrak{g}'$ leadsto to $\mathfrak{g}/\mathfrak{g}' \cong \mathfrak{h}$ abelian, hence $\mathrm{GCR}(\mathfrak{g}) \geq \dim \mathfrak{h}$. Next pick λ_0 such that $d_{\lambda_0} = d_\Lambda$ and consider

$$\mathfrak{i} = \ker(\lambda_0) \oplus \bigoplus_{\lambda \neq \lambda_0} \mathfrak{g}^\lambda \oplus \bigoplus_{\omega \in \Omega} \mathfrak{g}^\omega, \quad \pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}.$$

Then $\mathfrak{g}/\mathfrak{i}$ is almost abelian, therefore $\mathrm{GCR}(\mathfrak{g}) \geq \mathrm{GCR}(\mathfrak{g}/\mathfrak{i}) = \dim(\mathfrak{g}/\mathfrak{i}) = 1 + d_\Lambda$. Finally, pick ω_0 such that $d_{\omega_0} = d_\Omega$ and let

$$\mathfrak{i} = \ker(\omega_0) \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}^\lambda \oplus \bigoplus_{\omega \neq \omega_0} \mathfrak{g}^\omega, \quad \pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}.$$

Then $\mathfrak{g}/\mathfrak{i}$ is isomorphic to either $\mathbb{C}^{d_\Omega} \rtimes \mathbb{C}$, or $\mathbb{C}^{d_\Omega} \rtimes \mathbb{R}$. Thus $\mathrm{GCR}(\mathfrak{g}) \geq 1 + d_\Omega$ follows. □

Lemma 4.4 *There exist $B_1, \ldots, B_r$ with $\langle\langle B_1, \ldots, B_r \rangle\rangle = \mathfrak{g}$, so $GCR(\mathfrak{g}) \leq r$.*

Proof. First we pick a regular $H_1 \in \mathfrak{h}$, i.e.,

$$\alpha(H_1) \neq 0, \quad \text{and} \quad \alpha(H_1) \neq \beta(H_1) \quad \text{for all } \alpha, \beta \in \Lambda \cup \Omega,\ \alpha \neq \beta.$$

Next we choose $H_2, \ldots, H_{\dim \mathfrak{h}}$ to have a basis of $\mathfrak{h}$ and let $H_k = 0$ for $k > \dim \mathfrak{h}$. Similarly we choose an $\mathbb{R}$-basis X_k^λ of $\mathfrak{g}^\lambda$, a $\mathbb{C}$-basis U_k^ω of $\mathfrak{g}^\omega$, resp. $X_k^\lambda = 0$ for $k > d_\lambda$, $U_k^\omega = 0$ for $k > d_\omega$. Finally let

$$\begin{aligned} B_1 &= H_1 \\ B_k &= H_k + \sum_\lambda X_k^\lambda + \sum_\omega U_k^\omega, \quad k = 2, \ldots, r. \end{aligned}$$

Let $\mathfrak{a} = \langle\langle B_1, \ldots, B_r \rangle\rangle$. We claim that $\mathfrak{a} = \mathfrak{g}$. In order to prove this we define several maps from $\mathfrak{h}$ to $\mathfrak{gl}(\mathfrak{g})$:

$$\begin{aligned} p_\lambda(H) &= \mathrm{ad}(H) - \lambda(H)\mathbf{1}, \quad \lambda \in \Lambda \\ q_\omega(H) &= \mathrm{ad}(H)^2 - 2\,\mathrm{Re}(\omega(H))\,\mathrm{ad}(H) + |\omega(H)|^2\mathbf{1}, \quad \omega \in \Omega, \\ P_\mu(H) &= \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq \mu}} p_\lambda(H) \prod_{\substack{\omega \in \Omega \\ \omega \neq \mu}} q_\omega(H), \quad \mu \in \Lambda \cup \Omega. \end{aligned}$$

Then it is clear that $P_\mu(H_1)B_j \in \mathfrak{a}$. Now for $\mu \in \Lambda$ we obtain

$$P_\mu(H_1)B_{j+1} = \underbrace{\prod_{\lambda \neq \mu} (\mu(H_1) - \lambda(H_1))}_{\in \mathbb{R}\setminus\{0\}} \underbrace{\prod_{\omega \in \Omega} |\mu(H_1) - \omega(H_1)|^2}_{>0} X_j^\mu.$$

Since this is true for all j and $\mu \in \Lambda$, we deduce $\mathfrak{a} \supseteq \bigoplus_\lambda \mathfrak{g}^\lambda$. Similarly, for $\mu \in \omega$ we obtain

$$P_\mu(H_1)B_{j+1} = \underbrace{\prod_{\lambda\in\Lambda}(\mu(H_1)-\lambda(H_1))\prod_{\omega\neq\mu}(\mu(H_1)-\omega(H_1))\left(\mu(H_1)-\overline{\omega(H_1)}\right)}_{=:c\in\mathbb{C}\setminus\{0\}} U_j^\mu.$$

Since $\mu(H) \in \mathbb{C}\setminus\mathbb{R}$, $\mathbb{R}c + \mathbb{R}c\mu(H) = \mathbb{C}$, so

$$\mathbb{R}P_\mu(H)B_{j+1} + \mathbb{R}\,\mathrm{ad}(H)P_\mu(H)B_{j+1} = \mathbb{C}U_j^\mu \subseteq \mathfrak{a}.$$

Thus $\mathfrak{a} \supseteq \bigoplus_\omega \mathfrak{g}^\omega$ follows. Hence $\mathfrak{a} \supseteq \mathfrak{h}$, whence $\mathfrak{a} = \mathfrak{g}$. □

Lemma 4.5 *Let $i = \mathrm{GCR}(\mathfrak{g}/\Delta(\mathfrak{g}))$ and $d = r - i$, i.e.,*

$$\begin{aligned} i &= \max\{\dim\mathfrak{h},\, d_\Lambda + 1\}, \\ d &= \max\{0, d_\Omega + 1 - i\} = \begin{cases} 0, & \text{if } i = r, \\ d_\Omega + 1 - i, & \text{if } d_\Omega + 1 > i. \end{cases} \end{aligned}$$

Then a generic I_iD_d-system is controllable.

Proof. A generic system with $i + d = r$ generators is Lie-generating. Let $\pi\colon \mathfrak{g} \to \Delta(\mathfrak{g})$ denote the canonical projection. If $U \subseteq \mathfrak{g}$ with $\langle\langle U\rangle\rangle = \mathfrak{g}$, then U is controllable iff $\pi(U) \subseteq \pi(\mathfrak{g})$ is controllable. Thus if $\langle\langle A_1,\dots,A_d,B_1,\dots,B_i\rangle\rangle = \mathfrak{g}$ and $\langle\langle \pi(B_1),\dots,\pi(B_i)\rangle\rangle = \mathfrak{g}/\Delta(\mathfrak{g})$ then $\{A_1,\dots,A_d,\pm B_1,\dots,\pm B_i\}$ is controllable. Since $i + d = r = \mathrm{GCR}(\mathfrak{g})$ and $i = \mathrm{GCR}(\mathfrak{g}/\Delta\mathfrak{g})$, both conditions are generic, so an I_iD_d-system is generically controllable. □

We note that if $d = 0$, then a single drift term does not contribute at all to controllability. For $d > 0$ the situation changes drastically:

Proposition 4.6 *Let $r = \mathrm{GCR}(\mathfrak{g})$ and $\mathrm{GCT}(\mathfrak{g}) = (i,d)$. If $d > 0$, then $\mathfrak{g}$ is D_r-controllable.*

Proof. Consider $\pi\colon \mathfrak{g} \to \mathfrak{g}/\Delta(\mathfrak{g})$. Since $\mathrm{GCR}(\pi(\mathfrak{g})) = i = r - d < r$, we can find $A_1,\dots,A_r \in \mathfrak{g}$ such that

$$\langle\langle A_1,\dots,A_r\rangle\rangle = \mathfrak{g}, \quad \text{and} \quad \mathbb{R}^+\pi(A_1) + \cdots + \mathbb{R}^+\pi(A_r) = \pi(\mathfrak{g}).$$

Thus $\{A_1,\dots,A_r\}$ is controllable. □

Proposition 4.7 *Let $\mathrm{GCT}(\mathfrak{g}) = (i,d)$ with $d > 0$. Then $\mathfrak{g}$ is I_nD_{r-n}-controllable for all $n = 0,1,\dots,r$. For $n < i$ I_nD_{r-n}-controllability is not generic.*

Proof. The first claim follows from the previous proposition. If $n < i$, then we pass to an almost abelian factor $\mathfrak{g}/\mathfrak{i}$ with $\dim(\mathfrak{g}/\mathfrak{i}) > n$. It is clear then that we can find $B_1, \ldots, B_n, A_1, \ldots, A_{r-n} \in \mathfrak{g}$ such that $\{\pm\pi(B_1), \ldots, \pi(A_{r-n})$ is not controllable in $\pi(\mathfrak{g})$ and that this is true on a neighborhood of $(B_1, \ldots, B_n, A_1, \ldots, A_{r-n})$ in $\mathfrak{g} \times \mathfrak{g} \times \cdots \times \mathfrak{g} = \mathfrak{g}^r$. □

In particular for systems with a single drift we obtain:

Corollary 4.8 *Let $\mathfrak{g}$ be solvable, $r = \mathrm{GCR}(\mathfrak{g})$, and $(i, d) = \mathrm{GCT}(\mathfrak{g})$. Consider an I_nD_1-system. Then we have the following possibilities:*

$n+1<r$	$n+1=r$		$n+1>r$
	$d=0$	$d>0$	
noncontr.	*noncont.*	*gen. contr.*	*gen. contr.*

References

[1] Anselm Eggert. Über Liesche Semialgebren. Heft 204, Mitteilungen aus dem Mathematischen Seminar Giessen, 1991.

[2] Joachim Hilgert, Karl Heinrich Hofmann, and Jimmie D. Lawson. *Lie Groups, Convex Cones and Semigroups.* Oxford University Press, 1989.

[3] Joachim Hilgert and Karl-Hermann Neeb. *Lie Semigroups and their applications*, volume 1552 of *Lecture Notes in Mathematics.* Springer, Berlin Heidelberg New York Tokyo, 1993.

[4] Karl Heinrich Hofmann. Hyperplane-subalgebras of real Lie-algebras. *Geometriae Dedicata*, 36:207–224, 1990.

[5] Jimmie D. Lawson. Maximal semigroups of Lie groups that are total. *Proc. Edinburgh Math. Soc.*, 30:479–501, 1987.

[6] Dirk Mittenhuber. Controllability of solvable Lie algebras. *Journal of Dynamimcal and Control Systems*, 6:453–459, 2000.

[7] Yu. L. Sachkov. Controllability of hypersurface and solvable invariant systems. *Journal of Dynamimcal and Control Systems*, 2(1):55–67, 1996.

[8] Yu. L. Sachkov. Controllability of right-invariant systems on solvable Lie groups. *Journal of Dynamimcal and Control Systems*, 3(4):531–564, 1997.

[9] Yu. L. Sachkov. Controllability of invariant systems on Lie groups and homogeneous spaces. Technical Report 18/99/M, ISAS Trieste, February 1999.

[10] Yu. L. Sachkov. Classification of controllable systems on low-dimensional solvable Lie groups. *Journal of Dynamimcal and Control Systems*, 6:159–217, 2000.