Controllability of solvable Lie algebras

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Abstract

We call a Lie-algebra $\mathfrak{g}$ SID-controllable if there exist $A, B \in \mathfrak{g}$ such that the invariant control system on the simply connected group G given by $\dot{g} = g(A + uB)$, $u \in \mathbb{R}$ is controllable. This is tantamount to saying that the semigroup $S(A, \pm B) = \langle \exp(\mathbb{R}^+ A + \mathbb{R} B) \rangle \subseteq G$ is all of G.

In his papers [7, 4, 5, 6] Sachkov analyzed and characterized solvable SID-controllable Lie algebras. In this paper we improve on his results by proving the following simple reduction criterion: a solvable Lie algebra $\mathfrak{g}$ is SID-controllable iff the factor algebra $\mathfrak{g}/D^2\mathfrak{g}$ is SID-controllable.

For solvable $\mathfrak{g}$ this reduces the whole problem to metabelian algebras ($D^2\mathfrak{g} = 0$), a case which is completely solved in Sachkov's work.

Our result also generalizes immediately to systems with multiple inputs.

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1 Introduction

Let G be a group and $M \subseteq G$ an arbitrary subset. We denote by $\langle M \rangle$ the subsemigroup of G generated by M. We use the following definition which is due to Sachkov, cf. [7, Definition 16.1, p. 76]

Definition 1.1 A Lie group G is called *SID-controllable,* if there exist $A, B \in \mathfrak{g}$ such that $\langle \exp(\mathbb{R}^+ A + \mathbb{R} B) \rangle = G$.

Similarly, a Lie algebra $\mathfrak{g}$ is called *SID-controllable,* if the associated simply connected group is SID-controllable.

Remark 1.2 We prefer to say *SID-controllable* (**S**ingle **I**nput plus **D**rift) rather than just *controllable*. Similarly, one may then speak of MID_k*-controllability* if one can find $A, B_1, \ldots, B_k \in \mathfrak{g}$ such that $\mathbb{R}^+ A + \mathbb{R} B_1 + \cdots + \mathbb{R} B_k$ generates G. Finally, we say that G is $I_n D_m$*-controllable* if there exist n "inputs" $B_1, \ldots, B_n$ and m "drifts" $A_1, \ldots, A_m$ such that $\langle \exp(\mathbb{R}^+ A_1 + \cdots + \mathbb{R}^+ A_m + \mathbb{R} B_1 + \cdots + \mathbb{R} B_n) \rangle$ is all of G.

A characterization of solvable SID-controllable Lie algebras was given by Sachkov, cf. [6, 7]. But the conditions are still quite technical. An important observation of his is that for solvable $\mathfrak{g}$ SID-controllability is an intrinsic property of the Lie algebra, and it is generic. So if there exists one controllable pair $A, B \in \mathfrak{g}$, then the set of controllable pairs is open and dense. Sachkov also derives an exhaustive list of low-dimensional solvable SID-controllable Lie algebras, cf. [7, 4].

Dealing with the question whether a solvable Lie algebra $\mathfrak{g}$ is SID-controllable our main result is a purely algebraic reduction criterion:

Theorem 1.3 *Let $\mathfrak{g}$ be a solvable Lie algebra. Then the following statements are equivalent:*

(1) $\mathfrak{g}$ *is SID-controllable,*

(2) *Every factor algebra $\mathfrak{g}/\mathfrak{i}$ is SID-controllable where $\mathfrak{i} \lhd \mathfrak{g}$ is an ideal.*

(3) $\mathfrak{g}/D^2\mathfrak{g}$ *is SID-controllable.*

Moreover, if $\mathfrak{g}$ is SID-controllable, then the following statements are true:

(4) $\mathfrak{g}$ *has no ideals of codimension* 2*.*

(5) *The derived algebra $\mathfrak{g}'$ is a hyperplane, and it is the only hyperplane subalgebra of $\mathfrak{g}$.*

This reduces the controllability problem to metabelian algebras ($D^2\mathfrak{g} = 0$), a particularly simple case that was completely treated by Sachkov, see also Lemma 3.1.

An essential tool in the solution of our problem is the classification of maximal subsemigroups with nonempty interior in solvable Lie groups which is due to Lawson, cf. [3], or [1, V.5.42]. An abridged version that suffices for our purposes is:

Theorem 1.4 *Let $\mathfrak{g}$ be a solvable Lie algebra, G the corresponding simply connected group, and $U \subseteq \mathfrak{g}$ an arbitrary subset. Then the following statements are equivalent:*

(i) $\langle \exp(\mathbb{R}^+ U) \rangle = G$,

(ii) *U generates $\mathfrak{g}$ as a Lie algebra, and U is not contained in a halfspace $\mathfrak{h}^+ \subseteq \mathfrak{g}$ whose bounding hyperplane $\mathfrak{h}$ is a subalgebra.*

Since hyperplane subalgebras of real Lie algebras are quite well understood and classified, cf. [2], this theorem provides a testing device for controllability.

2 Auxiliary results

Before we prove Theorem 1.3, we need two auxiliary Lemmas. We use the following notation: for an arbitrary subset U of a Lie algebra $\mathfrak{g}$ we let $\langle\langle U\rangle\rangle$ denote the smallest Lie subalgebra of $\mathfrak{g}$ containing U.

Lemma 2.1 *Let $\mathfrak{n}$ be a nilpotent Lie algebra and $V \subseteq \mathfrak{n}$ a vector subspace such that $\mathfrak{n} = V + \mathfrak{n}'$. Then $\langle\langle V\rangle\rangle = \mathfrak{n}$, i.e., the smallest subalgebra containing V is $\mathfrak{n}$ itself.*

Proof. Consider the central descending series: $\mathcal{C}^0\mathfrak{n} = \mathfrak{n}, \mathcal{C}^k\mathfrak{n} = \mathrm{span}([\mathfrak{n}, \mathcal{C}^{k-1}\mathfrak{n}])$ for $k \in \mathbb{N}$. Since $\mathfrak{n}$ is nilpotent, there exists an $N \in \mathbb{N}$ such that $\mathcal{C}^N\mathfrak{n} = 0$.

Let $V_1 = V$ and define inductively $V_{k+1} = V_k + [V_k, V_k]$. Then it is clear that $\langle\langle V\rangle\rangle \supset V_k$ for all $k \in \mathbb{N}$. We use induction to prove that $V_N = \mathfrak{n}$. Indeed, $\mathfrak{n} = V_1 + \mathcal{C}^1\mathfrak{n}$ by assumption. Now suppose that $\mathfrak{n} = V_k + \mathcal{C}^k\mathfrak{n}$. Then every $X \in \mathfrak{n}$ can be written as $X = Y + Z$ with $Y \in V_k$, $Z \in \mathcal{C}^k\mathfrak{n}$. By definition of $\mathcal{C}^k\mathfrak{n}$ Z may be written as

$$Z = \sum_i [X_i, X_i'], \quad \text{with} \quad X_i \in \mathfrak{n}, \quad X_i' \in \mathcal{C}^{k-1}\mathfrak{n}.$$

Now decompose $X_i = Y_i + Z_i$, $X_i' = Y_i' + Z_i'$ with $Y_i, Y_i' \in V_k$ and $Z_i, Z_i' \in \mathcal{C}^k\mathfrak{n}$. Then

$$Z = \sum_i \underbrace{[Y_i, Y_i']}_{\in V_{k+1}} + \underbrace{[Y_i, Z_i'] + [Z_i, X_i']}_{\in \mathcal{C}^{k+1}\mathfrak{n}}.$$

Hence $\mathfrak{n} = V_{k+1} + \mathcal{C}^{k+1}\mathfrak{n}$ follows. This proves our claim. □

Corollary 2.2 *For a nilpotent Lie algebra $\mathfrak{n}$ the following statements are equivalent:*

(i) $\mathfrak{n}$ *is I_nD_m-controllable,*

(ii) $\mathfrak{n}/\mathfrak{n}'$ *is I_nD_m-controllable,*

(iii) $n \geq \dim(\mathfrak{n}) - \dim(\mathfrak{n}')$ *or* $n + m > \dim(\mathfrak{n}) - \dim(\mathfrak{n}')$.

In particular, there are no SID-controllable nilpotent Lie algebras of $\dim > 1$, *and the following statements are equivalent:*

(iv) $\mathfrak{n}$ *is* MID_k*-controllable,*

(v) $\mathfrak{n}$ *is I_kD_0-controllable,*

(vi) $n \geq \dim(\mathfrak{n}) - \dim(\mathfrak{n}')$.

In particular, I_nD_m-controllability is generic iff I_nD_0-controllability is generic, i.e., iff $n \geq \dim(\mathfrak{n}) - \dim(\mathfrak{n}')$.

Proof. The implication (i)$\Rightarrow$(ii) is trivial, and (ii)$\Rightarrow$(iii) is obvious because $\mathfrak{n}/\mathfrak{n}'$ is abelian, i.e., a vector space. In order to prove (iii)$\Rightarrow$(i) we pick $A_1, \ldots, A_m$, $B_1, \ldots, B_n$ such that $\mathbb{R}^+A_1 + \cdots + \mathbb{R}^+A_m + \mathbb{R}B_1 + \cdots + \mathbb{R}B_n$ is a vector subspace V of $\mathfrak{n}$ satisfying $V + \mathfrak{n}' = \mathfrak{n}$. By Lemma 2.1 $\langle\langle V \rangle\rangle = \mathfrak{n}$. So $\mathfrak{n}$ is I_nD_m-controllable.

Now the remaining statements follow immediately. □

Now we turn to solvable Lie algebras.

Lemma 2.3 *Let $\mathfrak{g}$ be a solvable Lie algebra and $V \subseteq \mathfrak{g}$ a vector subspace such that $\mathfrak{g} = V + D^2\mathfrak{g}$. Then $\langle\langle V \rangle\rangle = \mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is solvable, $\mathfrak{g}' =: \mathfrak{n}$ is nilpotent, and $D^2\mathfrak{g} = \mathfrak{n}'$. As $\mathfrak{g} = V + \mathfrak{n}'$, $\mathfrak{n} = (V \cap \mathfrak{n}) + \mathfrak{n}'$ follows. By Lemma 2.1 $\langle\langle V \cap \mathfrak{n} \rangle\rangle = \mathfrak{n}$, hence

$$\langle\langle V \rangle\rangle \supseteq V + \mathfrak{n} = \mathfrak{g}.$$

Thus our claim is proved. □

We call a subset U of a Lie algebra $\mathfrak{g}$ controllable if $\langle \exp(\mathbb{R}^+U) \rangle = G$ where G is the simply connected group associated with $\mathfrak{g}$.

Lemma 2.4 *Let $\mathfrak{g}$ be solvable and let $\pi \colon \mathfrak{g} \to \mathfrak{g}/D^2\mathfrak{g}$ denote the canonical projection. Then the following statements are equivalent:*

(1) *$U \subseteq \mathfrak{g}$ is controllable,*

(2) *$\pi(U)$ is controllable.*

Proof. The implication (1)$\Rightarrow$(2) is trivial since $\pi\left(\langle \exp(\mathbb{R}^+U) \rangle\right) = \langle \exp(\mathbb{R}^+\pi(U)) \rangle$.

Conversely, suppose that $\pi(U)$ is controllable. Let $\langle\langle U \rangle\rangle$ denote the subalgebra of $\mathfrak{g}$ generated by U. Since $\pi(U)$ is controllable, $\pi(\mathfrak{g}) = \langle\langle \pi(U) \rangle\rangle = \pi(\langle\langle U \rangle\rangle)$, hence $\langle\langle U \rangle\rangle + D^2\mathfrak{g} = \mathfrak{g}$, whence $\langle\langle U \rangle\rangle = \mathfrak{g}$ by Lemma 2.3.

If U were not controllable, then by Theorem 1.4 there would exist a halfspace $\mathfrak{h}^+ \subseteq \mathfrak{g}$ whose bounding hyperplane $\mathfrak{h}$ is a subalgebra such that $U \subseteq \mathfrak{h}^+$.

Let $\Delta(\mathfrak{g})$ denote the intersection of all hyperplane subalgebras of $\mathfrak{g}$, (this is the so-called Δ-radical, cf. [2, Definition 3]). Note that $\Delta(\mathfrak{g})$ is an ideal, and that $\Delta(\mathfrak{g}) \supseteq D^2\mathfrak{g}$, cf. [2, Remark 7(i)].

Hence $\mathfrak{h} \supseteq \ker \pi$, whence $\pi(\mathfrak{h})$ is a hyperplane subalgebra of $\mathfrak{g}/D^2\mathfrak{g}$. Thus $\pi(U) \subseteq \pi(\mathfrak{h}^+)$, a halfspace in $\mathfrak{g}/D^2\mathfrak{g}$ bounded by a hyperplane subalgebra. So $\pi(U)$ is not controllable. This contradiction proves that U must be controllable. □

3 Proof of the main theorem

Now we are ready to prove Theorem 1.3.
Proof. First we show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), then (2) $\Rightarrow$ (4) $\Rightarrow$ (5).

(1) $\Rightarrow$ (2) Immediate. Let $\mathfrak{i} \lhd \mathfrak{g}$ be an ideal. If $A, \pm B \in \mathfrak{g}$ is a controllable pair, then so is $\pi(A), \pm\pi(B)$ in $\mathfrak{g}/\mathfrak{i}$ where $\pi: \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ is the canonical projection.

(2) $\Rightarrow$ (3) Trivial since $D^2\mathfrak{g}$ is an ideal.

(3) $\Rightarrow$ (1) Let $\pi: \mathfrak{g} \to \mathfrak{g}/D^2\mathfrak{g}$ denote the canonical projection. Now suppose that $\mathfrak{g}/D^2\mathfrak{g}$ is SID-controllable. Then we can find $A, B \in \mathfrak{g}$ such that $\{\pi(A), \pm\pi(B)\}$ is controllable. Now Lemma 2.4 implies that $\{A, \pm B\} \subseteq \mathfrak{g}$ is controllable, hence $\mathfrak{g}$ is SID-controllable.

(2) $\Rightarrow$ (4) Two-dimensional Lie algebras are almost abelian. It is well-known that this implies that every proper wedge is a Lie wedge and noncontrollable, cf.[1, Thm. II.2.30]. Hence $\mathfrak{g}$ must not contain ideals of codimension 2.

(4) $\Rightarrow$ (5) Since $\mathfrak{g}'$ is an ideal, so is every subspace containing $\mathfrak{g}'$, whence $\dim(\mathfrak{g}) - \dim(\mathfrak{g}') = 1$ follows from (4).

Suppose that $\mathfrak{h}$ is another hyperplane subalgebra. The classification of hyperplane subalgebras [2, Proposition 2] tells us that there are only two possibilities.

First, if $\mathfrak{h}$ is not an ideal, then $\mathfrak{g}$ has an ideal $\mathfrak{i} \lhd \mathfrak{g}$ of codimension 2. This contradicts (4).

Second, if $\mathfrak{h}$ is an ideal, then $\mathfrak{h} \cap \mathfrak{g}'$ is an ideal, too. But this implies $\mathfrak{h} = \mathfrak{g}'$, for otherwise $\mathfrak{h} \cap \mathfrak{g}'$ would be an ideal of codimension 2 in $\mathfrak{g}$, again a contradiction to (4).

$\square$

Controllability of metabelian Lie algebras is easily characterized, cf. [7, Theorem 15.1]. Recall, that $\mathfrak{g}$ is metabelian iff $D^2\mathfrak{g} = \{0\}$. This means that $\mathfrak{g}'$ is abelian and $\mathfrak{g}$ is a semidirect product $V \rtimes \mathfrak{a}$ of a vector space V and an abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{gl}(V)$.

Lemma 3.1 *Let $\mathfrak{g}$ be a metabelian Lie algebra. Then the following statements are equivalent:*

(1) *$\mathfrak{g}$ is SID-controllable,*

(2) *$\mathfrak{g} \cong \mathbb{R}^k \rtimes_A \mathbb{R}$ where $A \in \mathfrak{gl}(k)$ has nonreal spectrum and every eigenvalue of A has geometric multiplicity 1, i.e., $\operatorname{spec}(A) \cap \mathbb{R} = \emptyset$ and for all $\lambda \in \mathbb{C}$ $\operatorname{rank}(A - \lambda I) \geq k - 1$.*

Remark 3.2 The multiplicity condition shows that SID-controllability of $\mathfrak{g}/\Delta(\mathfrak{g})$ does not imply that $\mathfrak{g}$ itself is SID-controllable. Simply take $\mathfrak{g} = \mathbb{R}^4 \rtimes_\phi \mathbb{R}$ where

$$\phi = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}, \qquad b \neq 0.$$

Then $\Delta(\mathfrak{g}) = \mathfrak{g}'$, and $\mathfrak{g}/\Delta(\mathfrak{g}) \cong \mathbb{R}$ is SID-controllable. But since $\operatorname{spec}(\phi) = \{a \pm ib\} = \{\lambda, \bar{\lambda}\}$, we have

$$0 = (\phi - \lambda I)(\phi - \bar{\lambda} I) = \phi^2 - 2\Re(\lambda)\phi + |\lambda|^2 I.$$

Hence one needs at least three elements in order to generate $\mathfrak{g}$ (as a Lie algebra). In fact, a system of the form $\mathbb{R}B + \mathbb{R}^+ A_1 + \mathbb{R}^+ A_2$ would generate all of $\mathfrak{g}$, so $\mathfrak{g}$ is $I_1 D_2$-controllable.

Thus we can conclude our characterization of SID-controllable solvable Lie algebras with:

Corollary 3.3 *A solvable Lie algebra $\mathfrak{g}$ is SID-controllable iff*

(6) $\mathfrak{g} \cong \mathfrak{n} \rtimes_\delta \mathbb{R}$, *where $\mathfrak{n}$ is a nilpotent Lie algebra and δ a derivation of $\mathfrak{n}$ with the following property: if ϕ denotes the endomorphism on $\mathfrak{n}/\mathfrak{n}'$ induced by δ, then* $\operatorname{spec}(\phi) \subseteq \mathbb{C} \backslash \mathbb{R}$, *and every eigenvalue of ϕ has geometric multiplicity* 1.

In particular, if $\mathfrak{g}$ is SID-controllable, then $\dim(\mathfrak{g}') - \dim(D^2\mathfrak{g}) \equiv 0 \pmod 2$.

We conclude with a statement about $I_n D_m$-controllability. The first statement is an immediate consequence of Lemma 2.4 while the last statement follows directly from Lemma 2.3.

Corollary 3.4 *For a solvable Lie algebra $\mathfrak{g}$ the following statements are equivalent:*

(1) $\mathfrak{g}$ *is $I_n D_m$-controllable,*

(2) $\mathfrak{g}/D^2\mathfrak{g}$ *is $I_n D_m$-controllable.*

In particular, if $n \geq \dim(\mathfrak{g}/D^2\mathfrak{g})$ or $n + m > \dim(\mathfrak{g}/D^2\mathfrak{g})$, then $\mathfrak{g}$ is $I_n D_m$-controllable.

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