Let $\phi \; : \; \Re \to \Re$ be a C^∞ map satisfying the following conditions: $\phi(x) \le 1/5$ and

$$\phi(x) = \begin{cases} 0 & x \le 0 \\ > 0 & x < 0 \end{cases}$$

Consider the control process

$$\begin{aligned}
&\dot{x}_0 = 1 \\
&\dot{\alpha} = u[\phi(-x_0) + \phi(x_0 - \alpha^2)] \\
&\dot{a} = v[\phi(-x_0) + \phi(x_0 - \alpha^2)] \\
&\dot{b} = w[\phi(-x_0) + \phi(x_0 - \alpha^2)] \\
&\dot{x} = a^4[\phi(x_0) + \phi(\alpha^2 - x_0)] \\
&\dot{y} = b^4[\phi(x_0) + \phi(\alpha^2 - x_0)] \\
&\dot{z} = a^2 b^2 \phi(x_0)\phi(\alpha^2 - x_0)
\end{aligned}$$

absolute value of u, v, w less than or equal to 1. Take as reference trajectory $t \mapsto \xi(t) = (t, 0, \ldots, 0)$ in the interval $[-1, 1]$ and as reference control the one identically equal to 0.

The reachable set from the initial point $(-1, 0, \ldots, 0)$ is contained in the set

$$\{\, x \ge 0,\ y \ge 0,\ z \ge 0,\ xy = z^2\};$$

moreover the following Lemma can be proved:

Lemma. For each positive number $\overline{\alpha}$ there exists $c(\overline{\alpha}) > \overline{\alpha}^2$ such that if

$$u(t) \equiv -1 \qquad\qquad t \in [\overline{\alpha}^2, c(\overline{\alpha})]$$

then the solution of

$$\begin{cases} \dot{x}_0 = 1 \\ \dot{\alpha} = u(t)[\phi(x_0) + \phi(x_t - \alpha^2)] \\ x_0(0) = 0 \\ \alpha(0) = \overline{\alpha} \end{cases}$$

is such that

$$\alpha(t) = \begin{cases} \overline{\alpha} & t \in [0, \overline{\alpha}^2] \\ 0 & t = c(\overline{\alpha}). \end{cases}$$

Let us remark that the number $c(\overline{\alpha})$ depends continuously on $\overline{\alpha}$ and goes to zero with $\overline{\alpha}$.

Let $\alpha(\epsilon)$ be the value at time 0 of the solution of

$$\begin{cases} \dot{\alpha} = [\phi(-t) + \phi(t - \alpha^2)] \\ \alpha(-\epsilon) = 0 \end{cases}$$

$0 < \alpha(\epsilon) < \epsilon/5$. Consider the needlelike control variation at $t = 0$

$$u(t) = \begin{cases} 1 \quad t \in [-\epsilon, 0] \\ -1 \quad t \in [\alpha^2(\epsilon), c(\alpha(\epsilon))] \\ 0 \quad \text{otherwise} \end{cases}$$

$$v(t) \equiv u(t)$$
$$w(t) \equiv 0$$

and let us compute the solution relative to this control variation at time $c(\alpha(\epsilon))$.

$$x_0 = c(\alpha(\epsilon)); \quad \alpha = a = b = y = z = 0; \qquad x = x(\epsilon)$$

with $x(\epsilon) > 0$. Therefore the vector $(0,0,0,0,1,0,0)$ is a needle-like variation at 0.

Analogously the control variation

$$u(t) = \begin{cases} 1 \quad t \in [-\epsilon, 0] \\ -1 \quad t \in [\alpha(\epsilon), c(\alpha(\epsilon))] \\ 0 \quad \text{otherwise} \end{cases}$$
$$v(t) \equiv 0$$
$$w(t) \equiv u(t)$$

produces the needlelike trajectory variation at 0, $(0,0,0,0,0,1,0)$. Let us consider the control variation

$$
u(t) = \begin{cases} 1 & t \in [-\epsilon, 0] \\ -1 & t \in [\alpha(\epsilon), c(\alpha(\epsilon))] \\ 0 & \text{otherwise} \end{cases}
$$

$$v(t) \equiv u(t)$$

$$w(t) \equiv u(t)$$

and let us calcolate the value at time $c(\alpha(\epsilon))$ of the solution relative to this control variation, $h(\epsilon)$.

$$x_0 = c(\alpha(\epsilon)); \quad \alpha = a = b = 0; \quad x = y = x(\epsilon); \quad z = x(\epsilon)^2.$$

Therefore

$$h(\epsilon) = (c(\alpha(\epsilon)), 0, 0, 0, 0, 0, 0) + (0, 0, 0, 0, x(\epsilon), x(\epsilon), 0) + o(x(\epsilon))$$

which proves that $1/2(0,0,0,0,1,0,0) + 1/2(0,0,0,0,0,1,0)$ is a needle-like variation at 0.

Notice that the three variations we have considered cannot belong to the interior of the cone given by the collection of needle-like variation at 0 because variations with the coordinate z negative cannot exists.