Partial synchronization: from symmetry towards stability

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Abstract

In this paper we study the existence and stability of linear invariant manifolds in a network of coupled identical dynamical systems. Symmetry under permutation of different units of the network is helpful to construct explicit formulae for linear invariant manifolds of the network, in order to classify them, and to examine their stability *a priori* through Lyapunov's direct method.

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1 Introduction

The high number of scientific contributions in the field of synchronization of coupled dynamical systems reflects the importance of this subject. The reason for this importance appears to be threefold: synchronization is common in nature, coupled dynamical systems display a very rich phenomenology and, finally, it can find applications.

First of all, many situations can be modelled as ensembles of coupled oscillators. Considering some examples from the natural world, there is evidence of synchrony among pulse-coupled biological oscillators [1], which relates to the observation that thousands of male fireflies can gather in trees and flash at unison (a very nice color picture of this event can be found in [2]). Synchronous activity has also been observed in many regions of the human brain,

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relative to behaviour and cognition, where neurons can electrically discharge synchronously in some known frequency ranges [3], a behaviour that is reproduced by different mathematical models [4]. The synchronous firing of cardiac pacemaker cells [5] in human hearts is another example where synchronized motion is reported [6]. Among other evidences of synchronous behaviour in the natural world, one can additionally consider the chorusing of crickets and the metabolic synchrony in yeast cell suspension [7].

The rich phenomenology constitutes another reason for the importance of these studies. Coupled dynamical systems have been shown to give rise to rather complex phenomena not previously observed, especially when chaotic motion is considered as output of a single dynamical system (see [8,9], for example, for two significant reviews). If the common output of several synchronized dynamical systems is chaotic, there is evidence that a new type of intermittent behaviour can result at the onset of the threshold of attractivity of the synchronized state [10], as well as the presence of very complex domains of attraction when several attracting solutions exist [11]. These two new phenomena are peculiar to those systems with invariant subsets, as suitably coupled identical dynamical systems are. A more detailed explanation of the onset of attractivity and stability of a synchronized state in coupled chaotic systems can be found in [12,13].

The importance of synchronized motion does not lie only in those situations in which synchronization can be found, but also where synchronous motion can be induced to ensure the proper functioning of a particular device [14,15]. Consider, for example, active integrated antennas [16] that can be built as arrays of multiple coupled oscillators to generate circular polarization [17]. In robotics, the problem of synchronization is usually referred to as coordination, or cooperation [18,19]. The problem appears when two or more robot-manipulators have to operate synchronously, especially in situations when some of them operate in hazardous environment, while others (that serve as reference) may be guided by human operation. Another interesting problem is to study (and control) spatiotemporal patterns in an ensemble of coupled systems. This problem is of interest in connection with many applications in communication engineering [20]. Synchronization is therefore important, so it is especially important to develop criteria that guarantee its *asymptotic* stability, if applications are sought.

In this paper we consider networks of identical systems coupled through diffusion, and we give conditions that guarantee asymptotic stability of a particular invariant manifold (*i.e.* a synchronous state) of a given network. Synchronous motion is most often understood as the equality of corresponding variables of two identical systems. In other words, the trajectories of two (or more) identical systems will follow, after some transient, the same path in time. This situation is not, of course, the only commonly understood situation of syn-

chronization. Other different relationships between coupled systems can be considered synchronous. More generally (and this is the viewpoint taken in [27] to draw a suitable definition of "synchronous behaviour" in a coordinate-independent way), synchronization with respect to some functional happens when this functional has always zero value, for all trajectories of the dynamical system on the synchronous state.

We study the existence and stability of linear invariant manifolds of a network of coupled systems, defined as the equality of some outputs of some systems only, a condition that is nowadays popular in the physical literature as *partial synchronization*. Considering that a network can be formed by an arbitrary number of units, the number of different existing linear invariant manifolds may be large. Therefore, several aspects should be taken into consideration for careful examination.

The first aspect is the *classification* of linear invariant manifolds. As previously mentioned, coupled dynamical systems can show a very rich phenomenology, so it is meaningful to find systematic ways to classify different invariant sets. Zhang *et al.*[21], for example, report the result of their studies of a chain of identical Rössler systems, showing that there are several possible synchronous states with different types of correspondences between the variables of different systems.

The second aspect to keep in mind is the possibility of *inclusion* of different invariant manifolds, an aspect studied in [22,23]. Briefly, it is possible that different synchronous states between coupled oscillators will appear by increasing some specified bifurcation parameter. This possibility defines a hierarchy, that arranges different invariant manifolds in a particular order.

Symmetry considerations are helpful to classify several invariant sets, and a possible hierarchy to accommodate them. Symmetry is a qualitative feature of a network of coupled units but, if quantified, it provides statements like "more symmetry" or "less symmetry" with a rigorous meaning. The symmetry generated by the coupling only has been termed *global*, to distinguish it from the additional symmetries brought upon by the dynamical systems modelling each unit, that has been termed *internal*. This terminology has been introduced in [24] where it is studied how these two groups of symmetries interact.

In this paper we exploit symmetry under permutation of a given network of dynamical systems coupled through diffusion in order to classify some linear invariant manifolds and investigate their stability. More specifically, we see that to any specific symmetry it is associated a linear invariant manifold, and we show how to construct a Lyapunov function to determine its stability, from the same specific symmetry. Therefore, under the conditions formulated in this work, stability in the network descends from its topology.

The paper is organized as follows. The problem statement is explained in Section 2, where the network dynamics is outlined. In Section 3 the association between symmetry and the linear invariant manifolds of the network is discussed. Section 4 begins with some background material from control theory, after which we propose a proof of asymptotic stability of a compact subset of a specified linear invariant manifold. Section 5 shows some relevant examples in connection to the theory.

Throughout the paper we use the following notations. I_k denotes the $k \times k$ identity matrix. The Euclidean norm in $\mathbb{R}^n$ is denoted simply as $|\cdot|$, $|x|^2 = x^\top x$, where $^\top$ defines transposition. The notation $\mathrm{col}(x_1, \ldots, x_n)$ stands for the column vector composed of the elements $x_1, \ldots, x_n$. This notation will also be used in case where the components x_i are vectors again. A function $V : X \to \mathbb{R}_+$ defined on a subset X of $\mathbb{R}^n$, $0 \in X$ is *positive definite* if $V(x) > 0$ for all $x \in X \setminus \{0\}$ and $V(0) = 0$. It is *radially unbounded* (if $X = \mathbb{R}^n$) or *proper* if $V(x) \to \infty$ as $|x| \to \infty$. If a quadratic form $x^\top P x$ with a symmetric matrix $P = P^\top$ is positive definite then the matrix P is called positive definite. For positive definite matrices we use the notation $P > 0$; moreover $P > Q$ means that the matrix $P - Q$ is positive definite. For matrices A and B the notation $A \otimes B$ (the Kronecker product) stands for the matrix composed of submatrices $A_{ij}B$, *i.e.*

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{pmatrix}$$

where A_{ij}, $i, j = 1 \ldots n$, stands for the ij-th entry of the $n \times n$ matrix A.

2 Problem statement

The subject of our research is the existence and stability of partial synchronization regimes in diffusive networks. To make the problem statement clearer, we start our discussion by introducing the concept of diffusive network. In 1976 Smale [25] proposed a model of two interacting cells based on two identical coupled oscillators, and noticed that diffusion, rather counterintuitively, does not necessarily smooth out differences between the two systems' outputs, giving the example of two stable systems that can display oscillations when connected via diffusive coupling. Taking inspiration from Smale's previous research, a *diffusive cellular network* describes a network composed of

identical dynamical systems coupled through diffusive coupling that cannot be decomposed into two or more disconnected smaller networks.

To put these statements into a more mathematical description, let us consider k identical systems of the form

$$\begin{cases} \dot{x}_j = f(x_j) + Bu_j \\ y_j = Cx_j \end{cases} \tag{1}$$

where $j = 1, \ldots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the j-th system, $u_j(t) \in \mathbb{R}^m$ and $y_j(t) \in \mathbb{R}^m$ are, respectively, the input and the output of the j-th system, and B, C are constant matrices of appropriate dimension. This representation of the dynamics of the elements of the network is most common in control theory, and it underlines that a description of a dynamical system is not complete unless inputs and outputs are specified. It also reflects the idea that a dynamical system can be viewed in a model-independent representation, through its input-output characteristics. For instance, interaction among cells, in the example quoted by Smale, can be viewed as a static relationship between inputs and outputs. In this representation we can say that the k systems (1) are *diffusively coupled* if the matrix CB is similar to a positive definite matrix, and the k systems are interconnected through mutual linear output coupling,

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \ldots - \gamma_{jk}(y_j - y_k) \tag{2}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ are constants such that $\sum_{j \neq i}^{k} \gamma_{ji} > 0$ for all $i = 1, \ldots, k$. With no loss of generality we assume in the sequel that CB *is* a positive definite matrix. Some results we present in this paper certainly hold for other types of coupling as well, but this special form of coupling, apart from being very naturally linked to the interaction of some real-life systems, allows us to obtain analytical results concerning the stability of some solutions of the whole network.

Define the symmetric $k \times k$ matrix Γ as

$$\Gamma = \begin{pmatrix} \sum_{i=2}^{k} \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1, i\neq 2}^{k} \gamma_{2i} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix} \tag{3}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ and all row sums are zero. The matrix Γ is symmetric and therefore all its eigenvalues are real. With the definition (3), the collection of k systems (1) with the feedback (2) can be rewritten in the more compact form

$$\begin{cases} \dot{x} = F(x) + (I_k \otimes B)u \\ y = (I_k \otimes C)x \end{cases} \tag{4}$$

with the feedback given by

$$u = -(\Gamma \otimes I_m)y, \tag{5}$$

where we denoted $x = \text{col}(x_1, \ldots, x_k)$, $F(x) = \text{col}(f(x_1), \ldots, f(x_k)) \in \mathbb{R}^{kn}$, $y = \text{col}(y_1, \ldots, y_k)$, and $u = \text{col}(u_1, \ldots, u_k) \in \mathbb{R}^{km}$. The matrix Γ in (3) displays some interesting properties: first, it is singular (all row sums are zero), so it has a zero eigenvalue and, according to Gerschgorin's theorem (see [26], for example), it is positive semidefinite, that is, all its eigenvalues are nonnegative. Additionally, if zero is a simple eigenvalue of Γ, the network of the diffusively coupled systems can not be divided into two or more disconnected networks.

The matrix Γ is a useful mathematical object in the study of solutions of (4,5), because it contains information on the topology of the network, with properties that hold independently of the particular dynamical system $f(\cdot)$ employed to model each of its elements. Consider, for instance,

- its size (the number of coupled systems, hence the dimension k of Γ),
- its interconnections (the dimension of kerΓ is the number of disconnected networks),
- the local density of interconnections (the number of systems connected to a specific one, that is, how many nonzero elements in a specific row or column),
- the strength of the interconnections (the constants γ_{ij}) and,
- last but not least, eventual symmetries the network may possess.

Dynamical systems constructed as in (1,2), specifically, using identical systems (the same $f(\cdot)$), with identical input and output forms (the same matrices B and C for all k systems), interconnected by diffusive coupling, allow solutions with equality of all (or some) of its states x_j. These situations can be described as fully and partially synchronized. We prefer not to dwell too much here on the suitability of a particular definition of synchronization, as reported in few research papers [27–29], but we simply define *full synchronization* the particular situation in which all states of all systems are identical (*i.e.* $x_i = x_j$, $\forall i, j = 1, ..., k$), and with *partial synchronization* all situations in which the states of some systems are identical, but not all of them (*i.e.* $x_i = x_j$, for some i, j). These are situations in which the overall dynamics of the network takes place on a linear invariant manifold, specified by the equality of some outputs.

All main points have now been introduced in order to formulate a clear prob-

lem statement. Can we exploit symmetry in the network to identify its linear invariant manifolds, and benefit from a representation of the system as (1,2), and/or (4,5), typical for control purposes, in order to give conditions that *guarantee* stability of some selected partial (or the full) synchronized states?

3 Symmetries and invariant manifolds

If a given network possesses a certain symmetry, this symmetry must be present in the matrix Γ. In particular, the network may contain some repeating patterns, when considering the arrangements of the constants γ_{ij}, hence the permutation of some elements will leave the network unchanged. The matricial representation of a permutation σ of the set $\{1, 2, ..., k\}$ is a permutation matrix $\Pi \in \mathbb{R}^{k\times k}$. Briefly, if $\varepsilon_1, ...\varepsilon_k$ denote the columns of I_k, the permutation matrix Π associated with σ is the matrix obtained from I_k by permuting its columns under σ, that is, the columns of Π are $\varepsilon_{\sigma(1)}, ...\varepsilon_{\sigma(k)}$. If S_k is the set of all permutations of $\{1, 2, ..., k\}$ it is possible to prove that the set of all $k \times k$ permutation matrices forms a group that is isomorphic to S_k [30]. Permutation matrices are orthogonal, *i.e.* $\Pi^\top\Pi = I_k$, and they form a group with respect to the multiplication, so for any two permutation matrices Π_i, Π_j of the same size, $\Pi_i\Pi_j$ is a permutation matrix too.

Rewrite the dynamics of (4,5) in the closed loop form

$$\dot{x} = F(x) + Gx \tag{6}$$

where $G = -(I_k \otimes B)(\Gamma \otimes I_m)(I_k \otimes C) \in \mathbb{R}^{kn\times kn}$, that can be simplified as $G = -\Gamma \otimes BC$. Let us recall here that given a dynamical system as (6), the linear manifold $\mathcal{A}_M = \{x \in \mathbb{R}^{kn} \ : \ Mx = 0\}$, with $M \in \mathbb{R}^{kn\times kn}$, is *invariant* if $M\dot{x} = 0$ whenever $Mx = 0$, that is, if at a certain time t_0 a trajectory is on the manifold, $x(t_0) \in \mathcal{A}_M$, then it will remain there for all time, $x(t) \in \mathcal{A}_M$ for all t. The problem can be summarized in the following terms: given G and $F(\cdot)$ find a solution M to

$$MF(x(t_0)) + MGx(t_0) = 0 \tag{7}$$

for all $x(t_0)$ for which $Mx(t_0) = 0$. There is no general solution to this, however, if these objects satisfy certain properties, it is possible to find a class of matrices M that solve (7). A natural way to do this is to exploit the symmetry of the network.

3.1 Global symmetries

In the representation (6), we can establish conditions to identify those permutations that leave a given network invariant. It can be easily derived that Π is a symmetry for the network (6) if

$$(\Pi \otimes I_n)G = G(\Pi \otimes I_n)$$

that is, if Π and Γ commute, $\Pi\Gamma = \Gamma\Pi$. For these symmetries, a first result follows directly from the properties of the permutation matrices. Let $\Sigma = \Pi \otimes I_n$ for simplicity, and assume that, at time t_0 it is $(I_{kn} - \Sigma)x(t_0) = 0$. Consider (6), and suppose that there is a permutation matrix Π commuting with Γ. Hence, also $\Sigma G = G\Sigma$, and since Π is a permutation matrix, it also follows that $\Sigma F(x) = F(\Sigma x)$. If we multiply both sides of (6) by $I_{kn} - \Sigma$, we obtain, at time t_0,

$$(I_{kn} - \Sigma)\dot{x}(t_0) = F(x(t_0)) - F(\Sigma x(t_0)) + G(I_{kn} - \Sigma)x(t_0) = 0$$

because we assumed $(I_{kn} - \Sigma)x(t_0) = 0$. Therefore, it is $(I_{kn} - \Sigma)x(t) = 0$ for all t, and we can reformulate this result as: *given a permutation matrix* Π *that commutes with* Γ*, the set*

$$\ker(I_{kn} - \Pi \otimes I_n) \tag{8}$$

is a linear invariant manifold for system (6).

Note that global symmetries have the property of leaving the system equations (4) and (5) invariant. One can see that, applying the linear transformation $x \to \bar{x} = (\Pi \otimes I_n)x$ to the network, if x and y are, respectively, state and output of (4), then $\bar{x}$ and $\bar{y} = (\Pi \otimes I_m)y$ are state and output of the same system. Additionally, if Π and Γ commute, also $\bar{u} = -(\Gamma \otimes I_m)\bar{y}$, where $\bar{u} = (\Pi \otimes I_m)u$. Hence, the triple $(\bar{x}, \bar{y}, \bar{u})$ satisfies the same equations (4), (5), as the original triple (x, y, u).

Example 1: Ring of 4 coupled oscillators

Consider the example of four coupled systems (1,2) in a ring, as shown schematically in Figure 1. In this Figure we have imposed the following symmetry in coupling constants: $\gamma_{12} = \gamma_{34} = K_0$, and $\gamma_{14} = \gamma_{23} = K_1$. The particular geometry of the coupling defines the following coupling matrix:

$$\Gamma = \begin{pmatrix} K_0+K_1 & -K_0 & 0 & -K_1 \\ -K_0 & K_0+K_1 & -K_1 & 0 \\ 0 & -K_1 & K_0+K_1 & -K_0 \\ -K_1 & 0 & -K_0 & K_0+K_1 \end{pmatrix}. \tag{9}$$

The four permutation matrices for which $\Pi\Gamma = \Gamma\Pi$ are

$$\Pi_1 = \begin{pmatrix} E & O \\ O & E \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} O & I_2 \\ I_2 & O \end{pmatrix}, \quad \Pi_3 = \begin{pmatrix} O & E \\ E & O \end{pmatrix}, \quad \Pi_4 = I_4 \tag{10}$$

where we denoted

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and O is the 2×2 zero matrix. Let us analyze what the action of these matrices Π is. The action of Π_1 is to switch *simultaneously* x_1 with x_2 and x_3 with x_4. One can easily notice from Fig. 1 that this operation leaves the network unchanged, with respect to its connections. Similar actions are brought upon by Π_2 and Π_3, while Π_4 is the identity, that leaves everything unchanged.

From our previous statements, we derive that the linear invariant manifolds associated with Π_1, Π_2 and Π_3 are, respectively,

$$\begin{aligned} \mathcal{A}_1 &= \{x \in \mathbb{R}^{4n} \ : \ x_1 = x_2, x_3 = x_4\} \\ \mathcal{A}_2 &= \{x \in \mathbb{R}^{4n} \ : \ x_1 = x_3, x_2 = x_4\} \\ \mathcal{A}_3 &= \{x \in \mathbb{R}^{4n} \ : \ x_1 = x_4, x_2 = x_3\}. \end{aligned} \tag{11}$$

The intersection of any two of these linear manifolds gives the linear manifold describing full synchronization (*i.e.* $x_1 = x_2 = x_3 = x_4$). These invariant manifolds (11) descend directly from the symmetries of Γ, hence they exist regardless of the particular form of $f(\cdot)$ chosen in (1).

3.2 Internal symmetries

Additional internal symmetries in the differential equations governing the dynamics of the elements of the network will lead to the existence of additional

linear invariant manifolds. Consider one uncoupled element of the network, $\dot{x}_j = f(x_j)$, with initial condition $x_j(0)$, generating the particular solution $x_j(t)$. It is easy to see that if

$$Jf(x_j) = f(Jx_j), \tag{12}$$

with $J \in \mathbb{R}^{n\times n}$ constant matrix, then $Jx_j(t)$ is a solution as well, generated by the initial condition $Jx_j(0)$. This property of $f(\cdot)$ defines an additional symmetry to the network, that originates additional invariant manifolds. As in the last argument, we can formulate the following, more general, statement: *suppose there is a permutation matrix* Π *commuting with* Γ*, and an* $n \times n$ *constant matrix* J *satisfying (12) for* $f(\cdot)$ *in (1), with* J *commuting with the* $n \times n$ *matrix* BC*. Then the set* $\ker(I_{kn} - \Pi \otimes J)$ *is a linear invariant manifold for system (6).* To prove this statement, let $\Sigma = \Pi \otimes J$ as before, for brevity, and assume $(I_{kn} - \Sigma)x(t_0) = 0$. This argument is the same used for the previous statement, it differs in just replacing I_n with J. If we multiply both sides of (6) by $I_{kn} - \Sigma$ we obtain

$$\begin{aligned}(I_{kn} - \Sigma)\dot{x}(t_0) &= (I_{kn} - \Sigma)F(x(t_0)) + (I_{kn} - \Sigma)Gx(t_0) = \\ &= F(x(t_0)) - F(\Sigma x(t_0)) + G(I_{kn} - \Sigma)x(t_0) = 0,\end{aligned}$$

where we made use of $\Sigma F(x) = F(\Sigma x)$ and assumed that J commutes with BC Therefore, if $(I_{kn} - \Sigma)x(t_0) = 0$, it will be $(I_{kn} - \Sigma)x(t) = 0$ for all t. A popular case is represented by the *anti-phase synchronization* that may take place in systems whose differential equations are odd functions of the state vector, that is, $f(-x_j) = -f(x_j)$, or $J = -I_n$.

Therefore, for any element Π from the group of global symmetries and for any element J, that commutes with BC, from the group of internal symmetries, the set

$$\ker(I_{kn} - \Pi \otimes J) \tag{13}$$

is a linear invariant manifold. The representation (13) reflects how two groups of symmetries contribute to the existence of invariant manifolds of a diffusive network.

If there is a positive integer L such that $J^L = I_n$, the J matrices form a finite group, hence the number of invariant manifolds generated by Π's and J's is finite.

Example 2: Four coupled Lorenz systems

Consider the network introduced in Example 1, and suppose the dynamics of each element of (1) is described by the Lorenz system, with $x_j =$

$(x_{j,1}, x_{j,2}, x_{j,3})^\top \in \mathbb{R}^3$,

$$\begin{cases} \dot{x}_{j,1} = \sigma(x_{j,2} - x_{j,1}) + u_j \\ \dot{x}_{j,2} = r x_{j,1} - x_{j,2} - x_{j,1} x_{j,3} \\ \dot{x}_{j,3} = -b x_{j,3} + x_{j,1} x_{j,2} \end{cases} \tag{14}$$

with output $y_j = x_{j,1}$. For Eq. (14), it is easy to see that if $(x_{j,1}(t), x_{j,2}(t), x_{j,3}(t))$ is a solution, $(-x_{j,1}(t), -x_{j,2}(t), x_{j,3}(t))$ is a solution too, so the matrix

$$J = \mathrm{diag}(-1 \quad -1 \quad 1) \tag{15}$$

is an internal symmetry for (14). In this example, $B = (1 \quad 0 \quad 0)^\top$ and $C = (1 \quad 0 \quad 0)$, and J commutes with BC. The three invariant manifolds given in (11) are associated with the matrices (10) as global symmetries, and (in absence of any information) I_3 as internal symmetry. In addition to (11), the linear invariant manifolds, defined as $\mathcal{A}'_i = \ker(I_{12} - \Pi_i \otimes J)$, associated with Π_i's in (10), and J in (15), are

$$\mathcal{A}'_1 = \{x \in \mathbb{R}^{12} \ : \ Jx_1 = x_2, Jx_3 = x_4\}$$
$$\mathcal{A}'_2 = \{x \in \mathbb{R}^{12} \ : \ Jx_1 = x_3, Jx_2 = x_4\}$$
$$\mathcal{A}'_3 = \{x \in \mathbb{R}^{12} \ : \ Jx_1 = x_4, Jx_2 = x_3\}$$

while the invariant manifold associated with $\Pi_4 = I_4$ is

$$\mathcal{A}'_4 = \{x \in \mathbb{R}^{12} \ : \ x_{j,1} = x_{j,2} = 0, \quad j = 1, ..., 4\},$$

and arbitrary $x_{j,3}$, $j = 1, ..., 4$.

Let us restrict for now with global symmetries only, that is, model-independent symmetries associated with Γ and represented by the matrices Π. For a given network we can "quantify" its symmetry simply by counting how many permutation matrices Π commute with the given Γ. Once the network is given, it is possible, in principle, to find out precisely how many invariant manifolds (associated with permutation symmetry) exist. Symmetry arguments solve the problem of classification of invariant manifolds, and they also give insights on the inclusion problem as well.

4 Stability analysis

Borrowing terminology from control theory, one can see the collection of k systems (1) as a *control system* with inputs u_j and outputs y_j, and the coupling

(2) as a *feedback*. This representation has already been useful to find a more compact way to identify linear invariant manifolds of the network, since specification of input and output is brought upon by matrices B and C. However, it also suggests a methodology when tackling a specific control goal. Consider systems of the form

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \tag{16}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^m$ is the output, $f(0,0) = 0$ and f, h are smooth enough to ensure existence and uniqueness of solutions. Suppose one needs to make the system (16) achieve some specific control goal (*e.g.* make the origin stable). The input-state-output representation is often more convenient when one intends to generalize the problem as to find a *class of possible feedbacks* which comply with the specified control goal (*e.g.* stabilize the system (16)). Any feedback u from this class is a solution to the control problem.

4.1 Passivity and semipassivity

We illustrate this approach by a simple example. Let the control goal be the stabilization of (16) to the origin. Suppose it is possible to find a scalar non-negative function V defined on $\mathbb{R}^n$, for which $V(0) = 0$, and whose derivative satisfies, along the solutions of (16), the inequality

$$\dot{V}(x,u) = \frac{\partial V(x)}{\partial x} f(x,u) \leq y^\top u. \tag{17}$$

The system (16) is then referred to as a *passive system* (see, *e.g.* [31]), the function V is called the *storage function* and the inequality (17) is called *dissipation inequality*. This inequality immediately defines a class of stabilizing feedbacks for the closed loop system (16). Additionally, if we search for the stabilizing feedback in the form $u = \phi(y)$, to prove that a given u belongs to the class of stabilizing feedbacks for (16) shifts to the verification of the inequality

$$y^\top \phi(y) \leq 0 \tag{18}$$

provided V is positive definite. To prove this statement one can simply consider the storage function V as a Lyapunov function candidate.

It is important to note that for a passive system with a positive definite storage function the dynamics corresponding to the constraint $y \equiv 0$ (that is commonly known as *zero dynamics*) is Lyapunov stable (see (17)). If the zero dynamics is asymptotically stable, systems of this sort are known as *minimum phase*, and it is an important class of systems for application design problems, since they possess some sort of internal stability.

Next, let us consider a different situation. Suppose that the dissipation inequality (17) is satisfied only for x lying outside some ball (or compact set). More rigorously, it can be represented as the following inequality

$$\dot{V}(x,u) \leq y^\top u - H(x) \tag{19}$$

where the function $H : \mathbb{R}^n \to \mathbb{R}$ is nonnegative outside some ball:

$$\exists \rho > 0,\ \forall |x| \geq \rho \Longrightarrow H(x) \geq \varrho(|x|) \tag{20}$$

for some continuous nonnegative function ϱ defined for $|x| \geq \rho$. In this case the system (16) is called *semipassive*. This notion was introduced in [32], while in [33] an equivalent notion was called *quasipassivity*. If the function H is positive outside some ball, *i.e.* (20) holds for some continuous positive function ϱ, then the system (16) is said to be *strictly semipassive*. In brief, a semipassive system behaves like a passive system for sufficiently large $|x|$.

It is important to observe that dissipation inequality (19) can be rewritten in an equivallent way as follows:

$$\frac{\partial V}{\partial x} f(x) \leq -H(x), \quad \frac{\partial V}{\partial x} B = x^\top C^\top.$$

In direct analogy with passive systems, from the inequality (19) one can specify the class of feedbacks which make the semipassive closed loop system (16) *ultimately bounded*, *i.e.* regardless of the initial conditions, all solutions of the closed loop system approach a compact set in a finite time and this compact set does not depend on the initial conditions.

Indeed, suppose that the system (16) is strictly semipassive and the storage function V satisfying the dissipation inequality (19) is *radially unbounded*, that is $V(x) \to \infty$ when $|x| \to \infty$, then any feedback $u = \phi(y)$ satisfying the inequality (18) makes the closed loop system ultimately bounded. This statement can be proven just by considering the storage function V as a Lyapunov function candidate.

This is an additional benefit of the representation of diffusive network equations as input-output systems. Under strict semipassivity, boundedness of solutions depends only on input-output relations of systems (1) and not on the particular interconnections within the network. The particular constants appearing in the feedback u are not crucial, as long as u satisfies inequality (19). For example, consider the input written in compact form $u = -(\Gamma \otimes I_n)y$ introduced in Section 3. $\Gamma \geq 0$, therefore u is a feedback that ensure boundedness of solutions of the network.

More precisely, *if the systems (1) are strictly semipassive with radially unbounded storage functions then all solutions of the coupled system (1), (2)*

exist for all $t \geq 0$ *and are ultimately bounded*. The technical proof of this statement and more general related results can be found in [32].

4.2 Convergent systems

Consider a dynamical system of the form

$$\dot{z} = q(z, w(t)), \tag{21}$$

with $z \in \mathbb{R}^l$, driven by the external signal $w(t)$ taking values from some compact set. This system is said to be *convergent* [34] if for any bounded signal $w(t)$ defined on the whole time interval $(-\infty, +\infty)$ there is a unique bounded, globally asymptotically stable solution $\bar{z}(t)$ defined on the same interval $(-\infty, +\infty)$, from which it follows

$$\lim_{t\to\infty} |z(t) - \bar{z}(t)| = 0, \tag{22}$$

for all initial conditions. In systems of this type the limit mode of them is solely determined by the external excitation $w(t)$, not by the initial conditions of z. From the existence of a unique mode $\bar{z}(t)$, it obviously descend that two identical copies of convergent system z_1 and z_2 (21) must synchronize, that is, if (22) holds,

$$\lim_{t\to\infty} |z_1(t) - z_2(t)| = 0$$

holds as well. Convergence is then closely related to synchronization, hence it is important to find conditions ensuring it. We present here a slight modification of a result obtained by Demidovich [34]: *if there is a positive definite symmetric* $l \times l$ *matrix* P *such that all eigenvalues* $\lambda_i(Q)$ *of the symmetric matrix*

$$Q(z, w) = \frac{1}{2}\left[P\left(\frac{\partial q}{\partial z}(z, w)\right) + \left(\frac{\partial q}{\partial z}(z, w)\right)^{\top} P\right] \tag{23}$$

are negative and separated from zero, i.e. there is $\delta > 0$ *such that*

$$\lambda_i(Q) \leq -\delta < 0, \tag{24}$$

with $i = 1, ..., l$ *for all* $z, w \in \mathbb{R}^l$*, then system (21) is convergent, and there exist a quadratic function* $W(\zeta) = \zeta^{\top} P \zeta$ *satisfying the inequality*

$$\frac{\partial W(z_1 - z_2)}{\partial \zeta}\left(q(z_1, w) - q(z_2, w)\right) \leq -\alpha |z_1 - z_2|^2, \tag{25}$$

for some $\alpha > 0$. This condition is a slight modification of the Demidovich theorem on convergent systems in the case $P = I_l$.

Example 3. Input-output properties of Lorenz system

Consider the Lorenz system (14)

$$\begin{cases} \dot{x}_{j,1} = \sigma(x_{j,2} - x_{j,1}) + u_j \\ \dot{x}_{j,2} = r x_{j,1} - x_{j,2} - x_{j,1} x_{j,3} \\ \dot{x}_{j,3} = -b x_{j,3} + x_{j,1} x_{j,2} \end{cases}$$

with σ, r, $b > 0$. Lorenz system with input u_j and output $y_j = x_{j,1}$ is strictly semipassive. To prove this statement, consider the following storage function candidate:

$$V(x_{j,1}, x_{j,2}, x_{j,3}) = \frac{1}{2}\left((x_{j,1})^2 + (x_{j,2})^2 + (x_{j,3} - \sigma - r)^2\right).$$

Calculating the derivative of this function along the solutions of the system (14) yields

$$\dot{V}(x_{j,1}, x_{j,2}, x_{j,3}, u) = x_{j,1} u - H(x_{j,1}, x_{j,2}, x_{j,3})$$

where

$$H(x_{j,1}, x_{j,2}, x_{j,3}) = \sigma(x_{j,1})^2 + (x_{j,2})^2 + b\left(x_{j,3} - \frac{\sigma + r}{2}\right)^2 - b\frac{(\sigma + r)^2}{4}.$$

It is easy to see that the condition $H = 0$ determines the ellipsoid which lies inside the ball

$$\Xi = \{x_{j,1}, x_{j,2}, x_{j,3} : (x_{j,1})^2 + (x_{j,2})^2 + (x_{j,3} - \sigma - r)^2 \leq L^2(\sigma + r)^2\} \quad (26)$$

in $\mathbb{R}^3$, with L derived in Appendix, outside which H is strictly positive. This fact proves strict semipassivity of the system (14). Hence, in a diffusive network of any number of Lorenz systems (take for example two systems)

$$\begin{cases} \dot{x}_{1,1} = \sigma(x_{1,2} - x_{1,1}) + u_1 \\ \dot{x}_{1,2} = r x_{1,1} - x_{1,2} - x_{1,1} x_{1,3} \\ \dot{x}_{1,3} = -b x_{1,3} + x_{1,1} x_{1,2} \end{cases} \qquad \begin{cases} \dot{x}_{2,1} = \sigma(x_{2,2} - x_{2,1}) + u_2 \\ \dot{x}_{2,2} = r x_{2,1} - x_{2,2} - x_{2,1} x_{2,3} \\ \dot{x}_{2,3} = -b x_{2,3} + x_{2,1} x_{2,2} \end{cases} \quad (27)$$

$$u_1 = -\gamma(y_1 - y_2), \quad u_2 = -\gamma(y_2 - y_1),$$

with outputs $y_1 = x_{1,1}$ and $y_2 = x_{2,1}$, all solutions are ultimately bounded.

Moreover, the same system, with input u_j and output $y_j = x_{j,1}$, is minimum phase. The dynamics corresponding to the constraint $y \equiv 0$ *(zero dynamics)* is simply

$$\begin{cases} \dot{x}_{j,2} = -x_{j,2} \\ \dot{x}_{j,3} = -bx_{j,3} \end{cases}$$

which is asymptotically stable. Minimum phaseness and convergence are closely related. If we think of the output y_j as a driving input for the remaining part of Lorenz system, we have

$$\begin{cases} \dot{x}_{j,2} = -x_{j,2} + ry_j - y_j x_{j,3} \\ \dot{x}_{j,3} = -bx_{j,3} + y_j x_{j,2}, \end{cases}$$

which is convergent. Applying Demidovich's result we see that, using $P = I_2$, the matrix $Q(x_j, y_j)$ in (23) is

$$Q(x_j, y_j) = \text{diag}(-1 \quad -b).$$

One final remark: after examination of these input-output properties of Lorenz system, it is not surprising that driving two identical Lorenz systems with the same $x_{j,1}$ signal leads to synchronization of the driven subsystems, even when their dynamics are chaotic [35,36]. The reader can easily check that using $x_{j,2}$ as drive works well too, and both cases are minimum phase, while using $x_{j,3}$ as input is not! Stability of response systems as referred in [36], with respective numerical evidence for different cases, is exactly convergent dynamics.

4.3 On global asymptotic stability of the partial synchronization manifolds

A permutation matrix Π commuting with Γ defines a linear invariant manifold of system (6), given by (8). This expression stands for a set of linear equations of the form

$$x_i - x_j = 0 \tag{28}$$

for some i and j that can be read off from the nonzero elements of the Π matrix under consideration. Therefore, we can identify a particular manifold associated with a particular matrix Π by the correspondent set $\mathcal{I}_\Pi$ of pairs i, j for which (28) holds.

Due to the Converse Lyapunov theorem the asymptotic stability of a set $\mathcal{A}$ is equivalent to the existence of a scalar smooth nonnegative function V which is zero only on $\mathcal{A}$ and decays along the system trajectories, when not on $\mathcal{A}$. In order to find a Lyapunov function which proves stability of the partial synchronization manifold, it is again possible to benefit from the representation of the closed loop system as a control system (1) with feedback (2), that is, one can seek a Lyapunov function candidate as a sum of two functions, the

first one dependent on the input-output relations of the systems (1) and the second one dependent on the way the systems interact via coupling. The best way to carry this out is to find a *globally* defined coordinate change that allows us to exploit minimum phaseness.

Let us first differentiate y_j,

$$\dot{y}_j = Cf(x_j) + CBu_j.$$

Then, choosing some $n - m$ coordinates z_j complementary to y_j it is possible to rewrite the system (1) in the form

$$\begin{cases} \dot{z}_j = q(z_j, y_j) \\ \dot{y}_j = a(z_j, y_j) + CBu_j \end{cases} \tag{29}$$

where $z_j \in \mathbb{R}^{n-m}$, and q and a are some vector functions. It is important to emphasize that the coordinate change $x_j \mapsto \mathrm{col}(z_j, y_j)$ can be linear, if CB is nonsingular. It is important to underline that, owing to the linear input-output relations, this transformation is globally defined. This transformation is explicitly computed in, for example, [37]. As the reader may expect, for more complicated input-output relations, this coordinate transformation may not be globally defined. Conditions on the existence of this *normal form* can be found in [38,31], for example. In the equation for z_j in (29), y_j acts as a forcing input, hence we can apply properties of convergent systems, if the matrix $Q(z, w)$ defined for q in (29) has negative eigenvalues, separated from zero.

The purpose of this section is to prove the following theorem:

Theorem 1 *Let λ' be the minimal eigenvalue of Γ under restriction that the eigenvectors of Γ are taken from the set* $\mathrm{range}(I_k - \Pi)$. *Suppose that*

i. Each free system (1) is strictly semipassive with respect to the input u_j and output y_j with a radially unbounded storage function.
ii. There exists a positive definite matrix P such that inequality (24) holds with some $\delta > 0$ for the matrix Q defined as in (23) for q as in (29).

Then for all positive semidefinite matrices Γ as in (3) all solutions of the diffusive cellular network (4, 5) are ultimately bounded and there exists a positive $\bar{\lambda}$ such that if $\lambda' > \bar{\lambda}$ the set $\ker(I_{kn} - \Pi \otimes I_n)$ *contains a globally asymptotically stable compact subset.*

Let us clarify the conditions imposed in the theorem. The first assumption on strict semipassivity ensures the ultimate boundedness of the solutions of the diffusive network. The second assumption requires some sort of stability of some internal dynamics of the system (29). It guarantees that the z-dynamics

of system (29) is convergent with y_j as an external signal. According to the Demidovich result previously mentioned, this assumption implies the existence of a quadratic function $W(\zeta)$ satisfying the inequality (25). From this inequality one can immediately see that, if $q(0,0) = 0$, the zero dynamics

$$\dot{z} = q(z, 0)$$

is globally asymptotically stable at the origin. Thus the second assumption necessarily implies that the system (29) is *minimum phase*. Recalling Example 3 about the Lorenz system, it means that the signal y_j acting as an input on z_j can make the dynamics z_j convergent. It is worth mentioning, however, that stability of the zero dynamics does not necessarily imply the stability of the free system ($u \equiv 0$). As one can see from the Example 3, the zero dynamics can be asymptotically stable (*i.e.* the system is minimum phase), while the free system is oscillatory.

In the rest of this section we sketch the proof of Theorem 1. To make the presentation more transparent we omitted some standard technical details which can be found in similar proofs, of related results, presented in [32,39]. In the proof we are mostly focused on the way to find the Lyapunov function guaranteeing stability of the partial synchronization mode. As we previously introduced the notation $y = \mathrm{col}(y_1, \ldots, y_k)$, let us denote with $z \in \mathbb{R}^{km}$ the vector $\mathrm{col}(z_1, \ldots, z_k)$. Since the derivative of z-variables in (29) does not depend on the coupling, while the derivative of y-variables does, we can search for a Lyapunov function in the form

$$V(z, y) = V_1(z) + V_2(y).$$

There is not a unique way of determine which Lyapunov function is best to investigate stability, although control theory can give guidelines to determine whether a Lyapunov function with a particular form exists. The existence of a Lyapunov function in the form above, structured as a sum of two independent parts is ensured by an important algebraic result known as *frequency theorem*, or *Kalman-Yakubovich-Popov lemma*. This theorem is one of the cornerstones of control theory, since it offers the necessary and sufficient conditions of the solvability of the Lyapunov and Riccati equations. However, further analysis of this theorem is beyond the scope of this paper, therefore we forward the interested reader to [40] for a review. The function V_1 expresses some internal stability properties of each subsystem, which will come from the input-output properties of (1,2), while the function V_2 shows how the coupling contributes to the stability of the partial invariant manifolds, where we will explicitly make use of the symmetries of the network, in the representation we examined.

Let us start with the function V_1. Suppose that there is a positive definite radially unbounded function $W(\zeta)$ defined on $\mathbb{R}^{n-m}$ which satisfies the partial differential inequality (25) for all $z_i, z_j \in \mathbb{R}^{n-m}$, $w \in \mathbb{R}^m$. Then we construct

the function V_1 as

$$V_1(z) = \sum_{(i,j)\in\mathcal{I}_\Pi} W(z_i - z_j).$$

Along the solutions of the closed loop system, the derivative of $V_1(z)$ satisfies

$$\begin{aligned}\dot{V}_1(z,y) &= \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} \left(q(z_i, y_i) - q(z_j, y_j)\right) \\ &\leq -\alpha \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j|^2 + \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} \left(q(z_j, y_i) - q(z_j, y_j)\right)\end{aligned}$$

The next step is to find the second part of the Lyapunov function, the function V_2. It is clear that if $x \in \ker(I_{kn} - \Pi \otimes I_n)$ then necessarily $y \in \ker(I_{km} - \Pi \otimes I_m)$. So, on this invariant manifold, the quantity $\xi(y) = (I_{km} - \Pi \otimes I_m)y$ is identically zero. We can therefore construct the function V_2 as

$$V_2(y) = \frac{1}{2}|\xi(y)|^2 = \frac{1}{2} y^\top (I_{km} - \Pi \otimes I_m)^\top (I_{km} - \Pi \otimes I_m) y = \frac{1}{2} \sum_{(i,j)\in\mathcal{I}_\Pi} |y_i - y_j|^2,$$

that is positive definite with respect to ξ, and zero on the set $\ker(I_{km} - \Pi \otimes I_m)$. Differentiating the function V_2 gives

$$\dot{V}_2(z,y) = \sum_{j=1}^{k} \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j) - U(y)$$

where using commutativity of Π and Γ,

$$U(y) = y^\top (I_{km} - \Pi \otimes I_m)^\top (\Gamma \otimes CB)(I_{km} - \Pi \otimes I_m) y$$

Since $\Gamma \geq 0$ and $CB > 0$ it follows that $U(y) \geq 0$. Moreover, since $\ker\Gamma \subset \ker(I_k - \Pi)$ (the kernel of Γ is a set of vectors with equal coordinates, as we have assumed that the defect of Γ is one) it follows that there is a positive constant κ such that the following inequality holds

$$U(y) \geq \kappa y^\top (I_{km} - \Pi \otimes I_m)^\top (I_{km} - \Pi \otimes I_m) y.$$

The number κ can be easily estimated as $\kappa \geq \lambda' \beta$, where β is the minimal eigenvalue of the matrix CB and λ' is the minimal eigenvalue of Γ under restriction that the eigenvectors of Γ are taken from the set $\mathrm{range}(I_k - \Pi)$.

We proceed now to evaluate the derivative of V. From the previous intermediate results it follows that

$$\dot{V}(z,y) \leq -\alpha \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j|^2 - 2\lambda'\beta V_2(y) + \sum_{j=1}^{k} \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j) + \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} \left(q(z_j, y_i) - q(z_j, y_j)\right).$$

Note that for any compact set Ω there exist some positive numbers C_1, C_2, C_3 such that the following estimates are valid on Ω:

$$\begin{aligned}\left|\sum_{j=1}^{k} \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j)\right| &= \left|\sum_{(i,j)\in\mathcal{I}_\Pi} (y_i - y_j)^\top \left(a(z_i, y_i) - a(z_j, y_j)\right)\right| \\ &\leq \left|\sum_{(i,j)\in\mathcal{I}_\Pi} (y_i - y_j)^\top \left(a(z_i, y_i) - a(z_i, y_j)\right)\right| \\ &+ \left|\sum_{(i,j)\in\mathcal{I}_\Pi} (y_i - y_j)^\top \left(a(z_i, y_j) - a(z_j, y_j)\right)\right| \\ &\leq C_1 V_2(y) + C_2 \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j||y_i - y_j|\end{aligned}$$

and

$$\left|\sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} \left(q(z_j, y_i) - q(z_j, y_j)\right)\right| \leq C_3 \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j| \cdot |y_i - y_j|.$$

Now we are going to use strict semipassivity of the systems forming the diffusive network. Recall that the strict semipassivity implies the ultimate boundedness of all the solutions, that is, all the solutions in a finite time approach some compact set Ω which can be chosen independently on Γ. On this compact set the derivative of V is a quadratic form with respect to $|z_i - z_j|$ and $|y_i - y_j|$. It is clear then that, if the value of λ' is large enough (that is, λ' is greater than a positive computable threshold $\bar{\lambda}$), due to inequality (25), the derivative of $V(z,y)$ is nonpositive on this set. After some algebra, an explicit formula for $\bar{\lambda}$ is derived as

$$\bar{\lambda} = \frac{1}{\beta}\left(\frac{C_1}{2} + \frac{(C_2 + C_3)^2}{4\alpha}\right). \tag{30}$$

This argument proves that the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable compact subset for $\lambda' > \bar{\lambda}$.

4.4 Remarks

Let us explain the result of Theorem 1.

1. Suppose we parameterize the matrix Γ as

$$\Gamma = \mu\Gamma_0$$

considering the positive parameter μ as a bifurcation parameter. Let λ_j, $j = 1, \ldots, k$ be the eigenvalues of Γ arranged in increasing order:

$$0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k.$$

Γ has only one zero eigenvalue, since we have assumed that the network can not be divided into two or more disconnected networks. Previous results [39], based on similar assumptions to those given in Theorem 1, show that for sufficiently large μ the full synchronization is proven to occur in the sense that the manifold $\mathcal{A} = \{x_j \in \mathbb{R}^n \ : \ x_1 = x_2 = \ldots = x_k\}$ contains a bounded closed globally asymptotically stable invariant subset. This situation occurs when the smallest nonzero eigenvalue λ_2 exceeds some computable threshold value.

The result formulated above allows to predict some additional bifurcations on the way to the full synchronization. Suppose that there is a permutation Π commuting with Γ that defines a partial synchronization manifold. For the given Π we can compute the number λ'. This number coincides with one of the nonzero eigenvalues of the matrix Γ, since $\ker\Gamma \subset \ker(I_k - \Pi)$ (the kernel of Γ is a set of vectors with equal coordinates, as we have assumed that the defect of Γ is one). Thus λ' takes value from the set $\{\lambda_2, \ldots, \lambda_k\}$. It can happen that λ' exceeds a synchronization threshold, while λ_2 does not. In this case a partial synchronization corresponding to the symmetry Π occurs.

Writing down all the admissible permutation matrices Π_i, $i = 1, \ldots, N$ (that define the sets $\mathcal{A}_i$) for the matrix Γ, and computing the corresponding values of λ'_i, $i = 1, \ldots, N$, one can predict the possible bifurcations occurring when μ is increasing from zero to infinity. Although the number N of permutation matrices can be large, the route towards full synchrony will show only a limited number of bifurcations, in the following sense. Consider the parametrization $\Gamma = \mu\Gamma_0$. As μ increases, suppose that a first invariant set $\mathcal{A}_1$ contains a globally asymptotically stable compact subset. Further increase of μ may result in an eigenvalue λ' corresponding to a different set $\mathcal{A}_2$ to meet the requirements set in the Theorem. Hence, the sets $\mathcal{A}_1$ and $\mathcal{A}_2$ will both contain globally asymptotically stable compact subsets, that is possible only if intersection $\mathcal{A}_1 \cap \mathcal{A}_2$ contains a globally asymptotically stable compact subset. From further increase of μ it may follow stability of subsequent intersections, so on until the set $\mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_l$ represents in fact full synchrony. If one considers a highly symmetrical network, the number of permutation matrices that leave the network unchanged can be large. However, Theorem 1 predicts a hierarchy with no more than $k - 1$ possible bifurcations towards full synchrony, since λ'_i is restricted to be equal to one of $k - 1$ positive eigenvalues

of the matrix Γ.

2. The proof as constructed before yields the estimate (30) of the threshold beyond which asymptotic stability of the compact subset, is guaranteed. Employing a different Lyapunov function may possibly result in an estimate that is less conservative than (30). A more general Lyapunov function can be chosen, for example, considering V_2 in the proof as

$$V_2(y) = \frac{1}{2}\xi(y)^\top P\xi(y) \tag{31}$$

for some positive definite symmetric matrix P. In this case, the sufficient condition for stability is given in terms of the eigenvalues of the matrix $(\Gamma P + P\Gamma)/2$ with all the other restrictions in the Theorem left intact. We illustrate this approach in a forthcoming example that also tests the conservativeness of the estimation.

3. Convergence (assumption *ii.*) is a strict but crucial assumption, since global asymptotic stability is sought but, in case of nonconvergent dynamics, no rigorous results expressed in terms of the eigenvalues of a coupling matrix are available, and asymptotic stability can only be conjectured. Consider, for example, a conjecture on synchronization criteria recently proposed in [41], that can be stated as follows: Consider two diffusively coupled networks with coupling matrices Γ' and Γ'', with equal smallest nonzero eigenvalues $\lambda_2' = \lambda_2''$. Then, conditions on global full synchronization for both networks are equivalent. This conjecture, called Wu-Chua conjecture, is wrong (as also pointed out by [42]), and a specific numerical counterexample can be found in [39]. However, under assumption *ii.* imposed in Theorem 1, the so called Wu-Chua conjecture is true, and sufficient conditions for synchronization can be formulated in terms of the eigenvalue λ_2.

5 Discussion and examples

5.1 Two coupled Lorenz systems

Consider the Example 3 where two Lorenz systems (27)

$$\begin{cases} \dot{x}_{1,1} = \sigma(x_{1,2} - x_{1,1}) + u_1 \\ \dot{x}_{1,2} = r x_{1,1} - x_{1,2} - x_{1,1}x_{1,3} \\ \dot{x}_{1,3} = -b x_{1,3} + x_{1,1}x_{1,2} \end{cases} \qquad \begin{cases} \dot{x}_{2,1} = \sigma(x_{2,2} - x_{2,1}) + u_2 \\ \dot{x}_{2,2} = r x_{2,1} - x_{2,2} - x_{2,1}x_{2,3} \\ \dot{x}_{2,3} = -b x_{2,3} + x_{2,1}x_{2,2} \end{cases}$$

with outputs $y_1 = x_{1,1}$ and $y_2 = x_{2,1}$ are diffusively coupled with input terms

$$u_1 = -\gamma(y_1 - y_2), \quad u_2 = -\gamma(y_2 - y_1)$$

respectively. Example 3 shows that Lorenz system has input-output properties that makes it suitable for synchronization via diffusion. How conservative is the estimate (30) for the threshold $\bar{\lambda}$, beyond which a particular linear invariant manifold contains a globally asymptotically stable subset? Let us consider the stability of the set $\mathcal{A} = \{x_{1,i} = x_{2,i}\}$. In this specific example the transformation to normal form (29) is simply given by $y_j = x_{j,1}$ and $z_j = (x_{j,2}, x_{j,3})$. In the explanatory remarks for the theorem, we suggested that the Lyapunov function

$$V = \frac{1}{2}\left(p(x_{1,1} - x_{2,1})^2 + (x_{1,2} - x_{2,2})^2 + (x_{1,3} - x_{2,3})^2\right),$$

where p is a positive number, may lead to a less conservative threshold than that obtained via the Lyapunov function used in the proof of the theorem. The time derivative reads

$$\begin{aligned}
\dot{V} &= -p(\sigma + 2\gamma)e_1^2 + (r + p\sigma - x_{1,3})e_1e_2 + x_{1,2}e_1e_3 - e_2^2 - be_3^2 \\
&= -p(\sigma + 2\gamma)e_1^2 + (r + p\sigma - x_{2,3})e_1e_2 + x_{2,2}e_1e_3 - e_2^2 - be_3^2 \\
&= -p(\sigma + 2\gamma)e_1^2 + (r + p\sigma - z)e_1e_2 + ye_1e_3 - e_2^2 - be_3^2
\end{aligned} \tag{32}$$

where $z = \max_j(x_{j,3})$ and $y = x_{i,2}$ for that i, for which the previous maximum is achieved.

This expression is negative definite with respect to e_1, e_2, e_3 if the following inequality is satisfied:

$$p(\sigma + 2\gamma) > \frac{1}{4b}(y^2(t) + b(r + p\sigma - z(t))^2) \tag{33}$$

According to Lemma 2 (see Appendix) it follows that if $2\sigma > b$ then $z(t) \geq 0$. Under assumption that $b \geq 1$ we can derive from Lemma 2 that

$$\limsup_{t\to\infty} \frac{1}{b}(y^2(t) + b(r + p\sigma - z(t))^2) \leq L^2r^2 + p^2\sigma^2 + 2rp\sigma$$

Therefore the stability condition is

$$2\gamma + \sigma > \frac{L^2r^2 + p^2\sigma^2 + 2rp\sigma}{4p}$$

Minimizing the right hand side of this inequality with respect to p yields the following stability condition

$$\gamma > \frac{1}{2}\sigma(Mr - 1), \quad M = \frac{L+1}{2}$$

Using this inequality and the values $\sigma = 10, r = 28, b = 8/3$ we obtain that $L^2 = 16/15$ (see Appendix), $M = 1.0164$ and the synchronization is globally stable as soon as $\gamma > 137.296$. This estimate serves as an upper bound for the threshold of global asymptotic stability of full synchrony between the two Lorenz systems.

We can obtain a lower estimate of the synchronization threshold by considering that systems (27) also present the additional invariant manifold $\mathcal{A}' = \{x_{1,1} = -x_{2,1}, x_{1,2} = -x_{2,2}, x_{1,3} = x_{2,3}\}$ resulting from the internal symmetry (15) of the Lorenz system discussed in Example 3. This situation is schematically represented in Figure 3. The picture shows manifolds $\mathcal{A}$ and $\mathcal{A}'$, and the invariant set Ω is the Lorenz attractor in $\mathcal{A}$. Dynamics on the manifold $\mathcal{A}'$ evolves according to

$$\begin{cases} \dot{x}_{1,1} = \sigma(x_{1,2} - x_{1,1}) - 2\gamma x_{1,1} \\ \dot{x}_{1,2} = r x_{1,1} - x_{1,2} - x_{1,1} x_{1,3} \\ \dot{x}_{1,3} = -b x_{1,3} + x_{1,1} x_{1,2}. \end{cases} \tag{34}$$

The intersection $\mathcal{A}' \cap \mathcal{A}$ is the stable manifold of the origin, since any trajectory initialized in this intersection evolves according to

$$\dot{x}_{1,3} = -b x_{1,3}, \quad \dot{x}_{2,3} = -b x_{2,3},$$

and converges to zero (since $b > 0$).

Consider the well known Lorenz chaotic attractor corresponding to parameters values $\sigma = 10, r = 28, b = 8/3$. This attractor contains the origin (see, e.g. [43]). From this it follows that, if Ω is an attractor in $\mathbb{R}^6$, all unstable manifolds of the origin must lie in $\mathcal{A}$. From (34) one can see that, if

$$\gamma < \frac{1}{2}\sigma(r-1)$$

the origin of (34) is unstable so, if $\gamma < 135$, $\mathcal{A}'$ contains an unstable manifold of the origin. Since O is a hyperbolic point, for $\gamma < 135$ Ω cannot be an attractor in $\mathbb{R}^6$. The value $\gamma = 135$ can thus be considered as a lower estimate of the synchronization threshold. Therefore, the true threshold for the coupling gain γ which ensures global asymptotic stability of the synchronized state lies in the interval

$$135 = \frac{\sigma(r-1)}{2} \leq \bar{\lambda} \leq \frac{\sigma(Mr-1)}{2} \approx 137.296. \tag{35}$$

Comparing these two bounds, we can conclude by emphasizing that a threshold that can be derived from the Lyapunov direct method is not so conservative (as often expected when employing Lyapunov functions). Computer simulations

show that for γ greater than (about) 5.0, randomly chosen initial conditions converge towards Ω, but from our estimate it follows that the synchronous mode in this case can be attractive *only* in the Milnor sense [12].

A recent result by Leonov quotes that the Lyapunov dimension of the Lorenz chaotic attractor is equal to the Lyapunov dimension of the origin (see Theorem 4 in [48]). This is a strong indication that the stability of the origin of (34) is a sufficient condition for *local* asymptotic stability of a compact subset of $\mathcal{A}$ which includes the Lorenz attractor. The interval we report in (35) makes plausible the conjecture that stability of the origin of (34) is actually a sufficient condition for *global* asymptotic stability of the same set.

5.2 Four coupled oscillators

Recall Example 1 of a ring of four coupled oscillators, whose arrangement is schematically shown in Figure 1. The eigenvalues of the matrix (9) are $\lambda_1 = 0$, $\lambda_2 = \min\{2K_0, 2K_1\}$, $\lambda_3 = \max\{2K_0, 2K_1\}$, $\lambda_4 = 2(K_0 + K_1)$. Hence, for the permutation described by Π_1 in (10) we have $\lambda' = 2K_0$. Similarly, $\lambda' = \min\{2K_0, 2K_1\}$ for Π_2 and $\lambda' = 2K_1$ for Π_3. According to Theorem 1, for large K_0 and small K_1 one can expect asymptotic stability of a subset of the set $\mathcal{A}_1$ in (11). For the permutation Π_3, for small K_0 and large K_1, leads to asymptotic stability of a subset of the set $\mathcal{A}_3$. Asymptotic stability of the full synchronization occurs for K_0 and K_1 both large enough. The subset of the set $\mathcal{A}_2$ is stable only as a stable intersection of $\mathcal{A}_1$ and $\mathcal{A}_3$, which describes full synchronization. Parameterizing Γ by one scalar parameter μ, it follows that the route to full synchrony can be either

$$\text{no synchrony} \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_1 \cap \mathcal{A}_2 \quad \text{(full synchrony)},$$

or

$$\text{no synchrony} \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_2 \cap \mathcal{A}_1 \quad \text{(full synchrony)},$$

depending on the ratio K_0/K_1. The diagram of asymptotic stability of the sets (11) in the (K_0, K_1) parameter space is schematically shown in Figure 2. It is worth mentioning that this stability diagram is model independent, that is, any dynamical system $f(\cdot)$ in (1) which satisfies the assumptions of Theorem 1 will yield this diagram, when coupled as in Example 1.

Computer simulations show that, in a ring of k systems (1,2), with identical coupling constants (*i.e.* when all γ_{ij}'s in (2,3) are identical), there is no evidence of asymptotically stable partially synchronized modes of oscillation. This evidence may be intuitively correct, considering that this highly symmetric case would not allow for a specific pattern of synchronized motion to arise. However, it is indeed remarkable, since a high number of permutations of the elements of the network are allowed symmetries (*i.e.* it commutes with Γ). In this case, if the common coupling constant is denoted by K, the Γ matrix is a *circulant* matrix

$$\Gamma = K\mathrm{circ}(2, -1, 0, \cdots, 0, -1)$$

and its eigenvalues can be calculated analytically:

$$\lambda_j = 2K\left(1 - \cos\left(\frac{2\pi j}{k}\right)\right), \quad j = 1, ..., k.$$

Properties of circulant matrices can be found in [44], or in the classical reference [45]. Among these, the property that any circulant matrix can be expressed in powers of the *shift* matrix θ,

$$\theta = \mathrm{circ}(0, 1, 0, \cdots, 0) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

In this case (note that $\theta^{k-1} = \theta^\top$ and $\theta^k = I_k$) the coupling matrix Γ can be represented in the following form

$$\Gamma = K(2I_k - \theta - \theta^\top).$$

It is not difficult to see that the eigenvalues of θ are the roots of the unity.

In conformity with numerical evidence, Theorem 1 does not predict the asymptotic stability of a partial synchronized mode, but it does not prove its instability either. Since we are looking for *global* asymptotic stability, the invariant manifold under consideration should be invariant under the transformation $x_i \mapsto x_{i+1}$. Therefore among all possible Π's commuting with Γ one has to consider only powers of θ.

According to Theorem 1 the global partial synchronization will happen if, for some integer s, there is a zero eigenvalue of $(1-\theta^s)$ with the eigenvector being

the eigenvector for the smallest nonzero eigenvalue of Γ. In other words, there should be a s such that the matrix θ^s has eigenvalue one with the eigenvector being the eigenvector corresponding to the eigenvalue $\exp(2\pi i/k)$ of θ. Since the matrices θ and θ^s have the same eigenvectors, it can happen only in case $s = 0$ modulo k. However, in this case $\Pi = I_k$ and $\ker(I_k - \Pi)$ is just the origin.

5.4 Application to systems with different input-output properties

Consider a diffusive network consisting of k coupled Chua's systems:

$$\begin{cases} \dot{x}_{j,1} = \alpha(-x_{j,1} + x_{j,2} - \varphi(x_{j,1})) + u_j \\ \dot{x}_{j,2} = x_{j,1} - x_{j,2} + x_{j,3} \\ \dot{x}_{j,3} = -\beta x_{j,2} \end{cases} \tag{36}$$

where $\varphi(\xi) = m_1\xi + 0.5(m_1 - m_0)(|\xi + 1| - |\xi - 1|)$, coupled to each other via diffusive coupling

$$u_j = \gamma_{1j}(y_1 - y_j) + \ldots + \gamma_{kj}(y_k - y_j)$$

with outputs $y_j = x_{j,1}$. This system is often investigated for parameter values (say, $\alpha = 9.0, \beta = 14.28, m_0 = -5/7, m_1 = -6/7$) which produce a known double-scroll chaotic attractor. System (36) with its input-output relationships, satisfies the convergence condition when $\beta > 0$, but is not semipassive. This statement can be demonstrated by computer simulation: together with a possible chaotic attractor the free system ($u_j \equiv 0$) can have unbounded trajectories. Hence we cannot apply Theorem 1 to Chua's system. Particularly, one cannot draw a conclusion that the partial synchronization manifold contains an asymptotically stable subset that is *compact* whose stability is *global*. However, since system (36) is convergent, predictions of Theorem 1 hold *locally*.

Next we consider a diffusive network of Rössler systems:

$$\begin{cases} \dot{x}_j = -y_j - e^{z_j} \\ \dot{y}_j = x_j + ay_j + u_j \\ \dot{z}_j = ce^{-z_j} + x_j - b \end{cases}$$

with output y_j, input u_j, $a, b, c > 0$ and the same type of coupling as in the previous example. This system is neither semipassive nor satisfies the convergence condition and, although it is possible to observe both full and partial synchronization in the diffusive network via computer simulation, the

synchronization conditions in this case do not depend on only one eigenvalue of the coupling matrix (see [39]).

6 Conclusion

In this paper we have demonstrated an approach, based on second Lyapunov method, to study partial synchronization in diffusively coupled (not necessarily locally) identical dynamical systems. In our approach we considered the diffusive coupling as a *feedback* that allowed us to borrow some useful control techniques. We have considered global symmetries in the network that can be represented by permutation matrices, in order to classify linear invariant manifolds different from the full synchronized state, with respect to each of the permutation matrices that commute with the given Γ matrix that represents the topology of the network. The additional advantage in using symmetry under permutation is that these permutation matrices can be used to construct Lyapunov function candidates for the stability test of the correspondent linear invariant manifolds.

Using this methodology we presented sufficient conditions guaranteeing global asymptotic stability of the partial synchronization regimes. The main limitation of our approach is that we require *minimum phaseness* of each system from the network. As a benefit, the synchronization test is relatively easy to check, at the same time, we believe that a study of the partial synchronization in an diffusive array of non minimum phase systems may constitute a challenge for the future work.

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Appendix. Bounds for trajectories of the Lorenz equations

Consider the Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases} \tag{37}$$

with σ, r, $b > 0$. Denote

$$L = \begin{cases} 1, & b \leq 2 \\ \dfrac{b}{2\sqrt{b-1}}, & b \geq 2. \end{cases}$$

We are going now to present a well known result (see e.g. [46], Lemma 5.6.1, Lemma 5.6.2; or Example 2 in [47]).

Lemma 1 *For an arbitrary solution of the system (37) $x(t), y(t), z(t)$, it follows that*

$$\limsup_{t\to\infty} \left(y^2(t) + (z(t) - r)^2\right) \leq L^2 r^2. \tag{38}$$

and if, additionally, $2\sigma > b$ then

$$\liminf_{t\to\infty} (z(t) - x^2/2\sigma) \geq 0 \tag{39}$$

The proof of this Lemma is based on the following Lyapunov function

$$V(y, z) = \frac{1}{2}\left(y^2 + (z - r)^2\right)$$

for (38) and

$$W(x, z) = \sigma z - x^2/2$$

for the estimate (39). For details, see e.g. [46].

From this lemma it follows that for any p the following inequality

$$y^2 + (z - p\sigma - r)^2 \leq L^2 r^2 + p^2\sigma^2 + 2rp\sigma \tag{40}$$

holds in the trapping region.

By analogy we can prove the following result.

Lemma 2 *Consider the diffusive network of k Lorenz systems (14) with outputs $y_j = x_{j,1}$ coupled via the coupling (2). If $2\sigma > b$ then for any solution*

$x_{j,1}(t), x_{j,2}(t), x_{j,3}(t)$, $j = 1, \ldots, k$ *it follows that*

$$\liminf_{t\to\infty} \sum_{j=1}^{k} x_{j,3} \geq 0$$

$$\limsup_{t\to\infty} \left(x_{j,2}^2 + (x_{j,3} - r)^2 \right) \leq L^2 r^2$$

The proof of this Lemma follows the same lines as in the previous Lemma with the following Lyapunov functions

$$V_j = \frac{1}{2} \left(x_{j,2}^2 + (x_{j,3} - r)^2 \right),$$

$$W = \sum_{j=1}^{k} \left(\sigma x_{j,3} - \frac{x_{j,1}^2}{2} \right)$$

making use of the fact that the matrix Γ is positive semidefinite.

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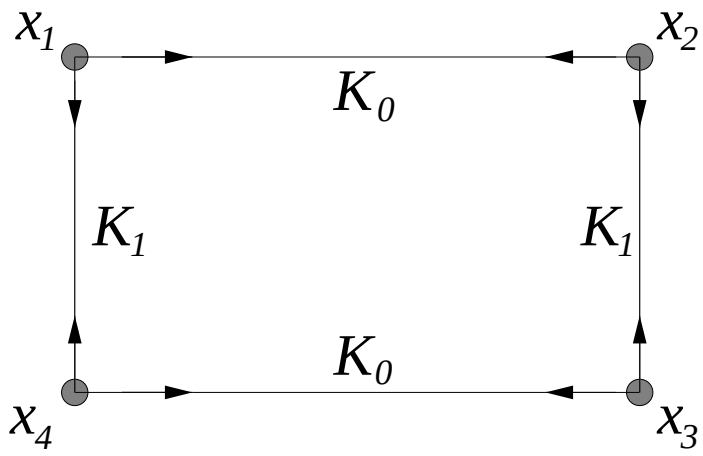

Figure 1. A network of four coupled identical systems with symmetric coupling at the opposite sides.

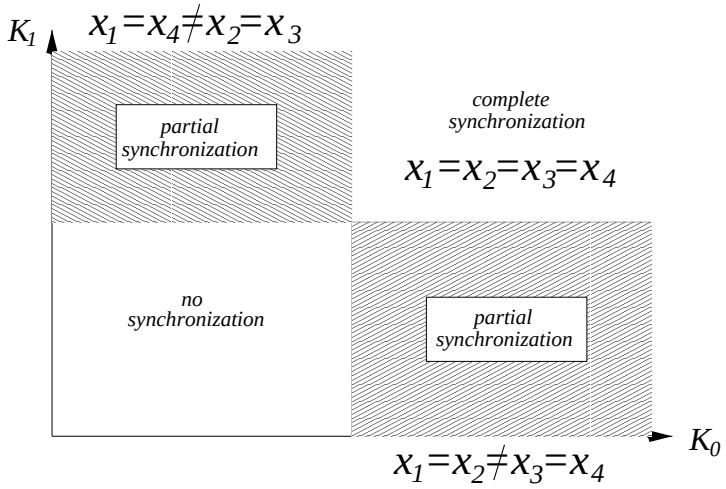

Figure 2. Stability diagram of different invariant manifolds in the ring of four dynamical systems coupled as in Figure 1.

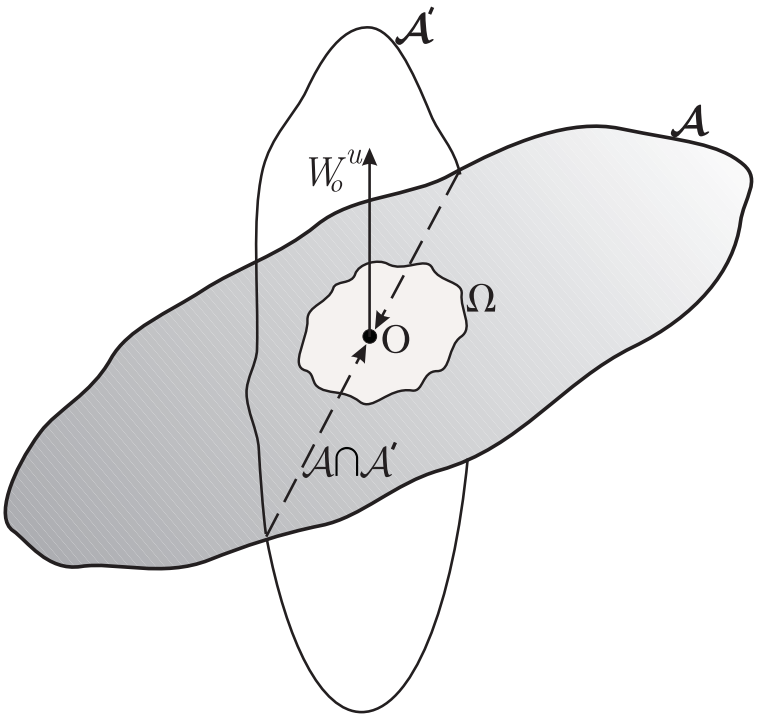

Figure 3. Schematic representation of the intersection of invariant sets $\mathcal{A}$ and $\mathcal{A}'$.