# REDUCTION FOR THE PROJECTIVE ARCLENGTH FUNCTIONAL

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Abstract. We consider the variational problem for curves in real projective plane defined by the projective arclength functional and discuss the integrability of its stationary curves in a geometric setting. We show how methods from the subject of exterior differential systems and the reduction procedure for Hamiltonian systems with symmetries lead to the integration by quadratures of the extrema. A scheme of integration is illustrated.

## 1. Introduction

Many geometric variational problems with one independent variable arise from functionals whose domain of definition consists of integral curves of an exterior differential system. Based on the 1922 seminal work of E. Cartan [5], Griffiths [8] developed a systematic geometric approach to the calculus of variations in one variable making use of the theory of exterior differential systems. Further developments are due to Bryant and Hsu [1, 12]. Griffiths constructs the Euler–Lagrange system for such functionals and by discussing several examples shows how to extend to this setting the rich geometric structures that are familiar in the case of classical mechanics. A comprehensive exposition and explanation of the classical calculus of variations of theoretical mechanics in the framework of modern differential geometry was presented by Goldschmidt–Sternberg [6]. In [3] Bryant–Griffiths extended to the setting of exterior differential systems the reduction procedure for Hamiltonian systems admitting a Lie group of symmetries to investigate the total squared curvature functional for immersed curves in a constant curvature surface.

In this article we deal with the functional

$$\mathcal{L}(\gamma) = \int_\gamma \mathrm{d}\tau_\gamma, \tag{1.1}$$

defined on parameterized curves $\gamma : I = (a, b) \to \mathbb{P}^2$ in real projective plane, where $\tau_\gamma$ is the projective arclength. The functional $\mathcal{L}$ is invariant under the projective action of $G = \mathrm{SL}(3, \mathbb{R})$ on $\mathbb{P}^2$. The associated variational problem was considered by E. Cartan [4]. In his study the stationary curves are determined by a third order linear ODE with doubly periodic coefficients and the equation is solved by a method developed by E. Picard [16] in terms of elliptic functions. In his analysis

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Cartan did not explicitly use methods from exterior differential systems, but rather derived the results by more ad hoc methods.

One motivation for the present paper is to gain insight into the geometric aspects of integrability of the extrema. To this end, we shall use methods from the theory of exterior differential systems and the reduction procedure for Hamiltonian systems with a Lie group of symmetries. By discussing in details a specific example, we hope to give an illustration of the common mechanism underlying integrability of several other constrained variational problems. Examples include the arclength functionals in conformal, pseudo-conformal, and affine geometry [15, 8], and the total squared curvature functional [3]. These examples as well as other in [13], belong to the class of so-called coisotropic variational problems which are discussed in [11].

We turn now to a more detailed description of the contents. The starting point of our study is the replacement of the original variational problem by a variational problem for integral curves of a Pfaffian system on $G$ with an independence condition. This is achieved by showing the existence of a preferred $G$-invariant frame along nondegenerate[1] curves. We then follow a general construction due to Griffiths [8] and carry out a calculation to associate to $\mathcal{L}$ a Pfaffian exterior differential system $\mathcal{J}$—the *Euler–Lagrange system*—whose integral curves are stationary for the associated functional. As a matter of fact, in this case all stationary curves arise as projections[2] of integral curves of $\mathcal{J}$. The Euler–Lagrange system is defined on $Y = G \times \mathfrak{p}$—the *momentum space*—for a suitable 3-dimensional affine subspace $\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of $G$. It turns out that the momentum space carries a contact structure, whose characteristic curves coincide with the integral curves of $\mathcal{J}$. Further, we show that the characteristic flow factors over the *phase* flow in $\mathfrak{p}$ and find a Lax formulation of its defining differential equation (infinitesimal Noether's conservation theorem). This implies that the moment map $\mu : Y \to \mathfrak{g}^*$ induced by the Hamiltonian action of $G$ on $Y$ is constant on solution curves to the Euler–Lagrange system. We then proceed to the description of the characteristic curves. First, we show that the characteristic curves through the points of $G \times \mathfrak{p}_s$, being $\mathfrak{p}_s$ the singular set of the phase flow, are orbits of one-parameter subgroups of $G$. Moreover, the corresponding stationary curves are nondegenerate curves with constant projective curvature. As for the other curves, we apply the reduction procedure for Hamiltonian systems with symmetries. In this case the determination of the integral curves of $\mathcal{J}$ reduces to determining a parameterization of the 1-dimensional Marsden–Weinstein reduced spaces (phase portraits) plus one more integration to lift these parameterized curves to the level sets of the moment mapping $\mu_{|Y_r}$. The last integration amounts to the construction of horizontal curves for a canonically defined connection on the Marsden–Weinstein principal fibration. As the structure group is abelian such an integration can be achieved by a single quadrature.

The paper is organized as follows. Section 2 gives the details of the construction of the preferred frame along nondegenerate curves using the method of moving frames and defines the Pfaffian differential system of nondegenerate projective curves. Section 3 studies the arclength functional and introduces the corresponding momentum space and Euler–Lagrange system. Section 4 focuses on the reduction procedure.

[1] i.e., without sextatic points (cf. Section 2). In particular, conics are not considered in the discussion.

[2] In this case all derived systems of $\mathcal{J}$ have constant rank.

The phase flow and the contact moment map associated to the Hamiltonian action of $G$ on the momentum space are studied and the general scheme of integration is presented. Finally, Section 5 briefly indicates how to implement the integration scheme.

## 2. Curves in projective plane by moving frames

**2.1. Projective frames along a curve.** The group $G = \mathrm{SL}(3,\mathbb{R})$ acts transitively and almost effectively on the real projective plane $\mathbb{P}^2$ by

$$A \cdot [x] = [Ax],$$

where $[x] \in \mathbb{P}^2$ and $A \in G$. Let $A_0, A_1, A_2$ denote the column vectors of $A \in G$. The projection map

$$A \in G \mapsto [A_0] \in \mathbb{P}^2$$

defines a principal $G_0$-bundle over $\mathbb{P}^2$; $G_0$ is the isotropy subgroup at $[(1,0,0)]$ and its elements are given by formula (2.1) below. The Maurer–Cartan 1-form $\omega = (\omega^i_j) = X^{-1}\mathrm{d}X$ of $G$ satisfies the structure equations

$$\mathrm{d}\omega^i_j = -\sum_{k=0}^{2} \omega^i_k \wedge \omega^k_j, \quad \omega^0_0 + \omega^1_1 + \omega^2_2 = 0.$$

*Remark* 2.1. If $f^i_j(t) : I \to \mathbb{R}$, $i,j = 0,1,2$, are smooth functions defined on an interval $I \subset \mathbb{R}$ and satisfying $f^0_0 + f^1_1 + f^2_2 = 0$, then, by the Cartan–Darboux integrability theorem [9, 14] there exists a smooth map $\mathbf{f} : I \to G$ such that $\mathbf{f}^*(\omega^i_j) = f^i_j \mathrm{d}t$. The map $\mathbf{f}$ is uniquely determined up to left multiplication.

A parameterized curve $(I, \gamma)$ in $\mathbb{P}^2$ consists of an open interval $I \subset \mathbb{R}$ and a smooth immersion $\gamma(t) : I \to \mathbb{P}^2$. Two curves $(I,\gamma)$ and $(\tilde{I}, \tilde{\gamma})$ are projectively equivalent if there exists a smooth diffeomorphism $h : \tilde{I} \to I$ and an element $A \in G$ such that $A \cdot \gamma \circ h = \tilde{\gamma}$, for each $t \in \tilde{I}$. A *projective frame field* along $\gamma$ is a smooth map

$$e : I \to G$$

such that $\gamma(t) = [e_0(t)]$, for each $t \in I$. For any such frame we put

$$\theta = e^*\omega = (\theta^i_j).$$

If $e, \tilde{e} : I \to G$ are two projective frames along $\gamma$, then $\tilde{e} = eb$ where $b : I \to G$ denotes the smooth $G$-valued function

$$b = \begin{pmatrix} \det(B)^{-1} & {}^t v \\ 0 & B \end{pmatrix}, \tag{2.1}$$

$B = (B^i_j)$, $i,j = 1,2$, ${}^t v = (v_1, v_2) \in \mathbb{R}^2$. If $\tilde{\theta} = \tilde{e}^*\omega$, then

$$\tilde{\theta} = b^{-1}\theta b + b^{-1}\mathrm{d}b. \tag{2.2}$$

In particular, this implies that

$$\begin{pmatrix} \tilde{\theta}^1_0 \\ \tilde{\theta}^2_0 \end{pmatrix} = \det(B)^{-1}(B)^{-1} \begin{pmatrix} \theta^1_0 \\ \theta^2_0 \end{pmatrix}. \tag{2.3}$$

The projective frame field $e : I \to G$ is of *first order* if

$$\theta^2_0 = 0. \tag{2.4}$$

From (2.3) it follows that first order frames do exist and any two of them are related by $b$ as in (2.1), where $B^2_1 = 0$. From (2.2) we have

$$\tilde{\theta}^1_0 = (B^1_1)^{-1}(B^2_2)^{-1}\theta^1_0, \quad \tilde{\theta}^2_1 = (B^2_2)^{-1}(B^1_1)\theta^2_1.$$

If we set

$$\theta^i_j = a^i_j\theta^1_0, \tag{2.5}$$

for suitable smooth functions $a^i_j : I \to \mathbb{R}$, then, in particular,

$$\tilde{a}^2_1 = (B^1_1)^3 a^2_1. \tag{2.6}$$

Let $\mathbb{P}^{2^*}$ be the dual projective plane (= planes in $\mathbb{R}^3$). Given a projective frame $e = (e_0, e_1, e_2) : I \to G$ along $\gamma$, let $e^* = (e^0, e^1, e^2)$ denote its dual frame. The *dual curve* $\gamma^* : I \to \mathbb{P}^{2^*}$ is defined by $\gamma^*(t) = [e^2(t)]$, for each $t \in I$, and is independent from the choice of the frame. The curve $\gamma$ is called *regular* if $\gamma^*$ is an immersion, that is, $\theta^2_{1|t} \neq 0$, for each $t \in I$ and every first order frame. By (2.6), each regular curve admits a first order frame such that $a^2_1 = 1$. First order frames are said of *second order* if they satisfy the condition $a^2_1 = 1$. If $e$ is a second order frame on $I$, then any other is given by $\tilde{e} = eb$, where $b$ is as in (2.1) with $B^1_1 = 0$ and $B^2_1 = 0$. It follows that

$$\tilde{\theta}^1_0 = (B^2_2)^{-1}\theta^1_0, \quad \tilde{\theta}^1_1 = \theta^1_1 + (v_1 - (B^2_2)^{-1}B^1_2)\theta^1_0,$$

which yields

$$\tilde{a}^1_1 = (B^2_2)(a^1_1 + v_1 - (B^2_2)^{-1}B^1_2).$$

This implies that there exist second order frame fields such that $a^1_1 = 0$. Second order frames are said of *third order* if they satisfy the condition

$$a^1_1 = 0. \tag{2.7}$$

If $e$ is a third order frame on $I$, then any other is given by $\tilde{e} = eb$, where $b$ is as in (2.1) with

$$B^1_1 = 0, \quad B^2_1 = 0, \quad v_1 = (B^2_2)^{-1}B^1_2.$$

From this it follows that

$$\tilde{\theta}^0_1 - \tilde{\theta}^1_2 = B^2_2(\theta^0_1 - \theta^1_2) + ((B^2_2)^{-1}(B^1_2)^2 - 2v_2)\theta^1_0,$$

which yields

$$\tilde{a}^0_1 - \tilde{a}^1_2 = (B^2_2)^2(a^0_1 - a^1_2) + (B^1_2)^2 - 2(B^2_2)^{-1}v_2.$$

Third order frames are said of *fourth order* if

$$a^0_1 - a^1_2 = 0. \tag{2.8}$$

If $e$ is a fourth order frame, then any other is given by $\tilde{e} = eb$, $b$ as in (2.1) with

$$B^1_1 = 0, \quad B^2_1 = 0, \quad v_1 = (B^2_2)^{-1}B^1_2, \quad v_2 = 1/2(B^2_2)(B^1_2)^2.$$

2.2. **The normal frame along a nondegenerate curve.** The projective space $\mathbb{P}^5$ corresponding to the 6-dimensional space of all symmetric bilinear forms on $\mathbb{R}^3$ naturally identifies with the space of conics in $\mathbb{P}^2$. If $\alpha\beta$ denotes the symmetric tensor product of $\alpha, \beta \in {\mathbb{R}^3}^*$, then the mapping

$$\Gamma : t \in I \mapsto [e^1e^1 - 2e^0e^2] \in \mathbb{P}^5$$

does not depend on the choice of the fourth order frame. $\Gamma$ is called the *osculating curve* of $\gamma$. Geometrically, $\Gamma(t)$ is a nondegenerate conic passing through the point $\gamma(t)$.

**Definition 2.2.** A regular curve $\gamma$ is said to be *nondegenerate* if its osculating curve is an immersion, i.e., $\gamma$ has no sextatic points.

Let $\gamma$ be a nondegenerate curve and $e$ be a fourth order frame along $\gamma$. Differentiating yields

$$\mathrm{d}(e^1e^1 - 2e^0e^2) = 2\theta^2_0 e^2e^2.$$

Thus, $\gamma$ is nondegenerate if and only if

$$\theta^2_{0\,|t} \neq 0, \text{ for each } t \in I.$$

Notice that $\theta^2_0$ vanishes if and only if $\Gamma$ is constant. Therefore, the condition $\theta^2_0 = 0$ characterizes nondegenerate conics among all plane curves.

Let $\gamma$ be a nondegenerate curve and $e, \tilde{e}$ be two fourth order frames along $\gamma$. We then have

$$\tilde{\theta}^1_0 = (B^2_2)^{-1}\theta^1_0, \quad \tilde{\theta}^0_2 = (B^2_2)^2\theta^0_2.$$

This implies

$$\tilde{a}^0_2 = (B^2_2)^3 a^0_2.$$

Since $\tilde{a}^0_2(t) \neq 0$ and $a^0_2(t) \neq 0$, for each $t \in I$, it follows that there always exist fourth order frames such that

$$a^0_2 = 0. \tag{2.9}$$

Fourth order frames are said of *fifth order* if they satisfy (2.9). If $e$ and $\tilde{e}$ are fifth order frames, then

$$\begin{aligned}
\tilde{e}_0 &= e_0\\
\tilde{e}_1 &= B^1_2 e_0 + e_1\\
\tilde{e}_2 &= \frac{1}{2}(B^1_2)^2 e_0 + B^1_2 e_1 + e_2,
\end{aligned}$$

where $B^1_2 : I \to \mathbb{R}$ is a smooth function. This yields

$$\tilde{\theta}^1_0 = \theta^1_0, \quad \tilde{\theta}^0_0 = \theta^0_0 - B^1_2\theta^1_0,$$

that is

$$\tilde{a}^0_0 = a^0_0 - (B^1_2).$$

Therefore there exists a unique fifth order frame along $\gamma$ such that $a^0_0 = 0$. Such a frame is called the *normal frame field of* $\gamma$.

We have proved the following:

**Theorem 2.3** (cf. Cartan [4]). *If $\gamma(t) : I \to \mathbb{P}^2$ is a nondegenerate curve, then there exists a unique lift $e : I \to G$, the normal frame field, satisfying the Pfaffian equations*

$$\theta_0^2 = \theta_1^2 - \theta_0^1 = \theta_1^1 = \theta_1^0 - \theta_2^1 = \theta_2^0 - \theta_0^1 = \theta_0^0 = 0 \tag{2.10}$$

*with the independence condition*

$$\theta^1_{0|t} \neq 0, \text{ for each } t \in I. \tag{2.11}$$

*Conversely, any smooth map $e : I \to G$ such that $e^*\omega$ satisfies (2.10) and (2.11) is the normal frame of the nondegenerate curve $\gamma : I \to \mathbb{P}^2$ defined by $\gamma(t) = [e_0]$.*

**Definition 2.4.** We set $\theta_0^1 = \mathrm{d}\tau$ and $\theta_1^0 = -\kappa \mathrm{d}\tau$. The differential $\mathrm{d}\tau$ is the *projective arc element* and the function $\kappa$ is the *projective curvature.*

*Remark* 2.5 (Intrinsic equations). Theorem 2.3 asserts that the normal frame field $e(\tau) : I \to G$ is the unique lift satisfying the following system of equations (the generalized Frenet system)

$$\begin{aligned} \frac{\mathrm{d}e_0}{\mathrm{d}\tau} &= e_1, \\ \frac{\mathrm{d}e_1}{\mathrm{d}\tau} &= -\kappa e_0 + e_2, \\ \frac{\mathrm{d}e_2}{\mathrm{d}\tau} &= e_0 - \kappa e_1. \end{aligned} \tag{2.12}$$

If $e(\tau), \tilde{e}(\tau) : I \to G$ are any two solutions of the above system, then there exists a unique $A \in G$ such that $\tilde{e} = Ae$ (cf. Remark 2.1). This shows that the projective curvature determines the curve up to projective equivalence. The system (2.12) reduces to the third order ODE:

$$\frac{\mathrm{d}^3 u}{\mathrm{d}\tau^3} + 2\kappa \frac{\mathrm{d}u}{\mathrm{d}\tau} + (\frac{\mathrm{d}\kappa}{\mathrm{d}\tau} - 1)u = 0. \tag{2.13}$$

Given a function $\kappa : I \to \mathbb{R}$ and three linearly independent solutions $u^j$, $j = 0, 1, 2$, of (2.13) we set

$$e_0 = {}^t(u^0, u^1, u^2), \quad e_1 = \frac{\mathrm{d}e_0}{\mathrm{d}\tau}, \quad e_2 = \frac{\mathrm{d}^2 e_0}{\mathrm{d}\tau^2} + \kappa e_0.$$

$e = (e_0, e_1, e_2)$ is a solution of (2.12) and since $W = \det(e_0, \frac{\mathrm{d}e_0}{\mathrm{d}\tau}, \frac{\mathrm{d}^2 e_0}{\mathrm{d}\tau^2}) = \text{const} \neq 0$ we may suppose that $W = 1$ so that $e(\tau) \in G$, for each $\tau \in I$. This means that $e$ is the normal frame of the curve defined by the $u^j$. The curve is parameterized by the projective arclength and $\kappa(\tau)$ is its projective curvature.

*Remark* 2.6 (Curves with constant projective curvature). The curves in $\mathbb{P}^2$ with constant projective curvature are obtained from the solutions of the linear equation with constant coefficients

$$\frac{\mathrm{d}^3 u}{\mathrm{d}\tau^3} + 2\kappa \frac{\mathrm{d}u}{\mathrm{d}\tau} - u = 0, \quad \kappa = \text{const}.$$

These curves are equivalent either to generalized parabolas, exponential curves, or logarithmic spirals, and are known classically as $W$-curves. They have the property of admitting a 1-dimensional subgroup of the projective group as group of symmetries.

2.3. **The projective Frenet system of nondegenerate curves.** According to Theorem 2.3, we shall identify nondegenerate curves in $\mathbb{P}^2$ with the integral curves of the Pfaffian differential system on $G$ with one independent variable defined by the sub-bundles $W, L$ of $T^*(G)$ given by

$$W = \mathrm{Span}(\eta^1, \dots, \eta^6), \quad L = \mathrm{Span}(\eta^1, \dots, \eta^6, \varphi),$$

where

$$\begin{aligned} &\eta^1 = \omega_0^2, \quad \eta^2 = \omega_1^2 - \omega_0^1, \quad \eta^3 = \omega_1^1, \\ &\eta^4 = \frac{1}{2}(\omega_1^0 - \omega_2^1), \quad \eta^5 = \omega_2^0 - \omega_0^1, \quad \eta^6 = \omega_0^0 \end{aligned} \tag{2.14}$$

$$\varphi = \omega_0^1. \tag{2.15}$$

We call $(G, W, L)$ the *projective Frenet system* of nondegenerate curves in $\mathbb{P}^2$ and denote it by $(\mathcal{F}, \varphi)$. Consider

$$\pi = \frac{1}{2}(\omega_1^0 + \omega_2^1)$$

so that $\{\eta^1, \dots, \eta^6, \varphi, \pi\}$ is a left-invariant trivialization of $T^*(G)$. By the structure equations of $G$ we get

$$\begin{aligned} \mathrm{d}\varphi &= (\eta^6 - \eta^3) \wedge \varphi + \eta^1 \wedge \pi - \eta^1 \wedge \eta^4, \\ \mathrm{d}\pi &= \frac{1}{2}\{\eta^2 \wedge \varphi - (\eta^3 + 2\eta^6) \wedge \pi + 3\eta^3 \wedge \eta^4 + \eta^2 \wedge \eta^5\}, \\ \mathrm{d}\eta^1 &= -\eta^2 \wedge \varphi - \eta^1 \wedge \eta^3 - 2\eta^1 \wedge \eta^6, \\ \mathrm{d}\eta^2 &= 3\eta^3 \wedge \varphi - 2\eta^1 \wedge \pi - 2\eta^2 \wedge \eta^3 - \eta^2 \wedge \eta^6, \\ \mathrm{d}\eta^3 &= 2\eta^4 \wedge \varphi + \eta^2 \wedge \pi - \eta^2 \wedge \eta^4, \\ \mathrm{d}\eta^4 &= \frac{1}{2}\{(\eta^2 + 2\eta^5) \wedge \varphi + 3\eta^3 \wedge \pi - \eta^3 \wedge \eta^4 + \eta^2 \wedge \eta^5 + 2\eta^4 \wedge \eta^6\}, \\ \mathrm{d}\eta^5 &= -3\eta^6 \wedge \varphi - (\eta^1 + 2\eta^4) \wedge \pi - \eta^1 \wedge \eta^4 - \eta^3 \wedge \eta^5 + 2\eta^5 \wedge \eta^6, \\ \mathrm{d}\eta^6 &= (\eta^1 - \eta^4) \wedge \varphi + \varphi \wedge \pi + \eta^1 \wedge \eta^2. \end{aligned} \tag{2.16}$$

*Remark* 2.7. From (2.16) we easily compute the derived flag (cf. [2]) of $(\mathcal{F}, \omega)$:

$$\mathcal{F}_6 = \{0\}, \quad \mathcal{F}_j = \{\eta, \dots, \eta^{6-j}\}, \quad j = 1, \dots, 5.$$

Thus all derived systems of $\mathcal{F}$ have constant rank.

## 3. The variational problem

3.1. **The projective arclength functional.** On the space $\mathcal{G}$ of nondegenerate plane curves we shall consider the projective arclength functional

$$\mathcal{L} : \gamma \in \mathcal{G} \mapsto \int_{I_\gamma} \mathrm{d}\tau_\gamma,$$

where $I_\gamma$ is the domain of definition of the curve and $\mathrm{d}\tau_\gamma$ is the corresponding projective arc element. By the preceding discussion, a curve $\gamma \in \mathcal{G}$ is a critical point of $\mathcal{L}$ if and only if its normal frame field $e_\gamma$ is a critical point of the variational problem on the space $\mathcal{V}(W,L)$ of all integral curves of the projective Frenet system $(\mathcal{F},\varphi)$ defined by the functional

$$\hat{\mathcal{L}} : e \in \mathcal{V}(W,L) \mapsto \int_{I_e} e^*\varphi,$$

where $I_e$ is the domain of definition of $e$. In the next section, following the work of Griffiths [8], based on the work of Cartan [5], we shall write the Euler–Lagrange equations of this functional as a Pfaffian exterior differential system on an affine sub-bundle of $T^*(G)$.

3.2. **The Euler–Lagrange system.** Consider the affine sub-bundle $Z = \varphi + W \subset T^*(G)$ and denote by $\pi_Z : Z \to G$ the bundle projection. Using $(\eta^1, \dots, \eta^6, \varphi)$, $Z$ may be identified with $G \times \mathbb{R}^6$ by posing

$$(e; \lambda_1, \dots, \lambda_6) \in G \times \mathbb{R}^6 \mapsto \varphi_{|e} + \lambda_\alpha {\eta^\alpha}_{|e} \in Z,$$

where $\lambda_1, \dots, \lambda_6$ are the fiber coordinates on $Z$. The restriction to $Z$ of the Liouville (tautological) 1-form of $T^*(G)$ takes the form

$$\psi = \varphi + \lambda_\alpha \eta^\alpha.$$

Let $\mathcal{C}(\Psi)$ be the Cartan system of the 2-form $\Psi = \mathrm{d}\psi$, i.e., the Pfaffian ideal on $Z$ generated by $\{i_X\Psi : X \in \Gamma(T(Z))\} \subset \Omega^1(Z)$ with the independence condition given by the pull-back of $\varphi$ via the projection $\pi_Z$. By (2.16), it follows that

$$\begin{aligned}
\Psi \quad = \quad & \mathrm{d}\lambda_\alpha \wedge \eta^\alpha - \lambda_6 \pi \wedge \varphi + [\eta^1 - 2\lambda_2\eta^1 + \lambda_3\eta^2 + \frac{3}{2}\lambda_4\eta^3 + \\
& -\lambda_5(\eta^1 + 2\eta^4)] \wedge \pi + [\eta^6 - \eta^3 - \lambda_1\eta^2 + 3\lambda_2\eta^3 + 2\lambda_3\eta^4 + \\
& + \frac{1}{2}\lambda_4(\eta^2 + 2\eta^5) - 3\lambda_5\eta^6 + \lambda_6(\eta^1 - \eta^4)] \wedge \varphi \quad \bmod \{\eta^\alpha \wedge \eta^\beta\}.
\end{aligned}$$

Contracting $\Psi$ with the vector fields

$$\left(\frac{\partial}{\partial \eta^1}, \dots, \frac{\partial}{\partial \eta^6}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \pi}, \frac{\partial}{\partial \lambda^1} \dots, \frac{\partial}{\partial \lambda^6}\right),$$

dual to the co-framing $(\eta^1, \dots \eta^6, \varphi, \pi, \mathrm{d}\lambda_1, \dots, \mathrm{d}\lambda_6)$ on $Z$, we find the 1-forms

$$\beta_1 = \mathrm{d}\lambda_1 - \lambda_6\varphi - (1 - 2\lambda_2 - \lambda_5)\pi, \tag{3.1}$$

$$\beta_2 = \mathrm{d}\lambda_2 + (\lambda_1 - \frac{1}{2}\lambda_4)\varphi - \lambda_3\pi, \tag{3.2}$$

$$\beta_3 = \mathrm{d}\lambda_3 + (1 - 3\lambda_2)\varphi - \frac{3}{2}\lambda_4\pi, \tag{3.3}$$

$$\beta_4 = \mathrm{d}\lambda_4 + (\lambda_6 - 2\lambda_3)\varphi + 2\lambda_5\pi, \tag{3.4}$$

$$\beta_5 = \mathrm{d}\lambda_5 - \lambda_4\varphi, \tag{3.5}$$

$$\beta_6 = \mathrm{d}\lambda_6 + (3\lambda_5 - 1)\varphi, \tag{3.6}$$

$$\rho_1 = \lambda_6\varphi, \quad \rho_2 = \lambda_6\pi, \tag{3.7}$$

and $\eta^1, \dots, \eta^6$, respectively. We have then

**Lemma 3.1.** *The Cartan system $\mathcal{C}(\Psi)$ associated to $(\mathcal{F},\varphi)$ is the differential ideal on $Z = \varphi + W$ generated by*

$$\{\eta^1, \dots \eta^6, \varphi, \rho_1, \rho_2, \beta_1, \dots, \beta_6\}$$

*and independence condition given by the Liouville form $\psi$.*

We now construct the involutive prolongation of the Cartan system (see Griffiths [8], pp. 78-83, for the details of this construction). Let $V_1(\Psi) \hookrightarrow \mathbb{P}[T(Z)] \to Z$ be the totality of all 1-dimensional integral elements of $\mathcal{C}(\Psi)$. In view of (3.7), we find that

$$V_1(\Psi)_{|(e,\lambda)} \neq \emptyset \iff \lambda_6 = 0.$$

Thus, the first involutive prolongation of $(\mathcal{C}(\Psi), \varphi)$, i.e., the image $Z_1 \subset Z$ of $V_1(\Psi)$ with respect to the natural projection $V_1(\Psi) \to Z$, is given by

$$Z_1 = \{(e,\lambda) \in Z : \lambda_6 = 0\} \cong G \times \mathbb{R}^5.$$

Next, the restriction of $\beta_6$ to $Z_1$ takes the form $(3\lambda_5 - 1)\varphi$. Thus the second involutive prolongation $Z_2$ is characterized by the equations

$$\lambda_6 = 0, \quad 3\lambda_5 - 1 = 0.$$

Considering then the restriction of $\beta_5$ to $Z_2$ yields the equations

$$\lambda_4 = 0, \quad \lambda_6 = 0, \quad 3\lambda_5 - 1 = 0,$$

which define the third involutive prolongation $Z_3$. Intrinsically, $Z_3 = (\varphi + \frac{1}{3}\eta^3) + \mathrm{Span}(\eta^1, \eta^2, \eta^3)$. The restriction $\mathcal{C}_3(\Psi)$ to $Z_3$ of $\mathcal{C}(\Psi)$ is generated by the 1-forms $\eta^1, \dots, \eta^6$ and

$$\begin{aligned}
\beta_1 &= \mathrm{d}\lambda_1 - 2(\frac{1}{3} - \lambda_2)\pi,\\
\beta_2 &= \mathrm{d}\lambda_2 + \lambda_1\varphi - \lambda_3\pi,\\
\beta_3 &= \mathrm{d}\lambda_3 + (1 - 3\lambda_2)\varphi,\\
\beta_4 &= -2\lambda_3\varphi + \frac{2}{3}\pi.
\end{aligned}$$

This implies that for each $p \in Z_3$ there exists an integral element $(p, E) \in V_1(\Psi)$, i.e., $V_1(\Psi)_{|(e,\lambda)} \neq \emptyset$, for each $(e,\lambda) \in Z_3$. In other words $\mathcal{C}_3(\Psi)$ is the involutive prolongation of $\mathcal{C}(\Psi)$.

**Definition 3.2.** Following Griffiths [8], $Y := Z_3 \subset Z$ is called the *momentum space* associated with the variational problem $\mathcal{L}$. The Pfaffian system $(\mathcal{J}, Y, \psi)$ on $Y$, generated by the involutive prolongation $\mathcal{C}(\Psi_Y)$ of the Cartan system $\mathcal{C}(\Psi)$, is the corresponding *Euler–Lagrange system.*

An easy computation also gives

$$\begin{aligned}
\Psi_Y = \mathrm{d}\psi_Y &= \mathrm{d}\lambda_1 \wedge \eta^1 + \mathrm{d}\lambda_2 \wedge \eta^2 + \mathrm{d}\lambda_3 \wedge \eta^3 + [-\lambda_1\eta^2 + (3\lambda_2 - 1)\eta^3 + \\
&+ 2\lambda_3\eta^4] \wedge \varphi + +[(\frac{2}{3} - 2\lambda_2)\eta^1 + \lambda_3\eta^2 - \frac{2}{3}\eta^4] \wedge \pi + (\lambda_1\eta^3 + \frac{4}{3}\eta^4 + \\
&+ \lambda_1\eta^6) \wedge \eta^1 + +(2\lambda_2\eta^3 + \lambda_3\eta^4 + \lambda_2\eta^6) \wedge \eta^2 - (\frac{1}{3}\eta^3 + \frac{2}{3}\eta^6) \wedge \eta^5.
\end{aligned}$$

The above discussion yields the following.

**Proposition 3.3.** *The momentum space $Y$ associated to the projective arclength functional $\mathcal{L}$ is the rank-3 affine sub-bundle*

$$Y = (\varphi + \frac{1}{3}\eta^3) + \mathrm{Span}(\eta^1, \eta^2, \eta^3) \subset T^*(G).$$

*The Euler–Lagrange system $(\mathcal{J}, Y)$ is generated by the 1-forms*

$$\{\eta^1, \dots \eta^6, \beta_1, \dots, \beta_4\}.$$

*The restriction of the Liouville form of $T^*(G)$ to $Y$ takes the form*

$$\psi_Y = \varphi + \frac{1}{3}\eta^5 + \lambda_1\eta^1 + \lambda_2\eta^2 + \lambda_3\eta^3$$

*and satisfies*

$$\psi_Y \wedge (\mathrm{d}\psi_Y)^5 \neq 0.$$

*Thus $\psi_Y$ is a contact form and the integral curves of the Euler–Lagrange system are the integral curves of the corresponding characteristic vector field*

$$\mathcal{V} = \frac{\partial}{\partial\varphi} + (2-6\lambda_2)\lambda_3\frac{\partial}{\partial\lambda_1} - (\lambda_1 - 3\lambda_3^2)\frac{\partial}{\partial\lambda_2} - (1-3\lambda_2)\frac{\partial}{\partial\lambda_3}.$$

*Remark* 3.4. Note that at each step of the prolongation process we end up with smooth algebraic subvarieties of $T^*(G)$. Let $\pi_Y$ be the restriction to $Y$ of the projection $\pi_Z$. The importance of the Euler–Lagrange system is in the fact that the projections of its integral curves are critical points of the action functional. The converse, in general, is not true. However, if all derived systems of the Pfaffian system have constant rank (as in the case under discussion), then all the extrema arise as projections of integral curves of the Euler–Lagrange system. We refer to [1], [8], and [12] for more details.

## 4. Reduction for the variational problem

4.1. **The Lax formulation.** We use the $\mathrm{Ad}(G)$-invariant bilinear form $\mathrm{tr}(XY)$ to identify $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ with its dual $\mathfrak{g}^*$. Accordingly, the adjoint and coadjoint representations become equivalent. Under this identification, the left invariant 1-form $\varphi + \frac{1}{3}\eta^5 + \lambda_1\eta^1 + \lambda_2\eta^2 + \lambda_3\eta^3 \in \mathfrak{g}^*$ goes over to

$$p(\lambda_1,\lambda_2,\lambda_3) = E_0 + \lambda_1E_1 + \lambda_2E_2 + \lambda_3E_3 = \begin{pmatrix} -\lambda_3 & 2-3\lambda_2 & 3\lambda_1 \\ 0 & 2\lambda_3 & 3\lambda_2 \\ 1 & 0 & -\lambda_3 \end{pmatrix} \in \mathfrak{g},$$

where $E_0 = E_{2,0} + 2E_{0,1}$, $E_1 = 3E_{0,2}$, $E_2 = 3(E_{1,2} - E_{0,1})$, $E_3 = 2E_{1,1} - E_{0,0} - E_{2,2}$, being $E_{i,j}$ $(0 \le i,j \le 2)$ the matrix with 1 in the $(i,j)$ place and 0 elsewhere.

Let $\mathfrak{p}$ be the 3-dimensional affine subspace

$$\mathfrak{p} = E_0 + \mathrm{Span}(E_1,E_2,E_3)$$

of $\mathfrak{g}$. Then, the mapping

$$(a, \varphi + \frac{1}{3}\eta^5 + \lambda_1\eta^1 + \lambda_2\eta^2 + \lambda_3\eta^3) \in Y \mapsto (a, p(\lambda_1,\lambda_2,\lambda_3)) \in G\times\mathfrak{p}$$

induces a bundle isomorphism $Y \cong G \times \mathfrak{p}$.

**Definition 4.1.** We call $\mathfrak{p}$ the *phase space* of the variational problem.

A curve $\Gamma : I \to Y$, $\Gamma(t) = (a(t), p(t))$, is a solution of the Euler–Lagrange system if and only if

$$\text{(i)}\ \Gamma^*\eta^j = 0,\ j = 1,\dots,6, \quad \text{(ii)}\ \Gamma^*\beta_4 = \Gamma^*(\pi - 3\lambda_3\varphi) = 0, \tag{4.1}$$

$$\text{(iii)}\ \Gamma^*\beta_1 = 0, \quad \text{(iv)}\ \Gamma^*\beta_2 = 0, \quad \text{(v)}\ \Gamma^*\beta_3 = 0 \tag{4.2}$$

with independence condition

$$\Gamma^*\varphi \neq 0. \tag{4.3}$$

If we assume (as always in the following) that the parameterization is such that $\Gamma^*\varphi = \mathrm{d}t$, then equations (i), (ii), and (4.3) tell us that

$$a^{-1}\dot{a} = \begin{pmatrix} 0 & 3\lambda_3(t) & 1 \\ 1 & 0 & 3\lambda_3(t) \\ 0 & 1 & 0 \end{pmatrix} =: Q(t) = Q_{p(t)}. \tag{4.4}$$

Here $Q : \mathfrak{p} \to \mathfrak{g}$ is the mapping defined by

$$p(\lambda_1, \lambda_2, \lambda_3) \mapsto Q_p = E_{1,0} + E_{1,1} + E_{2,1} + 3\lambda_3(E_{0,1} + E_{1,2}).$$

Using this, a direct calculation shows that equations (4.2) can be written in the form

$$\dot{p} = [p, Q_p] = \begin{pmatrix} 1 - 3\lambda_2 & -9\lambda_3^2 + 3\lambda_1 & 6\lambda_3 - 18\lambda_2\lambda_3 \\ 0 & 2 - 6\lambda_2 & 9\lambda_3^2 - 3\lambda_1 \\ 0 & 0 & 1 - 3\lambda_2 \end{pmatrix}. \tag{4.5}$$

We have established the following:

**Proposition 4.2.** *A smooth curve $\Gamma(t) = (a(t), p(t)) : I \to Y$ is an integral curve of the Euler–Lagrange system associated to the arclength variational problem if and only if*

$$\mathrm{d}t = \Gamma^*\varphi, \tag{4.6}$$

$$\dot{a} = aQ_p, \tag{4.7}$$

$$\dot{p} = -[Q_p, p]. \tag{4.8}$$

If $\Gamma = (a, p)$ is an integral curve of $(Y, \mathcal{J})$, then, according to Theorem 2.3, $a$ is a normal frame along the curve $\gamma = [a_0] : I \to \mathbb{P}^2$ and $p(t)$ can be written in terms of the projective curvature $\kappa$ of $\gamma$. In fact, equations (ii) and (iv), (v) yield, respectively,

$$\kappa = -3\lambda_3, \quad \text{and} \quad \lambda_1 = \frac{1}{9}\ddot{\kappa} + \frac{1}{3}\kappa^2, \quad \lambda_2 = -\frac{1}{9}\dot{\kappa} + \frac{1}{3}. \tag{4.9}$$

This together with (iii) gives[3]

$$\dddot{\kappa} + 8\kappa\dot{\kappa} = 0. \tag{4.10}$$

Conversely, if $\gamma : I \to \mathbb{P}^2$ is a nondegenerate curve parameterized by the projective arclength and $e_\gamma : I \to G$ is the corresponding normal frame, then the lift $\Gamma : I \to Y$ given by $\Gamma = (e_\gamma, \lambda_1, \lambda_2, \lambda_3)$, where the $\lambda$'s are given in terms of the projective curvature as above, is an integral curve of the Euler–Lagrange system if and only if (4.10) is satisfied.

We have then:

**Proposition 4.3.** *If a smooth curve $\Gamma = (a, p) : I \to Y$ is an integral curve of the Euler–Lagrange system $(\mathcal{J}, \varphi)$, then $a$ is the normal frame along $\gamma = [a_0]$ and the projective curvature of $\gamma$ is a solution to* (4.10)*. Conversely, any integral curve of the Euler–Lagrange system arises as a lift of an arclength parameterized curve in $\mathbb{P}^2$ whose projective curvature satisfies* (4.10).

[3]Equation (4.10) is the Euler–Lagrange equation for the critical points of (1.1). This was computed by Cartan [4].

*Remark* 4.4. If $\Gamma(t) = (a(t), p(t))$ is an integral curve of $\mathcal{J}$, (4.8) tells us that $p(t)$ is an integral curve of the vector field $\Phi$ defined on $\mathfrak{p}$ by

$$\Phi_{\mathcal{V}} : p \mapsto -[Q_p, p].$$

From Proposition 4.2 we also learn that the characteristic vector field $\mathcal{V}$ can be written in the form

$$\mathcal{V}_{|(a,p)} = Q_{p_{|a}} + \Phi_{\mathcal{V}}(p),$$

for each $(a,p) \in Y$. If $\mathfrak{p}_s$ denote the set of all singular points of $\Phi_{\mathcal{V}}$, then the integral curves of $\mathcal{J}$ passing through a point $(a,p) \in G \times \mathfrak{p}_s$ are orbits of the one-parameter subgroups of $G$ generated by $Q_P$. Moreover, from (4.5) and (4.9), it follows that they project to curves in $\mathbb{P}^2$ with constant projective curvature.

**Definition 4.5.** We shall denote by $\mathfrak{p}_r$ the complement of $\mathfrak{p}_s$ in $\mathfrak{p}$.

4.2. **The moment map.** The group $G$ induces a Hamiltonian action on $Y \subset T^*(G)$ by

$$g \cdot (a; \lambda_1, \lambda_2, \lambda_3) := (ga; \lambda_1, \lambda_2, \lambda_3), \quad \text{for each } g \in G, \text{ and } (a; \lambda_1, \lambda_2, \lambda_3) \in Y.$$

The *moment map* of this action is defined by

$$\mu : Y \to \mathfrak{g}^*, (a; \lambda_1, \lambda_2, \lambda_3) \mapsto \mathrm{Ad}^*(a)\lambda,$$

where $\mathrm{Ad}^*$ is the coadjoint action and where

$$\lambda = \varphi + \frac{1}{3}\eta^5 + \lambda_1\eta^1 + \lambda_2\eta^2 + \lambda_3\eta^3 \in \mathfrak{g}^*.$$

$\mu$ is an equivariant map. Under the identification $\mathfrak{g}^* \cong \mathfrak{g}$, the moment map takes the form

$$\mu : Y \to \mathfrak{g}, (a; p(\lambda_1, \lambda_2, \lambda_3)) \mapsto ap(\lambda_1, \lambda_2, \lambda_3)a^{-1}. \tag{4.11}$$

From (4.11), we obtain

$$\mathrm{d}\mu_{|(a;p)} = a(\mathrm{d}p + [\omega, p])a^{-1}. \tag{4.12}$$

As an immediate consequence of the Lax equation (4.8) we then have Noether's theorem

**Corollary 4.6.** *The moment map is constant on solution curves to the Euler–Lagrange system.*

In the following we will denote by $\mu_r : G \times \mathfrak{p}_r \to \mathfrak{g}$ the restriction of the moment map to the regular part of the momentum space. Note that $G \times \mathfrak{p}_s$ are the singular points of $\mu$.

For any integral curve $\Gamma$ of the Euler–Lagrange system and for each $r \in \mathbb{R}$ we have that $\mathrm{rank}(p(t) - r\mathrm{id}_3) \geq 2$. This implies that the eigenspaces of the constant element $\mu \circ \Gamma = m \in \mathfrak{g}$ are at most one dimensional. Therefore, $\mu \circ \Gamma$ takes values in either one of the following $\mathrm{Ad}(G)$-invariant subsets of $\mathfrak{g}$:

$$\begin{aligned}
\mathfrak{g}_{\mathrm{I.a}} &= \{X : X \text{ has one eigenvalue of multiplicity 3 and dimKer } X = 1\}, \\
\mathfrak{g}_{\mathrm{II.b}} &= \{X : X \text{ has two real eigenvalues, } X \text{ not simple}\}, \\
\mathfrak{g}_{\mathrm{III}} &= \{X : X \text{ has three real distinct eigenvalues}\}, \\
\mathfrak{g}_{\mathrm{IV}} &= \{X : X \text{ has one real and two complex conjugate eigenvalues}\}.
\end{aligned}$$

We now give a more detailed description of the various invariant subsets. A canonical form for the elements of $\mathfrak{g}_{\mathrm{I.a}}$ is

$$X_{\mathrm{I.a}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $\mathfrak{g}_{\mathrm{I.a}}$ coincides with the orbit through $X_{\mathrm{I.a}}$. The corresponding isotropy group $G_{\mathrm{I.a}}$ is the 2-dimensional group

$$G_{\mathrm{I.a}} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

and $\mathfrak{g}_{\mathrm{I.a}}$ is six dimensional.

In the case II.b, a canonical form for the elements is

$$X_{\mathrm{II.b}} = \begin{pmatrix} -2x & 0 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix}, \quad x \in \mathbb{R}.$$

The isotropy group is

$$G_{\mathrm{II.b}} = \left\{ \begin{pmatrix} v^{-2} & 0 & 0 \\ 0 & v & u \\ 0 & 0 & v \end{pmatrix} : v \neq 0, u \in \mathbb{R} \right\}$$

and the orbit through $X_{\mathrm{II.b}}$ is six-dimensional. The orbit space $\mathfrak{g}_{\mathrm{II.b}}/G \cong \mathbb{R}^*$.

In the case III, a canonical form for the elements of $\mathfrak{g}_{\mathrm{III}}$ is

$$X_{\mathrm{III}} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & -(x+y) \end{pmatrix}, \quad x, y \in \mathbb{R}, x > y.$$

The isotropy group is

$$G_{\mathrm{III}} = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & (uv)^{-1} \end{pmatrix} : u, v \in \mathbb{R}^* \right\}$$

and the orbit through $X_{\mathrm{III}}$ is six-dimensional. The orbit space $\mathfrak{g}_{\mathrm{III}}/G$ is the open cone $\{(x,y) \in \mathbb{R}^2 : x > y, x > -2y, x > 0\}$.

Finally the case IV. For any $X \in \mathfrak{g}_{\mathrm{IV}}$, let $-\frac{1}{2}x \pm iy$, $y > 0$, be the two complex conjugate eigenvalues. Then a canonical form for $X$ is

$$X_{\mathrm{IV}} = \begin{pmatrix} x & 0 & 0 \\ 0 & -\frac{x}{2} & -y \\ 0 & y & -\frac{x}{2} \end{pmatrix}, \quad x, y \in \mathbb{R},\ y > 0.$$

The corresponding isotropy group is

$$G_{\mathrm{IV}} = \left\{ \begin{pmatrix} (u^2+v^2)^{-1} & 0 & 0 \\ 0 & u & -v \\ 0 & v & u \end{pmatrix} : u^2 + v^2 \neq 0 \right\}.$$

The orbits are again six-dimensional and the orbit space $\mathfrak{g}_{\mathrm{IV}}/G$ is the upper half plane $\{(x,y) : y > 0\}$.

*Remark* 4.7. The elements of $R_\mu := \mathfrak{g}_{\mathrm{I.a}} \sqcup \mathfrak{g}_{\mathrm{II.b}} \sqcup \mathfrak{g}_{\mathrm{III}} \sqcup \mathfrak{g}_{\mathrm{IV}}$ are regular elements of the Lie algebra $\mathfrak{g}$ as well as regular values of the moment map $\mu_r$. In particular, the isotropy subgroup $G_m$ of any $m \in R_\mu$ is abelian.

Let $m \in R_\mu$ be a regular value of the moment map and let $G_m$ be the isotropy subgroup of $m$. From the above discussion we know that $G_m$ is abelian and $\dim G_m = \mathrm{rank}(G) = 2$. The level set $M_m = \mu_r^{-1}(m)$ is $G_m$-invariant and the quotient $Y_m = M_m/G_m$ is a (one-dimensional) manifold since $G_m$ acts properly without fixed points. The projection $\pi_m : M_m \to Y_m$ is a $G_m$-principal fibration. The characteristic vector field $\mathcal{V}$ is tangent to $M_m$ and invariant under $G_m$. Thus it is horizontal for $\pi_m$.

**Definition 4.8.** If $(a,p) \in M_m$, then $p \in \mathfrak{p}_r$ belongs to the orbit $\mathcal{O}_m$ of $m$, i.e., $p \in \mathcal{O}_m \cap \mathfrak{p}_r$. Let $P(m)$ be the connected component of $\mathcal{O}_m \cap \mathfrak{p}_r$ containing $p$. We call $P(m)$ the *phase portrait* of $m$

Besides the fibration $\pi_m$, we can consider the other $G_m$-principal fibration $\tilde{\pi}_m : M_m \to P(m)$ defined by sending $(a,p) \in M_m$ to $p \in P(m)$. Since $\tilde{\pi}_m$ is constant along the fibers of $\pi_m$, the reduced space $Y_m$ can be identified with the phase portrait $P(m)$. Observe that the vector field $\Phi_{\mathcal{V}}$ is tangent to $\mathcal{O}_m \cap \mathfrak{p}_r$ and that the restrictions of $\mathcal{V}$ and $\Phi_{\mathcal{V}}$ to $M_m$ and $P(m)$, respectively, are $\tilde{\pi}_m$-related. Moreover, it happens that, whenever $\Gamma(t) = (a(t), p(t))$ is an integral curve with moment $m$, $p(t)$ is a smooth parameterization of the phase portrait $P(m)$.

Summarizing, the integration of the extrema with moment $m \in R_\mu$ can be achieved by the following procedure:

- find a smooth parameterization $p(t) : I = (a,b) \to P(m)$ of the phase portrait $P(m)$;
- construct any map $x(t) : I \to G$ such that $(x(t), p(t)) : I \to Y$ is a cross section of the fibration $\tilde{\pi}_m : M_m \to P(m)$; this is a purely algebraic problem: $xpx^{-1} = m$;
- given $(x(t), p(t))$, any other lift $(a(t), p(t))$ is of the form $a(t) = b(t)x(t)$ for some $b : I \to G_m$. In order that $(a(t), p(t))$ be an integral curve, $b : I \to G_m$ must satisfy the equation $a^{-1}\dot{a} = Q_p$, that is,

$$b^{-1}\dot{b} = xQ_px^{-1} - \dot{x}x^{-1}. \tag{4.13}$$

*Remark* 4.9. Equation (4.13) expresses the fact that the curve $(a(t), p(t))$ is horizontal with respect to a connection defined on the principal $G_m$-fibration $\tilde{\pi}_m$. On a section $(x,p)$ the $\mathfrak{g}_m$-valued connection 1-form takes the form $xQ_px^{-1} - \mathrm{d}xx^{-1}$. According to Remark 4.4, it is not difficult to see that the integral curves of the characteristic vector field $\mathcal{V}$ are exactly the horizontal curves of $\eta$.

## 5. Integration of the extrema

In this last section we briefly indicate how to implement the scheme of integration. Let $\Gamma = (a,p)$ be an integral curve of $\mathcal{J}$. Then the image of the corresponding curve in the phase space is contained in the curve defined by the two equations

$$\begin{aligned} g_2 &= 12(\lambda_3^2 + \lambda_1), \\ g_3 &= 4(2\lambda_3^3 - 9\lambda_2^2 + 6\lambda_2 - 6\lambda_1\lambda_3 - 1), \end{aligned} \tag{5.1}$$

where $g_2 = G_2(\mu_r \circ \Gamma)$ and $g_3 = G_3(\mu_r \circ \Gamma)$ are the two invariants of $\Gamma$ defined by the $\mathrm{Ad}(G)$-invariant functions $G_2, G_3 : \mathfrak{g} \to \mathbb{R}$ :

$$G_2(X) = \mathrm{tr}(XX), \quad G_3(X) = 4(\det X - 1), \quad X \in \mathfrak{g}.$$

By (4.9), on integral curves of $\mathcal{J}$ these equations become

$$\ddot{\kappa} + 4\kappa^2 = \frac{3}{4}g_2,$$
$$(\dot{\kappa})^2 + \frac{8}{3}\kappa^3 - \frac{3}{2}g_2\kappa = -\frac{9}{4}g_3.$$

Setting $p == -\frac{2}{3}\kappa$, we obtain

$$(\dot{p})^2 = 4p^3 - g_2 p - g_3. \tag{5.2}$$

This equation can be solved by standard techniques in terms of elliptic functions. Consider the meromorphic function $q(z)$ with a double pole at the origin and residue zero satisfying equation (5.2) and let $\Delta(g_2, g_3) = -(1/27)g_2^3 + g_3^2$ be the discriminant of (5.2). If $\Delta(g_2, g_3) < 0$, then $q(z)$ is the Weierstrass elliptic function with invariants $g_2$ and $g_3$ and primitive half periods $\omega_1$ and $\omega_3$, respectively, real and pure imaginary. If $\Delta(g_2, g_3) > 0$, then $q(z)$ is the Weierstrass elliptic function with primitive half periods $\omega_1 \in \mathbb{R}$ and $\omega_3 = (1/2)(1 + iv)\omega_1$, $v > 0$. If $\Delta(g_2, g_3) = 0$, then equation (5.2) can be integrated explicitly in terms of elementary functions.

Let $\sigma(z)$, $\zeta(z)$ be the sigma and zeta functions of Weierstrass associated to $q(z)$. They satisfy $\sigma(0) = 0$, $\sigma'(0) = 1$, $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$, $q(z) = -\zeta'(z)$. We recall some well-known formulae:

$$\begin{aligned}
&\text{(i)} \quad q(v) - q(u) = \frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2},\\
&\text{(ii)} \quad \zeta(u+v) + \zeta(u-v) - 2\zeta(u) = \frac{q'(u)}{q(u) - q(v)},\\
&\text{(iii)} \quad \zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2}\frac{q'(u) - q'(v)}{q(u) - q(v)},\\
&\text{(iv)} \quad q(u+v) - q(v) = \frac{1}{2}\frac{q''(u)}{q(u) - q(v)} - \frac{1}{2}\frac{q'(u) - q''(v)}{(q(u) - q(v))^2}q'(v),\\
&\text{(v)} \quad q''(u) = 6q(u)^2 - \frac{1}{2}g_2.
\end{aligned} \tag{5.3}$$

Retaining the notation introduced above, we have

**Lemma 5.1.** *Let $\Gamma = (a, p) : I \to Y$ be an integral curve of $\mathcal{J}$. Then there exist an open domain $I \subset D \subset \mathbb{C}$, a holomorphic map $A(z) : D \to \mathrm{SL}(3, \mathbb{C})$ which extends $a$, and a meromorphic map $P(z) : D \to \mathfrak{sl}(3, \mathbb{C})$ which extends $p$ such that*

$$P(z) = \begin{pmatrix} -\frac{q(z)}{2} & \frac{2-q'(z)}{2} & \frac{g_2 - 3q(z)^2}{4} \\ 0 & q(z) & \frac{q'(z)+2}{2} \\ 1 & 0 & -\frac{q(z)}{2} \end{pmatrix}. \tag{5.4}$$

*Proof.* By the above discussion, the projective curvature $\kappa : I \to \mathbb{R}$ is the restriction of a meromorphic function $K : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. Let $D$ be a connected and simply connected domain containing $I$ such that $K_{|D}$ is regular. On $D$ we consider the system of holomorphic differential equations:

$$\mathrm{d}A_0 = A_1\mathrm{d}z, \quad \mathrm{d}A_1 = -KA_1\mathrm{d}z + A_2\mathrm{d}z, \quad \mathrm{d}A_2 = A_0\mathrm{d}z - KA_1\mathrm{d}z. \tag{5.5}$$

By the complex analogue of the Cartan–Darboux theorem ([9]), there exists a holomorphic map $A(z) : D \to \mathrm{SL}(3, \mathbb{C})$ whose columns $A_j(z)$, $0 \le j \le 2$, are solutions of (5.5). Moreover, $A(z)$ is unique up to left multiplication by a constant element of $\mathrm{SL}(3, \mathbb{C})$. The result follows by taking the solution satisfying the initial condition $A(t_0) = a(t_0)$, $t_0 \in I$. Further, $K(z)$ and $q(z)$ are related either by

$$K(z) = -\frac{3}{2}q(z), \quad \text{or} \quad K(z) = -\frac{3}{2}q(z + \omega_3). \tag{5.6}$$

In the latter case we just replace $A(z)$ by $A(z - \omega_3)$. By (5.6), (4.9), and (iii) of (5.3), $P$ extends to a meromorphic function $P(z) : D \to \mathfrak{sl}(3, \mathbb{C})$ as given by (5.4). □

This conclude the first step that consists in finding a parameterization of the phase portrait. Next, the eigenvalues of $P(z)$ are the roots $r_j$, $0 \le j \le 2$, of

$$-4\det(P(z) - sI_3) = 4s^3 - g_2 s - g_3 - 4 = 0. \tag{5.7}$$

By Corollary 4.6, the eigenvalues of $P(z)$ do not depend on $z$ and for each $r_j$, $0 \le j \le 2$, we can choose $z_j \in \mathbb{C}$ such that

$$q(z_j; g_2, g_3) = r_j, \quad q'(z_j; g_2, g_3) = -2.$$

If, for instance, we consider the case of curves with moment $m \in \mathfrak{g}_{\mathrm{III}}$, then a direct calculation using (5.3) shows that equation $XPX^{-1} = m$ can be solved by $X = Y^{-1}$, where

$$Y = \begin{pmatrix} \frac{q(z)+2r_1}{2} & \frac{q(z)+2r_2}{2} & \frac{q(z)+2r_3}{2} \\ -\frac{q'(z)+2}{2(q(z)-r_1)} & -\frac{q'(z)+2}{2(q(z)-r_2)} & -\frac{q'(z)+2}{2(q(z)-r_3)} \\ 1 & 1 & 1 \end{pmatrix}.$$

The integral curves with moment $m \in \mathfrak{g}_{\mathrm{III}}$ are then of the form $(bX, p)$ where $b$ is obtained by a single quadrature as a solution of (4.13) and is expressed in terms of Weierstrass functions. The other cases can be treated in a similar way.

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