. . **Construction of ergodic cocycles that are fundamental solutions to linear systems of special form**

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ABSTRACT.
Given a flow (Ω, T, m), a closed convex subset $S \subset L$ of a Lie algebra L and a continuous map $A : \Omega \to S$ consider the fundamental matrix solution X_A of the (time-dependent family) of linear system

$$x' = A(T_t\omega)x, \quad x \in G, \ \omega \in \Omega\,;$$

where G is a connected Lie group with Lie algebra L. In this paper we develop a new technique to show that the set of continuous maps $A : \Omega \to S$ for which cocycles X_A is ergodic is residual provided G is compact and S satisfies "strong accessibility property". We remark that the dimension of S can be much smaller than that of L and thus our result proves typicality of ergodicity even within very "thin" classes of cocycles. Our result covers a number of differential equations arising in Mahthematical Physics, and in particular applies to the widely studied example of the Rabi oscillator. As a consequence of our ergodicity result, it follows that the spectrum of the quasi-energy operator associated with the cocycle X_A is purely continuous for a typical A. In the case of the Rabi oscillator, this contrasts recent spectral results obtained using the K.A.M. technique.

§1. Introduction.

In this paper we develop a new technique to construct ergodic cocycles that arise as the fundamental matrix solutions to linear differential equations of a given form. These constructions are motivated by questions regarding spectral and stability of forced quantum oscillator systems (in particular the so called "Rabi oscillator" which models forced oscillations of a spin 1/2 particle moving under an external stationary stochastic field [BJL,DGP]). These questions lead to problems about ergodic properties of certain skew-product flows (see [NJ]). In [NS1] we had established that the cocycle arising from a generic Rabi oscillator is "minimal". In this paper we shall prove that in fact it is "ergodic". This will help us establish that typically the spectrum of the "quasi-energy operator" associated with such

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systems is only continuous. This result contrasts a result of [BJL] where the authors prove the existence of only discrete pure point spectrum using the KAM technique.

As in [NS1], our result will in fact be valid for a wide class of time dependent linear differential systems of "given specific form" where the time dependence is "recurrent" (in particular quasi-periodic). Such systems are given by specifying;

(1) a flow (Ω, T, m), where Ω is a compact metric space (with metric d) and $T = \{T_t\}_{r\in\mathbb{R}}$ is a one-parameter group of homeomorphisms of Ω that preserves a Borel probability measure m;
(2) a connected Lie group G with Lie algebra L; and
(3) a subset S of L.

Given a continuous map $A \in C(\Omega, S)$-the space of continuous S-valued maps on Ω, consider the family of linear differential equations (parametrized by points $\omega \in \Omega$),

$$x' = A(T_t\omega)x, \quad x \in G,\ \omega \in \Omega\,. \tag{1.1}$$

Here we identify L with the tangent space of G at the identity. Then the right hand side of (1.1) can be thought as the value at x of the right invariant vector field generated by $A(T_t\omega)$. Let $X_A(\omega, t)$ denote the fundamental matrix solution to (1.1). So for each $\omega \in \Omega$, the function $t \to X_A(\omega, t)$ is the solution of (1.1) such that $X_A(\omega, 0) = e_G$- the identity of G. Then the map $X_A : \Omega \times \mathbb{R} \to G$ is continuous and satisfies the following cocycle identity

$$X_A(\omega, t+s) = X_A(T_t(\omega), s)X_A(\omega, t) \qquad \text{for all } \omega \in \Omega,\ t, s \in \mathbb{R}. \tag{1.2}$$

A proper analogue of the flow generated by a family of time dependent differential equations such as (1.1) is the **skew**-*product flow* define on $G \times \Omega$ by

$$T_t^A(g, \omega) = (X_A(\omega, t)g, T_t(\omega)). \tag{1.3}$$

Notice that the product measure $\nu \times m$ (where ν is a left Haar measure on G) is invariant under the skew-product flow. We shall denote this skew-product system by $(G \times \Omega, \{T_t^A\}_{t\in\mathbb{R}}, m)$. When G is compact, we shall assume that $\nu(G) = 1$.

If G is a matrix Lie group, say $G \subseteq GL(n, \mathbb{C})$, one defines a representation $V^A = \{V_t^A\}_{t\in\mathbb{R}}$ on $L^2(\Omega, \mathbb{C}^n, m)$ by setting

$$V_t^A f(\omega) = X_A(\omega, t)^{-1} f(T_t(\omega)). \tag{1.4}$$

If G is a subgroup of the unitary group $U(n)$, this representation is unitary. In the Physics literature the infinitisimal generator of this representation is called the **quasi-energy operator** and the stability properties of evolution of such systems are studied through the spectral properties of this representation. Such a evolution is regarded **stable** if the spectrum of the quasi-energy operators is discrete pure point. In this paper we shall show that (Theorem B) within "very thin classes of linear differential systems of special form", those for which the quasi-energy operator has only purely continuous spectrum are generic.

The absence of point spectrum for a quasi-energy operator is very closely related to a problem of "lifting ergodicity". We recall that a flow $(X, \{T_t\}_{t\in\mathbf{R}}, \lambda)$ is *ergodic* if the λ-measure of any invariant Borel subset is either zero or one. The ergodicity lifting problem involves constructing ergodic skew-product flows $(G \times \Omega, \{T_t^A\}_{t\in\mathbb{R}}, m)$, where the map A is constrained by the special form of the underlying family of differential equations. Under suitable hypothesis on the "nature of constraints" we shall show (Theorem A) that for a generic map $A \in C(\Omega, S)$, the skew-product flow is ergodic. Then the spectral result (Theorem B) follows as a corollary.

Now we describe the assumption imposed on constraint conditions which give the system their special form. The special class of systems is described by specifying a fixed subset S of the Lie algebra L and then the special class results from the demand that the map A be S-valued. We illustrate this with the following example of a Rabi-oscillator.

Example Consider the system

$$i\frac{d\psi}{dt} = \begin{pmatrix} \lambda & f(t) \\ f(t)^* & -\lambda \end{pmatrix} \psi, \qquad \psi \in \mathbb{C}^2. \tag{1.5}$$

This is the Schrödinger equation which describes the dynamics of a "two level atom" or a spin 1/2 particle moving under external magnetic field $f(t)$. The function $f : \mathbb{R} \to \mathbb{C}$ is a complex-valued potential, typically quasiperiodic in t, and $\lambda \in \mathbb{R}$ is a fixed parameter.

The class of cocycles X_A arising from linear differential systems of a specific form is described via a constraint on the values of the map $A : \Omega \to L$. Namely, one considers only those maps with values in a fixed compact convex subset Q of the Lie algebra L. We shall impose a certain Lie algebraic condition (the condition of **strong accessibility**) on the constraint set Q. With this condition, the set Q can still be very "thin" in L in the sense that $dim(G)$-the dimension of G-will in general be higher than $dim(Q)$ which equals the number of free parameters that we are allowed to vary to generate a cocycle with the desired properties. For example, in the case of the Rabi oscillator the cocycle takes values in a three-dimensional Lie group, but we only have two real degrees of freedom, since the map f-which is our perturbation, takes values in a two-dimensional space of complex numbers.

To describe the condition "of strong accessibility", first, let $L(Q)$ be the Lie subalgebra of L generated by Q, and let $L_0(Q)$ be the ideal of $L(Q)$ generated by the difference set $Q - Q = \{x - y \mid x \in Q,\ y \in Q\}$.

1.1 Definition. *A subset Q of the Lie algebra L has the* **accessibility property** *if $L(Q) = L$, and the* **strong accessibility property** *if $L_0(Q) = L$.*

From now on, we choose an inner product $\langle \cdot, \cdot \rangle$ on L and let $|| \cdot ||$ be the corresponding norm.

Theorem A. *Let $(\Omega, \{T_t\}_{t\in\mathbb{R}}, m)$ be a flow on a compact metric space Ω, where m is a flow invariant Borel probability measure on Ω. Let G be a compact connected Lie group with Lie algebra L and let S be a subset of L. Let $C(\Omega, S)$ denote the metric space of all continuous maps from Ω to S with the supremum metric. Assume that*

(1) *the flow $(\Omega, \{T_t\}_{t\in\mathbb{R}}, m)$ is ergodic and aperiodic,*
(2) *S is closed and convex,*
(3) *S has the strong accessibility property.*

Then the set

$$C_{erg}(\Omega, S) = \{A \in C(\Omega, S) : (G \times \Omega, T^A, \nu \times m) \text{ is ergodic}\}, \tag{1.6}$$

is a residual subset of $C(\Omega, S)$.

Next, we state a consequence of Theorem A.

Theorem B. *In addition to the hypothesis of Theorem A, suppose that*

(4) *G is a closed subgroup of the unitary group $U(n)$ and*
(5) *G does not fix any ray in the complex projective n-space $P(\mathbb{C}^n)$.*

Let

$$\begin{aligned} &\mathcal{C}_{cont}(\Omega, S) \\ &\qquad = \{A \in C(\Omega, S) : \text{the representation } \{V_t^A\}_{t\in\mathbf{R}} \text{ has only continuous spectrum}\}. \end{aligned}$$

Then $\mathcal{C}_{cont}(\Omega, S)$ is a residual subset of $C(\Omega, S)$.

First, we show that Theorem B can be deduced from Theorem A via the following lemma.

1.2 Lemma. *Suppose that the group G satisfies conditions (4) and (5) of Theorem B and suppose that the skew-product flow $(G \times \Omega, \{T_t^A\}_{t\in\mathbb{R}}, m)$ is ergodic. Then the representation V^A does not admit any point spectrum.*

Proof. Suppose $f \in L^2(\Omega, \mathbb{C}^n, \mu)$ is an eigenvector of V^A. Since the representation is unitary, our supposition implies existence of $\lambda \in \mathbb{R}$ such that

$$(V_t^A f)(\omega) = e^{i\lambda t} f(\omega), \quad \text{a.e. } \omega \in \Omega \text{ and for all } t \in \mathbb{R}.$$

Consider the map $\rho : U(n) \times \Omega \to P(\mathbb{C}^n)$, defined by

$$\rho(g, \omega) = \pi(g^{-1} f(\omega)),$$

where $\pi : \mathbb{C}^n \to P(\mathbb{C}^n)$ is the canonical projection onto the projective space. Then,

$$\begin{aligned} \rho(T_t^A(g, \omega)) &= \rho\big(X_A(\omega, t) g, T_t\omega\big) \\ &= \pi\big(g^{-1} X_A(\omega, t)^{-1} f(T_t\omega)\big) \\ &= \pi\big(g^{-1} f(\omega)\big) \\ &= \rho(g, \omega). \end{aligned}$$

Thus ρ is T^A invariant and hence by the ergodicity assumption, it is constant a.e. (g,ω). Thus there is a non-zero vector $v \in \mathbb{C}^n$ such that

$$\pi(g^{-1}f(\omega)) = \pi(v) \quad \text{a.e. } (g,\omega).$$

In particular, this means that almost all $g \in G$, map the ray $\pi(v)$ into a fixed ray. This implies that the stabilizer of this fixed ray is a closed subgroup with full Haar measure and hence must be all of G. This contradicts assumption (5) of Theorem B. ∎

§2. PROOF OF THEOREM A.

To prove this result, first we introduce some notation. If $f \in L^2(G,\nu)$, we use $\langle f \rangle$ to denote the average of f over G with respect to ν, so $\langle f \rangle = \int_G f(g)\, d\nu(g)$. If f belongs to $L^2(G \times \Omega, \nu \times m)$ and $\omega \in \Omega$, we use f^ω to denote the function $G \ni g \mapsto f(g,\omega) \in \mathbb{C}$. Then $f^\omega \in L^2(G,\nu)$ for almost all $\omega \in \Omega$ and $\int_\Omega \|f^\omega\|^2_{L^2} dm(\omega) \le \|f\|^2_{L^2}$.

For $f \in L^2(G \times \Omega, \nu \times m)$ set

$$\hat{f}(g,\omega) = \langle f \rangle \overset{def}{=} \int_G f(g,\omega) d\nu(g).$$

Then $\hat{f}$ is in fact the orthogonal projection of f on the space of square-integrable functions on $G \times \Omega$ that are functions of ω only.

We use

$$\begin{aligned} \mathcal{H} &= L^2(G \times \Omega, \nu \times m)\,, \\ \mathcal{H}_0 &= \{f \in \mathcal{H} : \int_{G\times\Omega} f\, d(\nu \times m) = 0\}\,, \\ \mathcal{H}_{0,av} &= \{f \in \mathcal{H} : \hat{f} \equiv 0\}\,. \end{aligned}$$

Given $A \in C(\Omega, L)$ and $t \in \mathbb{R}$ define a unitary operator U_t^A on $\mathcal{H}$ by setting

$$U_t^A f = f \circ T_t^A. \tag{2.1}$$

Next, given a positive real number M, define the averaging operator $W_{M,A}$ on $\mathcal{H}$ by setting

$$W_{M,A} f = \frac{1}{M} \int_0^M U_t^A f dt. \tag{2.2}$$

We will then prove the following.

Theorem 2.1. *Let $(\Omega, \{T_t\}_{t\in\mathbb{R}}, m)$ be a flow on a compact metric space Ω, where m is a flow invariant Borel probability measure on Ω. Let G be a compact connected Lie group with Lie algebra L and let S be a subset of L. Assume that*

(1) *the flow $(\Omega, \{T_t\}_{t\in\mathbb{R}}, m)$ is ergodic and aperiodic,*
(2) *S is closed and convex,*
(3) *S has the strong accessibility property.*

Then the set

$$C_{erg,av}(\Omega, S) = \{A \in C(\Omega, S) : \lim_{M\to\infty} W_{M,A} f = 0 \quad \textit{for all} \quad f \in \mathcal{H}_{0,av}\},$$

is a residual subset of $C(\Omega, S)$.

PROOF THAT THEOREM 2.1 IMPLIES THEOREM A.

Assume that the conditions of Theorem A hold, so we can apply Theorem 2.1 as well. Let $A \in C_{erg,av}(\Omega, S)$. If $f \in \mathcal{H}_0$, then $\tilde{f} \stackrel{def}{=} f - \hat{f} \in \mathcal{H}_{0,av}$, so $W_{M,A}\tilde{f} \to 0$ as $M \to \infty$. Moreover, since the flow $(\Omega, \{T_t\}_{t\in\mathbb{R}}, m)$ is ergodic, the ergodic average $W_M h \stackrel{def}{=} \frac{1}{M}\int_0^M (h \circ T_t)dt$ converge to 0, where we let $h(\omega) \stackrel{def}{=} \langle f^\omega \rangle$, because $\int_\Omega h(\omega)\, dm(\omega) = 0$. Observe that

$$W_M h = W_{M,A}\hat{f}.$$

Therefore $W_{M,A} f = W_{M,A}\tilde{f} + W_{M,A}\hat{f} \to 0$ as $M \to \infty$. Since this is true for all $f \in \mathcal{H}_0$, we have shown that $(G \times \Omega, \{T_t^A\}_{t\in\mathbb{R}}, m)$ is ergodic and subsequently $C_{erg}(Y, G)$ is residual. □

PROOF OF THEOREM 2.1.

Given $f \in \mathcal{H}_{0,av}$, $\varepsilon > 0$ and $\bar{n} \in \mathbb{N}$, define a set $E(f, \varepsilon, \bar{n})$ as follows:

$$E(f, \varepsilon, \bar{n}) = \{A \in C(\Omega, S) : ||W_{n,A} f||_2 < \varepsilon \text{ for some } n \in \mathbb{N} \text{ such that } n > \bar{n}\}. \tag{2.3}$$

Lemma 2.2. *Let $A \in C(\Omega, S)$, and let $\mathcal{F}$ be a dense subset of $\mathcal{H}_{0,av}$. If $A \in E(f, \frac{1}{n}, \overline{n})$ for all $f \in \mathcal{F}$, $n \in \mathbb{N}$, $\overline{n} \in \mathbb{N}$, then $A \in C_{erg,av}(\Omega, S)$.*

Proof. Fix an $f \in \mathcal{F}$. By the L^2 ergodic theorem, the sequence $W_{n,A} f$ converges in $\mathcal{H}$ as $n \to \infty$ to some $f^* \in \mathcal{H}$. Since $A \in E(f, \frac{1}{n}, \overline{n})$ for all $n, \overline{n} \in \mathbb{N}$, f^* must be the zero function. So $W_{n,A} f \to 0$ for every $f \in \mathcal{F}$. Since $\mathcal{F}$ is dense in $\mathcal{H}_{0,av}$, and $\|W_{n,A}\| \leq 1$ for all n, it follows that $W_{n,A} f \to 0$ as $n \to \infty$ for all $f \in \mathcal{H}_{0,av}$, so $A \in C_{erg,av}(\Omega, S)$. □

In view of Lemma 2.2, Theorem 2.1 will follow from the Baire category theorem—together with the facts that (a) $C(\Omega, S)$ is a complete metric space, because G is a compact metric group, (b) $C(G \times \Omega) \cap \mathcal{H}_{0,av}$ is dense in $\mathcal{H}_{0,av}$, and (c) $C(G \times \Omega)$ is separable— if we prove

Lemma 2.3. *Let $f \in C(G \times \Omega) \cap \mathcal{H}_{0,av}$. Then $E(f, \frac{1}{n}, \overline{n})$ is open and dense in $C(\Omega, S)$.*

The openness of the sets $E(f, \varepsilon, \bar{n})$ is easily checked. We shall establish their density in the next section. The proof of density involves a series of constructions involving application of two general results. In the remainder of this section we shall state these results. The first result is about approximating the integral of a function over a compact group by the average of its translates and the second is a version of Rokhlin's lemma (due to D. Lind [L]) for flows. We begin by introducing the necessary notation.

Let $C(G)$ be the space of all continuous (complex valued) maps on G with the supremum metric. If $h \in C(G)$ and $g \in G$, we define functions $\tau_g h$ by setting

$$(\tau_g h)(x) = h(g^{-1}x), \qquad \text{for } x \in G.$$

Clearly, each τ_g is an isometric linear maps from $C(G)$ onto $C(G)$.

Proposition 2.4. *Let G be a compact metric group. If $\mathcal{K} \subseteq C(G)$ is compact and $\eta > 0$, then there exists a positive integer P, and members $\gamma_1, \dots, \gamma_P$ of G, such that*

$$\Big\| \langle h \rangle - \frac{1}{P} \sum_{r=1}^{P} \tau_{\gamma_r^{-1}} h \Big\|_{sup} \le \eta \text{ for all } h \in \mathcal{K}\,. \tag{2.4}$$

Proof. Since $\mathcal{K}$ is a compact subset of $C(G)$, it is bounded and equicontinuous. Hence there exist (a) a constant C such that $|h(x)| \le C$ whenever $x \in G$ and $h \in \mathcal{K}$, and (b) a neighborhood W of e_G such that $|h(x) - h(y)| < \frac{\eta}{2}$ whenever $xy^{-1} \in W$ and $h \in \mathcal{K}$. Partition G into measurable subsets $G_1, \dots, G_R$ such that each G_r is contained in a left translate $\xi_r W$ of W. Let $\tilde{p}_r = \nu(G_r)$.

If $h \in \mathcal{K}$ and $x \in G$, then

$$\begin{aligned} \langle h \rangle &= \int_G h(gx)\, d\nu(g) = \sum_{r=1}^{R} \int_{G_r} h(gx)\, d\nu(g) \\ &= \sum_{r=1}^{R} \int_{G_r} (h(gx) - h(\xi_r x))\, d\nu(g) + \sum_{r=1}^{R} \tilde{p}_r (\tau_{\xi_r^{-1}} h)(x)\,. \end{aligned}$$

If $g \in G_r$, then $(gx)(\xi_r x)^{-1} = g\xi_r^{-1} \in W$, so $|h(gx) - h(\xi_r x)| \le \frac{\eta}{2}$. Therefore

$$\Big| \langle h \rangle - \sum_{r=1}^{R} \tilde{p}_r \tau_{\xi_r^{-1}} h(x) \Big| \le \frac{\eta}{2}$$

for all $h \in \mathcal{K}$, $x \in G$. Now, let M be a positive integer, and write

$$\tilde{p}_r = \frac{p_r}{M} + \hat{p}_r\,,$$

where $0 \le \hat{p}_r < \frac{1}{M}$. Let $P = p_1 + \ldots + p_R$. Then $1 - \frac{R}{M} \le \frac{P}{M} \le 1$. Moreover,

$$\Big|\sum_{r=1}^{R} \hat{p}_r \tau_{\xi_r^{-1}} h(x)\Big| \le \frac{CR}{M}\,.$$

So

$$\Big|\langle h\rangle - \sum_{r=1}^{R} \frac{p_r}{M} \tau_{\xi_r^{-1}} h(x)\Big| \le \frac{\eta}{2} + \frac{CR}{M}\,.$$

Therefore

$$\Big|\langle h\rangle - \sum_{r=1}^{R} \frac{p_r}{P} \tau_{\xi_r^{-1}} h(x)\Big| \le \frac{\eta}{2} + \frac{2CR}{M}\,.$$

Now choose M such that $4CR < M\eta$. Then

$$\Big\|\langle h\rangle - \sum_{r=1}^{R} \frac{p_r}{P} \tau_{\xi_r^{-1}} h\Big\|_{sup} \le \eta \ \text{ for all } h \in \mathcal{K}\,.$$

If we now define

$$\begin{aligned}
\gamma_1 = \gamma_2 = \cdots = \gamma_{p_1} &= \xi_1\,,\\
\gamma_{p_1+1} = \gamma_{p_1+2} = \cdots = \gamma_{p_1+p_2} &= \xi_2\,,\\
\cdots\\
\gamma_{p_1+p_2+\cdots+p_{R-1}+1} = \gamma_{p_1+p_2+\cdot+p_{R-1}+2} = \cdots = \gamma_P &= \xi_R\,,
\end{aligned}$$

then

$$\Big\|\langle h\rangle - \frac{1}{P}\sum_{r=1}^{P} \tau_{\gamma_r^{-1}} h\Big\|_{sup} \le \eta \ \text{ for all } h \in \mathcal{K}\,.$$

$\square$

Next, we recall a result of D. Lind (see [L]) which extends the classical Rokhlin's lemma for aperiodic measure preserving transformations to aperiodic, measure preserving $\mathbb{R}$ actions. First we introduce some notation. Let $(X, \mathcal{B}, m)$ be a Lebesgue space and let $(x,t) \to T_t(x) : X \times \mathbb{R} \to X$ be a jointly measurable, m preserving flow (i.e. $\mathbb{R}$ action) on X. The flow $(X, \mathcal{B}, \{T_t\}_{t\in\mathbb{R}}, m)$ is *aperiodic* if there exists a null set $N \subset X$ such that if $x \notin N$ and $t \neq 0$, then $T_t x \neq x$.

Theorem 2.5. *Let $(X, \mathcal{B}, T, m)$ be an aperiodic measure preserving flow on a Lebesgue space. Then, given $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists a set $F \subset X$ such that*

(1) *F and $T_N(F) \overset{def}{=} \cup \{T_t(F) : 0 \leq t \leq N\}$ are measurable;*
(2) *$T_t F \cap T_s F = \emptyset$ for all $t \neq s$, $t, s \in [0, N]$;*
(3) *$m(T_N(F)) > 1 - \varepsilon$;*
(4) *the map $\varphi : F \times [0, N] \to T_N(F)$ defined by $\varphi(x, t) = T_t(x)$ is bijective, measurable and its inverse is measurable (i.e. a measurable isomorphism). Furthermore there exists a probability measure $\tilde{m}$ defined on (the measurable subsets) of F such that $\varphi^*(m|_{T_N(F)}) = \tilde{m} \times \frac{dl_N}{N}$, where $m|_{T_N(F)}$ is the normalized restriction of m to the set $T_N(F)$ and dl_N is the usual Lebesgue measure on $[0, N]$.*

The set F in the above theorem is a "transversal" to the flow and $\tilde{m}$ is a "transversal measure" induced by m on F. Also notice that if the flow $(X, \mathcal{B}, \{T_t\}_{t \in \mathbb{R}}, m)$ is ergodic and aperiodic, then $\tilde{m}$ is non-atomic.

In our set-up the underlying flow is actually a jointly continuous flow on a compact metric space. Since the measure space obtained from the completion of the Borel σ-algebra with respect to a given non-atomic Borel measure on compact spaces is a Lebesgue space [R], we can apply Lind's theorem in our setting. Furthermore, since the set $T_N(F)$ in above theorem can be approximated from inside by compact sets and since our flow is jointly continuous, without loss of generality in our setting, we can assume that the transversal F is compact.

§3. Density of $E(f, \varepsilon, \bar{n})$.

Now we turn to the proof of the density of $E(f, \varepsilon, \bar{n})$. For this purpose, we fix the following objects:

(1) $f \in C(G \times \Omega) \cap \mathcal{H}_{0,av}$;
(2) $A_0 \in C(\Omega, S)$;
(3) a positive number ε;
(4) a positive number δ and
(5) a positive integer $\bar{n}$.

To prove the density of $E(f, \varepsilon, \bar{n})$ we shall construct a map $A \in C(\Omega, S)$ such that

(1) $||A_0 - A|| < \delta$ and
(2) $A \in E(f, \varepsilon, \bar{n})$.

First we sketch-in general terms-the idea underlying the construction of map A. This map will be first defined on a "strip" or the Rokhlin tower $T_N(E)$ which is identified with the set $E \times [0, N]$ (see the notation in Theorem 2.5). Here E is the "base of the tower" and N is suitably chosen large positive integer (height of the tower). The existence of such a tower is guaranteed by D. Lind's theorm. Let n be a suitably chosen integer that is large but much smaller than N. The aperiodicity of the flow allows us to identify the piece of the orbit $\{T_t(\omega) : t \in [0, n]\}$ with the interval $[0, n]$ via the map $t \to T_t(\omega)$, for "most ω's" in the strip. Let $\gamma_{0,\omega}(t) \overset{def}{=} A_0(T_t(\omega))$. Then we want to construct an ε-perturbation γ_ω of $\gamma_{0,\omega}$

in the space $C([0,n],S)$ of continuous S valued functions so that the solution $\Gamma_\omega(t)$ of the Cauchy problem

$$g'_\omega(t) = \gamma_\omega(t)g(t), \quad g_\omega(0) = e_G,$$

has the property that the ergodic (time) average on the time interval $[0,n]$ of the given function $f \in C(G\times\Omega)\cap\mathcal{H}_{0,av}$ along the curve $(\Gamma_\omega(t), T_t\omega)$ is close to zero for "most of ω"s in the strip.

To solve this problem let us examine the Cauchy problem

$$g' = \gamma_\omega(t)g,$$

where γ_ω is a S-valued curve. This system can be regarded as a right invariant control system on the Lie group G (see [NS1] for a detailed discussion on this point view). The desired perturbation is then thought as a "control" function that stears the original trajectory (the one corresponding to $\gamma_{0,\omega}$) to suitable "targets". The demand that "the time average of the given f along the integral curves of this system is close to zero for most the ω's" can be translated into a "description of the targets". The approximation result - Proposition 2.4 - plays a role in this description. The general strategy of stearing trajectories to these targets involve "Property A^*" of the fiber group G. We refer the reader to the discussion on page 316 of [NS1] for a review of this strategy.

The crucial point in implementing this strategy is the existence of a neighbourhood W of e_G such that the "time 1 reachable set" of the control process $g'(t) = \gamma_\omega(t)g$, $g(0) = e_G$ contains some right translate of W. It is crucial that this W be independent of $\omega \in \Omega$. In the following (Theorem 3.1) we state this crucial fact precisely We begin by introducing the necessary notation.

Given a closed interval $[a,b]$, let $C^{pc}([a,b],S)$ denote the set of piecewise continuous maps $A : [a,b] \to S$ (i.e. continuous S-valued maps with a finite number of jump discontinuities). If $0 \in [a,b]$, $A \in C^{pc}([a,b],S)$, let Γ_A denote the unique G-valued absolutly continuous solution curve of the initial value problem

$$\begin{aligned} g' &= A(t)g(t), \qquad t \in [a,b], \\ g(0) &= e_G. \end{aligned}$$

If $A \in C^{pc}([a,b],S)$ and $\delta > 0$, define the set $N_S(A,\delta)$ as follows:

$$N_S(A,\delta) = \{B \in C^{pc}([a,b],S) : \sup_{t\in[a,b]} ||A(t) - B(t)|| < \delta\}.$$

The following result is proved in [NS1, Proposition 2.1].

Proposition 3.1. *Let G be a Lie group with Lie algebra L. Let $S \subseteq L$ be as in Theorem 1.1. Let $\tau > 0$ and $\mathcal{F} \subseteq C([0,\tau],S)$ be a compact subset. Then given $\delta > 0$ there exist a neighbourhood W of the identity of G, depending on δ but independent of $A \in \mathcal{F}$ such that the set $\{\Gamma_B(\tau) : B \in N_S(A,\delta)\}$ contains some right translate of W.*

Now we begin with the construction of the desired map.

Step 1 : Let $\mathcal{K}$ to be the closed convex hull of the set of all functions given by

$$\{\tau_g(f^\omega) : \omega \in \Omega\} \subset C(G).$$

Then $\mathcal{K}$ is a compact convex subset of $C(G)$ that is also invariant under τ_g for all $g \in G$. Furthermore $\langle h \rangle = 0$ for every $h \in \mathcal{K}$.

Next, we fix a finite subset $\mathcal{K}_0$ of $\mathcal{K}$ such that every $h \in \mathcal{K}$ satisfies the inequality

$$\|h - h_0\|_{sup} < \frac{\varepsilon}{16} \tag{3.1}$$

for some $h_0 \in \mathcal{K}_0$. We let $\hat{\kappa}$ be the cardinality of $\mathcal{K}_0$. We let

$$\beta = \frac{\varepsilon}{8\hat{\kappa}}\ . \tag{3.2}$$

Next, let $\delta_1 > 0$ be such that

$$\text{if } d(g_1, g_2) < \delta_1, \text{ then } |H(g_1) - H(g_2)| < \frac{\beta}{2},\ g_1, g_2 \in G \text{ and} \tag{3.3}$$

for all $H \in \mathcal{K}$.

Step 2 : Now applying Proposition 2.4 to the above set $\mathcal{K}$ and with $\eta = \beta$, we get a positive integer P and $\gamma_1, \cdots, \gamma_P \in G$ such that

$$\Big\| \frac{1}{P} \sum_{r=1}^{P} \tau_{\gamma_r^{-1}} h \Big\|_{sup} \leq \beta \text{ for all } h \in \mathcal{K}\,. \tag{3.4}$$

Step 3 : Corresponding to the given δ pick a neighbourhood W of e_G according to Proposition 3.1. (We apply Proposition 3.1 by taking $\tau = 1$ and $\mathcal{F} = \{(A_0)_\omega : \omega \in \Omega\} \subseteq C([0,1], S)$, where $(A_0)_\omega(t) = A_0(T_t(\omega))$, $t \in [0,1]$.) Hence for each $\omega \in \Omega$ there exists some $h(\omega) \in G$ satisfying

$$W h(\omega) \subseteq \{\Gamma_B(1) : B \in N_S((A_0)_\omega, \delta/3)\}. \tag{3.5}$$

Step 4 : Corresponding to W, pick $\kappa \in \mathbb{N}$ according to Property A^*.

Step 5 : Next, we choose $N = (\kappa + \mu)\lambda$ where μ and λ are large positive integers, the precise choice of which will be described later. Then using Theorem 2.5 pick a subset $\hat{E}$ of Ω with the following properties:

(1) $\hat{E}$ and $T_N(\hat{E}) \stackrel{def}{=} \cup \{T_t(\hat{E}) : t \in [0, N]\}$ are Borel sets;
(2) $T_s(\hat{E}) \cap T_t(\hat{E}) = \emptyset$ for all $t \neq s$, $t, s \in [0, N]$;
(3) $m(T_N(\hat{E})) \geq 1 - c_1$ where c_1 is a positive constant whose choice will be described later.

Let $\tilde{m}$ denote the induced measure on the transversal $\hat{E}$ as in Theorem (2.5).

Step 6 : Using uniformly continuity of f on $G \times \Omega$, choose

(1) a positive number c_2 such that $|f(g, y) - f(g, z)| \le \frac{\varepsilon}{8}$ whenever $g \in G$, $y, z \in \Omega$, and $d(y, z) \le c_2$;
(2) a neighborhood V of the identity e_G in G such that $|f(g, \omega) - f(h, \omega)| \le \frac{\varepsilon}{8}$ whenever $g, h \in G$, $\omega \in \Omega$, and $gh^{-1} \in V$.

Step 7 : Let c_3 be a positive constant such that

$$||A_0(T_t(\omega_1)) - A_0(T_t(\omega_2))|| < \frac{\delta}{3} \quad \text{if } d(\omega_1, \omega_2) < c_3 \text{ and } t \in [0, N].$$

Step 8 : Using continuity of the map T_t $(t \in [0, N])$, pick a positive c_4 such that

(1) $d(T_t y, T_t z) \le c_2$ whenever $d(y, z) \le c_4$ and $t \in [0, N]$,
(2) $c_4 < c_3$.

Step 9 : We partition Ω into finitely many Borel-measurable sets $\Omega_1, \dots, \Omega_s$ of diameter less than c_3, and let $\hat{E}_i = \hat{E} \cap \Omega_i$ for $i = 1, \dots, s$. Write $I = \{1, \dots, s\}$. Using the fact that m is regular, for each $i \in I$ pick compact sets E_i such that $E_i \subseteq \hat{E}_i$ and $m(\hat{E}_i \backslash E_i) < \frac{c_1}{sN}$. Note that $\{T_t E_i : i \in I,\ t \in [0, N]\}$ is a family of mutually disjoint compact sets and

$$m\left(\bigcup_{i \in I} \bigcup_{t \in [0, N-1]} T_t(E_i)\right) \ge m\left(\bigcup_{t \in [0, N-1]} T_t(\hat{E})\right)(1 - c_1) \ge (1 - c_1)^2 \ge 1 - 2c_1 \,. \tag{3.6}$$

We discard any E_i's that have $\tilde{m}$-measure zero, so $\tilde{m}(E_i) > 0$, for all $i \in I$.

Step 10 : Let $J = [0, 1, \cdots, N - 1]$. We pick points $\omega_i \in E_i$ for each $i \in I$, and then set $\omega_{ij} = T_j(\omega_i)$ for $j \in J$.

Step 11 : Now we are going to define the desired map A. For this purpose, we begin by dividing the integer interval $J = [0, 1, \cdots, N - 1]$ into λ blocks $J_1, \dots, J_\lambda$ of length $\kappa + \mu$, given by

$$J_\ell = \{(\ell - 1)(\kappa + \mu), (\ell - 1)(\kappa + \mu) + 1, \dots, \ell(\kappa + \mu) - 1\} \text{ for } \ell \in L \,,$$

where $L = \{1, 2, \dots, \lambda\}$. We then split each block J_ℓ into two blocks J_ℓ^1, J_ℓ^2, of lengths κ and μ, by letting

$$J_\ell^1 = \{(\ell - 1)(\kappa + \mu), (\ell - 1)(\kappa + \mu) + 1, \dots, \ell\kappa + (\ell - 1)\mu - 1\} \,,$$
$$J_\ell^2 = \{\ell\kappa + (\ell - 1)\mu, \ell\kappa + (\ell - 1)\mu + 1, \dots, \ell(\kappa + \mu) - 1\} \,.$$

For each $i \in \{1, \cdots, s\}$ and $j \in J_\ell^1$, where $\ell \in \{1, \cdots, \lambda\}$, consider the map $A_{ij} \in \mathcal{F}$ defined by

$$A_{i,j}(t) = A_0(T_t(\omega_{ij})), \qquad t \in [0, 1].$$

Letting $h_{i,j} = h(\omega_{ij})$, (3.5) yields

$$Wh_{ij} \subseteq \{\Gamma_B(1) : B \in N_S(A_{ij}, \delta/3)\}. \tag{3.7}$$

For each $(i,j) \in I \times J$ a choice of $g_{ij} \in G$ is made. This choice will be described later in detail. Then, using (3.7), select $\hat{A}_{ij} \in N_S(A_{ij}, \delta)$, (for each $(i,j) \in I \times J$) such that

$$g_{ij} = \Gamma_{\hat{A}_{ij}}(1). \tag{3.8}$$

Next, for each $\omega \in \cup_{i=1}^s E_i$ define $\tilde{A}_\omega : \mathbb{R} \to S$ by setting

$$\begin{aligned} \tilde{A}_\omega(t) &= A_{ij}(t-j) \quad \text{if } j \leq t \leq j+1,\ j \in J_\ell^1,\ l \in \{1, \cdots, \lambda\} \text{ and } \omega \in E_i, \\ &= A_0(T_t\omega) \quad \text{otherwise}. \end{aligned}$$

Note that the map $\tilde{A}_\omega$ remains the same as ω varies over E_i, (for each $1 \leq i \leq s$) and hence we shall also write $\tilde{A}_\omega = \tilde{A}_i$ if $\omega \in E_i$. Clearly for each $\omega \in \cup_{i=1}^s E_i$, $\tilde{A}_\omega \in C^{ps}(\mathbb{R}, S)$ and

$$||\tilde{A}_\omega(t) - A_0(T_t\omega)|| < \frac{\delta}{3} \quad \omega \in \cup_{i=1}^s E_i \text{ and } t \in [0, N]\,. \tag{3.9}$$

Step 12 : Now we construct a continuous approximation to the map $\tilde{A}_\omega$. A general approximation procedure involves defining the following family of maps (parametrized by a positive real parameter ν).

$$\tilde{A}_\omega^\nu(t) = \nu \int_t^{t+\frac{1}{\nu}} \tilde{A}_\omega(s)ds, \qquad \omega \in \cup_{i=1}^s E_i\,. \tag{3.10}$$

We claim that (see [NS1] for full details) by considering large enough values of ν, we can find a $\bar{\nu}$ such that

$$||\tilde{A}_\omega^{\bar{\nu}}(t) - A_0(T_t\omega)|| < \delta, \quad \text{for all } t \in [0, N],\ \omega \in \cup_{i=1}^s E_i\,, \quad \text{and} \tag{3.11}$$

$$\Gamma_{\tilde{A}_\omega}(t)\Gamma_{\tilde{A}_\omega^{\bar{\nu}}}(t)^{-1} \in V^*, \quad \text{for all } t \in [0, N],\ \omega \in \cup_{i=1}^s E_i\,, \tag{3.12}$$

where V^* is a neighbourhood of e_G such that $V^* \cdot V^* \subseteq V$ and $gV^*g^{-1} \subseteq V^*$ for all $g \in G$.

Step 13 : Set $A_\omega^* = \tilde{A}_\omega^{\bar{\nu}}$. Then the map $A^*(\omega, t) \to A_\omega^*(t)$ is continuous on $\cup_{i=1}^s E_i \times [0, N]$. (In fact $A_\omega^* \equiv \tilde{A}_i^{\bar{\nu}}$ if $\omega \in E_i$). We may regard A^* as a (continuous) map defined on the compact subset $\cup\{T_t(\omega) : \omega \in E_i,\ 1 \leq i \leq s,\ t \in [0, N]\}$; and then extend it continuously to a map $A : \Omega \to L$ satisfying

(1) $A(\omega) \in S$ for all $\omega \in \Omega$ and
(2) $||A(\omega) - A_0(\omega)|| < \delta$ for all $\omega \in \Omega$.

We refer to page 320 of [NS1] for the details regarding the construction of this extension.

Step 14 : Now we are now going to estimate the norm $\|W_{n,A}f\|_{L_2}$ for large n. We will choose $n = (\kappa + \mu)\rho$ such that

$$\frac{2}{n}\|f\|_{sup} < \frac{\varepsilon}{8},$$

where ρ is a large positive integer whose precise choice will be described later, but it is such that $\frac{\lambda}{\rho}$ is still a large positive integer. In fact the choice of μ, ρ and then λ (and N) will be made in this order.

Now we begin estimating $\|W_{n,A}f\|_{L_2}$. We let

$$C = \bigcup_{t\in[0,N-n-1]} \bigcup_{i=1}^{s} T_t(E_i)\,.$$

Since

$$m\Big(\bigcup_{t\in[N-n-1,N]} \bigcup_{i=1}^{s} T_t(E_i)\Big) \le \tilde{m}(\cup_{i=1}^{s} E)\cdot \frac{n+1}{N} \le \frac{n+1}{N} = \frac{\rho}{\lambda} + \frac{1}{N} < \frac{\rho+1}{\lambda}\,,$$

the set C satisfies

$$m(C) \ge 1 - c_1 - \frac{\rho+1}{\lambda}\,.$$

Outside $G\times C$, the function $W_{n,A}f$ is bounded pointwise by $\|f\|_{sup}$, so

$$\int_{G\times(\Omega\backslash C)} |W_{n,A}f(g,\omega)|^2 d\nu(g)\times m(\omega) \le \|f\|_{sup}^2 \left(c_1 + \frac{\rho+1}{\lambda}\right). \tag{3.13}$$

Now let $(g,\omega) \in G\times C$. Then ω is of the form $\omega = T_{j+t'}\tilde{\omega}$ for some $\tilde{\omega}\in E_i$ where $i\in\{1,\cdots,s\}$, $j\in J\equiv\{0,\cdots,N-1\}$ such that $j\le N-n-1$ and $t'\in[0,1]$ are uniquely determined by ω. Note that

$$\begin{aligned} W_{n,A}f(g,\omega) &= \frac{1}{n}\int_0^n f(X_A(\omega,s)g, T_s\omega)ds \\ &= \frac{1}{n}\int_0^n f(X_A(\omega,s)g, T_{s+j+t'}\tilde{\omega})ds \\ &\overset{\frac{\varepsilon}{8}}{\approx} \frac{1}{n}\int_0^n f(X_A(\omega,s)g, T_{s+j+t'}\omega_i)ds, \quad \text{by (1) of Step 6 and Step 8}\,; \end{aligned}$$

where we write $a \overset{\varepsilon}{\approx} b$ to mean $|a-b|<\varepsilon$. Thus, writing $w_{ij} \overset{def}{=} T_j\omega_i$ we get

$$W_{n,A}f(g,\omega) \overset{\frac{\varepsilon}{8}}{\approx} \frac{1}{n}\int_0^n f(X_A(\omega,s)g, T_{s+t'}\omega_{ij})ds = \frac{1}{n}\int_{t'}^{n+t'} f(X_A(\omega,s'-t')g, T_{s'}\omega_{ij})ds'\,.$$

The choice of n in Step 14 yields $\frac{2t'}{n}\|f\|_{sup} \leq \frac{2}{n}\|f\|_{sup} < \frac{\varepsilon}{8}$ and hence

$$\frac{1}{n}\int_{t'}^{n+t'} f(X_A(\omega, s'-t')g, T_{s'}\omega_{ij})ds' \overset{\frac{\varepsilon}{8}}{\approx} \frac{1}{n}\int_0^n f(X_A(\omega, s'-t')g, T_{s'}\omega_{ij})ds'\,.$$

Thus, using cocycle identity we obtain

$$\begin{aligned} W_{n,A}f(g,\omega) &\overset{\frac{\varepsilon}{4}}{\approx} \frac{1}{n}\int_0^n f(X_A(\omega, s'-t')g, T_{s'}\omega_{ij})ds' \\ &\overset{\frac{\varepsilon}{4}}{\approx} \frac{1}{n}\int_0^n f(X_A(T_j\tilde{\omega}, s)X_A(T_j\tilde{\omega}, t')^{-1}g, T_s\omega_{ij})ds\,. \end{aligned}$$

Now we claim that for any $\tilde{\omega} \in E_i$ and $j \in J$ such that $j \leq N-n-1$,

$$X_A(T_j\tilde{\omega}, s) = X_A(\omega_{ij}, s) \quad \text{for all } s \in [0,n]\,.$$

To prove this claim first notice that

$$A(T_t\tilde{\omega}) = A^*_{\tilde{\omega}}(t) = \tilde{A}^{\bar{\nu}}_{\tilde{\omega}}(t) = \tilde{A}^{\bar{\nu}}_{\omega_i}(t) = A^*_{\omega_i}(t) = A(T_t\omega_i)\,,$$

for all $t \in [0,N]$. Thus, $X_A(\tilde{\omega}, t) = X_A(\omega_i, t)$ for all $t \in [0,N]$. Now the claim follows by using the cocycle identity as follows;

$$\begin{aligned} X_A(T_j\tilde{\omega}, s) &= X_A(\tilde{\omega}, s+j)X_A(\tilde{\omega}, j)^{-1} \\ &= X_A(\omega_i, s+j)X_A(\omega_i, j)^{-1} \qquad (\text{since } s+j \in [0,N]) \\ &= X_A(\omega_{ij}, s)\,. \end{aligned}$$

Thus,

$$W_{n,A}f(g,\omega) \overset{\frac{\varepsilon}{4}}{\approx} \frac{1}{n}\int_0^n f(X_A(\omega_{ij}, s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{ij})ds. \tag{3.14}$$

Now fix $i \in I = \{1, \cdots, s\}$ and $j \in J$ such that $j \leq N-n-1$. Set

$$\begin{aligned} W_{n,A;i,j,t'}f(g) &\overset{def}{=} \int_0^n f(X_A(\omega_{ij}, s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{ij})ds \\ &= \sum_{k=0}^{n-1}\int_k^{k+1} f(X_A(\omega_{ij}, s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{ij})ds \\ &= \sum_{k=0}^{n-1}\int_0^1 f(X_A(\omega_{ij}, s'+k)X_A(\omega_{ij}, t')^{-1}g, T_{s'}\omega_{i(j+k)})ds'. \end{aligned}$$

As k varies from 0 to $n-1$, the index $j+k$ takes values in the integer interval

$$\Gamma_n(j) = \{j, j+1, \cdots, j+n-1\},$$

which is a subset of J because $j \leq N-n-1$. We can then write $\Gamma_n(j) = \Gamma_n^c(j) \cup \Gamma_n^b(j)$, where the "central part" $\Gamma_n^c(j)$ is a disjoint union of intervals J_ℓ and the "bad part" $\Gamma_n^b(j)$ has at most $2(\kappa+\mu)$ members. Let $\Delta_n(j)$ be the set of indices ℓ such that $J_\ell \subset \Gamma_n(j)$. Then

$$W_{n,A;i,j,t'}f(g) = W^c_{n,A;i,j,t'}f(g) + W^b_{n,A;i,j,t'}f(g)\,, \tag{3.15}$$

where $W^c_{n,A;i,j,t'}f(g)$, $W^b_{n,A;i,j,t'}f(g)$ are respectively the contributions to the right-hand side of (3.14) of the terms for which $k+j \in \Gamma_n^c(j)$ and those for which $k+j \in \Gamma_n^b(j)$. Thus, in particular,

$$\begin{aligned} W^c_{n,A;i,j,t'}f(g) &= \sum_{k+j\in\Gamma_n^c(j)} \int_0^1 f(X_A(\omega_{ij}, k+s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(j+k)})ds \\ &= \sum_{\ell\in\Delta_n(j)} \sum_{k+j\in J_\ell} \int_0^1 f(X_A(\omega_{ij}, k+s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(j+k)})ds, \end{aligned}$$

and

$$W^b_{n,A;i,j,t'}f(g) = \sum_{k+j\in\Gamma_n^b(j)} \int_0^1 f(X_A(\omega_{ij}, k+s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(j+k)})ds.$$

Since $\Gamma_n^b(j)$ has at most $2(\kappa+\mu)$ indices, we get the bound

$$\frac{1}{n}\Big|W^b_{n,A;i,j,t'}f(g)\Big| \leq \frac{2(\kappa+\mu)\|f\|_{sup}}{n} = \frac{2\|f\|_{sup}}{\rho}\,. \tag{3.16}$$

Now let us analyze the contribution to $W^c_{n,A;i,j,t'}f(g)$ by terms such as following.

$$\begin{aligned} W^c_{n,A;i,j,t';\ell}f(g) &\stackrel{def}{=} \sum_{k+j\in J_\ell} \int_0^1 f(X_A(\omega_{ij}, k+s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(j+k)})ds \\ &= \sum_{u\in J_\ell} \int_0^1 f(X_A(\omega_{ij}, u-j+s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{iu})ds\,. \end{aligned}$$

Let $J_\ell = \{a_\ell, a_\ell+1, \ldots, a_\ell+\kappa+\mu-1\}$. Rewriting the summation with $u = a_\ell+v$ and using the cocycle identity we get

$$\begin{aligned} W^c_{n,A;i,j,t';\ell}f(g) &= \sum_{v=0}^{\kappa+\mu-1} \int_0^1 f(X_A(\omega_{ij}, a_\ell+v-j+s)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(a_\ell+v)})ds \\ &= \sum_{v=0}^{\kappa+\mu-1} \int_0^1 f(X_A(T_{a_\ell-j}\omega_{ij}, v+s)X_A(\omega_{ij}, a_\ell-j)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(a_\ell+v)})ds \\ &= \sum_{v=0}^{\kappa+\mu-1} \int_0^1 f(X_A(\omega_{ia_\ell}, v+s)X_A(\omega_{ij}, a_\ell-j)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(a_\ell+v)})ds\,. \end{aligned}$$

Now we split the above sum once more, by separating out the first κ terms from the remaining μ ones. We get

$$W^{c}_{n,A;i,j;t';\ell}f(g) = W^{c,-}_{n,A;i,j,t';\ell}f(g) + W^{c,+}_{n,A;i,j,t';\ell}f(g)\,,$$

where

$$W^{c,-}_{n,A;i,j,t';\ell}f(g) = \sum_{v=0}^{\kappa-1}\int_0^1 f(X_A(\omega_{ia_\ell}, v+s)X_A(\omega_{ij}, a_\ell - j)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(a_\ell+v)})ds$$

and

$$\begin{aligned}W^{c,+}_{n,A;i,j,t';\ell}&f(g)\\ &= \sum_{q=0}^{\mu-1}\int_0^1 f(X_A(\omega_{ia_\ell}, \kappa+q+s)X_A(\omega_{ij}, a_\ell - j)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(a_\ell+\kappa+q)})ds\,.\end{aligned}$$

Note that each function $W^{c,-}_{n,A;i,j,t';\ell}f$ is pointwise bounded by $\kappa\|f\|_{sup}$. Since the number of members of $\Delta_n(j)$ is at most ρ, we have the estimate

$$\frac{1}{n}\Big|\sum_{\ell\in\Delta_n(j)} W^{c,-}_{n;A,i,j,t';\ell}f(g)\Big| \le \frac{\rho\kappa\|f\|_{sup}}{n} = \frac{\kappa\|f\|_{sup}}{\kappa+\mu}\,. \tag{3.17}$$

Next, we estimate $W^{c,+}_{n,A;i,j,t';\ell}f(g)$. Recall that

$$\begin{aligned}W^{c,+}_{n,A;i,j,t';\ell}&f(g)\\ &= \sum_{q=0}^{\mu-1}\int_0^1 f(X_A(\omega_{ia_\ell}, \kappa+q+s)X_A(\omega_{ij}, a_\ell - j)X_A(\omega_{ij}, t')^{-1}g, T_s\omega_{i(a_\ell+\kappa+q)})ds\,,\end{aligned}$$

First we compute the following expression using the cocycle identity,

$$\begin{aligned}X_A(\omega_{ia_\ell}, \kappa{+}q+s)&X_A(\omega_{ij}, a_\ell - j)X_A(\omega_{ij}, t')^{-1}\\ &= X_A(\omega_{i(a_\ell+\kappa)}, q+s)X_A(\omega_{ia_\ell}, \kappa)X_A(\omega_{ij}, a_\ell - j)X_A(\omega_{ij}, t')^{-1}\\ &= X_A(\omega_{i(a_\ell+\kappa)}, q+s)X_A(\omega_{ia_\ell}, \kappa)X_A(\omega_{ij}, a_\ell - j)X_A(\omega_{ij}, t')^{-1}\\ &= X_A(\omega_{i(a_\ell+\kappa)}, q+s)X_A(\omega_{ia_\ell}, \kappa)X_A(\omega_i, a_\ell)X_A(\omega_i, j)^{-1}X_A(\omega_{ij}, t')^{-1}\\ &= X_A(\omega_{i(a_\ell+\kappa)}, q+s)X_A(\omega_{ia_\ell}, \kappa)X_A(\omega_i, a_\ell)\big(X_A(\omega_{ij}, t')X_A(\omega_i, j)\big)^{-1}\\ &= X_A(\omega_{i(a_\ell+\kappa)}, q+s)X_A(\omega_{ia_\ell}, \kappa)X_A(\omega_i, a_\ell)X_A(\omega_i, j+t')^{-1}\,.\end{aligned}$$

Thus,

$$W^{c,+}_{n,A;i,j,t';\ell}f(g)$$
$$=\sum_{q=0}^{\mu-1}\int_0^1 f(X_A(\omega_{i(a_\ell+\kappa)},q+s)X_A(\omega_{ia_\ell},\kappa)X_A(\omega_i,a_\ell)X_A(\omega_i,j+t')^{-1}g,T_s\omega_{i(a_\ell+\kappa+q)})ds\,.$$

We now turn to the crucial estimate, namely, the bound for the functions $W^{c,+}_{n,A;i,j,t';\ell}f$. It is at this point that we will choose the g_{ij}. For this purpose, we will want to assign to each index $i\in I$ and each $\ell\in L$ an integer $r(i,\ell)\in\{1,\dots,P\}$. We let $F_{i,\ell}$ and $F^*_{i,\ell}$ be the functions on G defined by

$$F_{i,\ell}(g)=\frac{1}{\mu}\sum_{q=0}^{\mu-1}\int_0^1 f(X_A(\omega_{i(a_\ell+\kappa)},q+s)g,T_s\omega_{i(a_\ell+\kappa+q)})ds\quad\text{and}$$

$$F^*_{i,\ell}(g)=\frac{1}{\mu}\sum_{q=0}^{\mu-1}\int_0^1 f(X_{A_0}(\omega_{i(a_\ell+\kappa)},q+s)g,T_s\omega_{i(a_\ell+\kappa+q)})ds\quad\text{and}$$

Notice that the function $F^*_{i,\ell}$ does not depend at all on the choice of g_{ij}. Now (3.12), Step 6, along with the choice of V^* and the definition of A implies that

$$||F^*_{i,\ell}-F_{i,\ell}||_{sup}<\frac{\varepsilon}{8}\,.$$

Next, observe that

$$W^{c,+}_{n,A;i,j,t';\ell}f(g)=\mu F_{i,\ell}\big(X_A(\omega_{ia_\ell},\kappa)X_A(\omega_i,a_\ell)X_A(\omega_i,j+t')^{-1}g\big).$$

Note that for each $q\in\{0,1,\cdots,\mu-1\}$ the map

$$g\to\int_0^1 f(X_{A_0}(\omega_{i(a_\ell+\kappa)},q+s)g,T_s\omega_{i(a_\ell+\kappa+q)})ds$$

belongs to $\mathcal{K}$-the closed convex hull of the set $\{\tau_g(f^\omega)\}$. Subsequently each $F^*_{i,\ell}$ also belongs to $\mathcal{K}$. Thus, or each i,ℓ we can choose $H_{i,\ell}\in\mathcal{K}_0$ such that

$$\|F^*_{i,\ell}-H_{i,\ell}\|_{sup}\le\frac{\varepsilon}{8}.$$

Thus,

$$\|F_{i,\ell}-H_{i,\ell}\|_{sup}\le\frac{\varepsilon}{4}. \tag{3.18}$$

For each $i \in I, H \in \mathcal{K}_0$, let $A(i,H)$ be the set of those indices $\ell \in L$ such that $H_{i,\ell} = H$. Enumerate the members of $A(i,H)$, from left to right, so

$$A(i,H) = \{\alpha_{i,H,1}, \dots, \alpha_{i,H,\Lambda(i,H)}\},$$

with $\alpha_{i,H,1} < \alpha_{i,H,2} < \ldots < \alpha_{i,H,\Lambda(i,H)}$. Then for each $i \in I$, $\ell \in L$, there exists a unique $k = k(i,\ell) \in \{1, \dots, \Lambda(i, H_{i,\ell})\}$ such that $\ell = \alpha_{i,H_{i,\ell},k}$. We then define $r(i,\ell)$ to be the remainder of $k(i,\ell)$ modulo P, so $k(i,\ell) = \hat{k}(i,\ell)P + r(i,\ell)$, where $\hat{k}(i,\ell)$ and $r(i,\ell)$ are non-negative integers and $0 \le r(i,\ell) < P$.

We now define the g_{ij} for $i \in I$, $j \in J$. As explained above, we let $g_{ij} = e_G$ if $j \in \cup_\ell J^2_\ell$. We then define the family $\{g_{ij}\}_{i\in I, j\in J^1_\ell}$ for each $j \in J^1_\ell$, inductively with respect to ℓ.

Fix an ℓ, and assume that the g_{ij} have already been chosen for $j \in J^1_{\ell'}$ for all ℓ' such that $\ell' < \ell$, but not yet for $j \in J^1_\ell$. Then the g_{ij} are in fact determined for all j such that $j < a_\ell$. For any $i \in I$, we have

$$\begin{aligned}
&X_A(\omega_{i,a_\ell},\kappa)X_A(\omega_i,a_\ell)X_A(\omega_i,j+t')^{-1}\\
&\quad = X_A(\omega_{i,a_\ell+\kappa-1},1)X_A(\omega_{i,a_\ell+\kappa-2},1)\cdots X_A(\omega_{i,a_\ell},1)X_A(\omega_i,a_\ell)X_A(\omega_i,j+t')^{-1}\\
&\quad = g_{i,a_\ell+\kappa-1}\cdot z_{\kappa-1}\cdot g_{i,a_\ell+\kappa-2}\cdot z_{\kappa-2}\cdots g_{i,a_\ell}z_0\,,
\end{aligned} \tag{3.19}$$

where $z_k = X_{A_0}(\omega_{i,a_\ell+k},1)$ for $0 < k < \kappa$, and $z_0 = X_{A_0}(\omega_{i,a_\ell},1)X_A(\omega_i,a_\ell)X_A(\omega_i,j+t')^{-1}$. Note that z_0 is determined, because the g_{ij} have already been selected for $j < a_\ell$. Now, our choice of κ in Step 4 guarantees that $Wz_{\kappa-1}\cdot z_{\kappa-2}\cdots Wz_0 = G$. Thus we can choose $g_{i0}, g_{i1}, \cdots, g_{i(\kappa-1)}$ such that

$$X_A(\omega_{i,a_\ell},\kappa)X_A(\omega_i,a_\ell)X_A(\omega_i,j+t')^{-1} = \gamma_{r(i,\ell)}\,. \tag{3.20}$$

Then, for any i, j,

$$\begin{aligned}
\sum_{\ell\in\Delta_n(j)} W^{c,+}_{n,A;i,j,t';\ell} f(g) &= \mu \sum_{\ell\in\Delta_n(j)} F_{i,\ell}\big(\gamma_{r(i,\ell)X_A(\omega_i,j+t')^{-1}g}\big)\\
&= \mu \sum_{H\in\mathcal{K}_0}\ \sum_{\ell\in A(i,H)\cap\Delta_n(j)} \tau_{\gamma^{-1}_{r(i,\ell)}} F_{i,\ell}\big(X_A(\omega_i,j+t')^{-1}g\big).
\end{aligned}$$

Let

$$\mathcal{W}_{n,A;i,j}(g) = \mu \sum_{H\in\mathcal{K}_0}\ \sum_{\ell\in A(i,H)\cap\Delta_n(j)} \tau_{\gamma^{-1}_{r(i,\ell)}} H(g).$$

Since $\|H - F_{i,\ell}\|_{sup} \le \frac{\varepsilon}{4}$ whenever $\ell \in A(i,H)$, we have the bound

$$\Big|\mathcal{W}_{n,A;i,j}(X_A(\omega_i,j+t')^{-1}g) - \sum_{\ell\in\Delta_n(j)} W^{c,+}_{n,A;i,j,t';\ell} f(g)\Big| \le \frac{\mu\rho\varepsilon}{4}\,,$$

using the fact that $\Delta_n(j)$ has at most ρ members. Observing that $\frac{\mu\rho}{n} \leq 1$, we get

$$\frac{1}{n}\Big|\mathcal{W}_{n,A;i,j}(X_A(\omega_i, j+t')^{-1}g) - \sum_{\ell\in\Delta_n(j)} W^{c,+}_{n,A;i,j,t';\ell} f(g)\Big| \leq \frac{\varepsilon}{4}\,. \tag{3.21}$$

We now turn to the task of estimating $\mathcal{W}_{n,A;i,j}$. We write

$$\mathcal{W}_{n,A;i,j}(g) = \mu \sum_{H\in\mathcal{K}_0} \mathcal{W}_{n,A;i,j;H}(g),$$

where

$$\mathcal{W}_{n,A;i,j;H}(g) = \sum_{\ell\in A(i,H)\cap\Delta_n(j)} \tau_{\gamma^{-1}_{r(i,\ell)}} H(g). \tag{3.22}$$

Now suppose that $A(i,H)\cap\Delta_n(j)$ has $q_{i,j,H}P + p_{i,j,H}$ members, where $q_{i,j,H}$ and $p_{i,j,H}$ are integers such that $q_{i,j,H} \geq 0$ and $0 \leq p_{i,j,H} < P$. Then the sum of the first $q_{i,j,H}P$ terms of (3.22) is equal to $q_{i,j,H}\sum_{r=1}^{P} \tau_{\gamma_r^{-1}} H(g)$, whose absolute value is bounded by $P\beta q_{i,j,H}$, because $\langle H\rangle = 0$. The sum of the remaining $p_{i,j,H}$ terms is bounded by $p_{i,j,H}\|f\|_{sup}$. Thus

$$\Big|\mathcal{W}_{n,A;i,j;H}(g)\Big| \leq P\beta q_{i,j,H} + p_{i,j,H}\|f\|_{sup}\,.$$

Each number $q_{i,j,H}$ is bounded by $\frac{\rho}{P}$, and $p_{i,j,H} \leq P$. So

$$\Big|\mathcal{W}_{n,A;i,j;H}(g)\Big| \leq \rho\beta + P\|f\|_{sup}\,.$$

Therefore

$$\|\mathcal{W}_{n,A;i,j}\|_{sup} \leq \mu\hat{\kappa}\rho\beta + \mu\hat{\kappa}P\|f\|_{sup}\,. \tag{3.23}$$

If we combine (3.21) and (3.23), we find

$$\frac{1}{n}\Big\|\sum_{\ell\in\Delta_n(j)} W^{c,+}_{n,A;i,j,t';\ell} f(g)\Big\|_{sup} \leq \frac{\varepsilon}{4} + \frac{\mu\hat{\kappa}\rho\beta}{n} + \frac{\mu\hat{\kappa}P}{n}\|f\|_{sup}\,, \tag{3.24}$$

We now use (3.17) and get

$$\frac{1}{n}\Big|W^c_{n,A;i,j,t'}\Big|_{sup} = \frac{1}{n}\Big|\sum_{\ell\in\Delta_n(j)} W^c_{n,A;i,j,t';\ell}\Big|_{sup} \leq \frac{\kappa\|f\|_{sup}}{\kappa+\mu} + \frac{\varepsilon}{4} + \frac{\mu\hat{\kappa}\rho\beta}{n} + \frac{\mu\hat{\kappa}P}{n}\|f\|_{sup}\,.$$

Then (3.15) and (3.16) implies

$$|W_{n,A,i,j,t'}(g)| \leq \frac{2\|f\|_{sup}}{\rho} + \frac{\kappa\|f\|_{sup}}{\kappa+\mu} + \frac{\varepsilon}{4} + \frac{\mu\hat{\kappa}\rho\beta}{n} + \frac{\mu\hat{\kappa}P}{n}\|f\|_{sup}\,. \tag{3.25}$$

Finally, we use (3.14) and get the pointwise estimate

$$|W_{n,A}f(g,y)| \leq \frac{\varepsilon}{4} + \frac{2\|f\|_{sup}}{\rho} + \frac{\kappa\|f\|_{sup}}{\kappa+\mu} + \frac{\varepsilon}{4} + \frac{\mu\hat{\kappa}\rho\beta}{n} + \frac{\mu\hat{\kappa}P}{n}\|f\|_{sup}\,, \tag{3.26}$$

valid whenever $\bar{g} \in G$ and $y \in C$. Now note that depending on the initial given data (i.e. $A_0, f, \varepsilon, \delta$ and $\bar{n}$ we have so far chosen κ, $\hat{\kappa}$ and P. Next, we choose ρ and μ such that

$$n = (\kappa+\mu)\rho > \bar{n}\,, \quad \frac{\|f\|_{sup}}{\rho} \leq \frac{\varepsilon}{16}\,, \quad \frac{\kappa\|f\|_{sup}}{\mu} \leq \frac{\varepsilon}{16}\,, \quad \text{and} \quad \frac{P\hat{\kappa}\|f\|_{sup}}{\rho} \leq \frac{\varepsilon}{16}\,.$$

Then (3.26) and (3.2) implies that

$$|W_{n,A}f(\bar{g},y)| \leq \frac{7\varepsilon}{8} \quad \text{whenever} \quad g \in G \text{ and } y \in C\,. \tag{3.27}$$

This, together with (3.13), gives

$$\int_{G\times Y} |W_{n,A}f|^2 \leq \frac{49\varepsilon^2}{64} + \Big(2c_1 + \frac{\rho}{\lambda}\Big)\|f\|^2_{sup}\,.$$

Finally we choose λ and c_1 so that $(2c_1+\frac{\rho}{\lambda})\|f\|^2_{sup} \leq \frac{14\varepsilon^2}{64}$, and conclude that $\|W_{n,A}f\|_{L^2} < \varepsilon$.

Since $n > \bar{n}$, we have proved that $\psi \in E(f,\varepsilon,\bar{n})$. By construction, $||A - A_0|| < \delta$ and $A_0 \in E(f,\varepsilon,\bar{n})$, concluding the proof of Lemma 2.3 and subsequently the proof of Theorem 2.1 as well. $\square$

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