Generic continuous spectrum for quasi-energy operators arising from specific differential equations

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§1. Introduction.

This paper is motivated by recent developments in Mathematical Physics regarding spectral and stability questions concerning evolution of quantum systems. These questions lead to problems about ergodic properties of certain flows ([N3], [NJ]). As an example of the general situation to be studied here, consider the so called "Rabi oscillator". The equation of this system is

$$i\frac{d\psi}{dt} = \begin{pmatrix} \lambda & f(t) \\ f(t)^* & -\lambda \end{pmatrix}\psi, \qquad \psi \in \mathbf{C}^2. \tag{1.1}$$

This is the Schrödinger equation which describes the dynamics of a "two level atom" or a spin $1/2$ particle moving under external magnetic field $f(t)$. The function $f : \mathbf{R} \to \mathbf{C}$ is a complex-valued potential, typically quasiperiodic in t, and $\lambda \in \mathbf{R}$ is a fixed parameter. In the past few years this system has been extensively investigated by physicists using numerical techniques, cf.[B], [BM], [DGP], [SK]. First rigorous results regarding existence of quasi-periodic solutions and pure point spectrum (a sure sign of stability of the quantum evolution) have been proved in [BJL] using the K.A.M. technique. In contrast, we shall prove that that generically the quantum evolution is "unstable". More precisely, the unitary representation generated by the "quasi-energy operator" associated to equation (1.1) has only purely continuous spectrum for a generic choice of f.

Our result will in fact be valid for differential equations far more general than (1.1). Our differential equations are time-dependent and typically the time dependence of the coefficient functions is "recurrent", in particular almost periodic. Such functions are best described by thinking of them as values of some given function, evaluated along a trajectory of some given flow with the same recurrence properties. Thus we are given a flow $(\Omega, \{T_t\}_{t\in\mathbf{R}})$ where Ω is a compact metric space and $T_t : \Omega \to \Omega,\ t \in \mathbf{R}$ is a one-parameter group of homeomorphisms of Ω. The coefficient functions of the differential equation are described by a continuous function $A : \Omega \to L$, where L is some fixed Lie algebra of a connected Lie group G. Then the family of differential equations, parametrized by a point $\omega \in \Omega$, so obtained is described as follows,

$$x' = A(T_t\omega)x, \quad x \in G,\ \omega \in \Omega. \tag{1.2}$$

Here we identify L with the tangent space of G at the identity. Then the right hand side of (1.2) can be thought as the value at x of the right invariant vector field generated by $A(T_t\omega)$. A single equation of the form $x' = A(t)x$ generates such a family of equations by taking Ω to be the "hull" of the map A and then extending the map A to all of Ω. Thus, the above set-up is fairly general.

Let $X_A(\omega, t)$ denote the fundamental matrix solution to (1.2), (so for each $\omega \in \Omega$, the function $t \to X_A(\omega, t)$ is the solution of (1.2) such that $X_A(\omega, 0) = e_G$- the identity of G). Then the map $X_A : \Omega \times \mathbf{R} \to G$ is continuous and satisfies the cocycle identity

$$X_A(\omega, t+s) = X_A(T_t(\omega), s)X_A(\omega, t) \qquad \text{for all } \omega \in \Omega,\ t, s \in \mathbf{R}. \tag{1.3}$$

Notice that in the case of equation (1.1), $G = SU(2,\mathbf{C})$ and for more general finite dimensional quantum systems, G will be a subgroup of the unitary group.

The proper analogue of the flow generated by a family of the time dependent differential equations such as (1.2) is the **skew- product flow** $(G\times\Omega,\{T_t^A\}_{t\in\mathbf{R}})$ define on $G\times\Omega$ by

$$T_t^A(g,\omega) = (X_A(\omega,t)g, T_t(\omega)). \tag{1.4}$$

A flow $(\Omega,\{T_t\}_{t\in\mathbf{R}})$ with a given flow invariant, regular Borel probability measure μ on Ω will be denoted by the triple $(\Omega,\{T_t\}_{t\in\mathbf{R}},\mu)$. Given such a system, notice that the product measure $\nu\times\mu$ (where ν is a left Haar measure on G) is invariant under the skew-product flow. If G is compact, we shall assume that $\nu(G) = 1$.

If G is a matrix Lie group, say $G\subseteq GL(n,\mathbf{C})$, one defines a representation $V^A = \{V_t^A\}_{t\in\mathbf{R}}$ on $L^2(\Omega,\mathbf{C}^n,\mu)$ by setting

$$V_t^A f(\omega) = X_A(\omega,t)^{-1} f(T_t(\omega)). \tag{1.5}$$

If G is a subgroup of the unitary group $U(n)$ (as is the case in equations describing evolution of finite dimensional quantum systems), this representation is unitary and in the Physics literature the infinitisimal generator of this representation is called the **quasi-energy operator**. Physicist study the stability properties of the quantum evolution (with time-dependent Hamiltonian) through the spectral properties of this representation. Following the intuition based on classical mechanics, a quantum evolution is regarded as **stable** if the spectrum of the quasi-energy operators is discrete pure point. In this paper, under suitable hypothesis, we shall show that (Theorem B) within very "thin" classes of differential systems (for example, equations of the form (1.1)), those systems whose quasi-energy operator has only purely continuous spectrum are generic. In particular a generic quantum evolution, even within very special classes of systems is highly unstable.

In fact the absence of point spectrum of the quasi-energy operators is very closely related to a problem of "lifting ergodicity". We recall that a flow $(X,\{T_t\}_{t\in\mathbf{R}},\lambda)$ is **ergodic** if the λ-measure of any invariant Borel subsets is either zero or one. The ergodicity lifting problem involves constructing ergodic skew-product flows $(\Omega,\{T_t^A\}_{t\in\mathbf{R}},\nu\times\mu)$ where the map A is constrained by the special form of the underlying family of differential equations. Under suitable hypothesis we shall solve this problem (Theorem A) and then derive the spectral result (Theorem B) as a corollary.

Before stating the theorems precisely, we need to discuss the constraining condition on the classes of cocycles-the strong accessibility condition as well as other assumptions on the base flow.

The strong accessibility condition.

The class of cocycles X_A arising from linear differential systems of a specific form is described via a constraint on the values of the map $A:\Omega\to L$. Namely, one considers only those maps with values in a fixed compact convex subset Q of the Lie algebra L. We shall impose a certain Lie algebraic condition (the condition of **strong accessibility**) on the constraint set Q, however the set Q can still be very "thin" in L in the sense that the dimension $dim(G)$ of G will in general be considerably higher than the number $dim(Q)$ of free parameters that we are allowed to vary to generate cocycle with desired properties. For example, in the case of the Rabi oscillator the cocycle takes values in a three-dimensional Lie group, but we only have two real degrees of freedom, since the map f which is our perturbation takes values in a two-dimensional space of complex numbers.

To describe the condition "of strong accessibility", first, let $L(Q)$ be the Lie subalgebra of L generated by Q, and let $L_0(Q)$ be the ideal of $L(Q)$ generated by the difference set $Q - Q = \{x - y \mid x \in Q,\ y \in Q\}$.

1.1 Definition. *A subset Q of the Lie algebra L has the* **accessibility property** *if $L(Q) = L$, and the* **strong accessibility property** *if $L_0(Q) = L$.*

Later (in Section 5) we shall introduce and study these notions in a more general set up of control systems on a general analytic manifold.

Next, we briefly discuss the conditions we need to impose on the base flow.

Flows with nice sections.

Basically we need our base flow $(\Omega, \{T_t\}_{t\in\mathbf{R}}, \mu)$ to admit either a global or 'sufficiently many' nice local sections. We begin by introducing basic notion of a section to a flow.

1.2 Definition. *(I) A* **flow box** *Σ^T to a given flow $(\Omega, \{T_t\}_{t\in\mathbf{R}})$ consists of a subset (transverse section) $\Sigma \subset \Omega$ and $T > 0$ such that*

(1) *Σ is compact and*
(2) *the map $(\sigma, t) \to T_t(\sigma)$ defines a homeomorphism from $\Sigma \times [0, T]$ onto its image (denoted by Σ^T) such that for all $s \neq t$, $s, t \in [0, T]$ one has $T_s\Sigma \cap T_t\Sigma = \emptyset$.*

(II) A flow box (Σ, T) is **nice** *if there exists a continuous ' first return time' map $\tau : \Sigma \to (0, \infty)$. That is for each $\sigma \in \Sigma$, $\tau(\sigma)$ is the smallest positive real number r such that $T_r(\sigma) \in \Sigma$.*

Let $P(\sigma) = T_{\tau(\sigma)}(\sigma) : \Sigma \to \Sigma$ be the corresponding Poincaré map. We shall refer to the pair (Σ, P) as the "Poincaré section" to the flow.

1.3 Definition. *A flow $(\Omega, \{T_t\}_{t\in\mathbf{R}}, \mu)$ satisfies Hypothesis-(H) if either (1) it is a flow built by a continuous ceiling function over a continuous, one to one transformation on a compact metric space or (2) there exists a countable family $\mathcal{F} = \{({\Sigma_n}^{T_n} \mid n \in \mathbf{N}\}$ of nice flow boxes such that $\{\Sigma_n \mid n \in \mathbf{N}\}$ generate the σ algebra of Borel subsets of Ω.*

The important class of examples of flows to which our results apply are suspention flows. We recall that such a flow is obtained from a discrete dynamical system (Y, S, m) (where Y is compact metric, $S : Y \to Y$ is a homeomorphism, preserving a Borel propbability measure m) and a continuous ceiling function $f : Y \to (0, \infty)$. The phase space of the flow is $Y_f \equiv \hat{Y}_f / \sim$, where

$$\hat{Y}_f = \{(y, t) \mid y \in Y,\ 0 \leq t \leq f(x)\}$$

and the equivalence relation $\sim$ identifies $(y, f(y))$ with $(Sy, 0)$. The flow is obtained by 'moving vertically with unit speed on the fiber over $y \in Y$' and identifying $(y, f(y))$ with $(Sy, 0)$. The measure $\mu \equiv \int \lambda_y dm(y)$ is an invariant measure for the flow, where λ_y is the usual Lebesgue measure on the fiber $[0, f(y)]$.

For example, the irrational rotation flow $(\mathbf{T}^n, T^\alpha)$ on the n-torus given by

$$T^\alpha(x, y) = (x + \overline{\alpha}t, y + t) \bmod 1$$

can be regarded as a suspention flow over the irrational rotation transformation $x \to x+\overline{\alpha}$ on the $\mathbf{T}^{n-1}$ torus, (where $\overline{\alpha} = (\alpha_1, \cdots, \alpha_{n-1})$ and α_i's are rationally independent).

More generally flow generated by a C^1 vector field $(f_1(x_1, x_2), f_2(x_1, x_2))$ on the 2-torus $\mathbf{T}^2$ with $f_2 > 0$ can be represented as suspention flows with a continuous ceiling function (see [CFS]) and hence satisfy Hypothesis-(H). Furthermore fixed point free flow generated by any C^1 (C^2) vector field on $\mathbf{T}^2$ is a suspention flow over an orientation preserving diffeomorphism (irrational rotation) of the circle (see [KH page 458]).

Now we state our main theorems.

A Theorem. *Let $(\Omega, \{T_t\}_{t\in\mathbf{R}}, \mu)$ be a flow on a compact metric space Ω. Let G be a compact connected Lie group with Lie algebra L, and let Q be a subset of L. Let $C(\Omega, Q)$ denote the metric space of all continuous maps from Ω to Q with the supremum metric. Assume the following:*

1. *The flow $(\Omega, \{T_t\}_{t\in\mathbf{R}}, \mu)$ is ergodic, the measure μ is not supported on a single orbit and $\mu(U) > 0$ for any non-empty open set U.*
2. *The flow $(\Omega, \{T_t\}_{t\in\mathbf{R}}, \mu)$ satisfies Hypothesis-(H);*
3. *Q is compact and convex;*
4. *Q has the strong accessibility property.*

For each $A \in C(\Omega, Q)$, consider the skew-product flow $(G \times \Omega, \{T_t^A\}_{t\in\mathbf{R}}, \nu \times \mu)$ defined by (1.5). Let

$$\mathcal{C}_{erg}(\Omega, Q) = \{A \in C(\Omega, Q) : (G \times \Omega, \{T_t^A\}_{t\in\mathbf{R}}, \nu \times \mu) \text{ is ergodic}\}.$$

Then $\mathcal{C}_{erg}(\Omega, Q)$ is a residual subset of $C(\Omega, Q)$.

B Theorem. *In addition to the hypothesis of Theorem A, suppose that*

5. *G is a closed subgroup of the unitary group $U(n)$ and*
6. *G does not fix any ray in the complex projective n-space $P(\mathbf{C}^n)$.*

Let

$$\mathcal{C}_{cont}(\Omega, Q) = \{A \in C(\Omega, Q) : \text{the representation } \{V_t^A\}_{t\in\mathbf{R}} \text{ have only continuous spectrum}\}.$$

Then $\mathcal{C}_{cont}(\Omega, Q)$ is a residual subset of $C(\Omega, Q)$.

We remark that any suspention flow over a discrete ergodic system (Y, S, m) (in particular the irrational rotation flows on n-torus) satisfies conditions (1) and (2) of Theorem

A, where the ergodic measure m is non- atomic and supported on all of Y. Conditions (1) and (2) also hold when the base flow on 2-torus is generated by a divergence free C^1 vector field (f_1, f_2) with $f_2 > 0$, where μ is taken to be the normalized Lebesgue measure.

We end this section with a remark about the general notation used throughout the paper. Letter d will be used to denote metrics on various topological spaces and will be assumed to be bi-invariant if the underlying space is a compact topological group. Letter D will denote the supremum metric on any space of continuous bounded functions between any two given metric spaces.

§2. The strategy to prove Theorem A.

First, we note that Theorem B can be deduced from Theorem A via the following lemma.

2.1 Lemma. *Suppose that the group G satisfies conditions (5) and (6) of Theorem B and suppose that the skew-product flow $(G \times \Omega, \{T_t^A\}_{t\in\mathbf{R}}, \nu \times \mu)$ is ergodic. Then the representation V^A does not admit any point spectrum.*

Proof. Suppose $f \in L^2(\Omega, \mathbf{C}^n, \mu)$ is an eigenvector of V^A. Since the representation is unitary, our supposition implies existence of $\lambda \in \mathbf{R}$ such that

$$(V_t^A f)(\omega) = e^{i\lambda t} f(\omega), \quad \text{a.e. } \omega \in \Omega \text{ and for all } t \in \mathbf{R}.$$

Consider the map $\rho : U(n) \times \Omega \to P(\mathbf{C}^n)$, defined by

$$\rho(g, \omega) = \pi(g^{-1} f(\omega)),$$

where $\pi : \mathbf{C}^n \to P(\mathbf{C}^n)$ is the canonical projection onto the projective space. Then,

$$\begin{aligned}
\rho(T_t^A(g, \omega)) &= \rho\big(X_A(\omega, t)g, T_t\omega\big) \\
&= \pi\big(g^{-1} X_A(\omega, t)^{-1} f(T_t\omega)\big) \\
&= \pi\big(g^{-1} f(\omega)\big) \\
&= \rho(g, \omega).
\end{aligned}$$

Thus ρ is T^A invariant and hence by the ergodicity assumption, it is constant a.e. (g, ω). Thus there is a non-zero vector $v \in \mathbf{C}^n$ such that

$$\pi(g^{-1} f(\omega)) = \pi(v) \quad \text{a.e. } (g, \omega).$$

In particular, this means that almost all $g \in G$, map the ray $\pi(v)$ into a fixed ray. This implies that the stabilizer of this fixed ray is a closed subgroup with full Haar measure and hence must be all of G. This contradicts assumption (6) of Theorem B. $\square$

The proof of Theorem A consists of the following major steps.

Step I : First, (in Section 3) we prove a generic lifting of ergodicity in the class of **all** continuous cocycles for the **integer actions** (Theorem 3.1). This result by itself is new and is of independent interest in Ergodic Theory. For compact **non-abelian** group extensions no such result was known untill now. The obstructions to lifting ergodicity in the class of all continuous cocycles and the reasons for the inadequecy of the previously known "ergodicity lifting techniques" will be discussed later in detail.

Step II : Next, (in section 4) we reduce the ergodicity lifting problem in the (constrained) class $C(\Omega, Q)$ of cocycles for real action on Ω, to the corresponding problem in the class of **all** cocycles for the integer action generated by the Poincaré maps on the Poincaré section to the flow on Ω.

An important ingradient in this reduction is the 'almost open mapping theorem' (Theorem 4.2). To explain it more precisely, consider a Poincaré section (Σ, P) to the flow on Ω, (see Definition 1.2). Given $B \in C(\Omega, Q)$, restriction of the cocycle X_B to (Σ, P) generates a continuous map φ_B, i.e. $\varphi_B(\sigma) = X_B(\sigma, \tau(\sigma))$ can be viewed as a cocycle for the integer action generated by P on Σ. Let $\Psi : C(\Omega, q) \to C(\Sigma, G)$ be the map $\Psi(B) = \varphi_B$.

First we note that a given ergodic measure μ on Ω induces a P invariant, ergodic measure m (the transversal measure) on Σ. Now let $A_0 \in C(\Omega, Q)$ be given. Then the result obtained in *Step I* will allow us to approximate φ_{A_0} (arbitrarily closely in the supremum norm) by a continuous map $\varphi : \Sigma \to G$ that generates an ergodic (with respect to $\nu \times m$) skew-product on $G \times \Sigma$. Now a natural question is can we find $A \in C(\Omega, Q)$ 'sufficiently close' to A_0 such that $X_A = \varphi$. Well, this is not exactly true, nevertheless we prove that A_0 can be approximated arbitrarily closely by maps A such that the map Ψ at A is 'almost open', (i.e. the image of a ball (in $C(\Omega, Q)$) centered at A under Ψ contains $\Psi(A)$ in its interior). This result along with a criterion for ergodicity in terms of the Mackey Range of a cocycle (see [Sch]) allows us to prove Theorem A..

Step III : Now remains the problem of proving the 'almost open mapping theorem mentioned in *Step II*. More precisely, we want to show that given $A_0 \in C(\Omega, Q)$ and $\varepsilon > 0$ we can find $A \in C(\Omega, Q)$ within ε of A_0 and a $\delta > 0$ such that if φ is within δ of φ_A then there exists $\tilde{A} \in C(\Omega, Q)$ within ε of A such that $\Psi(\tilde{A}) = \varphi$. This is what we refer to as the 'almost open mapping theorem' (Theorem 4.2).

To prove Theorem 4.2, we shall use Control Theoretic ideas. At this point for better exposition and to explain why our previous method to construct minimal skew-products (see [NS]) does not yield ergodic or even smooth (C^1) minimal skew-products, we shall go into some details about the need for a statement like that of Theorem 4.2.

Basically our problem is a perturbation problem. Given a map $A_0 \in C(\Omega, Q)$, we want to obtain its perturbation (with the constraint of being Q valued and close to A_0) such that the trajectories of the differential equation corresponding to A have certain 'dynamical properties'. We approach (and solve) this problem by viewing it as a controllability problem where the desired perturbation is thought as a control function which is to be chosen so that the corresponding trajectory 'hits a specified target'. A more precise and (more general) framework for such problems is as follows:

Let M be a smooth manifold and let $f_0, f_1, \cdots, f_m$ be smooth vector fields on M.

A **control process** $(M, f_0, f_1, \cdots f_m)$ is a system given by the following differential equation;

$$x' = f_0(x) + \sum_{i=1}^{m} u_i f_i(x),$$

where the map $\overline{u} = (u_1, u_2, \cdots, u_m) : [0, \infty] \to \mathbf{R}^m$ is called the control function. Now fix any $T > 0$ and we shall consider the class W_T of measurable and essentially bounded control functions on the interval $[0, T]$. Assume that for each $x \in M$ and $\overline{u} \in W_T$ the solution $\xi_{x,\overline{u}}(t)$ exists on $[0, T]$. Define the evaluation map $E_x : W_T \to M$ by setting

$$E_x(\overline{u}) = \xi_{x,\overline{u}}(T).$$

With this notation, the central theme of our approach to the perturbation problem is to view it as a control problem described above, where the 'drift term' (i.e. the vector field f_0) comes from the given function $A_0 \in C(\Omega, Q)$ and the perturbation we want is given by a suitable control function $\overline{u}$. Of course for $\omega_0 \in \Omega$ one identifies the interval $[0, T]$ with the piece of the orbit $\{T_t(\omega_0) \mid t \in [0, T]\}$ and then the control function will generate a perturbation of A_0 at least on this piece of the orbit(s), (actually, for us the values of the perturbation outside this piece of the orbit(s) do not matter in general). In order to lift minimality (see [NS]) it is enough to make such a perturbation just on one orbit (i.e. orbit of a single chosen point). However for lifting ergodicity we need to make the desired perturbations on a set of orbits with **positive measure**. This is what makes the problem much harder. Since we have no apriori control on the lenghts of the orbits (i.e. T above) on which it is necessary to make these perturbations and during this time interval these orbits can enter the set of points (of positive measure) whose orbits need to be perturbed the desired way. These orbits can also come very close to their initial starting point within this lenght of time, making it impossible to have any control on the growth of the derivatives of the perturbation defined on them (as a function of $\omega \in \Omega$). This explains why the technique developed in [NS] is completely inadequete to lift ergodicity or to obtain even minimality lifting results in the category of smooth cocycles.

In terms of the control problem described above, to lift minimality we had to show that for a given x the image of a ball centered at $\overline{0}$ (in W_T) under the evaluation map E_x contains some open ball centered at $E_x(\overline{0})$. To lift ergodicity (or to obtain lifting results in the smooth category) we need to study the **first variation** of the family of maps $\{E_x\}_{x \in M}$ as a function on W_T.

Thus, in *Step III* (Section 5) we shall convert our perturbation problem to a control problem and reduce the proof of Theorem 4.2 to proving (a uniform) submersivity of the (family of) evaluation maps E_x (Theorem 5.4).

Step IV : We need to prove that the family of maps $\{E_x\}_{x \in K}$ is 'uniformly submersive' where $K \subset M$ is a given compact set. This 'uniformity' is crucial in proving Theorem 4.2. The problem of uniform submersivity for the control system described above is then converted into a problem of 'detectibility' for another control system (obtained from above by considering its first variation). Given a smooth map $h : M \to \mathbf{R}$, the problem of 'detecting' $x \in M$ involves finding a function $\overline{u}$ such that $h(\xi_{x,\overline{u}})(t) \neq 0$ for some $t \in [0, T]$. Our need for the 'uniformity' gets translated into a demand that 'the same $\overline{u}$ should work for all x in M. Such $\overline{u}$'s are called 'universal detectors' in Control Theory literature . Futhermore to obtain results in the smooth category we need to obtain such a $\overline{u}$ with arbitrarily small derivatives. In *Step IV* (Section 6) we show that all of this can be done. Theorem 6.6 proves the existence of the desired universal detectors in its 'infinitisimal form' i.e. as suitable jets. Then Corollary 6.8 gives us the desired control functions (perturbations) necessary to prove Theorem 5.4.

Above discussion shows that in the course of the proof Theorem A, we develop new techniques and obtain new results that are of independent interest in Ergodic Theory (Theorem 3.1) and Control Theory (Theorem 5.4).

§3 Ergodicity lifting result for discrete flows.

Let (Y, S, m) be a discrete flow. In this section first we shall prove a generic ergodicity lifting result in the class $C(Y, G)$ of all continuous G valued maps (Theorem 3.1). We shall be applying this result to the discrete flow generated by the Poincaré map on a (suitable) Poincaré section to the given continuous flow on Ω.

3.1 Theorem. *Let (Y, S, m) be a discrete dynamical system and G be a metric topological group. Suppose that*

(1) *(Y, S, m) is ergodic and m is non-atomic, and*
(2) *G is a compact, connected Lie group.*

Then the set

$$C_{erg}(Y, G) = \{\varphi \in C(Y, G) \mid \textit{the transformation } (g, y) \to (\varphi(y)g, Sy) \textit{ is ergodic}\},$$

is a residual subset of $C(Y, G)$. Here, the ergodicity is with respect to the product measure $\nu \times m$.

It is known that ergodicity can be lifted generically in the class of closure of coboundaries [N1], (a precise definition of a coboundary will be given shortly). But we need to prove such a generic result in the class of **all** continuous cocycles. The discussion in Section 2 points out that given $A_0 \in C(\Omega, Q)$, we are looking for $A \in C(\Omega, Q)$ such that A is close to A_0 and the cocycle $\Psi(A)$ on the Poincaré section lifts ergodicity to the discrete skew-product flow it generates. Notice that there is no reason why the cocycle $\Psi(A)$ (on the Poincaré section) should be in the closure of coboundaries. Thus (for discrete systems) proving a generic ergodicity lifting result in the class of **all** cocycles is absolutely crucial. For non-abelian compact connected groups (or for that matter, even for non-abelian Lie groups) no such result (in the class $C(Y, G)$) was known.

The non-commutativity of the fiber group G is the main obstacle in extending the "closure of coboundaries technique" to yield a generic lifting theorem in the class $C(Y, G)$ of all continuous cocycles. In this paper we develop a technique to overcome this problem. The first step involves introducing more "general skew-products", where one multiplies the elements of the fiber group G on both, left as well as the right side by the skewing function (i.e. by the cocycle). Our technique to analyze the generic ergodicity lifting in such extensions consists of two ingradients : (a) 'modified closure of coboundaries technique' (which proves Theorem 3.3) and (b) an approximation procedure, which will allow us to derive Theorem 3.1 from Theorem 3.3. This procedure consists of approximating the average of a given function under the "usual skew-product" flow by the average of its translate under the generalized skew-product flow". We now give the precise details.

We begin by refining the notion of a skew-product transformation.

3.2 Definitions. *Let (Y, S, m) be a discrete dynamical system and Γ be a topological group. Let e_Γ denote the identity of the group Γ.*
(1) Any continuos map from Y to Γ will be refered to as a Γ-valued cocycle. Thus the set $C(Y,\Gamma)$ will be identified with the set of all Γ-valued cocycles. The constant map $y \to e_\Gamma$ is called the identity cocycle and suppressing its dependence on the underlying group Γ, will be denoted by 1.
(2) Given a function $\psi \in C(Y,\Gamma)$, set

$$1^\psi(y) = \psi(Sy)\psi(y)^{-1}. \tag{3.1}$$

The map 1^ψ is called the coboundary generated by ψ. Let $B(Y,\Gamma)$ denote the subset of $C(Y,\Gamma)$ consisting all coboundaries, i.e.

$$B(Y,\Gamma) = \{1^\psi \mid \psi \in C(Y,\Gamma)\}.$$

(3) Given a cocycle $\varphi \in C(Y,\Gamma)$ and a coboundary $1^\psi \in B(Y,\Gamma)$, define a cocycle $\varphi \cdot 1^\psi$ by setting

$$\varphi \cdot 1^\psi(y) = \psi(Sy)\varphi(y)\psi(y)^{-1}, \quad y \in Y. \tag{3.2}$$

(4) Two cocycles φ_1 and φ_2 are cohomologous if $\varphi_2 = \varphi_1 \cdot 1^\psi$ for some $\psi \in C(Y,\Gamma)$.
(5) Suppose the group Γ acts on the left of another topological group G by a jointly continuous action;

$$(\gamma, g) \to \gamma * g, \quad \gamma \in \Gamma,\ g \in G.$$

Then given a map $\varphi \in C(Y,\Gamma)$ one defines the skew-product transformation $S^\varphi : G\times Y \to G \times Y$ by setting

$$S^\varphi(g,y) = (\varphi(y) * g, Sy). \tag{3.3}$$

In fact the map φ uniquely determines a cocycle (again denoted by) $\varphi : Y \times \mathbf{Z} \to \Gamma$ by the formula

$$(S^\varphi)^n(g,y) = (\varphi(y,n)g, S^n y), \quad n \in \mathbf{Z}. \tag{3.4}$$

This map $\varphi : Y \times \mathbf{Z} \to \Gamma$ satisfies the discrete version of the cocycle identity (1.3). *This justifies our terminology.*

(6) In the rest of the paper the group Γ *will be the product group* $G \times G$ *and its action on* G *(on the left) is given by*

$$(h,k) * g = hgk^{-1}, \quad h,g,k \in G.$$

Thus, given a $G \times G$*-valued cocycle* $\varphi \equiv (\varphi_1,\varphi_2) \in C(Y, G \times G)$*, the skew-product map* $S^\varphi \equiv S^{(\varphi_1,\varphi_2)}$ *is given by*

$$S^{(\varphi_1,\varphi_2)}(g,y) = \big(\varphi_1(y) g \varphi_2(y)^{-1}, Sy\big). \tag{3.5}$$

The discrete flow $(G \times Y, S^{(\varphi_1,\varphi_2)}, \nu \times m)$ *is called the* **generalized skew-product** *transformation.*
(7) The usual or - what we shall refer to as the **standard skew-product** *transformation is obtained as a special case when* $\varphi \equiv (\varphi_1, 1)$.
(8) Notice that,

$$\begin{aligned} B(Y, G \times G) &= \{1^\psi \equiv 1^{(\psi_1,\psi_2)} \mid \psi_1,\psi_2 \in C(Y,G)\} \\ &= \{(\varphi_1,\varphi_2) \mid \varphi_1,\varphi_2 \in B(Y,G)\}. \end{aligned}$$

Note that, for $(\alpha_1,\alpha_2),(\psi_1,\psi_2) \in C(Y, G \times G)$*, we have*

$$(\alpha_1,\alpha_2) \cdot 1^{(\psi_1,\psi_2)} = (\alpha_1 \cdot 1^\psi, \alpha_2 \cdot 1^{\psi_2}).$$

In terms of this notation, Theorem 3.1 states that the set of ergodic transformations is residual in the class $\{S^{(\xi,1)} \mid \xi \in C(Y,G)\}$ of skew-product maps. To prove this, first we shall show that given any $\alpha \in C(Y,G)$ the set of ergodic transformations is residual in the class $\{S^{(\alpha,\beta)} \mid \beta \in \overline{B(Y,G)}\}$, (Theorem 3.3). Here $\overline{B(Y,G)}$ denotes the closure of the set $B(Y,G)$ as a subset of $C(Y,G)$. The proof of Theorem 3.1 is then obtained from Theorem 3.3 by developing an "approximation procedure" based on a "stacking construction" and the use of Rokhlin's lemma.

A modified closure of coboundary technique.

As mentioned above, as a first step towards Theorem 3.1 we prove the following

3.3 Theorem. *Let* (Y,S,m) *be a discrete dynamical system and* G *be a metric, topological group and let* $\alpha \in C(Y,G)$ *be a given cocycle. Suppose*

(1) (Y,S,m) *is ergodic and* m *is non-atomic and*
(2) G *is compact and connected.*

Then the set

$$\{\beta \in \overline{B(Y,G)} \mid (G \times Y, S^{(\alpha,\beta)}, \nu \times m) \text{ is ergodic}\}$$

is a residual subset of $\overline{B(Y,G)}$.

Proof of Theorem3.3

The technique employed to prove this theorem is a modification of what is known as "the closure of coboundaries technique". Those who are familiar with this technique, will realize that the essential feature of this modification is following Proposition 3.5. We begin with the usual steps followed in this procedure (see [N2]).

Let $\mathcal{H} = L^2(G \times Y, \nu \times m)$ and $\mathcal{H}_0 = \{f \in \mathcal{H} \mid \int_{G\times Y} f d(\nu \times m) = 0\}$. Let $|| \ ||_2$ be the L^2 norm on $\mathcal{H}$. Given $\gamma, \beta \in C(Y, G)$, define a unitary operator $U_{(\gamma,\beta)}$ on $\mathcal{H}$ by setting

$$U_{(\gamma,\beta)} f = f \circ S^{(\gamma,\beta)}. \tag{3.6}$$

Furthermore, let

$$W_n^{(\gamma,\beta)} = \frac{1}{n} \sum_{i=0}^{n-1} U^i_{(\gamma,\beta)}. \tag{3.7}$$

Given $f \in \mathcal{H}_0, \varepsilon > 0$ and $r \in \mathbf{N}$, define a set $E(f, \varepsilon, r)$ as follows:

$$E(f,\varepsilon,r) = \{\beta \in \overline{B(Y,G)} \mid \text{there exists } M \in \mathbf{N},\ M > r \text{ such that } ||W_M^{(\alpha,\beta)} f||_2 < \varepsilon\}.$$

3.4 Lemma. *Let $\{f_j \mid j \in \mathbf{N}\}$ be a dense subset of $\mathcal{H}_0$. If $\beta \in E(f_j, \frac{1}{n}, r)$ for all $j, n, r \in \mathbf{N}$, then $(G \times Y, S^{(\alpha,\beta)}, \nu \times m)$ is ergodic.*

Proof. Fix any $j \in \mathbf{N}$. By the L^2 ergodic theorem, as $n \to \infty$ the sequence $W_n^{(\alpha,\beta)} f_j$ converges in the L^2 norm to some $f_j^* \in \mathcal{H}$. Since $\beta \in E(f_j, \frac{1}{n}, r)$ for all $n, r \in \mathbf{N}$, f_j^* must be the zero function. Since j was arbitrary and $\{f_j \mid j \in \mathbf{N}\}$ is dense in $\mathcal{H}_0$, it follows that for each $f \in \mathcal{H}_0$, $W_n^{(\alpha,\beta)} f \to 0$, as $n \to \infty$. This proves ergodicity of $S^{(\alpha,\beta)}$. □

Thus, once openness and density of each $E(f, \varepsilon, r)$ in $\overline{B(Y,G)}$ is established, the proof of Theorem 3.3 follows from the Baire category theorem . Openness of each $E(f, \varepsilon, r)$ can be easily checked, so we turn to the proof of its density.

Density of $E(f, \varepsilon, r)$:

We continue with the "usual steps" of the "closure of coboundaries technique" in our more general set-up. Given a $\psi \in C(Y, G)$, define a $\nu \times m$ preserving map $\hat{R}_\psi : G \times Y \to G \times Y$ by setting

$$\hat{R}_\psi(g, y) = (g\psi(y), y).$$

Let R_ψ be the corresponding unitary operator induced on $\mathcal{H}$, i.e.

$$R_\psi f = f \circ \hat{R}_\psi, \qquad f \in \mathcal{H}.$$

Now, the following identities can be easily verified:

$$\begin{aligned} \hat{R}_\psi \circ S^{(\alpha,1^\psi)} &= S^{(\alpha,1)} \circ \hat{R}_\psi, \\ U_{(\alpha,1^\psi)} &= R_\psi \circ U_{(\alpha,1)} \circ R_\psi^{-1}, \\ W_n^{(\alpha,1^\psi)} &= R_\psi \circ W_n^{(\alpha,1)} \circ R_\psi^{-1}, \qquad \text{for all } n \in \mathbf{N}. \end{aligned}$$

Since R_ψ is unitary, $1^\psi \in E(f,\varepsilon,r)$ if and only if $1 \in E(R_\psi^{-1}f,\varepsilon,r)$. Thus to prove the density of $E(f,\varepsilon,r)$ in $\overline{B(Y,G)}$, it is enough to prove that given any $f \in \mathcal{H}_0$, $\varepsilon > 0$, $r \in \mathbf{N}$ and $\delta > 0$, there exists a $\psi \in C(Y,G)$ such that

(1) $D(1^\psi,1) \equiv \sup_{y\in Y} d(1^\psi(y),e_G) < \delta$, and

(2) $||W_M^{(\alpha,1^\psi)}f||_2 < \varepsilon$ for some $M > r$.

Now observe that,

$$||W_M^{(\alpha,1^\psi)}f||_2 = ||R_\psi \circ W_M^{(\alpha,1)} \circ R_\psi^{-1}f||_2 = ||W_M^{(\alpha,1)} \circ R_\psi^{-1}f||_2.$$

This computation shows that we need to analyze the ergodic averages of functions under the transformation $S^{(\alpha,1)}$. This is done via the following result, which describes the ergodic components of the invariant measure $\nu \times m$ in terms of the "Mackey range"-G_α of the cocycle α (see [Sch],[Z]] for a proof).

3.5 Proposition. *Let (Y,S,m) be ergodic and let G be a compact group. Let $\alpha : Y \to G$ be a cocycle (in fact, α can be any Borel measurable map). Then there exists a closed subgroup $G_\alpha \subseteq G$ and a Borel measurable map $\xi : Y \to G$ such that*

(1) $\alpha \cdot 1^\xi \equiv \xi(Sy)\alpha(y)\xi(y)^{-1} \in G_\alpha$, a.e. $y \in Y$ and

(2) $(G_\alpha \times Y, S^{(\alpha\cdot 1^\xi,1)}, \nu_\alpha \times m)$ is ergodic,

where ν_α is the normalized Haar measure on G_α.

Applying this proposition to our given map α, we get a closed subgroup G_α and a Borel measurable map $\xi : Y \to G$ satisfying the conclusion of Proposition 3.5.

Next, define a map $\hat{L}_\xi$ on $G \times Y$ by setting

$$\hat{L}_\xi(g,y) = (\xi(y)^{-1}g, y)$$

and let L_ξ be the corresponding unitary operator induced on $\mathcal{H}$. Then the following identities can be easily verified:

$$\begin{aligned} S^{(\alpha,1)} &= \hat{L}_\xi \circ S^{(\alpha\cdot 1^\xi,1)} \circ \hat{L}_\xi^{-1}, \\ U_{(\alpha,1)} &= L_\xi^{-1} \circ U_{(\alpha\cdot 1^\xi,1)} \circ L_\xi \\ W_M^{(\alpha,1)} &= L_\xi^{-1} \circ W_M^{(\alpha\cdot 1^\xi,1)} \circ L_\xi. \end{aligned}$$

Returning to the proof of Theorem 3.3, we have

$$\begin{aligned} ||W_M^{(\alpha,1^\psi)}f||_2 &= ||W_M^{(\alpha,1)} \circ R_\psi^{-1}f||_2 \\ &= ||L_\xi^{-1}W_M^{(\alpha\cdot 1^\xi,1)}L_\xi \circ R_\psi^{-1}f||_2 \\ &= ||W_M^{(\alpha\cdot 1^\xi,1)}\big(L_\xi \circ R_\psi^{-1}f\big)||_2. \end{aligned}$$

Now by the ergodicity of the discrete flow $(G_\alpha \times Y, S^{(\alpha\cdot 1^\xi,1)}, \nu_\alpha \times m)$, we know that the sequence of operators $n \to W_n^{(\alpha\cdot 1^\xi,1)}$ converges strongly to the projection operator P_α given by

$$(P_\alpha \rho)(g,y) = \int_Y \int_{G_\alpha} \rho(kg,y)d\nu_\alpha(k)dm(y), \qquad \rho \in \mathcal{H}.$$

Thus as $M \to \infty$,

$$W_M^{(\alpha\cdot 1^\xi,1)}\big(L_\xi(R_\psi^{-1}f)\big) \to \int_Y \int_{G_\alpha} f\big(\xi(y)^{-1}kg\psi(y)^{-1},y\big)d\nu_\alpha(k)dm(y).$$

Set,

$$f^*(g,y) = \int_{G_\alpha} f\big(\xi(y)^{-1}kg,y\big)d\nu_\alpha(k),$$

then $f^* \in \mathcal{H}_0$ and as $M \to \infty$

$$W_M^{(\alpha\cdot 1^\xi,1)}\big(L_\xi(R_\psi^{-1}f)\big) \to \int_Y f^*(R_\psi^{-1}(g,y))dy = \int_Y f^*(g\psi(y)^{-1},y)dm(y).$$

Hence, if we can choose a $\psi \in C(Y,G)$ such that
(1) $D(1^\psi,1) < \delta$ and
(2) $|| \int_Y f^*(g\psi(y)^{-1},y)dm(y)||_2 < \frac{\varepsilon}{2}$,
then by choosing M large enough, we can make $||W_M^{(\alpha,1^\psi)}f||_2 < \varepsilon$. Thus we have reduced the proof of Theorem 3.3 to proving the following lemma.

3.6 Lemma. *Under the hypothesis of Theorem 3.3 and with the above notation, given any $f^* \in \mathcal{H}_0$, $\varepsilon > 0$ and $\delta > 0$, there exists a map $\psi \in C(Y,G)$ such that*

(1) *$D(1^\psi,1) < \delta$, and*
(2) *$|| \int_Y f^*(g\psi(y),y)d\mu(y)||_2 < \varepsilon$.*

This lemma is proved in [N1] (Lemma 3.9). In fact, in the continuous case the proof is given for far more general ergodic dynamical systems (Y,S,m) where the acting group S is allowed to be any "reasonable ameanable group". This completes the proof of Theorem 3.3. $\square$

The approximation procedure and the Proof of Theorem 3.1.

Now we develop the approximation technique to derive Theorem 3.1 from Theorem 3.3. Let $\mathcal{H}$ and $\mathcal{H}_0$ be as before. Define operators U_γ and W_n^γ by setting

$$U_\gamma \equiv U_{(\gamma,1)} \quad \text{and}$$
$$W_n^\gamma \equiv W_n^{(\gamma,1)},$$

where the operators on the right hand side are defined by equations (3.6) and (3.7). As before, given $f \in \mathcal{H}_0$, $\varepsilon > 0$ and $r \in \mathbf{N}$, define the set $F(f,\varepsilon,r)$ by setting

$$F(f,\varepsilon,r) = \{\varphi \in C(Y,G) \mid \text{there exists } M \in \mathbf{N},\ M > r \text{ such that } ||W_M^\varphi f||_2 < \varepsilon\}.$$

Once again the arguments of Lemma 3.4 remain valid and the proof reduces to showing that each $F(f,\varepsilon,r)$ is dense in $C(Y,G)$.

3.7 Lemma. *Each $F(f,\varepsilon,r)$ is dense in $C(Y,G)$.*

Proof. Since the set of continuous maps is dense in $\mathcal{H}_0$, there is no loss of generality in assuming that the map $f : G \times Y \to \mathbf{R}$ is continuous. Let $\alpha \in C(Y,G)$ and $\delta > 0$ be given. We want to construct a map $\psi \in C(Y,G)$ such that

(1) $D(\alpha, \alpha\psi) \equiv D(1,\psi) < \delta$, and
(2) $\alpha\psi \in F(f,\varepsilon,r)$.

Now, we carry out this construction in a series of steps.
Step 1 First, select $\delta_1 > 0$ such that

$$d(h, e_G) < \delta_1,\ h \in G \text{ implies that } d(ghg^{-1}, e_G) < \delta, \text{ for all } g \in G.$$

Step 2 Then applying Theorem 3.3, we get a map $\beta \in C(Y,G)$ such that
(1) $D(1,\beta) < \delta_1$ and
(2) $(G \times Y, S^{(\alpha,\beta)}, \eta \times m)$ is ergodic.

If G is commutative, then $S^{(\alpha,\beta)} = S^{(\alpha\beta^{-1},1)}$ and β^{-1} will be our desired function ψ. When G is non-abelian, the key idea in the construction of the desired ψ is to make sure that the averages of f over sufficiently long pieces of orbits of "most of the points" $(h,y) \in G \times Y$ under the actions of $S^{(\alpha\psi,1)}$ are close to the averages of translates of f under the action of $S^{(\alpha,\beta)}$. This is a sort of "approximation" procedure which will be carried out by a "stacking construction" using Rokhlin's lemma.

Before we continue, we make a note of the following formulae;

$$(3.8)\qquad \left(S^{(\alpha\psi,1)}\right)^n(g,y) = (\alpha(S^{n-1}y)\psi(S^{n-1}y)\cdots\alpha(y)\psi(y)g, S^n y), \quad n \in \mathbf{N},$$

$$(3.9)\qquad \left(S^{(\alpha,\beta)}\right)^n(g,y) = (\alpha(S^{n-1}y)\cdots\alpha(y)g\beta(y)^{-1}\cdots\beta(S^{n-1}y)^{-1}, S^n y), \quad n \in \mathbf{N}.$$

Step 3 Given $h \in G$, define $f_h \in \mathcal{H}_0$ by setting,

$$f_h(g,y) = f(gh,y).$$

Since $\{f_h \mid h \in G\}$ is a compact subset of $\mathcal{H}_0$, by the ergodicity of the flow $(G \times Y, S^{(\alpha,\beta)}, \nu \times m)$ we know that as $n \to \infty$, the sequence $W_n^{(\alpha,\beta)}(f_h)$ converges to 0 in the L^2 norm, uniformly in $h \in G$. Furthermore, since f is bounded, the family $\{f_h \mid h \in G\}$ is uniformly bounded in the supremum norm. Thus we can find a subsequence $n_k \to \infty$ and a subset $P \subseteq G \times Y$ of full measure (i.e. $(\nu \times m)(P) = 1$) such that

$$\left(W_{n_k}^{(\alpha,\beta)} f_h\right)(g,y) \to 0, \quad \text{uniformly in both } h \in G, \text{ and } (g,y) \in P.$$

By the Fubini's theorem there exists a measurable set $Y_1 \subseteq Y$ with $\mu(Y_1) = 1$, and for each $y \in Y_1$ a measurable set $G_y \subseteq G$ with $\nu(G_y) = 1$ such that as $n_k \to \infty$,

$$\left(W_{n_k}^{(\alpha,\beta)} f_h\right)(g,y) \to 0, \quad \text{uniformly in both } h \in G, \text{ and } (g,y) \in G_y \times \{y\},\ y \in Y_1.$$

Since the fiber group G is compact, it admits a bi-invariant metric. Hence

$$d\big((S^{(\alpha,\beta)})^k(g_1,y),(S^{(\alpha,\beta)})^k(g_2,y)\big)=d(g_1,g_2),\quad \text{for all } g_1,g_2\in G, y\in Y \text{ and } k\in\mathbf{N}.$$

This fact, along with continuity of f implies that, as $n_k\to\infty$,

$$\big(W_{n_k}^{(\alpha,\beta)}f_h\big)(g,y)\to 0,\quad \text{uniformly in } h,g\in G \text{ and } y\in Y_1.$$

In particular, taking $g=e_G$, we can select $N\in\mathbf{N}$ such that

$$\frac{1}{N}<\min\{\frac{1}{r},\frac{\varepsilon^2}{2K}\},\ \text{and}$$

$$\big|\big(W_N^{(\alpha,\beta)}f_h\big)(e_G,y)\big|<\frac{\varepsilon}{2}\quad \text{for all } h\in G \text{ and } y\in Y_1, \tag{3.10}$$

where $K=\sup_{(g,y)\in G\times Y}|f(g,y)|$.

Step 4 Now we construct $\psi\in C(Y,G)$ such that for "most of the points" the average of f over the first N iterates under $S^{(\alpha\psi,1)}$ can be approximated by average of its translate under $S^{(\alpha,\beta)}$. To do so, using the Rokhlin's lemma, select a Borel set $E\subset Y$ such that the sets $E,SE,\cdots,S^{N^2-1}E$ are pairwise disjoint and

$$\mu\big(\cup\{S^iE\mid 0\le i\le N^2-1\}\big)>1-\frac{\varepsilon^2}{4K}. \tag{3.11}$$

Furthermore, without loss of generality, we can assume that $E\subset Y_1$ and since μ is regular, we shall also assume that E is compact.

Step 5 Now we shall define ψ on $E,SE,\cdots,S^{N^2-1}E$ succesively by the requirement that

$$\big(S^{(\alpha\psi,1)}\big)^n(e_G,y)=\big(S^{(\alpha,\beta)}\big)^n(e_G,y),\quad \text{for all } y\in E \text{ and } n=0,1,\cdots N^2.$$

At this point we recall the notation introduced in (5) of Definition 3.2. If $(\varphi_1,\varphi_2)\in C(Y,G\times G)$, then $(\varphi_1,\varphi_2)(y,n)\in G\times G$ denotes the value of the cocycle it generates at a point $(y,n)\in Y\times\mathbf{Z}$. Taking $g=e_G$ in equations (3.8) and (3.9) and rewritting them in this notation, we get the following equations: For each $y\in Y$ and $n=0,1,\cdots,N^2$;

$$\big(S^{(\alpha\psi,1)}\big)^n(e_G,y)=\big((\alpha\psi,1)(y,n),S^ny\big),\ \text{and}$$
$$\big(S^{(\alpha,\beta)}\big)^n(e_G,y)=\big((\alpha,\beta)(y,n),S^ny\big).$$

Thus, we define ψ on E by requiring

$$(\alpha\psi,1)(y,n+1)=(\alpha,\beta)(y,n+1),\quad \text{for all } y\in E \text{ and } n=0,1,\cdots,N^2-1. \tag{3.12}$$

This is equivalent to requiring

$$\alpha(S^ny)\psi(S^ny)\big(\alpha\psi,1\big)(y,n)=\alpha(S^ny)\big(\alpha,\beta\big)(y,n)\beta(S^ny)^{-1},$$

for all $y \in E$ and $n = 0, 1, \cdots, N^2 - 1$. This allows us to define ψ inductively on $\cup\{S^n E \mid 0 \leq n \leq N^2 - 1\}$ by first setting

$$\psi(y) = \beta(y)^{-1}, \text{if } y \in E,$$

and then defining

$$\psi(S^n y) = \big[(\alpha, \beta)(y, n)\big]\beta(S^n y)^{-1}\big[(\alpha\psi, 1)(y, n)\big]^{-1}, \tag{3.13}$$

for all $y \in E$ and $n = 1, \cdots, N^2 - 1$. This equation along with our choice of β in Step (2) and δ_1 in step (1) implies that

$$d(\psi(y), e_G) < \delta, \quad \text{for all } y \in S^k E,\ 0 \leq k \leq N^2 - 1. \tag{3.14}$$

Now extend ψ continuously to all of Y so that

$$d(\psi(y), e_G) < \delta, \qquad \text{for all } y \in Y.$$

This is possible by viewing the δ neighbourhood of e_G as a Euclidean ball and then applying the Tietze's extension theorem. Thus we have constructed ψ in $C(Y, G)$ such that conclusion (1) of this lemma is satisfied.

Now, we verify that it also satisfies the condition (2) of this lemma. We begin by observing that if $y \in E$, $k \in [0, N^2 - N]$ and $n \in [0, N - 1]$, then equation (3.13) along with cocycle identity yields,

$$\begin{aligned}
\big(S^{(\alpha\psi,1)}\big)^n(e_G, S^k y) &= \big((\alpha\psi, 1)(S^k y, n), S^n y\big)\\
&= \Big(\big[(\alpha\psi, 1)(y, k+n)\big]\big[(\alpha\psi, 1)(y, k)\big]^{-1}, S^n y\Big)\\
&= \Big(\big[(\alpha, \beta)(y, k+n)\big]\big[(\alpha, \beta)(y, k)\big]^{-1}, S^n y\Big)\\
&= \big((\alpha\beta, 1)(S^k y, n), S^n y\big)\\
&= \big(S^{(\alpha,\beta)}\big)^n(e_G, S^k y).
\end{aligned}$$

Thus, we have shown that

$$\big(S^{(\alpha\psi,1)}\big)^n(e_G, y) = \big(S^{(\alpha,\beta)}\big)^n(e_G, y), \quad \text{for all } 0 \leq n \leq N - 1, \text{ and } y \in S^{N^2 - N}E.$$

Let $F = \cup\{S^n E \mid 0 \leq n \leq N^2 - N\}$, then

$$\begin{aligned}
\mu(Y - F) &\leq \frac{\varepsilon^2}{4K} + (N - 1)\mu(E) \quad \text{(by 3.11)}\\
&< \frac{\varepsilon^2}{4K} + (N - 1)\Big(\frac{1}{N^2}\Big)\\
&< \frac{\varepsilon^2}{4K} + \frac{\varepsilon^2}{2K}, \qquad \text{(by 3.10)}\\
&< \frac{3\varepsilon^2}{4K}.
\end{aligned} \tag{3.15}$$

Thus, for $(h,y) \in G \times F$,

$$\begin{aligned}
\big(W_N^{(\alpha\psi,1)} f\big)(h,y) &= \frac{1}{N}\sum_{j=0}^{N-1} f_h\big(\big(S^{(\alpha\psi,1)}\big)^n(e_G,y)\big) \\
&= \frac{1}{N}\sum_{j=0}^{N-1} f_h\big(\big(S^{(\alpha,\beta)}\big)^n(e_G,y)\big) \\
&= \big(W_N^{(\alpha,\beta)} f_h\big)(e_G,y).
\end{aligned}$$

Since $E \subseteq Y_1$, equation (3.10) implies that

$$\big|W_N^{(\alpha\psi,1)} f(h,y)\big| < \frac{\varepsilon}{2}, \quad \text{for all } y \in F \text{ and } h \in G. \tag{3.16}$$

Hence,

$$\begin{aligned}
||W_N^{(\alpha\psi,1)} f||_2^2 &= \int_{G\times Y} \big|W_N^{(\alpha\psi,1)} f(h,y)\big|^2 d\nu(h) \times dm(y) \\
&\leq \int_{G\times F} \big|W_N^{(\alpha\psi,1)} f(h,y)\big|^2 d\nu(h) \times dm(y) + Km(Y-F) \\
&< \frac{\varepsilon^2}{4} + \frac{3\varepsilon^2}{4}, \qquad \text{(by 3.15 and 3.16)} \\
&< \varepsilon^2.
\end{aligned}$$

Since $N > r$, (see (3.10)), above computation shows that $\alpha\psi \in F(f,\varepsilon,r)$. Thus, the proof of Lemma 3.7 and subsequently that of Theorem 3.1 is complete. $\square$

§4. Discretization and reduction to an 'almost open mapping theorem'.

In this section we shall reduce the proof of Theorem A to results developed in Section 3 via the "almost open mapping theorem"- Theorem 4.2. We begin by developing some notation.

Fix a nice flow box (see Definition 1.2) (Σ^T, P) where $P : \Sigma \to \Sigma$ is the Poincaré map $P(\sigma) = T_{\tau(\sigma)}(\sigma)$ and $\tau : \Sigma \to \Sigma$ is the first return-time map. When the flow is the suspention flow over a discrete system (Y,S), we shall take $\Sigma = Y$, $P = S$ and choose any T such that $0 < T < \inf_{y\in Y} f(y)$ where f is the ceiling function.

4.1 Definition. *Define a maps $\Psi, \Psi^T : C(\Omega, Q) \to C(\Sigma, G)$ by setting*

$$\begin{aligned}
\Psi(A)(\sigma) &= X_A(\sigma, \tau(\sigma)), \quad \sigma \in \Sigma, \\
\Psi^T(A)(\sigma) &= X_A(\sigma, T), \quad \sigma \in \Sigma
\end{aligned}$$

The next two sections of the paper are devoted to proving the following crucial theorem.

4.2 Theorem. *Assume the hypothesis of Theorem A and let Σ^T be a nice flow box. Let $A_0 \in C(\Omega, Q)$ and $\varepsilon > 0$ be given.*
(I) Then there exists $A \in C(\Omega, Q)$ and a $\delta > 0$ such that

(1) $D(A, A_0) < \varepsilon$ *and*
(2) *given any $\varphi \in C(\Sigma, G)$ with $D(\varphi, \Psi^T(A)) < \delta$, there exists $\tilde{A} \in C(\Omega, Q)$ with $D(\tilde{A}, A) < \varepsilon$ and $\Psi^T(\tilde{A}) = \varphi$.*

(II) In part (I) Ψ^T can be replaced with Ψ.
(III) Both parts (I) and (II) are valid in the smooth setting as well. More precisely, if $(\Omega, \{T_t\}_{t\in\mathbf{R}})$ and (Σ, P) are C^r smooth dynamical systems, then the maps Ψ and Ψ^T are well defined on $C^r(\Omega, Q)$ and in the statement of Part I of this theorem the space $C(\Omega, Q)$ and the suprimum metric D can be replaced by the space $C^r(\Omega, Q)$ and the C^r metric D_r respectively.

Proof of Theorem A

Let $\Sigma^T \equiv \Sigma \times [0, T]$ be a nice flow box. Since μ is positive on open sets, $\mu(\Sigma^T) > 0$. Let m be the projection of μ on Σ (under the 'canonical projection' $(\sigma, t) \to \sigma : \Sigma^T \to \Sigma$). The invariance and ergodicity of μ under the flow implies the invariance and ergodicity of m under the Poincaré map P. Thus we get an ergodic (discrete) dynamical system (Σ, P, m) to which resuls of Section 3 will be applied. Note that since μ is not supported on a single orbit, m is non-atomic.

Now given a nice flow box (Σ, T, m), $g \in G$ and an ε ball $N_\varepsilon(g)$ centered at g, $(\varepsilon > 0)$, define

$$\begin{aligned} \mathcal{E}_{\Sigma,g,\varepsilon} = \{A \in C(\Omega, Q) \mid \text{such that} \\ \mu\big(\omega \in \Sigma^T \mid T_t^A(\omega) \in Int(N_\varepsilon(g) \times \Sigma^T) \text{ for some } t \equiv t(\omega) \neq 0\big) > \frac{\mu(\Sigma^T)}{2}\}. \end{aligned}$$

Let C be a countable dense subset of G. Note that if

$$A \in \cap\{\mathcal{E}_{\Sigma,g,\frac{1}{k}} \mid \Sigma \in \mathcal{F}, g \in C, k \in \mathbf{N}\},$$

then by Hypothesis (H), given any Borel set $B \subseteq \Omega$ with $\mu(B) > 0$, $g \in G$ and $\varepsilon > 0$ we have

$$\mu\big(\omega \in B \mid \text{there exists a } t \equiv t(\omega) \neq 0,\ T_t^A(\omega) \in N_\varepsilon(g) \times B\big) > 0.$$

This implies that the Mackey range of the cocycle X_A is all of G and in particular the skew-product flow $(G \times \Omega, \{T_t^A\}_{t\in\mathbf{R}}, \nu \times \mu)$ is ergodic, (see [Sch]). Thus the proof will be complete once we show that each $\mathcal{S}_{\Sigma,g,\varepsilon}$ is open and dense in $C(Y, Q)$. Openness can be easily verified. We shall establish the density as a consequence of Theorem 3.1 and Theorem 4.2.

So let $A_0 \in C(\Omega, Q)$ and $\varepsilon > 0$ be given. Then there exists a $A \in C(\Omega, Q)$ and a $\delta > 0$ such that (1) and (2) of Theorem 4.2 are satisfied. By Thorem 3.1 we can pick a map $\varphi \in C(\Sigma, G)$ within δ of $\Psi(A)$ such that the skew product flow $(G \times \Sigma, P^{(\varphi,1)}, \nu \times m)$

is ergodic. By Theorem 4.2, there exists $\tilde{A} \in C(\Omega, Q)$ within ε of A (and hence within 2ε of A_0 such that $\Psi(A) = \varphi$. We shall show that $A \in \mathcal{E}_{\Sigma,g,\varepsilon}$. This will establish the density of $\mathcal{E}_{\Sigma,g,\varepsilon}$.

Since $(G \times \Sigma, P^{(\varphi,1)}, \nu \times m)$ is ergodic, $m(\tilde{\Sigma}) = 1$, (see [Sch]) where

$$\tilde{\Sigma} = \{\sigma \in \Sigma \mid \text{for any given } g \in G \text{ there is some } n \in \mathbf{N} \text{ such that } \varphi(\sigma, n) \in N_\varepsilon\}.$$

Now notice that

(1) $\varphi(\sigma, n) = X_{\tilde{A}}(\sigma, t)$, where $t = \tau^n(\sigma)$-the time of n^{th} return of σ to Σ.

(2) Writing a typical element $\omega \in \Sigma^T$ as $\omega = T_r\sigma$ for some $r \in [0, T]$ observe that

$$\begin{aligned} X_{\tilde{A}}(\omega, t) &= X_{\tilde{A}}(T_r\sigma, t) \\ &= X_{\tilde{A}}(\sigma, t + r) X_{\tilde{A}}(\sigma, r)^{-1} \\ &= X_{\tilde{A}}(T_t\sigma, r) X_{\tilde{A}}(\sigma, t) X_{\tilde{A}}(\sigma, r)^{-1}. \end{aligned}$$

Thus, using the bi-invariance of the metric on G, given any $g \in G$ we have

$$d(g, X_{\tilde{A}}(\omega, t)) = d(g', X_{\tilde{A}}(\sigma, t)),$$

where $g' = X_{\tilde{A}}(T_t\sigma, r)^{-1} g X_{\tilde{A}}(\sigma, r)$.

With these observations it follows that given any $g \in G$, if $\sigma \in \tilde{\Sigma}$ then for any $r \in [0, T]$ we can find some $t \neq 0$ (t may depend on σ and r) such that $T_t^{\tilde{A}}(T_r(\sigma)) \in N_\varepsilon(g) \times \Sigma$. Since $m(\tilde{\Sigma}) = 1$, it follows that $X_{\tilde{A}} \in \mathcal{E}_{\Sigma,g,\varepsilon}$ and the proof is complete. $\square$

§5. Reduction of Theorem 4.2 to a Control Theoretic result.

We begin by sketching the proof of Theorem 4.2 (I) in the 'continuous category' (i.e. in $C(\Omega, Q)$). Recall that $T > 0$ is fixed and $A_0 \in C(\Omega, Q)$ and $\varepsilon > 0$ is given. We want to find some map in $C(\Omega, Q)$ that is ε close to A_0 and a $\delta > 0$ such that the δ ball centered at $\Psi^T(A)$ is contained in the image of the ε ball centered at A under the mapping Ψ^T. We shall construct such a map in a series of steps.

Step 1: First approximate A_0 by a map $A_1 \in C(\Omega, Q)$ so that

(1) $D(A_0, A_1) < \frac{\varepsilon}{4}$ and

(2) A_1 takes values in $Int(Q)$-the interior of Q relative to itself.

Step 2: Let $m = \dim$ (affine hull of Q). Fix a point $F_0 \in Int(Q)$ and then fix a basis $F_1, F_2, \cdots, F_m$ of the subspace spanned by the set $\{F - F_0 \mid F \in Q\}$. Then the system

$$X' = AX, \quad (A \in Q),$$

can be written as

$$X' = \big(F_0 + \sum_{i=1}^{m} u_i F_i\big) X. \tag{5.1}$$

Thus, given any $A \in C(\Omega, Q)$, for each $\omega \equiv (\sigma, t) \in \Sigma \times [0, T] \subset \Omega$, the value $A(\omega) \equiv A(\sigma, t) \in Q$ determines a point $\overline{u}^A(\sigma, t) \equiv (u_1^A(\sigma, t), \cdots, u_m^A(\sigma, t)) \in \mathbf{R}^m$ such that

$$A(\sigma, t) = F_0 + \sum_{i=1}^{m} u_i^A(\sigma, t) F_i. \tag{5.2}$$

Since A is continuous and the flow on Ω is (jointly) continuous, the map $\sigma \to \overline{u}^A(\sigma, t)$ is continuous (uniformly in $t \in [0, T]$).

Step 3: Using the Stone-Weierstrass theorem, approximate A_1 by $A_2 \in C(\Omega, Q)$ so that

(1) $D(A_1, A_2) < \frac{\varepsilon}{4}$, A_2 also takes values in $Int(Q)$ and the function $\overline{u}^{A_2}$ defined by the representation (5.2) is given by

(2) $\overline{u}^{A_2}(\sigma, t) = \sum\limits_{j=0}^{k} \overline{\theta}_j(\sigma) t^j \equiv \sum\limits_{j=0}^{k} (\theta_{j,1}, \cdots, \theta_{j,m}) t^j$,

where $\overline{\theta}_j = (\theta_{j,1}, \cdots, \theta_{j,m}) : \Sigma \to \mathbf{R}^m, \quad 0 \le j \le k$, for some $k \in \mathbf{N}$. Note that each map $\overline{\theta}_j$ is continuous.

Consider the map $\overline{\theta} \equiv (\overline{\theta}_0, \cdots, \overline{\theta}_k) : \Sigma \to \mathbf{R}^{m(k+1)}$ and set

$$K = \{\overline{\theta}(\sigma) \mid \sigma \in \Sigma\} \subset \mathbf{R}^{m(k+1)}.$$

Then K is a compact subset of $\mathbf{R}^{m(k+1)}$. Thus, with the above notation the system

$$X' = A_2(T_t \sigma) X, \quad (\sigma, t) \in \Sigma \times [0, T],$$

can be written as

$$X' = \Big(F_0 + \sum_{l=1}^{m} \big(\sum_{j=0}^{k} \theta_{j,l}(\sigma) t^j\big) F_l\Big) X.$$

This family of (non-autonomous) differential equations depending on a parameter $\sigma \in \Sigma$ can be rewritten as a (autonomous) system on $G \times \mathbf{R}^{m(k+1)} \times \mathbf{R}$ as follows

$$\begin{aligned} X' &= G_0(z, \tau) X \\ \overline{z}' &= 0 \\ \tau' &= 1, \end{aligned}$$

where $G_0(z, \tau) = \Big(F_0 + \sum\limits_{l=1}^{m} \big(\sum\limits_{j=0}^{k} z_{j,l} \tau^j\big) F_l\Big)$, $X \in G$, $\overline{z} = (\overline{z}_0, \cdots, \overline{z}_k) \in \mathbf{R}^{m(k+1)}$ and $\overline{z}_j = (z_{j,1}, \cdots, z_{j,m}) \in \mathbf{R}^m$.

Step 4: Now we convert our perturbation problem into a control problem. We want to consider small perturbations of the map A_2 that are $Int(Q)$ valued and within ε of A_0. Notice that since A_2 takes values in $Int(Q)$ according to representation (5.2) this can be guaranteed if we restrict $\overline{u}$ in (5.2) to a suitably small ball B centered at 0 in $\mathbf{R}^m$.

The radius of this ball will depend only on Q, ε and our choice of $F_0, F_1, \cdots, F_m$. Thus according to representation (5.2), the systems given by a Q-valued small perturbations of the system corresponding to A_2 can be regarded as the following control system;

$$\begin{aligned} X' &= G_0(z,\tau)X + \sum_{i=1}^{m} u_i F_i X \\ z' &= 0 \\ \tau' &= 1. \end{aligned} \tag{5.3}$$

Regarding the control function $\overline{u} = (u_1, \cdots, u_m) \in B \subset \mathbf{R}^m$ as a perturbation of the map A_2 on the interval $[0,T]$, we will be able to convert the of problem constructing a "suitable perturbation" to the problem of constructing a "suitable control function". In this way we shall reduce the proof of Theorem 4.2 to proving the following Control Theoretic result (Theorem 5.4).

At this point we shall introduce a more general Control Theoretic framework of which (5.3) will be a special case and formulate a more general geometric problem regarding submersivity properties of a certain evaluation map. The solution to this problem will yield proof of Theorem 4.2. Our more general approach will make the exposition clearer and will indicate exactly what properties of system (5.3) are crucial to our arguments. Besides, in the process of doing so, we shall obtain a result that is of independent interest in Control Theory. Thus, temporarily abondoning the system (5.3), we shall consider more general control systems and develop some useful notation.

5.1 Definition. *Let M be a C^ω (i.e. analytic) manifold. Let $f_0, f_1, \cdots, f_m$ be analytic vector fields on M. A* **control process** *$(M, f_0, f_1, \cdots f_m)$ is a system given by the following differential equation;*

$$x' = f_0(x) + \sum_{i=1}^{m} u_i f_i(x). \tag{5.4.$\overline{u}$}$$

The control function $\overline{u} = (u_1, u_2, \cdots, u_m)$ belongs to the set $U^m \equiv \cup_{r\geq 0} W_r^m$ where W_r^m is given by

$$W_r^m = L^\infty([0,r], \mathbf{R}^m).$$

In addition, if we are given an analytic **observable**, *i.e. a non-constant analytic map $h : M \to \mathbf{R}$, then the process will be denoted by $(M, f_0, f_1, \cdots, f_m, h)$.*

Now fix any $T > 0$, (in proving Theorem 4.2 this T will be the one that comes from the flow box (Σ, T)). In the following discussion our standing hypothesis (Hypothesis-C_T) is:

$$\begin{aligned} &Hypothesis - C_T : \text{for every } (x, \overline{\eta}) \in M \times W_T^m \text{ the solution } \xi_{x,\overline{\eta}}(t) \\ &\qquad \text{to } (5.4.\overline{\eta}) \text{ exists for all } t \in [0,T]. \end{aligned}$$

For a fixed $x \in M$, consider the evaluation map:

$$E_x : W_T^m \to M : E_x(\overline{\eta}) = \xi_{x,\overline{\eta}}(T).$$

In our analysis understanding the variation of this map is very crucial. Thus, (for a fixed $x \in M$), consider $dE_x(\overline{\eta})$ - the Frechet differential of the map E_x at $\overline{\eta} \in W_T^m$. Note that $dE_x(\overline{\eta})$ is a linear map

$$dE_x(\overline{\eta}) : W_T^m \to T_{E_x(\overline{\eta})}M,$$

where $T_m M$ denotes the tangent space to M at m. Given a $\overline{v} = (v_1, v_2, \cdots, v_m) \in W_T^m$, the value $dE_x(\overline{\eta})(\overline{v})$ of $dE_x(\overline{\eta})$ at $\overline{v}$ can be described by the following control process (on the tangent bundle TM) obtained as the first vatiation of the process $(5.4.\overline{\eta})$ along the solution $\xi(t) \equiv \xi_{x,\overline{\eta}}(t)$:

$$(5.5.\overline{\eta}.\overline{v}) \qquad \begin{aligned} \xi' &= f_0(\xi) + \sum_{i=1}^m \eta_i(t) f_i(\xi) \\ \mathcal{Y}' &= \Big(\frac{\partial f_0}{\partial x} + \sum_{i=1}^m \eta_i(t) \frac{\partial f_i}{\partial x}\Big)\Big|_{(\xi(t))} \mathcal{Y}(t) + \sum_{i=1}^m v_i(t) f_i(\xi(t)), \end{aligned}$$

where $\eta = (\eta_1, \cdots, \eta_m)$. Let $\mathcal{Y}_{x,\overline{\eta},\overline{v}}(t)$ be the solution of above differential equation with initial condition $\xi(0) = x$ and $\mathcal{Y}_{x,\overline{\eta},\overline{v}}(0) = 0$. Then,

$$dE_x(\overline{\eta})(\overline{v}) = \mathcal{Y}_{x,\overline{\eta},\overline{v}}(T). \tag{5.6}$$

Now we consider a non-linear version of the stong accessibility condition introduced in Definition 1.1.

5.2 Definition. *Let M be a C^∞ manifold, $x \in X$ and $\mathcal{F}$ be a family of C^∞ vector fields on M. Let L be the Lie algebra generated by elements of $\mathcal{F}$ and let L_0 be the ideal of L generated by the 'difference set' $\mathcal{F} - \mathcal{F} \equiv \{f - g \mid f, g \in \mathcal{F}\}$.*

(I) The family $\mathcal{F}$ is said to have **accessibility property** *(AP) at x if $L_x = T_x M$, and the* **strong accessibility property** *(SAP) at x if $(L_0)_x = T_x M$.*

(II) A control process given by $(5.4.\overline{u})$ is said to have strong accessibility property (or SAP) if

$$(L_0)_x = T_x M, \quad \textit{for all } x \in M,$$

where L_0 is the ideal of L generated by the vector fields $f_1, f_2, \cdots, f_m$ and L is the Lie algebra generated by the vector fields $f_0, f_1, f_2, \cdots, f_m$.

5.3 Remark (1) Notice that if the control process $(5.4.\overline{u})$ has SAP, then the family $\mathcal{F} = \{f_0 + \sum_{i=1}^m u_i f_i \mid u_i \in \mathbf{R}\}$ of vector fields on M has SAP. Thus the notions defined in (I) and (II) above are consistent.

(2) The assumption in Theorem (A) that the set Q has SAP implies that the system (5.3) (to which we are going to apply our control theoretic results) also has SAP. To see this, observe (going back to step (2)) that each F_i ($i \leq i \leq m$) belongs to the vector space generated by $Q - Q$ and on the other hand every element of $Q - Q$ is in the span of F_i's. Thus the ideal generated by $Q - Q$ is same as the one generated by F_i's.

With the above notation, we state the main control theoretic result that will yield Theorem 4.2

5.4 Theorem. *Consider a analytic control process given by* (5.4.$\overline{u}$) *and satisfying Hypothesis -C_T. Suppose this process has SAP. Then given any $\varepsilon > 0$, and a compact set $K \subset M$, there exists a C^∞ control $\overline{u} = (u_1, u_2, \cdots, u_m) \in W_T^m$ with $Sup|\overline{u}| < \varepsilon$ and a finite set $V \subset W_T^m$ of smooth functions such that*

$$Span\{\mathcal{Y}_{z,\overline{u},\overline{v}}(T) \mid \overline{v} \in V\} = T_{E_z(\overline{u})}M \quad \textit{(uniformly) for all } z \in K.$$

Proof of Theorem 4.4

We recall that we are given a map $A_0 \in C(\Omega, Q)$ and $\varepsilon > 0$. First, we shall only consider perturbing A_0 on the set $\Sigma \times [0, T]$ which we have identified with a subset of Ω. In step (4) we have already obtained a perturbation A_2 of A_0 that is within $\frac{\varepsilon}{2}$ of A_0 and such that any further perturbation of A_2 (and hence subsequently that of A_0) can be viewed as a (non-autonomous) control process

$$\begin{aligned} X' &= G_0(z,\tau)X + \sum_{i=1}^{m} u_i(t)F_iX \\ z' &= 0 \\ \tau' &= 1, \end{aligned} \tag{5.3}$$

where a control function $\overline{u} = (u_1, \cdots, u_m)$ with values in the ball $B \subset \mathbf{R}^m$ can be thought as a perturbation that is within ε of A_0 (of course on the set Σ).

Now we apply Theorem 5.4 to the control system (5.3). First notice that the system (5.3) satisfies Hypothesis-C_T because it is "essentially" a right invariant system on a Lie group. Next, taking $M = G \times \mathbf{R}^{m(k+1)} \times \mathbf{R}$ and the compact subset to be $\{e_G\} \times K \times \{0\} \equiv \{e_G\} \times \{\overline{\theta}(\sigma) \mid \sigma \in \Sigma\} \times \{0\}$, by applying Theorem 5.4 we get a C^∞ control function $\overline{u}(t) = (u_1(t), \cdots, u_m(t)) \in W_T^m$ with $\sup|\overline{u}| < \frac{\varepsilon}{4}$ and a finite set $V = \{\overline{v}^1, \overline{v}^2, \cdots, \overline{v}^q\} \subset W_T^m$ of C^∞ functions (for some $q \in \mathbf{N}$), such that the conclusion of Theorem 5.4 holds.

Now for $(\alpha_1, \cdots, \alpha_q) \in \mathbf{R}^q$ and $z = \overline{\theta}(\sigma) \in K$ consider the following initial value problem:

$$\begin{aligned} X' &= G_0(\xi,\tau)X + \sum_{i=1}^{m} u_i(t)F_iX + \Big[\sum_{i=1}^{q}\sum_{l=1}^{m} \alpha_i v_l^i(t)F_l\Big]X, \\ X(0) &= e_G,\ \xi(0) = z,\ \tau(0) = 0, \end{aligned} \tag{5.7}$$

where $\overline{v}^i(t) = (v_1^i(t), \cdots, v_m^i(t)) \in \mathbf{R}^m$. Note that $A_2(T_t(\sigma)) = G_0(\overline{\theta}(\sigma), t)$. Now set

$$A_3(T_t(\sigma)) = A_2(T_t(\sigma)) + \sum_{i=1}^{m} u_i(t) F_i, \quad (\sigma, t) \in \Sigma \times [0, T].$$

We shall see that A_3 will be our desired perturbation (i.e. perturbation A in Theorem 4.2). Since $\sup |\overline{u}| < \frac{\varepsilon}{4}$, $|\overline{u}|$ is B valued and hence A_3 is $Int(Q)$ valued and within ε of A_0- atleast over $\Sigma \times [0, T]$. The values outside this set are not going to matter anyway. Of course we need to keep it Q valued and within ε of A_0 on the rest of Ω.

Let $t \to X(\alpha_1, \cdots, \alpha_q, \sigma)(t)$ denote the solution to the initial value problem (5.7), that is to the problem

$$X' = A_3(T_t(\sigma))X + \big[\sum_{i=1}^{q} \sum_{l=1}^{m} \alpha_i v_l^i(t) F_l\big] X$$
$$X(0) = e_G.$$

For a fixed $\sigma \in \Sigma$, define a map $\rho_\sigma : B \to G$ by setting

$$\rho_\sigma(\alpha_1, \cdots, \alpha_q) = X(\alpha_1, \cdots, \alpha_q, \sigma)(T),$$

where B is the ball chosen in step (4). Notice that,

$$\frac{\partial \rho_\sigma}{\partial \alpha_j}(0, \cdots, 0) = \mathcal{Y}_{\sigma, \overline{u}, \overline{v}^j}(T), \quad 1 \leq j \leq q,$$

i.e. the linear transformation $D\rho_\sigma(\overline{0})$ is onto for all $\sigma \in \Sigma$.

Let W_r denote the open ball centered at $\overline{0}$ in $\mathbf{R}^q$. Next, pick $\varepsilon_1 > 0$ such that the following two conditions hold.

(1) the perturbation $(\sigma, t) \to A_3(T_t(\sigma)) + \sum_{i=1}^{q} \sum_{l=1}^{m} \alpha_i v_l^i(t) F_l$ remains $Int(Q)$ valued for all $(\sigma, t) \in \Sigma \times [0, T]$ and $\overline{\alpha} \in W_{\varepsilon_1}$;
(2) $D\rho_\sigma(\overline{\alpha})$ is onto for all $\overline{\alpha} \in W_{\varepsilon_1}$.

We have to show that A_3 satisfies (2) of Theorem 4.2. Before giving the precise details, first we sketch the idea behind the proof. Suppose we are given a $\varphi \in C(\Sigma, G)$ 'sufficiently close' to $\varphi_0 \equiv \Psi^T(A_3)$, (i.e. $\varphi_0(\sigma) = \rho_(\sigma)(\overline{0})$). We want to construct a perturbation A_4 of A_3 such that $\Psi^T(A_4) = \varphi$.

For each $\sigma \in \Sigma$, set

$$X_\sigma = \log\ (\varphi(\sigma)\varphi_0(\sigma)^{-1}). \tag{5.8}$$

Then X_σ is an element of the Lie algebra L and can be thought as a right invariant vector field on G. Now consider the map $\mathcal{P}_\sigma : W_{\varepsilon_1} \subset \mathbf{R}^q \to G$ defined by

$$\mathcal{P}_\sigma(\overline{\alpha}) = \rho_\sigma(\overline{\alpha})\rho_\sigma(\overline{0})^{-1}, \quad \sigma \in \Sigma.$$

Notice that

(1) $\mathcal{P}_\sigma(\overline{0}) = e_G$ for all $\sigma \in \Sigma$ and
(2) $D\mathcal{P}_\sigma(\overline{\alpha}) : T_{\overline{\alpha}}\mathbf{R}^q \to T_{\mathcal{P}_\sigma(\overline{\alpha})}(G)$ is onto for all $\sigma \in \Sigma$.

The second fact allows us to define (for each $\sigma \in \Sigma$) a smooth vector field Y_σ on W_{ε_1} by setting

$$Y_\sigma(\overline{\alpha}) = inv(D\mathcal{P}_\sigma)(\alpha), \tag{5.9}$$

where $inv(D\mathcal{P}_\sigma)(\alpha) \equiv (D\mathcal{P}_\sigma)(\alpha)^*\left[(D\mathcal{P}_\sigma)(\alpha)(D\mathcal{P}_\sigma)(\alpha)^*\right]^{-1}$ is the so called Moore Penrose psudo-inverse of the operator $(D\mathcal{P}_\sigma)(\alpha)$. (We recall that given an onto linear transformation $A : V_1 \to V_2$ between finite dimensional vector spaces V_1 and V_2, $inv(A) \equiv A^*(AA^*)^{-1}$ is a well defined operator from V_2 to V_1 such that $A(inv(A)) = Id$-the identity map on V_2.) This vector field Y_σ 'projects' onto the vector field X_σ on G under the derivative map $D\mathcal{P}_\sigma$, $(\sigma \in \Sigma)$, i.e

$$(D\mathcal{P}_\sigma)(Y_\sigma) = X_\sigma. \tag{5.10}$$

Then the map $\zeta : \Sigma \to W_{\epsilon_1} \subset \mathbf{R}^q$ defined by

$$\zeta(\sigma) = \overline{0}e^{Y_\sigma} \tag{5.11}$$

has the property

$$\mathcal{P}_\sigma(\zeta(\sigma)) = e_G e^{X_\sigma}. \tag{5.12}$$

That is

$$\rho_\sigma(\zeta(\sigma))\rho_\sigma(\overline{0})^{-1} = e_G e^{X_\sigma} = \varphi(\sigma)\varphi_0(\sigma)^{-1} = \varphi(\sigma)\rho_\sigma(\overline{0})^{-1}.$$

In other words,

$$\rho_\sigma(\zeta(\sigma)) = \varphi(\sigma). \tag{5.13}$$

This shows that if $\zeta(\sigma)_i$ is the ith component of the vector $\zeta(\sigma) \in \mathbf{R}^q$, then the perturbation A_4 defined by

$$A_4(T_t(\sigma)) = A_3(T_t(\sigma)) + \Big[\sum_{i=1}^{q}\sum_{l=1}^{m}\zeta(\sigma)_i v_l^i(t)F_l\Big]$$

satisfies $\Psi^T(A_4) = \varphi$. This is the general outline of the proof. In the following we shall provide the precise technical details of the above argument.

First select $\nu > 0$ such that if $Y \equiv Y_{\overline{\alpha}}$ is any (smooth) vector field on the ball W_{ϵ_1} with $||Y_{\overline{\alpha}}|| < \nu$, then $\overline{0}e^Y \in W_{\epsilon_1}$.

Next, select $\kappa > 0$ such that the linear transformation $inv(D\mathcal{P}_\sigma)(\overline{\alpha})$ maps any vector of norm less than κ into a vector of norm less than ν for all $\overline{\alpha} \in W_{\epsilon_1}$ and for all $\sigma \in \Sigma$.

Finally select $\delta > 0$ such that the δ-ball N_δ centered at the identity e_G of G has the following properties:

(1) $N_\delta \subseteq \mathcal{P}_\sigma(W_{\epsilon_1})$ for all $\sigma \in \Sigma$. (Since $(D\mathcal{P}_\sigma)(\overline{0})$ is onto, this follows from (a uniform version) of the rank theorem).

(2) $g \to log\ g$ is well defined on N_δ and $||log(g)|| < \kappa$ for all $g \in N_\delta$.

Now let $\varphi \in C(\Sigma, G)$ be a given map such that $D(\varphi, \phi_0) < \delta$ where $\varphi_0 = \Psi^T(A_3)$. Then by our choice of δ, the vector field X_σ on G defined by (5.8) is actually well defined (for each $\sigma \in \Sigma$). Furthermore the vector field Y_σ (defined by (5.9)) is also well defined and $\sigma \to Y_\sigma$ is continuous. Subsequently, the map $\zeta : \Sigma \to W_{\epsilon_1}$ (defined by 5.11) is also well defined, continuous and satisfies (5.12) and (5.13). Now all that remains is to extend, first A_3 and then A_4 continuously to all of Ω, (keeping them Q valued and within ε of A_0 and A_3 respectively and this can be done with Tietze's extension theorem. $\square$

§6. Universal detectors and the Proof of Theorem 5.4.

The conclusion in Theorem 5.4 is about "uniform" submersivity of the endpoint map (uniformity is in the sense that we want the conclusion to hold for all $z \in K$). Now we shall introduce a Control Theoretic concept of "a detectible state". It turns out that submersivity of the endpoint map can be deduced from the information about detectible states of a suitable control process. We begin with the following definition.

6.1 Definition. *Consider the analytic control process $(M, f_0, \cdots, f_m, h)$ given by $(5.4.\overline{u})$. We shall assume Hypothesis -C_T for some fixed $T > 0$, and use the notation developed thereafter.*
(1) A point $x \in M$ is **detectable** *by a control $\overline{\eta} \in W_T^m$ if there is some $t \in [0, T]$ such that $h(\xi_{x,\overline{\eta}}(t)) \neq 0$. In this case we say that $\overline{\eta}$* **detects** *x.*
(2) A control $\overline{\eta} \in W_T^m$ is called a **universal detector** *for the Control Process $(M, f_0, \cdots, f_m, h)$ if it detects every detectable x.*

We shall show that for analytic control processes having SAP, universal detectors not only exist but are in fact 'generic'. This is proved by first introducing a "localized version" of detectable states and then reducing the problem to finding a "localized version" of universal detectors. The localized version of the concepts introduced in Definition 6.1 is formulated using the notion of k-jets as follows.

6.2 Definition. *(1) Let $J^{k,m}$ denote the set of k-jets at 0 of smooth maps from $\mathbf{R}$ to $\mathbf{R}^m$, where $k \in \{0\} \cup \mathbf{N} \cup \{\infty\}$. We shall identify $J^{k,m}$ with $\mathbf{R}^{(k+1)m}$. At times we may allow k to be -1 with the understanding that the only -1-jet is $\emptyset$.*
(2) For $l \geq k$, let $\pi^{l,k} : J^{l,m} \to J^{k,m}$ and $\pi^l : J^{\infty,m} \to J^{l,m}$ denote the canonical projection, i.e.

$$\pi^{l,k}(\overline{\beta}) \equiv \pi^{l,k}(\beta_1, \cdots \beta_k, \beta_{k+1}, \cdots \beta_l) = (\beta_1, \cdots \beta_k),$$
$$\pi^l(\overline{\beta}) = (\beta_1, \cdots \beta_l).$$

Given $l \geq k$, $\alpha \in J^{k,m}$ and $\beta \in J^{l,m}$, we shall write $\beta \geq \alpha$ to mean $\pi^{l,k}(\beta) = \alpha$.
(3) Given a real analytic control process $(M, f_0, \cdots, f_m, h)$ as in $(5.4.\overline{u})$, $\alpha \in J^{k,m}$ and $x \in M$, let $\sigma_{k,m}(\alpha, x)$ denote the $k+1$-jet at 0 determined by the map $t \to h(\xi_{x,\overline{\eta}}(t))$, where $\overline{\eta}$ is a smooth input map whose k-jet at 0 is α. Notice that $\sigma_{k,m} : J^{k,m} \times M \to J^{k+1,m}$ is well defined.
(4) A point $x \in M$ is **detectable** *by a jet $\alpha \in J^{k,m}$ if $\sigma_{k,m}(\alpha, x) \neq \overline{0}$. In this case we say that α* **detects** *x.*

(5) A jet $\alpha \in J^{k,m}$ is called a **universal detector** *for the Control Process $(M, f_0, \cdots, f_m, h)$ if it detects every detectable x.*

6.3 Proposition. *Consider a real analytic control process $(M, f_0, f_1, \cdots, f_m, h)$ satisfying Hypothesis-C_T. Let $x \in M$ and suppose that the proces has SAP at x. Then there exists a control $\overline{\eta} \in W_T^m$ that detects x.*

To prove Proposition 6.3 we need to recall the following lemma (a proof of which will be given in the appendix).

6.4 Lemma. *Let M be a connected C^∞ manifold. Let $x \in X$ and $T > 0$ be given. Let $\mathcal{F}$ be a family of C^∞ vector fields on M satisfying SAP at x. Then there exists a $\mu \in \mathbf{N}$ and a μ-tuple of vector fields $X = (X_1, \cdots, X_\mu)$, where $X_i \in \mathcal{F}$ such that the map $\theta_T^X : H_\mu \to M$ defined by*

$$\theta_T^X(\overline{t}) \equiv \theta_T^X(t_1, \cdots, t_\mu) = xe^{t_1 X_1} \cdots e^{t_\mu X_\mu}$$

is submersive for almost all $\overline{t} \in H_\mu$, where $H_\mu = \{\overline{t} = (t_1, \cdots, t_\mu) \mid \sum_{i=1}^{m} t_i = T\}$.

Proof of Proposition 6.3

Fix $x_0 \in X$. Consider the family of vector fields $\mathcal{F} = \{f_0 + \sum_{i=1}^{m} u_i f_i \mid u_i \in \mathbf{R}\}$. Since the process process has SAP at x_0, by Remark 5.3 (1) $\mathcal{F}$ has SAP at x_0. Thus by the previous lemma we can pick $\overline{t} = (\tilde{t}_1, \cdots, \tilde{t}_\mu) \in H_T$ with $\tilde{t}_j > 0$ and $\overline{u}^j \equiv (u_1^j, \cdots, u_\mu^j) \in \mathbf{R}^\mu$ for $1 \leq j \leq m$, such that if $X_j = f_0 + \sum_{i=1}^{m} u_i^j f_i$ and $X = (X_1, \cdots, X_\mu)$, then the derivative $d(\theta_T^X)$ at $\overline{t}$ is an onto linear transformation.

Now for any $\overline{s} = (s_1, \cdots, s_\mu) \in H_T$ near $\overline{t}$, define a piecewise constant control $\eta_{\overline{s}}$ as follows: First divide the interval $[0, T]$ into μ succesive subintervals of lenght s_j. on the interval of lenght s_j define $\eta_{\overline{s}} : [0, T] \to \mathbf{R}^m$ to be the constant map taking value $(u_1^j, \cdots, u_m^j)$. Notice that if $r \to \gamma^{\overline{s}}(r)$ denotes the trajectory of our control process corresponding to the control $\eta_{\overline{s}}$ and with initial condition x_0, then the $\theta_T^X(\overline{s}) = \gamma^{\overline{s}}(T)$. Hence by submersivity of the map θ_T^X at $\overline{t}$, we conclude that the image (under θ_T^X) of some open neighbourhood-say V_1 of $\overline{t}$ contains an open neighbourhood-say V_2 of the point $\gamma^{\overline{t}}(T)$. Since h is non-constant analytic, it is not identically zero on V_2 and hence there exists some $\overline{s} \in V_1$ such that $h(\gamma^{\overline{s}}(T)) \neq 0$, i.e. $\eta^{\overline{s}}$ detects x_0. $\square$

Now we shall "localize" the conclusion of the above Proposition in the following lemma.

6.5 Lemma. *Consider a real analytic control process as in Proposition 6.3. Let $x \in M$ be detectable by some input $\overline{\eta} \in W_T^m$. Then given any $\alpha \in J^{k,m}$, there exists some $l \geq k$ and $\beta \in J^{l,m}$ such that*

(1) *$\alpha \leq \beta$ and*
(2) *β detects x.*

Proof. The proof is based on a simple approximation argument. Let $\overline{\eta}_j \in W_T^m$ be polynomials such that $\overline{\eta}_j \to \overline{\eta}$ in $L^1[0,T]$. Let $\psi : [0,T] \to \mathbf{R}^m$ be a polynomial whose k-jet at 0 is α. Let $\varphi(t) = e^{-t^{2(k+1)}}$. Consider the sequence of inputs in W_T^m defined by,

$$\tilde{\eta}_j(t) = \varphi(jt)\psi(t) + (1 - \varphi(jt))\overline{\eta}_j, \quad j \in \mathbf{N}.$$

Then $\tilde{\eta}_j \to \overline{\eta}_j$ in $L^1[0,T]$ and each $\tilde{\eta}_j$ is a real analytic input map whose k-jet at 0 is α. Hence by the Gronwall's lemma the trajectory $\gamma_j(t) \equiv \xi_{x,\tilde{\eta}_j}(t)$ converges to $\gamma(t) \equiv \xi_{x,\overline{\eta}}(t)$ uniformly on $[0,T]$. Since $h(\gamma(t_0)) \neq 0$ for some $t_0 \in [0,T]$, it follows that $h(\gamma_{j_0}(t_0)) \neq 0$ for sufficiently large j_0. Now, since the input map $\tilde{\eta}_j$ is real analytic, it follows that the l-jet of the map $t \to h(\gamma_{j_0}(t))$ is not (identically) zero for some $l \geq k$. So if β is the l jet of $\tilde{\eta}_{j_0}$, then $\alpha \leq \beta$ and β detects x in the 'jet sense'. □

Now we prove the main result about universal detectors.

6.6 Theorem. *Consider a real analytic control process $(M, f_0, \cdots, f_m, h)$ satisfying Hypothesis-C_T and having SAP. Then given any $\alpha \in J^{\infty,m}$, $r, k \in \mathbf{N}$ ($r < k$), and $\varepsilon > 0$ there exists a $\beta \in J^{\infty,m}$ such that*

(1) $\pi^r(\beta) = \pi^r(\alpha)$,
(2) $|\beta_j - \alpha_j| < \varepsilon_j$ *for all* $1 \leq j \leq k$ *and*
(3) β *detects every detectable state in* M.

We begin the proof of Theorem 6.6 with the following lemma.

6.7 Lemma. *Let $(M, f_0, \cdots, f_m, h)$ be a real analytic control process that satisfies Hypothesis-C_T and has SAP. Let $\gamma \in J^{\infty,m}$, $r, k \in \mathbf{N}$ ($r < k$), $\varepsilon > 0$ and a compact subanalytic set $K \subset M$ be given. For any $l > r$ set*

$$\tilde{S}^l = \{\beta \in J^{l,m} \mid \pi^{l,r}(\beta) = \pi^r(\gamma) \textit{ and } |\gamma_i - \beta_i| < \varepsilon \textit{ for } i = r+1 \textit{ to } l\} \textit{ and}$$

$$S^l = \{(\beta, x) \in J^{l,m} \times K \mid \beta \in \tilde{S}^l \textit{ and } \sigma_{l,m}(\beta, x) = 0\}.$$

Then there exists a $k' \geq k$ such that $\tilde{S}^{k'} \backslash p_1(S^{k'})$ is open and dense in $\tilde{S}^{k'}$, where $p_1 : J^{k',m} \times M \to J^{k',m}$ is the projection on the 'first co-ordinate' (i.e. $p_1(\beta, x) = \beta$).

Proof. It suffices to show that there exists a $k' \geq k$ such that $\text{co-dim}_{J^{k',m} \times M} S^{k'}$-the codimension of $S^{k'} \times M$ in $J^{k',m} \times M$ is (strictly) greater than $\text{co-dim}_{J^{k,m} \times M} S^k$.

Since S^k is subanalytic, write it as a union

$$S^k = S_1^k \cup \cdots \cup S_q^k,$$

where each S_i^k is a connected, embedded analytic submanifold of $J^{k,m} \times M$.

For each $i \in \{1, \cdots q\}$ pick $(\beta^i, x^i) \in S_i^k$ and then find $k'(i)$ and $\tilde{\beta}^i \in J^{k'(i),m}$ such that

(1) $\tilde{\beta}^i$ is obtained by extending β^i, i.e. $\pi^{k',k}(\tilde{\beta}^i) = \beta^i$) and
(2) $\tilde{\beta}^i$ detects x^i.

This is possible by Lemma 6.5. Let $k' = \text{Max}(k'(1), \cdots, k'(q))$. Now we show that this k' satisfies the conclusion of the lemma.

The condition (2) above, guarantees that the set

$$\{(\beta, x) \in J^{k',m} \times M \mid \pi^{k',k}(\beta) \in \tilde{S}_i^k \text{ and } \sigma_{k',m}(\beta, x) = 0\}$$

is a **proper** analytic subset of $p_1^{-1}(S_i^k)$. Hence,

$$\dim\big(S^{k'} \cap p_1{}^{-1}(S_i^k)\big) < \dim\big(p_1{}^{-1}(S_i^k)\big).$$

Furthermore,

$$S^{k'} \subseteq p_1{}^{-1}(S^{k'}) = \cup\{p_1{}^{-1}(S_i^k) \mid 1 \le i \le q\}.$$

Thus,

$$\dim(S^{k'}) < \dim\big(p_1^{-1}(S^k)\big),$$

and subsequently,

$$\text{co-dim}_{J^{k',m} \times M}(S^{k'}) > \text{co-dim}_{J^{k',m} \times M} p_1^{-1}(S^k) = \text{co-dim}_{J^{k,m} \times M}(S^k).$$

The last equality follows from the fact that $p_1 : J^{k',m} \times M \to J^{k,m} \times M$ is the canonical projection. Now the proof follows. $\Box$

Proof of Theorem 6.6

For a fixed $r \in \mathbf{N}$ and $\alpha \in J^{r,m}$ consider the set

$$\mathcal{J}(\alpha, r) = \{\beta \in J^{\infty,m} \mid \pi^r(\beta) = \alpha\} \subseteq J^{\infty,m}.$$

Let $K \subset M$ be a compact subanalytic set. Since in Lemma 6.7 $r \in \mathbf{N}$, $\gamma \in \mathcal{J}$ and $\varepsilon > 0$ are arbitrary, we have actually shown that $\mathcal{D}(K)$ is dense in $\mathcal{J}$, where

$$\mathcal{D}(K) = \{\beta \in \mathcal{J} \mid \beta \text{ detects every detectable } x \in K\}.$$

Now Let $\{K_n\}_{n \in \mathbf{N}}$ be a sequence of compact subanalytic subsets of M such that

(1) $K_n \subseteq \text{Int}(K_{n+1})$, for all $n \in \mathbf{N}$ and
(2) $\cup_{n \in \mathbf{N}} K_n = M$.

From the Baire category theorem it follows that the set $\cap_{n \in \mathbf{N}} \mathcal{D}(K_n)$ is residual (in particular dense) in $\mathcal{J}$. Note that any $\beta \in \cap_{n \in \mathbf{N}} \mathcal{D}(K_n)$ detects every detectable state in M. $\Box$

6.8 Corollary. *Let $(M, f_0, \cdots, f_m, h)$ be a real analytic control process that satisfies Hypothesis-C_T and having SAP. Let given any $\varepsilon > 0$ and $r \in \mathbf{N}$, there exists a C^r universal detector $\overline{\eta}$ such that $||\overline{\eta}||_{C^r} < \varepsilon$, where $||\cdot||_{C^r}$ denotes the C^r norm on $C^r[0,T]$.*

Proof of Theorem 5.4

Consider the control process defined on $(TM)^n$ (-the n-fold 'fiber product' of the tangent bundle TM) given by the following equations:

$$\begin{array}{c} x' = f_0(x) + \sum\limits_{i=1}^{m} u_i(t) f_i(x) \\ \mathcal{Y}_1' = \left(\frac{\partial f_0}{\partial x} + \sum\limits_{i=1}^{m} u_i(t)\frac{\partial f_i}{\partial x}\right)\Big|_{(\xi(t))} \mathcal{Y}_1(t) + \sum\limits_{i=1}^{m} v_i^1(t) f_i(\xi(t)) \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{Y}_n' = \left(\frac{\partial f_0}{\partial x} + \sum\limits_{i=1}^{m} u_i(t)\frac{\partial f_i}{\partial x}\right)\Big|_{(\xi(t))} \mathcal{Y}_n(t) + \sum\limits_{i=1}^{m} v_i^1(t) f_i(\xi(t)) \end{array},$$

where $n = \dim(M)$ and $\overline{u} = (u_1, \cdots, u_m), \overline{v}^i = (v_1^1, \cdots, V_m^i)$, $1 \leq i \leq n$, are the control functions. Consider the observable $h : (TM)^n \to \mathbf{R}$ defined by

$$h(x, \mathcal{Y}_1, \cdots, \mathcal{Y}_n) = det(\mathcal{Y}_1, \cdots, \mathcal{Y}_n).$$

Notice that since the process $(5.4.\overline{u})$ has SAP, theabove control process has SAP at $(x, 0, \cdots, 0)$ for every $x \in K$. Hence by Proposition 6.3 every point $(x, 0, \cdots, 0)$, $x \in K$, is detectable by some control. Thus by Corollary 6.8 there exists a universal detector $(\overline{\eta}, \overline{\omega}^1, \cdots, \overline{\omega}^n)$ that detects every point $(x, o, \cdots, 0)$, $x \in K$ with $\mathrm{Sup}||\overline{\eta}|| < \varepsilon$. Thus for each $x \in K$, the set $\{\mathcal{Y}_{x,\overline{\eta},\overline{\omega}^j}(t) \mid 1 \leq j \leq n\}$ is a linearly independent set of vectors for some $t_0 \in [0,T]$. Let $\overline{\hat{\omega}}^j = \overline{\omega}^j \chi_{[0,t_0]}$. Then it follows that, for each $x \in K$ the set $\mathcal{Y}_{x,\overline{\eta},\overline{\hat{\omega}}^j}(T) \mid 1 \leq j \leq n\}$ is a linearly independent set of vectors. Approximating the $\overline{\hat{\omega}}^j$'s by polynomials, we now have for each $x \in K$ a finite set $\mathcal{C}_x$ of polynomials such that the set of vectors $\{\mathcal{Y}_{x,\overline{\eta},\overline{\omega}(T)} \mid \overline{\omega} \in \mathcal{C}_x\}$ span $T_{E_x(\eta)}M$. Using continuity of the map $x \to dE_x(\eta)$ and compactness of K, the result follows. $\square$

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